Semilinear Parabolic Differential Inclusions with One-sided Lipschitz Nonlinearities

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Abstract

We present an existence result for a partial differential inclusion with linear parabolic principal part and relaxed one-sided Lipschitz multivalued nonlinearity in the framework of Gelfand triples. Our study uses discretizations of the differential inclusion by a Galerkin scheme, which is compatible with a conforming finite element method, and we analyze convergence properties of the discrete solution sets.

Key words. Partial differential inclusion, one-sided Lipschitz condition, Galerkin method.
AMS subject classification. primary: 35R70, 65M60; secondary: 35K20, 35K91, 49J53.

1 Introduction

We consider the inital value problem for semilinear partial differential inclusions of the form

$$u'(t) + Au(t) \in F(t, u(t)) \text{ for } t \in (0,T), \quad u(0,\cdot) = u_0, \tag{1}$$

in a Gelfand triple $V \subseteq H \subseteq V^*$ with a strongly positive, linear, and bounded operator $A : V \to V^*$ and a genuinely set-valued nonlinearity

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$F : V \rightrightarrows H$ with closed, bounded, and convex images. In contrast to partial differential equations with maximal monotone principle part (see [2, 7]), differential inclusions of type (1) possess a nontrivial solution set. They may be considered the deterministic counterparts of stochastic partial differential equations since they model deterministic uncertainty by a set-valued operator. Likewise, they provide a framework for the analysis of control systems of partial differential equations with control constraints (see e.g. [6, 25]).

In the present paper, we discuss a weak reformulation

$$\langle u'(t), v\rangle + a(u(t), v) = (f(t), v) \quad \dot\forall\, t \in (0, T),\ \forall\, v \in V, \tag{2a}$$
$$f(t) \in F(t, u(t)) \quad \dot\forall\, t \in (0, T), \tag{2b}$$
$$u(0) = u_0, \tag{2c}$$

of inclusion (1), where $(\cdot, \cdot) : H \times H \to \mathbb{R}$ is the inner product in H, the duality pairing between V^* and V is denoted by $\langle \cdot, \cdot\rangle : V^* \times V \to \mathbb{R}$, the bilinear form associated with the operator A is $a : V \times V \to \mathbb{R}$, and the symbol '$\dot\forall$' means 'for Lebesgue-almost every'. For an overview of existence and uniqueness results for differential inclusions of type (2), we refer to [19]. Given a Galerkin scheme $(V_N)_{N\in\mathbb{N}}$ of finite-dimensional subspaces of V, the approximate Galerkin inclusion

$$\langle u_N'(t), v\rangle + a(u_N(t), v) = (f_N(t), v) \quad \dot\forall\, t \in (0, T),\ \forall\, v \in V_N, \tag{3a}$$
$$f_N(t) \in F(t, u_N(t)) \quad \dot\forall\, t \in (0, T), \tag{3b}$$
$$u_N(0) = u_{N,0} \in V_N, \tag{3c}$$

is compatible with standard conforming finite element approaches for parabolic partial differential equations.

Our goal is to study the convergence of the solution set of inclusion (3) to the solution set of inclusion (2) with respect to the Hausdorff metric in $L^2(0, T; H)$. The main novelty of our approach is to establish such a result for a multivalued nonlinearity satisfying a relaxed one-sided Lipschitz property (see assumption (A4) for details). This property, which goes back to Donchev (see e.g. [11]), is much weaker than standard Lipschitz or dissipativity conditions treated in [19], and it found many applications in the theory and numerical analysis of differential inclusions, see e.g. [3, 4, 12, 13, 21, 22].

2 Problem setting and main results

In this section we present the main results and specify the analytical setting underlying the differential inclusion (2) and its Galerkin approximation (3).

2.1 Preliminaries from set-valued analysis

We refer to the monographs [1] and [18] for general notions from set-valued analysis. In the following we specify some notation that will be used throughout this paper. In the following, let X and Y be normed spaces.

Definition 1. *For any $x \in X$ and any subset $M \subseteq X$, we define the distance of x to M by*

$$\operatorname{dist}(x, M)_X = \inf\{\|x - y\|_X : y \in M\}$$

and the proximal set by

$$\operatorname{proj}(x, M)_X := \{y \in M : \|x - y\|_X \leq \|x - z\|_X \ \forall z \in M\}.$$

Recall that $\operatorname{proj}(x, M)_X$ is a singleton in case M is convex and X is a Hilbert space. Moreover, by a common abuse of notation, we write

$$\|M\|_X := \sup_{x \in M} \|x\|_X \quad \text{for } M \subseteq X \text{ bounded.}$$

By $\mathcal{CBC}(X)$ we denote the set of all closed, bounded, and convex subsets of X. There are various ways of defining a topology on $\mathcal{CBC}(X)$ which in general are not equivalent. We will use convergence in the Hausdorff and in the Kuratowski sense.

Definition 2. *For any two sets $M, \widetilde{M} \subseteq X$, the Hausdorff semi-distance and the Hausdorff distance are defined by*

$$\begin{aligned} &\operatorname{dist}(M, \widetilde{M})_X := \sup_{x \in M} \operatorname{dist}(x, \widetilde{M})_X, \\ &\operatorname{dist}_{\mathcal{H}}(M, \widetilde{M})_X := \max\{\operatorname{dist}(M, \widetilde{M})_X, \operatorname{dist}(\widetilde{M}, M)_X\}. \end{aligned} \tag{4}$$

It is well-known that $\operatorname{dist}_{\mathcal{H}}$ defines a metric on $\mathcal{CBC}(X)$.

Definition 3. *A sequence $\{M_n\}_{n\in\mathbb{N}}$ of sets $M_n \subseteq X$ is said to converge to a set $M \subseteq X$ in Kuratowski sense, which is denoted by* $\operatorname{Lim}_{n\to\infty} M_n = M$*, if*

$$\text{Lim sup}_{n\to\infty} M_n \subseteq M \subseteq \text{Lim inf}_{n\to\infty} M_n,$$

where the upper and lower Kuratowski limits of a sequence $(M_n)_{n\in\mathbb{N}}$ are given by

$$\text{Lim sup}_{n\to\infty} M_n := \{x \in X : \liminf_{n\to\infty} \operatorname{dist}(x, M_n)_X = 0\},$$
$$\text{Lim inf}_{n\to\infty} M_n := \{x \in X : \lim_{n\to\infty} \operatorname{dist}(x, M_n)_X = 0\}.$$

An important situation where convergence in the sense of Kuratowski implies Hausdorff convergence is given in the following Lemma, which is a slight variation of [18, Chapter 7, Proposition 1.19].

Lemma 4. *Let $B \subseteq X$ be relatively compact and let $M_n, M \subseteq B$ for all $n \in \mathbb{N}$. If $\operatorname{Lim}_{n\to\infty} M_n = M$, then $\operatorname{dist}_{\mathcal{H}}(M_n, M)_X \to 0$ as $n \to \infty$.*

Proof. If $\operatorname{dist}(M_n, M)_X \to 0$ as $n \to \infty$ is false, there exist $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in M_n$ such that $\operatorname{dist}(x_n, M)_X > \varepsilon$ along a subsequence $\mathbb{N}' \subset \mathbb{N}$. But $x_n \in B$ for all $n \in \mathbb{N}$ implies that there exists a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $x \in \bar{B}$ such that $x_n \to x$ as $\mathbb{N}'' \ni n \to \infty$. By the Kuratowski upper limit property, we have $x \in M$, which is a contradiction.

If $\operatorname{dist}(M, M_n)_X \to 0$ as $n \to \infty$ is false, there exist $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ such that $\operatorname{dist}(x_n, M_n)_X > \varepsilon$ along a subsequence $\mathbb{N}' \subset \mathbb{N}$. By the relative compactness of M, there exists a subsequence $\mathbb{N}'' \subseteq \mathbb{N}'$ such that $x_n \to x \in \bar{M}$ as $\mathbb{N}'' \ni n \to \infty$. By the Kuratowski lower limit property, for every $n \in \mathbb{N}''$ there exist sequences $\{x_k^n\}_{k\in\mathbb{N}}$ with $x_k^n \in M_k$ such that $x_k^n \to x_n$ as $k \to \infty$. In particular, there exist $k_n \in \mathbb{N}$ with $k_n \to \infty$ as $n \to \infty$ and $\|x_{k_n}^n - x_n\|_X \le \frac{1}{n}$. But then $x_{k_n}^n \to x$ as $n \to \infty$, and hence

$$\operatorname{dist}(x_{k_n}, M_{k_n})_X \le \|x_{k_n} - x_{k_n}^n\|_X \le \|x_{k_n} - x\|_X + \|x - x_{k_n}^n\|_X \to 0,$$

which is a contradiction. □

This statement will be crucial for proving uniform convergence of Galerkin solution sets.

We adopt the notion of measurability from [1, Def. 8.1.1].

Definition 5. *Let $F : X \rightrightarrows Y$ be a multivalued mapping with closed images. Then F is called measurable if the preimage*

$$F^{-1}(M) := \{x \in X : F(x) \cap M \neq \emptyset\} \subseteq X$$

of any open set $M \subseteq Y$ is a Borel set in X.

In a finite-dimensional space, the following notion of upper semicontinuity is equivalent to the concepts in [1] and [8], as can be shown by an elementary argument and [8, Proposition 1.1]. In particular, Proposition 1.4.9 from [1] and Theorem 5.2 from [8] hold for mappings with this property.

Definition 6. *A set-valued mapping $G : \mathbb{R}^N \to \mathcal{CBC}(\mathbb{R}^N)$ is called upper semicontinuous if $x = \lim_{n\to\infty} x_n$ for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ implies*

$$\operatorname{dist}(G(x_n), G(x))_{\mathcal{CBC}(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty.$$

For further definitions and elementary facts concerning measurability of multivalued mappings, we refer to [1]. The present paper deviates from this source inasmuch as a multivalued mapping $F : X \to \mathcal{CBC}(Y)$ will be called continuous at $x \in X$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} \|x_n - x\|_X = 0$ we have

$$\operatorname{dist}_{\mathcal{H}}(F(x), F(x_n))_Y \to 0 \text{ as } n \to \infty.$$

2.2 Preliminaries from the theory of differential inclusions

The following result is an adapted version of the existence theorem [8, Theorem 5.2] for the initial value problem for a finite-dimensional differential inclusion

$$u' \in G(t, u) \quad \dot{\forall}\, t \in (0, T), \quad u(0) = u_0. \tag{5}$$

In this setting, solutions are elements of the space $\mathrm{AC}([0, T]; \mathbb{R}^N)$ of absolutely continuous functions.

Theorem 7. *Let $G : [0, T] \times \mathbb{R}^N \to \mathcal{CBC}(\mathbb{R}^N)$ be a set-valued map with the following properties:*

(i) The mapping $t \mapsto G(t, v)$ is measurable for all $v \in \mathbb{R}^N$.

(ii) *The mapping* $v \mapsto G(t,v)$ *is upper semicontinuous for almost every* $t \in (0,T)$.

(iii) *There exists* $K \geq 0$ *such that all solutions* $u \in \mathrm{AC}([0,T'];\mathbb{R}^N)$ *of inclusion* (5) *on* $(0,T')$ *with* $0 < T' \leq T$ *satisfy the a priori bound* $\|u\|_{L^\infty(0,T';\mathbb{R}^N)} \leq K$.

Then there exists a global solution $u \in \mathrm{AC}([0,T];\mathbb{R}^N)$ *of* (5).

Note that Theorem 7 differs from [8, Theorem 5.2], which requires a linear bound

$$\|G(t,v)\| \leq c(t)(1+\|v\|) \quad \dot{\forall}\, t \in (0,T),\, \forall\, v \in \mathbb{R}^N.$$

This bound is, however, exclusively used to prove a-priori bound (iii), which we assume in our setting.

We also use the following refined version of Gronwall's Lemma.

Lemma 8. *Let* $s \in \mathrm{AC}([0,T];\mathbb{R}_+)$, *let* $\kappa \in L^1((0,T);\mathbb{R})$, *and let* $\rho \in L^1((0,T);\mathbb{R}_+)$ *be such that*

$$s(t)s'(t) \leq \kappa(t)s(t)^2 + \rho(t)s(t) \quad \dot{\forall}\, t \in (0,T). \tag{6}$$

Then we have

$$s'(t) \leq \kappa(t)s(t) + \rho(t) \quad \dot{\forall}\, t \in (0,T), \tag{7}$$

$$s(t) \leq s(0)e^{\int_0^t \kappa(\tau)d\tau} + \int_0^t e^{\int_\sigma^t \kappa(\tau)d\tau}\rho(\sigma)d\sigma \quad \forall\, t \in [0,T]. \tag{8}$$

Proof. Consider the set $Z := \{t \in [0,T] : s(t) = 0\}$, and let $\tilde{Z} \subset Z$ be the subset of all density points of Z. Then Z and $\tilde{Z}$ have the same Lebesgue measure due to the Lebesgue density theorem [20, Theorem 2.2.1]. If $t \in [0,T] \setminus Z$, then inequality (7) holds, because both sides of inequality (6) can be divided by $s(t)$. If $t \in \tilde{Z}$ and $s'(t)$ exists, then $s'(t) = 0$ and inequality (7) holds, because $s(t) = 0$ and $\rho(t) \geq 0$. Thus s' satisfies inequality (7) almost everywhere in $(0,T)$, and the Gronwall Lemma yields the estimate (8). □

2.3 Function spaces

Our standing assumptions on the underlying spaces are as follows.

(S1) Let $(V, \|\cdot\|_V)$ be a separable Hilbert space, densely and compactly embedded in a Hilbert space $(H, (\cdot,\cdot), \|\cdot\|_H)$.

In particular, there exists $c_{VH} > 0$ such that

$$\|v\|_H \leq c_{VH}\|v\|_V \quad \forall\, v \in V.$$

The dual space $(V^*, \|\cdot\|_{V^*})$ of V is equipped with the norm

$$\|f\|_{V^*} = \sup_{\|v\|_V=1} |\langle f, v\rangle|,$$

where $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the duality pairing. We identify H with its dual so that $V \subseteq H \subseteq V^*$ form a Gelfand triple with

$$\langle u, v\rangle = (u, v) \quad \forall\, u \in H,\, v \in V.$$

Our assumptions on the Galerkin scheme for the inclusion (2) are summarized below.

(S2) Let $\{V_N\}_{N\in\mathbb{N}}$ be a nested sequence of finite-dimensional subspaces of V such that for all $v \in V$

$$\operatorname{dist}(v, V_N)_V \to 0 \text{ as } N \to \infty.$$

(S3) There exists $C_P > 0$ such that the H-orthogonal projection $P_N : H \to V_N$ onto V_N satisfies $\|P_N v\|_V \leq C_P\|v\|_V$ for all $v \in V$ and $N \in \mathbb{N}$.

Condition (S3) is crucial for the convergence result in Theorem 12. It holds for various types of finite element spaces under suitable geometric conditions, see [5, 15] and the references therein.

For a Banach space X, we denote the spaces of all Bochner measurable and all continuous functions from $[0, T] \subset \mathbb{R}$ to X by $\mathcal{M}(0, T; X)$ and $\mathcal{C}([0, T], X)$, respectively. We refer to [9, 23] for the general theory of Bochner-Lebesgue spaces. Several equivalent notions of measurability are discussed in [10, p.93, Proposition 12] and [9, p.42]. By $L^r(0, T; X)(1 \leq r < \infty)$ we denote the functions in $\mathcal{M}(0, T; X)$ with finite norm

$$\|u\|_{L^r(0,T;X)} = (\int_0^T \|u(t)\|_X^r dt)^{1/r}.$$

In case $X = \mathbb{R}$ we write $L^r(0, T)$ instead of $L^r(0, T; \mathbb{R})$. A function $u \in L^1_{\text{loc}}(0, T; V^*)$ has a weak derivative $u' \in L^1_{\text{loc}}(0, T; V^*)$ provided

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \forall\, \varphi \in C_c^\infty(0, T),$$

where $\varphi \in C_c^\infty(0,T)$ denotes the space of all infinitely many times continuously differentiable functions on $(0,T)$ with compact support. This relation may be written equivalently as

$$\int_0^T \varphi(t)\langle u'(t), v\rangle dt = -\int_0^T \varphi'(t)\langle u(t), v\rangle dt \;\; \forall\, v \in V, \varphi \in C_c^\infty(0,T). \quad (9)$$

Following [24, Lemma 19.1], we introduce the space

$$W_+ = L^2(0,T;V^*) + L^1(0,T;H),$$

with norm (cf. [16, Kap.IV, §1])

$$\begin{aligned}\|f\|_{W_+} = \inf\{\max(\|g\|_{L^2(0,T;V^*)}, \|h\|_{L^1(0,T;H)}) : f = g + h \\ g \in L^2(0,T;V^*),\, h \in L^1(0,T;H)\}.\end{aligned}$$

The dual of W_+ can be identified with

$$W_+^* = L^2(0,T;V) \cap L^\infty(0,T;H)$$

equipped with the norm

$$\|v\|_{W_+^*} = \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;V)}$$

and duality pairing $\langle\langle \cdot, \cdot \rangle\rangle : W_+^* \times W_+ \to \mathbb{R}$ given by

$$\langle\langle v, f\rangle\rangle = \int_0^T \Big(\langle g(t), v(t)\rangle + (h(t), v(t))\Big) dt \quad (10)$$

(cf. [16, Kap.I, §5 and Kap.IV, §1]), where $v \in W_+^*$ and $f = g + h$ with $g \in L^2(0,T;V^*)$ and $h \in L^1(0,T;H)$. We look for solutions of (2) in the space

$$W = \{u \in L^2(0,T;V) : u' \text{ exists and lies in } W_+\}$$

with norm given by

$$\|u\|_W = \|u\|_{L^2(0,T;V)} + \|u'\|_{W_+}. \quad (11)$$

Indeed, one may show that

$$W = \{u \in W_+^* : u' \in W_+\}. \quad (12)$$

From [24, Lemma19.1, p.114] one further obtains the continuous embedding

$$W \subseteq C([0,T], H), \quad (13)$$

which shows that the initial condition (2c) makes sense. Moreover, by the Lions-Aubin Theorem (see [23, Lemma 7.7]) and the compact embedding of V in H, we have for all $1 \le r < \infty$

$$W \text{ is compactly embedded into } L^r(0,T;H). \tag{14}$$

Next we consider function spaces for the Galerkin approximations. Since V_N is finite dimensional we need not distinguish topologies in the image spaces and choose our solutions to be in the space of absolutely continuous functions

$$W_N := \mathrm{AC}([0,T];V_N).$$

Note that $u_N \in W_N$ implies $u_N'(t) \in V_N$ for almost every $t \in (0,T)$ and

$$\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_H^2 = (u_N'(t), u_N(t)) \quad \dot{\forall} t \in (0,T). \tag{15}$$

Equation (3a) may now be written as

$$(u_N'(t), v) + a(u_N(t), v) = (f_N(t), v) \qquad \dot{\forall} t \in (0,T),\ \forall\, v \in V_N. \tag{16}$$

2.4 Problem data

We state our main assumptions on a and F.

(A1) The bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is strongly positive and bounded, i.e. there exist constants $c_a, C_a > 0$ such that for all $v, w \in V$

$$c_a\|v\|_V^2 \le a(v,v) \quad \text{and} \quad a(v,w) \le C_a\|v\|_V\|w\|_V. \tag{17}$$

(A2) The set-valued mapping $F : [0,T]\times V \to \mathcal{CBC}(H)$ is Carathéodory, i.e. the mapping $t \mapsto F(t,v) : [0,T] \to \mathcal{CBC}(H)$ is measurable for any $v \in V$, and for almost every $t \in (0,T)$, the mapping $v \mapsto F(t,v) : (V, \|\cdot\|_V) \to (\mathcal{CBC}(H), \mathrm{dist}_{\mathcal{H}}(\cdot,\cdot)_H)$ is continuous.

(A3) There exist a function $\alpha \in L^1(0,T)$ and a constant $c_F > 0$ such that for almost every $t \in (0,T)$ and all $u, v \in V$, we have bounds

$$\begin{aligned}&\|F(t,0)\|_H \le \alpha(t),\\ &\mathrm{dist}_{\mathcal{H}}(F(t,u),F(t,v))_H \le c_F(1 + \|u\|_V + \|v\|_V)\|u - v\|_H.\end{aligned}$$

(A4) The mapping F is relaxed one-sided Lipschitz in its second argument, i.e. there exists $\ell \in L^1(0,T)$, such that for almost every $t \in (0,T)$, all $v, \tilde{v} \in V$, and all $g \in F(t,v)$, there exists some $\tilde{g} \in F(t,\tilde{v})$ such that

$$(g - \tilde{g}, v - \tilde{v}) \leq \ell(t)\|v - \tilde{v}\|_H^2.$$

(A5) The initial values satisfy $u_{N,0} = P_N u_0$ for all $N \in \mathbb{N}$.

Note that the Lipschitz condition in (A3) is of local type and implies a stronger continuity property of $v \mapsto F(t,v)$ than (A2), namely

$$\begin{aligned} &v, v_k \in V,\ \|v_k\|_V \leq C(k \in \mathbb{N}),\ \|v_k - v\|_H \to 0 \quad \text{as} \quad k \to \infty \\ &\implies \quad \operatorname{dist}_{\mathcal{H}}(F(t,v_k), F(t,v))_H \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Moreover, condition (A3) with $u = 0$ implies the growth estimate

$$\|F(t,v)\|_H \leq \alpha(t) + c_F(1 + \|v\|_V)\|v\|_H \quad \dot{\forall} t \in (0,T),\ \forall v \in V. \tag{18}$$

2.5 Main results

The definition of solutions to (2) is straightforward.

Definition 9. *A function $u \in W$ is called a solution of* (2) *(or a weak solution of* (1)*) if u satisfies* (2c) *and there exists $f \in L^1(0,T;H)$ satisfying* (2a)*,* (2b) *for almost every $t \in [0,T]$.*

Similarly, we define a weak solution of (3).

Definition 10. *Let $N \in \mathbb{N}$ be fixed. A function $u_N \in W_N$ is called a solution of* (3)*, (or a Galerkin solution of* (2)*) if u satisfies* (3c) *and there exists some $f_N \in L^1(0,T;H)$ satisfying* (3a)*,* (3b) *for almost every $t \in [0,T]$.*

Our first result is a uniform a priori bound for solutions of the differential inclusion and of the Galerkin inclusion.

Proposition 11. *There exists a constant $K > 0$ such that all solutions $u \in W$ and $u_N \in W_N$ of inclusions* (2) *and* (3) *satisfy*

$$\|u\|_W \leq K, \quad \|u_N\|_W \leq K \quad \forall\, N \in \mathbb{N}. \tag{19}$$

The constant K depends only on the values $\|u_0\|_H$ and $\sup_{N\in\mathbb{N}} \|u_{0,N}\|_H$ and on the bounds from (S1)-(S3) and (A1)-(A4).

The proof will show that the same bounds hold for all solutions of inclusions (2) and (3) on any subinterval $[0, T']$ with $0 < T' \leq T$. Note also that the bound in W implies a bound in $L^\infty(0,T;H)$ according to (12) and (13).

From now on we denote by $\mathcal{S} = \mathcal{S}(u_0)$ the set of all solutions of (2) and by $\mathcal{S}_N = \mathcal{S}_N(P_N u_0)$ the set of all solutions of (3) with initial data given by (A5). By Proposition 11 and (S2) these sets are uniformly bounded for u_0 fixed.

For our second main result, we prove existence of solutions to (3) and use the bounds from Proposition 11 to extract a weakly convergent subsequence. The limit of this sequence is a solution to (2). This is essentially sufficient to conclude convergence of $\mathcal{S}_N$ to $\mathcal{S}$ in the upper Kuratowski sense in $L^2(0,T;H)$.

Theorem 12. *The solution sets $\mathcal{S}$ and $\mathcal{S}_N$ for all $N \in \mathbb{N}$ are non-empty, and we have* $\operatorname{Lim\,sup}_{N\to\infty} \mathcal{S}_N \subset \mathcal{S}$ *in $L^2(0,T;H)$.*

Our second main result may be viewed as convergence in the lower Kuratowski sense in W_+^*.

Theorem 13. *For every solution $u \in \mathcal{S}$, there exists a sequence $\{u_N\}_{N\in\mathbb{N}}$ with $u_N \in \mathcal{S}_N$ for all $N \in \mathbb{N}$ and*

$$\|u - u_N\|_{W_+^*} \to 0 \quad \textit{as} \quad N \to \infty.$$

As a consequence of Theorems 12 and 13, we obtain the next main result of this paper.

Theorem 14. *The sets $\mathcal{S}_N$ converge to $\mathcal{S}$ in the sense that*

$$\operatorname{dist}_{\mathcal{H}}(\mathcal{S}, \mathcal{S}_N)_{L^2(0,T;H)} \to 0 \quad \textit{as} \quad N \to \infty.$$

3 A priori estimates

Our first step is an a priori bound for solutions of the differential inclusions (2) and (3).

Proof of Proposition 11. Recall that estimate (19) requires to show

$$\|u_N\|_{L^2(0,T;V)} \leq K_0, \quad \|u_N'\|_{L^2(0,T;V^*)+L^1(0,T;H)} \leq K_0' \tag{20}$$

for suitable constants K_0, K_0' depending on $C_0 := \sup_{N\in\mathbb{N}} \|u_{N,0}\|_H$. If $u_N \in \mathrm{AC}([0,T]; V_N)$ solves inclusion (3), then according to (16), there exists a function $f_N \in L^1(0,T;H)$ with

$$(u_N'(t), v) + a(u_N(t), v) = (f_N(t), v) \quad \dot\forall t \in (0,T),\ \forall v \in V_N, \tag{21}$$
$$f_N(t) \in F(t, u_N(t)) \quad \dot\forall t \in (0,T). \tag{22}$$

Setting $v = u_N(t)$ and using (A1) and (15) gives the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_H^2 + c_a\|u_N(t)\|_V^2 \le (f_N(t), u_N(t)) \quad \dot\forall\, t \in (0,T). \tag{23}$$

Next we apply condition (A4) for fixed $t \in [0,T]$ with $v = u_N(t)$, $\tilde v = 0$, $g = f_N(t)$ and find an element $\tilde g \in F(t,0)$ such that

$$(f_N(t) - \tilde g, u_N(t)) \le \ell(t)\|u_N(t)\|_H^2.$$

When combined with (18) and (23), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_N(t)\|_H^2 + c_a\|u_N(t)\|_V^2 \le \ell(t)\|u_N(t)\|_H^2 + \alpha(t)\|u_N(t)\|_H. \tag{24}$$

Then the Gronwall Lemma 8 leads to the estimate for $t \in [0,T]$

$$\begin{aligned}\|u_N(t)\|_H \le& \|u_{N,0}\|_H e^{\int_0^t \ell(\tau)d\tau} + \int_0^t \alpha(\sigma) e^{\int_\sigma^t \ell(\tau)d\tau} d\sigma \\ \le& e^{\|\ell\|_{L^1(0,T)}}(C_0 + \|\alpha\|_{L^1(0,T)}) =: K_1.\end{aligned} \tag{25}$$

This proves $\|u_N\|_{L^\infty(0,T;H)} \le K_1$. In the next step we use this estimate and integrate (24) to obtain

$$c_a\|u_N\|_{L^2(0,T;V)}^2 \le \frac{C_0^2}{2} + K_1(K_1\|\ell\|_{L^1[0,T]} + \|\alpha\|_{L^1[0,T]}) =: C_1, \tag{26}$$

so that the first part of (20) follows with $K_0^2 = \frac{C_1}{c_a}$. We use the duality relation

$$\|u_N'\|_{W_+} = \sup_{\varphi \in W_+^*,\, \|\varphi\|_{W_+^*}=1} \langle\!\langle \varphi, u_N' \rangle\!\rangle$$

to estimate the derivative $u_N' \in L^1(0,T;V_N)$. Because of (10) and (21), we find

$$\begin{aligned}\langle\!\langle \varphi, u_N' \rangle\!\rangle =& \int_0^T (\varphi(t), u_N'(t))dt = \int_0^T (u_N'(t), P_N\varphi(t))dt \\ =& -\int_0^T a(u_N(t), P_N\varphi(t))dt + \int_0^T (f_N(t), P_N\varphi(t))dt.\end{aligned} \tag{27}$$

By (18) and (26), we arrive at the estimate

$$\begin{aligned}\|f_N\|_{L^1(0,T;H)} \leq& \|\alpha\|_{L^1(0,T)} + c_F(\sqrt{T} + \|u_N\|_{L^2(0,T;V)})\|u_N\|_{L^2(0,T;H)} \\ \leq& \|\alpha\|_{L^1(0,T)} + c_F c_{VH}(\sqrt{T} + K_0)K_0 =: C_2.\end{aligned} \tag{28}$$

Further, using (A1) and $\|P_N\varphi(t)\|_V \leq C_P\|\varphi(t)\|_V$ from (S3), we deduce from (27) and (28) that

$$\begin{aligned}|\langle\langle\varphi, u_N'\rangle\rangle| \leq& \int_0^T C_a C_P \|u_N(t)\|_V \|\varphi(t)\|_V dt + \int_0^T \|f_N(t)\|_H \|P_N\varphi(t)\|_H dt \\ \leq& C_a C_P K_0 \|\varphi\|_{L^2(0,T;V)} + C_2\|\varphi\|_{L^\infty(0,T;H)} \leq K_0'\|\varphi\|_{W_+^*}\end{aligned}$$

with $K_0' := C_a C_P K_0 + C_2$, which proves our assertion.

The bound for $\mathcal{S}$ can be obtained essentially in the same way, because according to [24, Ch. 20], functions $u \in W$ satisfy

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 = \langle u'(t), u(t)\rangle \quad \dot{\forall}\, t \in (0,T).$$

□

4 Existence of Galerkin solutions

The following Filippov-type result measures the minimal distance from a given $v_N \in W_N$ to $\mathcal{S}_N$. It is convenient to introduce the operator

$$A_N : V_N \to V_N, \quad (A_N v, w) = a(v,w) \quad \forall\, v, w \in V_N.$$

Proposition 15. *For any $v_N \in W_N$, any $u_{N,0} \in V_N$, and any function $\delta_N \in L^1(0,T)$ with*

$$\operatorname{dist}(v_N'(t) + A_N v_N(t), F(t, v_N(t)))_H \leq \delta_N(t) \quad \dot{\forall}\, t \in (0,T),$$

there exists a solution $u_N \in W_N$ of (3) satisfying

$$\|v_N - u_N\|_{L^\infty(0,T;H)} \leq C_\ell(\|v_N(0) - u_{N,0}\|_H + \|\delta_N\|_{L^1(0,T)}), \tag{29}$$

$$\begin{aligned}c_a\|v_N - u_N\|_{L^2(0,T;V)}^2 \leq& \|\delta_N\|_{L^1(0,T)}|v_N - u_N\|_{L^\infty(0,T;H)} \\ &+ \|\ell_+\|_{L^1(0,T)}\|v_N - u_N\|_{L^\infty(0,T;H)}^2 + \|v_N(0) - u_{N,0}\|_H^2,\end{aligned} \tag{30}$$

with $C_\ell = \sup_{0\leq s\leq t\leq T} \exp\left(\int_s^t \ell(\tau)d\tau\right)$ and $\ell_+(t) = \max(0, \ell(t))$.

Proof. We construct a multivalued right-hand side in such a way that any solution of the corresponding differential inclusion is also a solution of the Galerkin inclusion (3) and satisfies bounds (29) and (30). Then we invoke Theorem 7 to ensure the existence of such a solution.

Since V_N is finite-dimensional, there exists $\beta_N > 0$ such that $\|v\|_V \leq \beta_N \|v\|_H$ for all $v \in V_N$. Hence the operator $A_N : V_N \to V_N$ is continuous with bound $\|A_N v\|_H \leq \beta_N C_a \|v\|_V$. By [1, Theorem 8.2.8], the mapping $t \mapsto F(t, v_N(t))$ is measurable, and [1, Corollary 8.2.13] guarantees measurability of the function

$$f : [0,T] \to H, \quad f(t) := \operatorname{proj}(v_N'(t) + A_N v_N(t), F(t, v_N(t)))_H.$$

Consider the mapping $L_N : [0,T] \times V_N \rightrightarrows H$ given by

$$L_N(t,v) := \{g \in H : (f(t) - g, v_N(t) - v) \leq \ell(t)\|v_N(t) - v\|_H^2\}.$$

By [1, Theorem 8.2.9], the mapping $t \mapsto L_N(t,v)$ is measurable for all $v \in V$, and it is easy to see that $v \mapsto L_N(t,v)$ has closed graph in $(V_N, \|\cdot\|_V) \times (H, \|\cdot\|_H)$ for almost every $t \in (0,T)$. In particular, its images are closed, and they are convex by construction.

Now consider the map $P_N F : [0,T] \times V_N \rightrightarrows V_N$. Since the projection P_N is linear and has operator norm 1 w.r.t. $\|\cdot\|_H$, the mapping $P_N F$ has convex images and inherits growth bound (18) and continuity in v from F. By [1, Theorem 8.2.8], the mapping $t \mapsto P_N F(t,v)$ is measurable for all $v \in V_N$.

To show that $P_N F(t,v)$ is H-closed for almost every $t \in (0,T)$ and all $v \in V_N$, fix $t \in (0,T)$ and $v \in V_N$ and consider a sequence $\{g_k\}_{k\in\mathbb{N}} \subset P_N F(t,v)$ with $\lim_{k\to\infty} \|g_k - g\|_H = 0$ for some $g \in H$. There exists$\{f_k\}_{k\in\mathbb{N}} \subset F(t,v)$ such that $g_k = P_N f_k$ for all $k \in \mathbb{N}$. By Mazur's theorem, and since $F(t,v) \in \mathcal{CBC}(H)$, there exist $\tilde{f} \in F(t,v)$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $f_k \rightharpoonup \tilde{f}$ in H as $\mathbb{N}' \ni k \to \infty$. As a consequence, we have $g_k = P_N f_k \rightharpoonup P_N \tilde{f}$ in H as $\mathbb{N}' \ni k \to \infty$, so uniqueness of the weak limit yields $g = P_N \tilde{f} \in P_N F(t,v)$.

Since $\dim V_N < \infty$, the images $P_N F(t,v)$ are compact in H for almost every $t \in (0,T)$ and all $v \in V_N$.

Finally, the mapping

$$F_N : [0,T] \times V_N \rightrightarrows H, \quad F_N(t,v) := P_N F(t,v) \cap L_N(t,v),$$

has nonempty images for almost every $t \in (0,T)$ according to assumption (A4). By the above, we have $F_N(t,v) \in \mathcal{CBC}(H)$ for almost every

$t \in (0,T)$ and every $v \in V_N$, and F_N inherits growth bound (18) from $P_N F$. By [1, Theorem 8.2.4], the mapping $t \mapsto F_N(t,v)$ is measurable for all $v \in V$, and by [1, Proposition 1.4.9], the mapping $v \mapsto F_N(t,v)$ is upper semicontinuous for almost every $t \in (0,T)$.

In particular, the right-hand side of the differential inclusion

$$u_N'(t) \in F_N(t, u_N(t)) - A_N u_N(t) \quad \dot{\forall}\, t \in (0,T), \quad u_N(0) = u_{N,0}, \tag{31}$$

is measurable in time and upper semicontinuous in the second argument with compact and convex images in $(V_N, \|\cdot\|_H)$. For any $u_N \in \mathrm{AC}([0,T]; V_N)$ solving inclusion (31), the function

$$f_N : [0,T] \to H, \quad f_N(t) := u_N'(t) + A_N u_N(t),$$

is an element of $L^1(0,T;H)$. Hence u_N and f_N solve the Galerkin inclusion (3), and Proposition 11 provides an a priori bound for u_N.

Now all assumptions of Theorem 7 are satisfied, so there exists indeed a solution $u_N \in \mathrm{AC}([0,T]; V_N)$ of the inclusion (31), and hence of inclusion (3). Using the definition of f and L_N we estimate the distance between u_N and v_N as follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_N(t) - u_N(t)\|_H^2 + c_a \|v_N(t) - u_N(t)\|_V^2 \\
&\le (v_N'(t) - u_N'(t), v_N(t) - u_N(t)) + a(v_N(t) - u_N(t), v_N(t) - u_N(t)) \\
&= \big((v_N'(t) + A_N v_N(t) - f(t)) + (f(t) - f_N(t)), v_N(t) - u_N(t)\big) \\
&\le \delta_N(t)\|v_N(t) - u_N(t)\|_H + \ell(t)\|v_N(t) - u_N(t)\|_H^2.
\end{aligned} \tag{32}$$

By Lemma 8, we obtain

$$\|v_N(t) - u_N(t)\|_H \le e^{\int_0^t \ell(s)ds}\|v_N(0) - u_N(0)\|_H + \int_0^t e^{\int_s^t \ell(\tau)d\tau}\delta_N(s)ds,$$

which yields (29). Integrating inequality (32) yields estimate (30). □

5 Convergence of solution sets

In the following, we prove upper and lower Kuratowski convergence of the sets $\mathcal{S}_N$ of Galerkin solutions to the set $\mathcal{S}$, from which we deduce Hausdorff convergence.

5.1 Upper limit of Galerkin solution sets

In this section, we show that the set $\mathcal{S}$ of all exact solutions contains the upper Kuratowski limit of the sets $\mathcal{S}_N$ in $L^2(0,T;H)$.

Proof of Theorem 12. a) The sets $\mathcal{S}_N$ are nonempty.
This follows from Proposition 15 applied to the function $v_N = 0$, since (A3) implies

$$\delta_N(t) := \mathrm{dist}(0, F(t,0))_H \leq \|F(t,0)\|_H \leq \alpha(t).$$

b) Extraction of convergent subsequences.
Let $\{u_N\}_{N\in\mathbb{N}}$ be a sequence with $u_N \in \mathcal{S}_N$ for all $N \in \mathbb{N}$. Consider a subsequence $\mathbb{N}' \subset \mathbb{N}$, and let $f_N \in L^1(0,T;H)$ satisfy (3a) and (3b) for all $N \in \mathbb{N}'$. By Proposition 11 the sequence $(u_N)_{N\in\mathbb{N}}$ is bounded in W. Therefore, by the compact embedding (14), there exist a subsequence $\mathbb{N}'' \subseteq \mathbb{N}'$ and some $u \in L^2(0,T;H)$ satisfying

$$\|u_N - u\|_{L^2(0,T;H)} \to 0 \quad \text{as} \quad \mathbb{N}'' \ni N \to \infty. \tag{33}$$

According to Proposition 11, there exists a subsequence $\mathbb{N}''' \subseteq \mathbb{N}''$ such that $(u_N)_{N\in\mathbb{N}'''}$ is weak-$*$ convergent in $L^2(0,T;V) \cap L^\infty(0,T;H)$ (cf. [26, Ch.21.8]). Due to the uniqueness of the weak-$*$ limit, we have

$$u_N \overset{*}{\rightharpoonup} u \in L^2(0,T;V) \cap L^\infty(0,T;H) \quad \text{as} \quad \mathbb{N}''' \ni N \to \infty. \tag{34}$$

The estimates (18) and (19) show that $\{f_N\}_{N\in\mathbb{N}'''}$ is bounded in $L^1(0,T;H)$. By the Dunford-Pettis theorem [9, Thm. IV 2.1, p. 101] the convergence (33) ensures uniform integrability with respect to the Lesbesgue measure μ, i.e. for any Lebesgue measurable set $E \subseteq (0,T)$ one has

$$\sup_{N\in\mathbb{N}'''} \int_E \|u_N(t)\|_H^2 dt \to 0 \quad \text{as} \quad \mu(E) \to 0.$$

Using bounds (18) and (19), we obtain for $\mu(E) \to 0$

$$\sup_{N\in\mathbb{N}'''} \| \int_E f_N(t)dt\|_H \leq \sup_{N\in\mathbb{N}'''} \int_E \Big(\alpha(t) + c_F(1 + \|u_N(t)\|_V)\|u_N(t)\|_H\Big)dt$$
$$\leq \int_E \alpha(t)dt + c_F(\sqrt{T} + K) \sup_{N\in\mathbb{N}'''} (\int_E \|u_N(t)\|_H^2 dt)^{\frac{1}{2}} \to 0.$$

Therefore, the set $\{f_N : N \in \mathbb{N}'''\}$ is uniformly integrable and bounded in $L^1(0,T;H)$, and balls in H are relatively weakly compact. Then, again by the Dunford-Pettis theorem, there exists a subsequence $\mathbb{N}^{(4)} \subset \mathbb{N}'''$ such that

$$f_N \rightharpoonup f \text{ in } L^1(0,T;H) \quad \text{as} \quad \mathbb{N}^{(4)} \ni N \to \infty. \tag{35}$$

c) The limits solve the differential inclusion.
By assumption (A2) and estimate (18), the sets

$$\mathcal{F} := \{g \in L^1(0,T;H) : g(t) \in F(t,u(t)) \;\dot{\forall}\, t \in [0,T]\},$$
$$\mathcal{F}_\varepsilon := \{g \in L^1(0,T;H) : \operatorname{dist}(g,\mathcal{F})_{L^1(0,T;H)} \le \varepsilon\},$$

are closed, bounded and convex subsets of $L^1(0,T;H)$. Note that the map $t \mapsto F(t,u(t))$ is measurable due to [1, Theorem 8.2.8], so that measurability of the map $t \mapsto \operatorname{dist}(g(t), F(t,u(t))$ follows from [1, Corollary 8.2.13] by the continuity assumption of (A2). Further, Assumption (A3), the bound (19) and statement (33) imply

$$\begin{aligned}&\int_0^T \operatorname{dist}(f_N(t), F(t,u(t)))_H dt \\ &\le 2c_F(\sqrt{T}+K)\|u_N - u\|_{L^2(0,T;H)} \to 0 \quad \text{as} \quad \mathbb{N}^{(4)} \ni N \to \infty.\end{aligned} \tag{36}$$

Hence for every $\varepsilon > 0$ there exists $N_0(\varepsilon)$ such that $f_N \in \mathcal{F}_\varepsilon$ for all $N \in N^{(4)}$ with $N \ge N_0(\varepsilon)$. By Mazur's theorem, the set $\mathcal{F}_\varepsilon$ is weakly closed in $L^1(0,T;H)$, so that (35) implies $f \in \mathcal{F}_\varepsilon$. Since this holds for every $\varepsilon > 0$, we have proved $f \in \mathcal{F}$ and hence inclusion (2b).

Since the Galerkin spaces V_N are nested, we obtain from (3) and (9) for all $v_M \in V_M$, all $\varphi \in C_c^\infty(0,T)$ and all $N \ge M$ that

$$\int_0^T (u_N(t), v_M)\varphi'(t)dt = \int_0^T \Big(a(u_N(t), v_M)\varphi(t) - (f_N(t), v_M)\varphi(t)\Big)dt.$$

Let $N \to \infty$ in this equality, and use the convergences (33), (34) and (35) for $v_M\varphi' \in L^2(0,T;H)$, $v_M\varphi \in L^2(0,T;V)$, $v_M\varphi \in L^\infty(0,T;H)$, to obtain

$$\int_0^T (u(t), v_M)\varphi'(t)dt = \int_0^T \Big(a(u(t), v_M)\varphi(t) - (f(t), v_M)\varphi(t)\Big)dt.$$

Since $\bigcup_{M\in\mathbb{N}} V_M$ is dense in V, we have

$$\int_0^T (u(t), v)\varphi'(t)dt = \int_0^T \Big(a(u(t), v) - (f(t), v)\Big)\varphi(t)dt$$

for all $v \in V$ and $\varphi \in C_c^\infty(0,T)$. Hence u has a weak derivative in $W_+ = L^2(0,T,V^*) + L^1(0,T;H)$, given by the two terms on the right-hand side. Thus we have shown that $u \in W$ satisfies statements (2a) and (2b).

d) The solution assumes the initial data
It remains to verify that the initial value condition (2c) is satisfied. Consider again $N \geq M$ and an arbitrary element $v_M \in V_M$. Using the weak differentiability of u and the absolute continuity of u_N, we obtain

$$-(u_{N,0} - u(0), v_M) = \int_0^T \frac{d}{dt}\Big[(u_N(t) - u(t), \tfrac{T-t}{T} v_M)\Big] dt$$
$$= \int_0^T (u_N'(t), \tfrac{T-t}{T} v_M) - \langle u'(t), \tfrac{T-t}{T} v_M \rangle - \frac{1}{T}(u_N(t) - u(t), v_M) dt$$
$$= \int_0^T \Big(- a(u_N(t) - u(t), \tfrac{T-t}{T} v_M) + (f_N(t) - f(t), \tfrac{T-t}{T} v_M)\Big) dt$$
$$- \int_0^T \frac{1}{T}(u_N(t) - u(t), v_M) dt.$$

In the last step we used the differential inclusions (2a) and (3a). Equation (33) shows that the last term converges to 0 as $\mathbb{N}'' \ni N \to \infty$. The first two integral terms converge to zero due to the weak$*$-convergence (34), the weak convergence (35) and the fact that the test function $t \mapsto \frac{T-t}{T} v_M$ is in $L^2(0,T;V) \cap L^\infty(0,T;H)$. From assumption (A5), we conclude

$$(u_0 - u(0), v_M) = 0 \quad \forall\, v_M \in V_M.$$

Now the density of $\bigcup_M V_M$ in H shows $u(0) = u_0$.

e) Upper Kuratowski convergence of $\mathcal{S}_N$ to $\mathcal{S}$
If $\operatorname{Lim\,sup}_{N\to\infty} \mathcal{S}_N \subseteq \mathcal{S}$ is false in $L^2(0,T;H)$, there exist a subsequence $\mathbb{N}' \subseteq \mathbb{N}$, elements $u_N \in \mathcal{S}_N$ for all $N \in \mathbb{N}'$, and some $u \in L^2(0,T;H)$ with $u \notin \mathcal{S}$ and $\|u_N - u\|_{L^2(0,T;H)} \to 0$. But by the above, there exist a subsequence $\mathbb{N}'' \subseteq \mathbb{N}'$ and some $\tilde{u} \in \mathcal{S}$ such that $\|u_N - \tilde{u}\|_{L^2(0,T;H)} \to 0$ as $N \to \infty$ in $\mathbb{N}''$, which is a contradiction. □

5.2 Lower limit of Galerkin solution sets

In this section we show that solutions in $\mathcal{S}$ are limits of solutions in $\mathcal{S}_N$ with respect to the norm of $L^2(0,T;V) \cap L^\infty(0,T;H)$. In principle, we project every $u \in \mathcal{S}$ to W_N and invoke Proposition 15 to generate a Galerkin solution close to u.

Lemma 16. *For any $u \in W$ and $N \in \mathbb{N}$, there exists a unique solution $v_N \in W_N$ of the linear problem*

$$(v_N'(t), v) + a(v_N(t), v) = \langle u'(t), v\rangle + a(u(t), v) \quad \dot\forall t \in (0,T),\, \forall v \in V_N,$$
$$v_N(0) = P_N u(0),$$

and the sequence $\{v_N\}_{N\in\mathbb{N}}$ of these solutions satisfies

$$\|v_N - u\|_{W_+^\star} \to 0 \quad as \quad N \to \infty. \tag{37}$$

Proof. This is the usual Galerkin approximation of $w' + Aw = f$ in V_N with $f \in L^2(0,T;V^*) + L^1(0,T;H)$ given by $f(t) := u'(t) + Au(t)$ for almost every $t \in (0,T)$. Existence and uniqueness of a solution as well as convergence (37) are shown in [26, Theorem 23.A]. □

Now we prove lower Kuratowski convergence of $\mathcal{S}_N$ to $\mathcal{S}$.

Proof of Theorem 13. Let $u \in \mathcal{S}$ be a solution and let $f \in L^1(0,T;H)$ be as in (2a) and (2b). For all $N \in \mathbb{N}$ and v_N as in Lemma 16, we have

$$\begin{aligned}&(v_N'(t), v) + a(v_N(t), v) = \langle u'(t), v\rangle + a(u(t), v)\\ &= (f(t), v) = (P_N f(t), v) \quad \dot\forall\, t \in (0,T),\, \forall\, v \in V_N.\end{aligned}$$

Therefore, the distance

$$\delta_N(t) := \operatorname{dist}(v_N'(t) + A_N v_N(t), F(t, v_N(t)))_H$$

satisfies

$$\delta_N(t) \le \|P_N f(t) - f(t)\|_H + \operatorname{dist}(f(t), F(t, v_N(t)))_H.$$

The first term converges to 0 pointwise by assumption (S2) and has the integrable bound $\|f(t)\|_H, t \in (0,T)$, hence $\|P_N f - f\|_{L^1(0,T;H)} \to 0$ as $N \to \infty$ by Lebesgue's theorem. The second term is estimated by invoking assumption (A3)

$$\begin{aligned}&\int_0^T \operatorname{dist}(f(t), F(t, v_N(t)))_H dt \le \int_0^T \operatorname{dist}(F(t, u(t)), F(t, v_N(t)))_H dt\\ &\le c_F(T + \|u\|_{L^2(0,T;V)} + \|v_N\|_{L^2(0,T;V)})\|u - v_N\|_{L^2(0,T;H)},\end{aligned} \tag{38}$$

which converges to 0 as $N \to \infty$ due to (37). Hence, by Proposition 15, there exist solutions $u_N \in W_N (N \in \mathbb{N})$ of inclusion (3), with $u_{N,0} = P_N u(0)$ which satisfy

$$\|u_N - v_N\|_{W_+^*} \leq C_\ell \|\delta_N\|_{L^1(0,T)} \to 0 \quad \text{as} \quad N \to \infty.$$

Finally, statement (37) implies

$$\|u - u_N\|_{W_+^*} \leq \|u - v_N\|_{W_+^*} + \|v_N - u_N\|_{W_+^*} \to 0 \quad \text{as} \quad N \to \infty.$$

□

Remark. *Note that convergence in Theorem 13 holds in the strong norm of $L^2(0,T;V) \cap L^\infty(0,T;H)$, while Theorem 12 assures convergence only in the weaker norm of $L^2(0,T;H)$ (or in $L^r(0,T;H)$ with a fixed $r \in [1,\infty)$).*

5.3 Hausdorff convergence of solution sets

We finally combine upper and lower Kuratowski convergence to prove convergence with respect to the Hausdorff metric in $L^2(0,T;H)$.

Proof of Theorem 14. Theorems 12 and 13 imply Kuratowski convergence $\operatorname{Lim}_{N\to\infty} \mathcal{S}_N = \mathcal{S}$ in $L^2(0,T;H)$. According to Proposition 11, there exists a ball $B \subset W$ with $\mathcal{S} \subset B$ and $\mathcal{S}_N \subset B$ for all $N \in \mathbb{N}$. Therefore, invoking Lemma 4 with $Y = L^2(0,T;H)$ yields the desired convergence statement, because B is relatively compact in $L^2(0,T;H)$ by the compact embedding (14). □

5.4 An extension

Condition (A3) seriously limits the polynomial growth of nonlinearities to which Theorems 12 and 13 apply, see Section 6. Therefore, we present in this section a weaker assumption under which our conclusions still hold. The type of condition is similar to [14, Assumption B] where it is used to derive a priori estimates for variable time-step discretizations of evolution equations.

Instead of (A3) we require
(A3') There exist functions $\alpha \in L^1(0,T)$, $b \in C(\mathbb{R})$ and constants

$\beta \in [0,2)$, $\gamma \in (0,1]$ such that for almost every $t \in (0,T)$ and all $u, v \in V$, the following estimates hold

$$\|F(t,0)\|_H \le \alpha(t),$$
$$\mathrm{dist}_{\mathcal{H}}(F(t,u), F(t,v))_H \le b(\|u\|_H + \|v\|_H)(1 + \|u\|_V^\beta + \|v\|_V^\beta)\|u-v\|_H^\gamma.$$

For $\beta = \gamma = 1$ and a constant function b this coincides with condition (A3). Without loss of generality we assume the function b to be non-negative and monotone increasing. In the following we indicate the changes in the proofs of our results without providing all details.

The growth estimate (18) now reads

$$\|F(t,v)\|_H \le \alpha(t) + b(\|v\|_H)(1 + \|v\|_V^\beta)\|v\|_H^\gamma \quad \dot{\forall} t \in (0,T),\ \forall v \in V. \tag{39}$$

The modified L^1-estimate (28) is obtained from (39), (25) and (26) via Hölder's inequality as follows

$$\|f_N\|_{L^1(0,T;H)} \le \|\alpha\|_{L^1(0,T)} + b(K_1)K_1^\gamma(T + K_0^\beta T^{1-\frac{\beta}{2}}). \tag{40}$$

In a similar way, the L^1-bound and uniform integrability of the sequence $f_N \in L^1(0,T;H)$ from part b) of the proof of Theorem 12 follows from (39)

$$\sup_{N \in \mathbb{N}'''} \| \int_E f_N(t)dt\|_H \le \int_E \alpha(t)dt + b(K)K^\gamma(\mu(E) + K^\beta \mu(E)^{1-\frac{\beta}{2}}).$$

Moreover, since the compact embedding (14) holds for all $1 \le r < \infty$ we select the subsequence $\mathbb{N}''$ such that one has, instead of (33)

$$\|u_N - u\|_{L^r(0,T;H)} \to 0 \quad \text{as} \quad \mathbb{N}'' \ni N \to \infty, \quad r := \frac{2\gamma}{2-\beta}. \tag{41}$$

Using Hölder's inequality the estimate (36) is replaced by

$$\begin{aligned}
&\int_0^T \mathrm{dist}(f_N(t), F(t,u(t)))_H dt \\
&\le b(2K)\left\{\int_0^T (1 + \|u_N(t)\|_V^\beta + \|u(t)\|_V^\beta)^{\frac{2}{\beta}} dt\right\}^{\frac{\beta}{2}} \|u_N - u\|_{L^r(0,T;H)}^\gamma \\
&\le Cb(2K)(1 + \|u_N\|_{L^2(0,T;V)} + \|u(t)\|_{L^2(0,T;V)})^\beta \|u_N - u\|_{L^r(0,T;H)}^\gamma
\end{aligned}$$

with a suitable constant C. Finally, with r from (41), the estimate (38) is modified in a similar way:

$$\begin{aligned}
&\int_0^T \mathrm{dist}(F(t,u(t)), F(t,v_N(t))_H dt \\
&\le Cb(2\|u\|_{W_+^*})(1 + \|u\|_{L^2(0,T;V)} + \|v_N\|_{L^2(0,T;V)})^\beta \|u - v_N\|_{L^r(0,T;H)}^\gamma.
\end{aligned}$$

6 Example

Let $\Omega \subset \mathbb{R}^1$ be a bounded open interval, and consider the Gelfand triple $V \subseteq H \subseteq V^*$ with spaces $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V^* = H^{-1}(\Omega)$, which satisfy assumption (A1). Let $(V_N)_{N\in\mathbb{N}} \subset V$ be spaces of piecewise linear functions subject to successively refined equidistant grids. This standard construction is known to satisfy (S2) and under some additional conditions also satisfies (S3), see [5].

Consider the partial differential inclusion

$$u'(t) - \Delta u(t) \in F(u(t)).$$

It is well-known that $-\Delta : V \to V^*$ induces a bilinear form which satisfies assumption (A1). By the Sobolev embedding theorem, we have $V \subset L^\infty(\Omega)$, and there exists $C_\infty > 0$ with

$$\|v\|_{L^\infty(\Omega)} \le C_\infty \|v\|_V \quad \forall\, v \in V.$$

The function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(\eta) = \eta(1 - |\eta|) \quad \forall\, \eta \in \mathbb{R},$$

clearly does not have linear growth. It satisfies

$$\Big(\int_\Omega g(v(x))^2 dx\Big)^{\frac{1}{2}} \le \|v\|_{L^4(\Omega)}^2 + \|v\|_{L^2(\Omega)},$$

and hence induces a Nemytskii operator

$$N_g : L^4(\Omega) \to L^2(\Omega), \quad N_g(v)(x) := g(v(x)).$$

In view of [17, Theorem 3.4.4], the operator N_g is continuous. Let $h \in L^2(\Omega)$ be nonnegative. Then the multivalued nonlinearity

$$\begin{aligned} &F : [0,T] \times V \to \mathcal{CBC}(H), \\ &F(t,v) = \{f \in \mathcal{M}(\Omega;\mathbb{R}) : |N_g(v)(x) - f(x)| \le h(x)\ \dot\forall\, x \in \Omega\} \end{aligned}$$

does not depend on t and clearly satisfies the measurability condition of (A2). Continuity will follow from (A3) which we verify next. Since

$$\|F(t,0)\|_{L^2(\Omega)} = \|h\|_{L^2(\Omega)},$$

the first inequality holds with $\alpha(t) \equiv \|h\|_{L^2(\Omega)}$. For $\xi, \eta \in \mathbb{R}$ we have

$$|g(\xi) - g(\eta)| = |\xi - \eta + |\xi|(\eta - \xi) + \eta(|\eta| - |\xi|)| \le (1 + |\xi| + |\eta|)|\xi - \eta|,$$

hence for $u, v \in L^2(\Omega)$

$$\begin{aligned}
&\operatorname{dist}(F(t,u),F(t,v))_H^2 \le \int_\Omega |g(u(x)) - g(v(x))|^2 dx \\
&\le \int_\Omega (1 + |u(x)| + |v(x)|)^2 (u(x) - v(x))^2 dx \\
&\le (1 + \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)})^2 \|u - v\|_{L^2(\Omega)}^2 \\
&\le \max(1, C_\infty)^2 (1 + \|u\|_V + \|v\|_V)^2 \|u - v\|_H^2.
\end{aligned}$$

Therefore, the second inequality of assumption (A3) holds with $c_F = \max(1, C_\infty)$. One easily verifies that

$$(g(\xi) - g(\eta))(\xi - \eta) \le (\xi - \eta)^2 \quad \forall\, \xi, \eta \in \mathbb{R},$$

and it follows that

$$\begin{aligned}
(N_g(u) - N_g(v), u - v) &= \int_\Omega (g(u(x)) - g(v(x)))(u(x) - v(x)) dx \\
&\le \int_\Omega (u(x) - v(x))^2 dx.
\end{aligned}$$

This implies assumption (A4) with $\ell(t) \equiv 1$. All in all, Theorem 14 applies and yields

$$\operatorname{dist}_{\mathcal{H}}(\mathcal{S}, \mathcal{S}_N)_{L^2(0,T;H)} \to 0 \quad \text{as} \quad N \to \infty.$$

Let us finally note that the weakened assumption (A3') from section 5.4 allows to treat more general nonlinearities, for example

$$g(\eta) = \eta(1 - |\eta|^{2-\varepsilon}) \quad \text{for some } 0 < \varepsilon \le 2.$$

However, it remains as an open problem whether our results extend to the standard cubic nonlinearity with $\varepsilon = 0$.

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