The tame dimension vectors for stars

§1: Problem

wild quivers: have families of representations with arbitrary many parameters

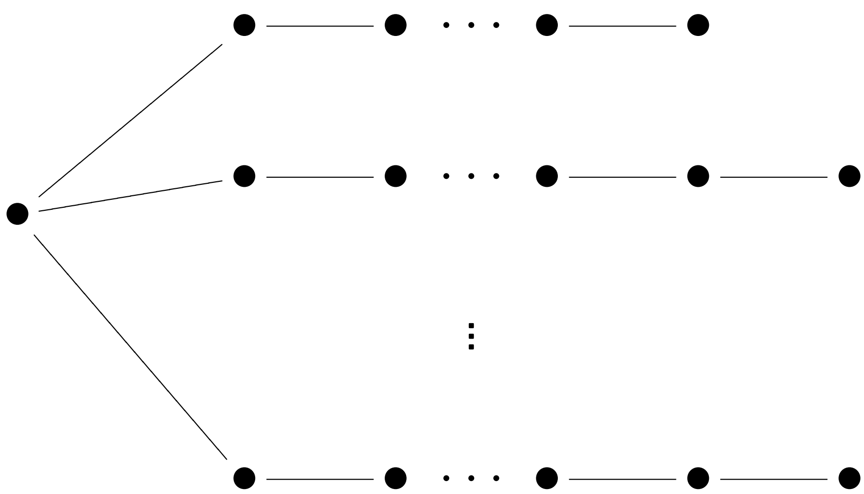
Aim: Describe those dimension vectors for which all families of representations depend on a *single* parameter.

$K = \overline{K}$ alg. closed field.

Definition
A dimension vector $\mathbf{d}$ is called **tame** if
1) there is a one parameter family of indecomposable representations for $\mathbf{d}$, and
2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2} \Rightarrow$ there is no m-parameter family of indecomposable representations for $\mathbf{d_1}$ or $\mathbf{d_2}$ with $m \geq 2$.

Restrict ourselves to stars:

star: quiver whose underlying graph is of the following shape



k arms with (maybe different) lengths p_i, $i = 1, \ldots, k$

§2: The s-tame dimension vectors for stars

Consider stars with subspace orientations.

Definition

A dimension vector $\mathbf{d}$ is called **s-tame** if

1) there is a one parameter family of indecomposable subspace representations for $\mathbf{d}$, and

2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ decomposition into a sum of dimension vectors of subspace representations $\Rightarrow$ there is no m-parameter family of indecomposable subspace representations for $\mathbf{d_1}$ or $\mathbf{d_2}$ with $m \geq 2$.

Lemma

$\mathbf{V}$ indecomposable representation of a star with subspace orientation with central dimension $\neq 0 \Rightarrow \mathbf{V}$ subspace representation

So:

$$\boxed{\textbf{tame} \Rightarrow \textbf{s-tame}}$$

But:

$$\boxed{\textbf{s-tame} \not\Rightarrow \textbf{tame}}$$

Decompose

$$\begin{array}{ccc} & 3 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{array}$$

into

$$\begin{array}{ccc} & 2 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{array} \oplus \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \\ & 0 & \end{array}.$$

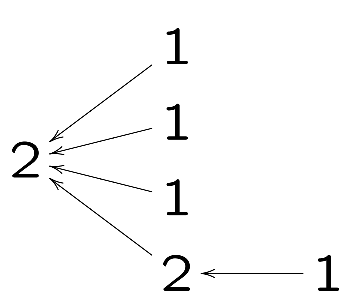
For the first dimension vector there is an indecomposable two parameter family of representations, but the family of representations constructed in this way is *not* a family of *subspace* representations.

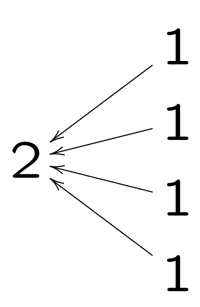
Classification Theorem
(for s-tame dimension vectors for stars)

A dimension vector $\mathbf{d} \neq 0$ for a star is s-tame if and only if
1) $q(\mathbf{d}) = 0$, and
2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ a sum of dimension vectors of subspace representations $\Rightarrow q(\mathbf{d_1}) \geq 0$ and $q(\mathbf{d_2}) \geq 0$,
where q denotes the Tits form corresponding to the star.

Identifying subspace representations with isomorphisms along their arms with the corresponding ones without isomorphisms, there is also a "finite list" of all s-tame dimension vectors.

Example





§3: The (s-)hypercritical dimension vectors for stars

Definition

A dimension vector $\mathbf{d}$ is called **(s-)hypercritical** if

1) there is an n-parameter family of indecomposable (subspace) representations for $\mathbf{d}$ with $n \geq 2$, and

2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ a non trivial decomposition into a sum of dimension vectors (of subspace representations) $\Rightarrow$ there is no m-parameter family of indecomposable (subspace) representations for $\mathbf{d_1}$ or $\mathbf{d_2}$ with $m \geq 2$.

Have that:

hypercritical $\Rightarrow$ s-hypercritical

Classification Theorem
(for s-hypercritical dimension vectors for stars)

A dimension vector $\mathbf{d} \neq 0$ for a star is s-hypercritical if and only if
1) $q(\mathbf{d}) < 0$, and
2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ a non trivial sum of dimension vectors of subspace representations $\Rightarrow$ $q(\mathbf{d_1}) \geq 0$ and $q(\mathbf{d_2}) \geq 0$,
where q denotes the Tits form corresponding to the star.

There is also a "finite list" of all s-hypercritical dimension vectors.

We also have a list of dimension vectors of subspace repns. with restrictions on minimal dimension jumps and arm lengths:

($*$) Tits form is positive
($\star$) Tits form is non negative
($\circ$) Tits form is negative for all dimension vectors with smallest possible central dimension

Lemma
A dimension vector for a star is s-hypercritical if and only if it occurs in a ($\circ$)-case and has smallest possible central dimension.

A dimension vector $\mathbf{d}$ for a star is s-tame if and only if it occurs in a ($\star$)-case and has Tits form $q(\mathbf{d}) = 0$.

Kac's Theorem
If $q(\mathbf{d}) \geq 1$, then there is no family of indecomposable representations.

With these considerations:

We obtain all hypercritical dimension vectors for stars by comparison of the s-hypercritical dimension vectors.

§4: The tame dimension vectors for stars

Now we can obtain the list of all tame dimension vectors for stars:

Take the list of all s-tame dimension vectors for stars. If it is not possible to split off a hypercritical dimension vector, then it is tame, otherwise not.

Check if this is also o.k. for the dimension vectors which we obtain from those of the list by a prolongation of an arm!

We also have a characterisation of the tame dimension vectors for stars by their Tits forms and those of their decompositions:

Theorem

A dimension vector $\mathbf{d} \neq 0$ for a star is tame if and only if

1) $q(\mathbf{d}) = 0$, and

2) $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ a sum of dimension vectors $\Rightarrow q(\mathbf{d_1}) \geq 0$ and $q(\mathbf{d_2}) \geq 0$,

where q denotes the Tits form corresponding to the star.

§5: Families of indecomposable representations for the tame dimension vectors for stars

We know all families of indecomposable representations for the s-tame dimension vectors for stars explicitly (H, 2003).

Lemma

- BGP-reflections along the arms result in changes of dimension *jumps* along the arms.
- A change of dimension jumps along the arms does not change the property of being s-tame.

Proposition

For every tame dimension vector of a star with any orientation there are sequences of admissible reflections such that the obtained dimension vector is s-tame.

Proof
By the previous Lemma, we only have to show that for an arbitrary star there are sequences of admissible reflections within the arms such that the new star carries the subspace orientation.

It is enough to show that we can obtain any orientation of a quiver of type $\mathbb{A}_n$ from the quiver with linear orientation by admissible reflections without reflecting at the sink.

$$1 \to 2 \to \cdots \to n$$

The reflection $r_i \circ \ldots \circ r_1$ is admissible as long as the subquiver containing $1, 2, \ldots, i$ is ordered linearly increasing, changes the orientation of

$$i \to i + 1$$

and leaves all other orientations of the arrows as before.

□