Limit sets and asymptotic methods in Operator Theory Topological Transformation Groups and Dynamical Systems

Habilitationsschrift

A. Manoussos

Fakultät für Mathematik
Universität Bielefeld
2010

Contents

Introduction **1**

1 Limit sets and asymptotic methods in Topological Transformation Groups **5**

2 Operator Theory **11**

3 Dynamical Systems **15**

Bibliography **16**

Acknowledgements

Most of the work that I present here has been done during my stay as a guest and as member of the Collaborative Research Center 701 in the University of Bielefeld. I am grateful to people of CRC 701 and to people of the Department of Mathematics of the University of Bielefeld for their hospitality and generosity. Especially, I would like to express my deep gratitude to Prof. Dr. H. Abels for his valuable support and the unlimited time he spent in fruitful discussions with me. I would like also to thank Prof. Dr. F. Götze for his interest and his support.

Introduction

The main characteristic of my work is the use of asymptotic methods coming from general topology and the theory of limit sets to study problems in Topological Transformation Groups, Operator Theory and Dynamical Systems. The present Habilitationsschrift is cumulative, in the sense that we expose results from papers which has been recently submitted or accepted for publication. We divided them in three categories. Namely, we present some results from

- the theory of Topological Transformation Groups related to properly discontinuous, proper and isometric actions.
- Operator Theory related to topologically transitive and locally topologically transitive (J-class) operators, hypercyclic operators and dynamics of commuting tuples of matrices.
- the theory of Dynamical Systems related to analytic crossed products.

In what follows X will denote a Hausdorff locally compact space or a (complex or real) Hilbert or Banach space and G will denote a locally compact group acting on X or G will denote the semigroup of non-negative integers generated by a continuous map or a bounded linear operator on X. For $x \in X$ the *limit set* $L(x)$ is defined by

$$L(x) = \{y \in X \,|\, \text{there exists a divergent net}\{g_i\}_{i\in I} \text{ in } G \text{ such that } \{g_i x\}_{i\in I} \\ \text{converges to } y\}$$

and the *extended (prolongational) limit set* $J(x)$ is defined by

$$J(x) = \{y \in X \,|\, \text{there exist a divergent net}\{g_i\}_{i\in I} \text{ in } G \text{ and a net } \{x_i\}_{i\in I} \\ \text{in } X \text{ converging to } x \text{ such that } \{g_i x_i\}_{i\in I} \text{ converges to } y\}.$$

So we can say that limit sets describe the limit behavior of an orbit and generalized limit sets describe the asymptotic behavior of the orbits of nearby points to $x \in X$. In the cases we study the limit and the generalized limit sets are closed and invariant sets.

Limit and extended limit sets have their roots in the Qualitative Theory of Dynamical Systems when they are used mainly to describe the Lyapunov and the asymptotic stability of an equilibrium point or, more generally, of a compact minimal set. They, also, "encode" information which allows us to connect the global structure of the underlying space with local properties. Such an example are the parallelizable flows. In this case we assumed that the extended limit sets are empty and as a consequence the underlying space is equivariantly isomorphic to a cartesian product of the form $\mathbb{R} \times S$, where S is global continuous section for the system. Using the same asymptotic topological methods as in the Qualitative Theory of Dynamical Systems and basic properties of the structure of the underlying space (e.g. Hilbert space geometry and the linearity of an operator or connectedness and local compactness for the case of a topological space) we study several problems concerning the dynamic behavior of the systems we investigate and the structure of the underlying space. To make this more clear let us describe briefly the methods and the tools we used to show the main results of the presentation at hand. In Chapter 1 we give a characterization of proper actions in terms of the geometry of the underlying space. A proper action has the property that all the extended limit sets are empty. In [1] we showed that a locally compact group G acts properly on a locally compact σ-compact metrizable space X if and only if there exists a G-invariant proper (Heine-Borel) compatible metric on X. The construction of such a metric is based on the existence of an open fundamental set for a proper action (which is a basic tool in this theory) and the use of special coverings of the space created by this set. Using a similar approach in [11] we constructed a dynamic invariant for properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces by looking the dynamic behavior of such an action at infinity (i.e. by embedding such an action in a suitable zero-dimensional compactification and looking at the cardinality of the remainder of our space in it). Proper and isometric actions are closely related as we showed in [1] but in general isometric actions are not proper. In [8] we studied the dynamic behavior of the action of the group of isometries of a locally compact metric space. Since such an action is not necessarily proper the idea is to look for "thick" (i.e. closed-open) invariant subsets of the underlying space where the action behaves like a proper one. In Chapter 2 we deal mainly with topologically transitive operators on Hilbert spaces. The notion of a topologically transitive operator can be viewed as the opposite of the notion of a proper action. Topologically transitive operators have the property that all the extended limit sets are the whole space in contrast with proper actions where all the extended limit sets are empty. In the main result in [6] we used information of local nature (the generalized set of a cyclic vector has non-empty interior) and we got, as a result, the global behavior of an operator (that it is topologically transitive). Precisely, we showed that if x is a cyclic vector for an operator $T : X \to X$ and the set $J(x)$ has non-empty interior then $J(y) = X$ for every $y \in X$, hence T is topologically transitive. This result gave us the idea to "localize" the notion of a topological transitive operator by introducing and studying a new class of operators called locally topologically transitive or J-class operators. This class of operators is characterized by the property that there exists a non-zero vector $x \in X$ with $J(x) = X$. The arguments we used in this work are quite similar to those we used to study isometric actions plus the additional structure of linearity. In Chapter 3 we present an answer we gave to a long standing question asked by W. B. Arveson and K. B. Josephson in 1969 concerning the description of the radical of the analytic crossed product of a classical dynamical system in terms of the dynamic behavior of the system. The analytic crossed product of a classical dynamical system is a non self adjoint algebra of operators that characterizes the dynamical system. Two dynamical systems are topologically conjugate if and only if the corresponding analytic crossed products are isomorphic as algebras. The basic ideas in the proof of the main theorem in [7] came again from the theory of topological dynamics. Firstly, we showed that any monomial in the Jacobson radical has Fourier coefficient that vanishes on the recurrent points of the dynamical system (a point is called recurrent if $x \in L(x)$). And secondly, we showed that a monomial in the Jacobson radical which has Fourier coefficient with support contained in an open set of points with the property $x \notin J(x)$ generates a two-sided ideal whose square is 0. Using this as the first step of a transfinite induction and using a procedure similar to the one used to find the Birkhoff center of a dynamical system in the theory of topological dynamics (i.e. a procedure with successively "peeling off" the parts of the dynamical system which $x \notin J(x)$) we showed that the Jacobson radical consists of all elements with Fourier coefficients which vanish on the set of recurrent points of the dynamical system and the zero Fourier coefficient is 0.

The structure of the present text has two parts. In the first part, we have divided the results and the methods we used into three categories, Topological Transformation Groups, Operator Theory and Dynamical Systems. In each category we give a brief description of the results in specific chapters. The second part consists of copies of the papers we analyze, again divided into three categories. For the economy of space, since we have included copies of all papers at the second part, when we want to refer, e.g. to the reference [2] in the paper [1] of the bibliography we will just write [1, reference 2].

Chapter 1

Limit sets and asymptotic methods in Topological Transformation Groups

References for this chapter are the following papers put in the same order as we present them in the following.

[1] H. Abels, G. Noskov and A. Manoussos, *Proper actions and proper invariant metrics*, SFB preprint 08-011.

[11] A. Manoussos and P. Strantzalos, *On embeddings of proper and equicontinuous actions in zero-dimensional compactifications*, SFB preprint 07-054, Trans. Amer. Math. Soc. **359** (2007), 5593-5609.

[10] A. Manoussos and P. Strantzalos, *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), 7-12.

[8] A. Manoussos, *On the action of the group of isometries on a locally compact metric space: closed-open partitions and closed orbits*, SFB preprint 09-026.

[2] H. Abels and A. Manoussos, *A group of isometries with non-closed orbits*, SFB preprint 09-064.

[9] A. Manoussos, *The group of isometries of a locally compact metric space with one end*, SFB preprint 09-066.

One of the most important notions related to the limit and to the extended limit sets in the theory of Topological Transformation Groups is the notion of a proper action. Proper actions are characterized by the property $J(x) = L(x) = \emptyset$ for every $x \in X$. In case G is a locally compact group we have the usual definition: an action is proper if for every $x, y \in X$ there exist neighborhoods U and V of x and y, respectively, such that the set $\{g \in G \,|\, gU \cap V \neq \emptyset\}$ has compact closure in G. The next interesting class of actions related to the limit and generalized limit sets is the class where $J(x) = L(x)$ holds for every $x \in X$ but $J(x)$ may not be empty. This class contains the isometric actions.

In [1] we characterized proper actions in terms of the geometry of the underlying space. Namely, we showed that a locally compact group G acts properly on a locally compact σ-compact metrizable space X if and only if there exists a G-invariant proper (Heine-Borel) compatible metric on X. A few words concerning terminology. A σ-compact space is a topological space that can be written as a countable union of compact sets. For locally compact metrizable spaces this is equivalent to separability. By a proper (or Heine-Borel) metric we mean a metric such that all balls of bounded radius have compact closures. In other words the previous result says that we can consider the group G, modulo the kernel of the action, as a closed subgroup of the group of isometries of a locally compact σ-compact metrizable space. Removing the assumption about metrizability for X we generalized the previous result as follows. If a locally compact group G acts properly on a locally compact σ-compact space X then there is a family of G-invariant proper continuous finite-valued pseudometrics which induces the topology of X. We showed also a converse result: let X be a topological space and let $\mathcal{D}$ be a family of proper continuous finite-valued pseudometrics on X, which induces the topology of X. Let G be the group of all bijective maps $X \to X$, leaving every $d \in \mathcal{D}$ invariant. If we endow G with the compact-open topology then G is a locally compact topological group and acts properly on X.

There is a remarkable invariant concerning the cardinality of the ends of a locally compact and connected space with the "property Z" which admits a proper action of a non-compact group. "Property Z" ia a certain technical connectedness assumption: a space X has "property Z" if every compact subset of X is contained in a compact and connected one, for instance every locally compact connected and locally connected space has "property Z". When we say ends we mean the remainder of X in the end-point (Freudenthal) compactification εX of X. As it is proved in [11, reference 2] X has at most two or infinitely many ends. In [11] we provided a tool for studying properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces, by embedding such an action in a suitable zero-dimensional compactification (i.e. a compactification such that X has compact totally disconnected remainder) of the underlying space with pleasant properties. Precisely, given such an action we constructed a zero-dimensional compactification μX of X which is the maximal (in the ordering of zero-dimensional compactifications of X) with respect to the following properties: (a) the action has a continuous extension on μX, (b) if μL denotes the set of the limit points of the orbits of the initial action in μX, the restricted action of G on $\mu X \setminus \mu L$ remains properly discontinuous, is equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX (so all the information concerning the invariants is contained in the set μL) and (c) the action is indivisible, i.e. if $\lim g_i x_0 = e \in \mu L$ for some $x_0 \in \mu X \setminus \mu L$ and a net $\{g_i\}$ in G, then $\lim g_i y = e$ for every $x \in \mu X \setminus \mu L$ (so actually there is a correspondence between divergent nets in G and limit points in μL). As we showed by an example there is a locally compact, connected and paracompact space not having the "property Z" for which our compactification is different from the end point compactification. So, if X doesn't have the "property Z" εX may fail to have the above mentioned properties. The construction of the compactification μX stated above relies on a new construction: The action of G on μX is obtained by taking the initial action as an equivariant inverse limit of properly discontinuous G-actions on polyhedra, which are constructed via G-invariant locally finite open coverings of X, generated by locally finite coverings of (always existing) suitable fundamental sets of the initial action. As an application of the previously mentioned construction we have that μL consists of at most two or infinitely many points. Another result is that if X has the "property Z" then μX coincides with the end point compactification εX of X. Finally, we gave an application concerning the cardinality of the ends of X. To be more precise, let X be a locally compact, connected and paracompact space, and G be a non-compact group acting properly on X such that either G_1, the connected component of the neutral element of G, is non-compact, or G_1 is compact and G/G_1 contains an infinite discrete subgroup. Then X has at most two or infinitely many ends, and has at most two ends, if G_1 is not compact.

Another important class of transformation groups is the class in which $J(x) = L(x)$ holds for every $x \in X$. As we mentioned before this class contains the isometric actions. One of the first problems studied in this direction was the problem of the local compactness of the group of isometries and the way it acts on the underlying space. A classic result is the theorem of D. van Dantzig and B. L. van der Waerden which says that the group G of isometries of a connected, locally compact metric space X is locally compact (with respect to the compact-open topology) and acts properly on X (via the natural action $(g, x) \mapsto g(x)$ $g \in G$, $x \in X$). Combining this result with the result mentioned before about the cardinality of the ends of the space we have the following remarkable implication. For locally compact locally connected and connected metric space (e.g. a finite dimensional manifold) with finitely many but more than two ends the group of isometries is compact. In [10] we generalized the results of D. van Dantzig and B. L. van der Waerden for the case of a locally compact metric space which has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components). In particular it is shown that the group of isometries of X is locally compact but may fail to act properly on X even for the case that X has only two connected components.

The paper [8] can be considered as a first step towards the study of the natural action of the group of isometries G on a locally compact metric space (X, d) without the assumption that G is a locally compact group. We gave an answer to the following question: Assume that there is a pair of points $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$. What can we say about the convergence of $\{g_i\}$? The answer is that the net $\{g_i\}$ (or a subnet of it) converges pointwise on a closed open subset of X which contains the pseudo-component of x. This result shows also "what is behind the lines of the proofs" for all the already well known results when G is locally compact and so, we can recover them using a unifying approach. Moreover, it leads to a simple decomposition of X into closed-open invariant disjoint sets that are related to various limit properties of the orbits in X. More precisely, we showed that if G is locally compact and not compact and $CL = \{x \in X \mid L(x)$ is not empty and compact$\}$, $NCL = \{x \in X \mid L(x)$ is not compact$\}$ and $P = \{x \in X \mid L(x)$ is the empty set$\}$, then the sets CL, NCL and P are closed-open G-invariant disjoint, their union is X and each one of them is a union of pseudo-components (for the definition of a pseudo-component, introduced by S. Gao and A. S. Kechris, see [8, reference 8] and [8]). In case P is not empty we have a very interesting result concerning its structure. If G is not compact and has compact space of connected components (or the connected component of the identity of G is not compact) then, P is homeomorphic to a product of the form $\mathbb{R}^n \times M$ for some $n \in \mathbb{N}$ where M is a closed subset of P. Actually one can take as n the same n if we write the group G as a homeomorphic image of the product $\mathbb{R}^n \times K$ where K is a maximal compact subgroup of G in Malcev-Iwasawa's decomposition theorem for G. We showed, also, that the sets CL, NCL and P may coexist in any combination.

In [2, reference 3] S. Gao and A. S. Kechris asked the following question. Let (X, d) be a locally compact complete metric space with finitely many pseudo-components or connected components. Does its group of isometries have closed orbits? This is the case if X is connected since then the group of isometries acts properly by the result of van Dantzig and van der Waerden we mentioned above and hence all of its orbits are closed. The above question arose in the following context. Suppose a locally compact group with a countable base acts on a locally compact space with a countable base. Then the action has locally closed orbits (i.e. orbits which are open in their closures) if and only if there exists a Borel section for the action (see [2, reference 4], [2, reference 2]) or, in other terminology, the corresponding orbit equivalence relation is smooth. For isometric actions it is easy to see that an orbit is locally closed if and only if it is closed. In this paper we gave a negative answer to the question of Gao and Kechris. Our space is a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. In the course of the construction we gave an example of a 2-dimensional manifold with two connected components one compact and one non-compact and a complete metric whose group G of isometries also has non-closed dense orbits on the compact component. The difference is that G contains a subgroup of index 2 which is isomorphic to $\mathbb{R}$. Finally in [9] we studied the action of the group of isometries G of a locally compact metric space X with one end. Using technics we developed in [8] , we showed that X has only finitely many pseudo-components exactly one of which is not compact and G acts properly on this pseudo-component. The complement of the non-compact component is a compact subset of X and G may fail to act properly on it.

Chapter 2

Operator Theory

In this chapter we present results from Operator Theory related to topologically transitive, locally topologically transitive operators (J-class), hypercyclic operators and dynamics of commuting tuples of matrices. Before, we present our results, and for our convenience, let us recall some definitions. A topologically transitive operator is a bounded linear operator T on a Banach space X such that $J(x) = X$ for every $x \in X$ or, in other words, for every pair of non-empty open sets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \emptyset$. A bounded linear operator on a separable Banach space is hypercyclic if it has the property that $L(x) = X$ for some non-zero vector $x \in X$ (i.e. the orbit of x is dense in X). Actually the existence of one (non-zero) vector $x \in X$ such that $L(x) = X$ is enough to ensure that the set of vectors with this property is a dense G_δ subset of X. Obviously in the case of a hypercyclic operator $T : X \to X$, the space X must be separable and T is a topologically transitive operator. For separable spaces the converse is also true: Birkhoff's Transitivity Theorem says that a topologically transitive operator on a separable Banach space is hypercyclic. In [6] we introduced and studied a new class of operators called locally topologically transitive or J-class operators. This class of operators is characterized by the property that there exists a non-zero vector $x \in X$ with $J(x) = X$, so, J-class operators can be viewed as a "localization" of the notion of topologically transitive and hypercyclic operators.

References for this chapter are the following papers put in the same order as we present them in the following.

[6] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, SFB preprint 07-028, to appear in J. Operator Theory.

[5] G. Costakis and A. Manoussos, *J-class weighted shifts on the space of bounded sequences of complex numbers*, SFB preprint 07-029, Integral Equations Operator Theory **62** (2008), 149-158.

[4] G. Costakis, D. Hadjiloucas and A. Manoussos, *On the minimal number of matrices which form a locally hypercyclic, non-hypercyclic tuple*, SFB preprint 09-035, J. Math. Anal. Appl. **365** (2010), 229-237.

[3] G. Costakis, D. Hadjiloucas and A. Manoussos, *Dynamics of tuples of matrices*, SFB preprint 08-032, Proc. Amer. Math. Soc. **137** (2009), 1025-1034.

The study of dynamics of linear operators is a rapidly growing research area in Analysis and Geometry. In general we look for the dynamics created by the iterates of a bounded linear operator $T : X \to X$ on a complex or real Banach or Hilbert space X ore more generally on a Fréchet space X (that is a locally convex topological vector space whose topology is defined by a translation invariant complete metric). One of the first classes of operators studied were the classes of topologically transitive and hypercyclic operators. Some examples of hypercyclic operators are the following (a) the translation operator $T_\alpha : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $T_\alpha(f) = f(z+\alpha)$, where $z \in \mathbb{C}$, α is a non-zero complex number and $H(\mathbb{C})$ is the space of holomorphic functions on $\mathbb{C}$ (G. D. Birkhoff 1929), (b) the differentiation operator on $H(\mathbb{C})$ (G. R. MacLane 1952) and (c) for every scalar λ of modulus greater than 1 the operator λB on $l^p(\mathbb{N})$ for each $1 < p < +\infty$ where B is the backward shift on $l^p(\mathbb{N})$ (S. Rolewicz 1969). Actually the hypercyclic operators in the previous examples have also the additional property that the set of periodic points is dense and they are chaotic (in the sense of R. L. Devaney).

In [6] we introduced and studied a new class of operators called locally topologically transitive or J-class operators. Recall that an operator is called J-class if there exists a non-zero vector $x \in X$ with $J(x) = X$. The reason we excluded the zero vector is to avoid certain trivialities, as for example the multiples of the identity operator acting on a finite or infinite dimensional space. This class can be viewed as a "localization" of the notion of topologically transitive and hypercyclic operators. Hypercyclic and J-class operators can occur only in infinite dimensional spaces. As it turns out this new notion of operators although different from the notion of hypercyclic operators shares some similarities with the behavior of hypercyclic operators. No compact, positive or normal operators can be J-class. We would like to stress that some non-separable Banach spaces, like the space $l^\infty(\mathbb{N})$ of bounded sequences, supports J-class operators (in [6] we showed that the operator λB for every scalar λ of modulus greater than 1 is J-class, where B is the backward shift on $l^\infty(\mathbb{N})$), while it is known [6, reference 3] that the space $l^\infty(\mathbb{N})$ does not support topologically transitive operators. A connection between hypercyclic and J-class operators is given in the main theorem of [6]. We showed that if x is a cyclic vector for an operator $T : X \to X$

and the set $J(x)$ has non-empty interior then $J(x) = X$ and, in addition, T is hypercyclic without x being necessarily a hypercyclic vector (i.e a vector with dense orbit). An important implication of this theorem is that it gives the Bourdon-Feldman Theorem as a corollary. Bourdon-Feldman's Theorem [6, reference 11] says that somewhere dense orbits are everywhere dense and plays an important role in the theory of hypercyclic operators. Finally we showed that if T is a bilateral or a unilateral weighted shift on the space of square summable sequences then T is hypercyclic if and only if T is a J-class operator. At this point, we would like to mention that in a recent book of F. Bayart and É. Matheron (*Dynamics of linear operators*, Cambridge Tracts in Mathematics, 179. Cambridge University Press, Cambridge, 2009) which is actually the first published book concerning dynamics of linear operators, they referred to J-class operators and they used our asymptotic technics to simplify lengthy proofs of old results.

In [5] we provided a characterization of J-class unilateral weighted shifts on $l^\infty(\mathbb{N})$ in terms of their weight sequences and we described the set of the J-vectors (i.e. vectors $x \in l^\infty(\mathbb{N})$ such that $J(x) = l^\infty(\mathbb{N})$). In contrast to the previously mentioned result we showed that a bilateral weighted shift on $l^\infty(\mathbb{Z})$ cannot be a J-class operator. As we mentioned before, hypercyclic and J-class operators can occur only in infinite dimensional spaces. This is in contrast with the case of hypercyclic and J-class commuting tuples of matrices. In [4] we extended the notion of a J-class operator to that of a J-class tuple of operators. We then showed that the class of hypercyclic tuples of operators forms a proper subclass to that of J-class tuples of operators. What is rather remarkable is that in every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, a pair of commuting matrices exists which forms a J-class non-hypercyclic tuple. This comes in direct contrast to the case of hypercyclic tuples where the minimal number of matrices required for hypercyclicity is related to the dimension of the vector space. Finally in [4], as also in [3], we gave some complementing results concerning hypercyclic and J-class commuting pairs of matrices in diagonal or in upper triangular form.

Chapter 3

Dynamical Systems

In this chapter we present an answer we gave to a long standing question asked by W. B. Arveson and K. B. Josephson in 1969 concerning the problem of the description of the radical of the analytic crossed product of a classical dynamical system in terms of the dynamic behavior of the system. The analytic crossed product of a classical dynamical system is a non self adjoint algebra of operators that characterizes the dynamical system. Two dynamical systems are topologically conjugate if and only if the corresponding analytic crossed products are isomorphic as algebras. Reference for this chapter is the following paper.

[7] A. P. Donsig, A. Katavolos and A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. **187** (2001), 129-145.

There is a rich interplay between operator algebras and dynamical systems, going back to the founding work of F. J. Murray and J. von Neumann in the 1930's. Crossed product constructions continue to provide fundamental examples of von Neumann algebras and C^*-algebras as also remarkable results in the theory of dynamical systems. Comparatively recently, W. B. Arveson in 1967 introduced a non-selfadjoint crossed product construction, called the analytic crossed product or the semi-crossed product, which has the remarkable property of capturing all of the information about the dynamical system. By this we mean that two analytic crossed product algebras are isomorphic as complex algebras if and only if the underlying dynamical systems are topologically conjugate, i.e. there is a homeomorphism between the spaces that intertwines the two actions. The construction of an analytic crossed product starts with a dynamical system, i.e. a locally compact Hausdorff space X and a continuous, proper surjection $\phi : X \to X$. Consider the algebra generated by $C_0(X)$ (i.e. the space of continuous functions of X that vanish at infinity) and a symbol U, where U satisfies the relation $fU = U(f \circ \phi)$, $f \in C_0(X)$. The elements F of this algebra can be viewed as noncommutative polynomials in U of the form $F = \sum_{n=0}^{N} U^n f_n$, $f_n \in C_0(X)$, $N \in \mathbb{N}$. Let us call this algebra $\mathcal{A}_0$. We formed the Banach Algebra $l_1(\mathcal{A}_0)$ by providing a norm to elements F as above by setting $\|F\|_1 = \sum_{n=0}^{N} \|f_n\|_{C_0(X)}$ and then completing $\mathcal{A}_0$ in this norm. On the other hand, we can define the class of covariant representations of $\mathcal{A}_0$ and complete $\mathcal{A}_0$ in the resulting norm. Either approach yields the same analytic crossed product $C_0(X) \times_\phi \mathbb{Z}_+$. By a covariant representation of $\mathcal{A}_0$ we mean a homomorphism π of $\mathcal{A}_0$ into the bounded operators of a Hilbert space, which is a $*$-representation when restricted to $C_0(X)$, viewed as a subalgebra of $\mathcal{A}_0$, and such that $\pi(U)$ is an isometry. Let us denote an element of the analytic crossed product by $\sum_{n=0}^{+\infty} U^n f_n$, $f_n \in C_0(X)$ and let us call the sequence $\{f_n\}$ the corresponding Fourier coefficients. A long standing question asked by W. B. Arveson and K. B. Josephson in 1969 was to characterize the Jacobson radical of the analytic crossed product in terms of the dynamic behavior of the system. Recall that the Jacobson radical of an algebra is the intersection of all primitive ideals, i.e. the intersection of kernels of all irreducible representations of the algebra. If the Jacobson radical is zero then the algebra is called semisimple. In [7] we solved this problem. We showed that the Jacobson radical consists of all elements of the form $\sum_{n=1}^{+\infty} U^n f_n$ such that each Fourier coefficient f_n vanishes on the set of recurrent points of the dynamical system (a point $x \in X$ is called recurrent if $x \in L(x)$). We generalized also this result for the case of a multivariable dynamical system, that is a locally compact Hausdorff space with a d-tuple of commuting proper surjections. In this case we need a modification of the notion of a recurrent point (as also a modification of the notion of the Birkhoff center of the dynamical system we used in the case of one variable). Namely, let $I \subset \{1, 2, \ldots, d\}$. A point $x \in X$ is called I-recurrent if there is a sequence $\{\mathbf{n_k}\} \subset \mathbb{N}^d$ such that the i-th entry of $\mathbf{n_{k+1}}$ is greater than the i-th coordinate of $\mathbf{n_k}$ for every $i \in I$ such that $\phi_{\mathbf{n_k}} x \to x$. In this case the Jacobson radical is the closed ideal generated by all monomials of the form $U_{\mathbf{n}} f$, $\mathbf{n} \neq \mathbf{0}$ where f vanishes on the set of recurrent point corresponding to the support of $\mathbf{n}$. Some interesting corollaries of the previous results are the following: (a) The analytic crossed product is semisimple if and only if it is semiprime and (b) The prime radical of the analytic crossed product coincides with the Jacobson radical if and only if it is closed. Recall that the prime radical is the intersections of all prime ideals and the algebra is called semiprime if the prime radical is zero or, equivalently, if there are no (non-zero) nilpotent ideals.

Bibliography

[1] H. Abels, G. Noskov and A. Manoussos, *Proper actions and proper invariant metrics*, SFB preprint 08-011.

[2] H. Abels and A. Manoussos, *A group of isometries with non-closed orbits*, SFB preprint 09-064.

[3] G. Costakis, D. Hadjiloucas and A. Manoussos, *Dynamics of tuples of matrices*, SFB preprint 08-032, Proc. Amer. Math. Soc. **137** (2009), 1025-1034.

[4] G. Costakis, D. Hadjiloucas and A. Manoussos, *On the minimal number of matrices which form a locally hypercyclic, non-hypercyclic tuple*, SFB preprint 09-035, J. Math. Anal. Appl. **365** (2010), 229-237.

[5] G. Costakis and A. Manoussos, *J-class weighted shifts on the space of bounded sequences of complex numbers*, SFB preprint 07-029, Integral Equations Operator Theory **62** (2008), 149-158.

[6] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, SFB preprint 07-028, to appear in J. Operator Theory.

[7] A. P. Donsig, A. Katavolos and A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. **187** (2001), 129-145.

[8] A. Manoussos, *On the action of the group of isometries on a locally compact metric space: closed-open partitions and closed orbits*, SFB preprint 09-026.

[9] A. Manoussos, *The group of isometries of a locally compact metric space with one end*, SFB preprint 09-066.

[10] A. Manoussos and P. Strantzalos, *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), 7-12.

[11] A. Manoussos and P. Strantzalos, *On embeddings of proper and equicontinuous actions in zero-dimensional compactifications*, SFB preprint 07-054, Trans. Amer. Math. Soc. **359** (2007), 5593-5609.

Papers on
Topological Transformation Groups

Proper actions and proper invariant metrics

H. Abels, A. Manoussos and G. Noskov *

Abstract

We show that if a (locally compact) group G acts properly on a locally compact σ-compact space X then there is a family of G-invariant proper continuous finite-valued pseudometrics which induces the topology of X. If X is furthermore metrizable then G acts properly on X if and only if there exists a G-invariant proper compatible metric on X.

Subject classification [2000]: Primary 37B05, 54H15; Secondary 54H20, 54D45.

Keywords: Proper action, group of isometries, proper metric, proper pseudometric, Heine-Borel metric.

1 Introduction

We establish a close connection between proper group actions and groups of isometries. There is an old result in this direction, proved in 1928 by van Dantzig and van der Waerden It says that for a locally compact connected metric space (X, d) its group $G = \mathrm{Iso}(X, d)$ of isometries is locally compact and acts properly. That the action is proper is no longer true in general, if X is not connected, although G is sometimes still locally compact, see [13]. Concerning properness of the action, Gao and Kechris [6] proved the following result. If (X, d) is a proper metric space, then G (is locally compact and) acts properly on X. Recall that a metric d on a space X is called proper if balls of bounded radius have compact closures.

There is the following converse result. If a locally compact group G acts properly on a locally compact σ-compact metrizable space X, then there is a compatible G-invariant metric d on X [12]. In this paper we prove that under these hypotheses there is actually a compatible G-invariant *proper* metric on X. We call a metric on a topological space

*During this research the second and the third named authors were fully supported by the SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. They are grateful for its generosity and hospitality. The paper was finished while the first named author was staying at MSRI in Berkeley. He wishes to thank the MSRI for its hospitality and support, as well as the SFB 701 for its support.

compatible if induces its topology. Note that a proper metric space is σ-compact. For the records, here is one version of our main result, namely the one for metrizable spaces (see also theorem 4.2).

Theorem 1.1. *Suppose the (locally compact) topological group G acts properly on the metrizable locally compact σ-compact topological space X. Then there is a G-invariant proper compatible metric on X.*

These results raise the question if they generalize to the non-metrizable case. We give a complete answer as follows. Recall that a pseudometric on X is a function d on $X \times X$ which has all the properties of a metric, except that its value may be ∞ and that $d(x, y) = 0$ may not imply that $x = y$. For a precise definition see below definition 2.1. A locally compact space is σ-compact if and only if has a proper finite-valued continuous pseudometric, as is easily seen, see e.g. below, the proof of corollary 5.3. It then actually has a family of such pseudometrics which induces the topology of X. The corresponding statement for the equivariant situation is the following version of the main result of our paper, namely for not necessarily metrizable spaces (see also theorem 4.1).

Theorem 1.2. *Let G be a (locally compact) topological group which acts properly on a locally compact σ-compact Hausdorff space X. Then there is a family of G-invariant proper finite-valued continuous pseudometrics on X which induces the topology of X.*

The connection of theorem 1.1 and theorem 1.2 is given by the following result. We are in the case of theorem 1.1 iff there is a countable family as in theorem 1.2. For a precise statement see corollary 4.4.

Note that continuity of the pseudometrics follows from the other properties, see remark 5.5.

This theorem may be considered as the converse of the following theorem, see below theorem 3.1.

Theorem 1.3. *Let X be a topological space and let $\mathcal{D}$ be a family of proper continuous finite-valued pseudometrics on X, which induces the topology of X. Let G be the group of all bijective maps $X \to X$, leaving every $d \in \mathcal{D}$ invariant. Endow G with the compact–open topology. Then G is a locally compact topological group and acts properly on X.*

The main result of our paper has been proved already for the special case of a smooth manifold. Namely Kankaanrinta proved in [9] that if a Lie group G acts properly and smoothly on a smooth manifold M, then M admits a complete G-invariant Riemannian metric. A consequence of our main result for the metrizable case is the following result of Haagerup and Przybyszewska [7]: Every second countable locally compact group has a left invariant compatible proper metric which generates its topology, see below corollary 9.5. Proper G–invariant metrics have been used in several fields of mathematics, see [8] and [11]. For more information about related work, open questions and miscellaneous remarks see the last chapter of this paper.

2 Preliminaries

Pseudometrics

Definition 2.1. *A pseudometric d on a set X is a function $d : X \times X \to [0, +\infty]$ which fulfills for $x, y, z \in X$ the following properties*

i) $d(x, x) = 0$,

ii) $d(x, y) = d(y, x)$,

iii) $d(x, y) + d(y, z) \geq d(x, z)$.

Thus, loosely speaking, a pseudometric is a metric except that its values may be $+\infty$ and $d(x, y) = 0$ does not imply $x = y$. A family $\mathcal{D}$ of pseudometrics on X induces a topology on X, for which finite intersections of balls $B_d(x, r) := \{y \in X; d(x, y) < r\}$ with $x \in X$, $d \in \mathcal{D}$ and $r \in [0, \infty)$ form a basis. This topology is the coarsest topology for which every $d \in \mathcal{D}$ is a continuous function on $X \times X$. The topology of a topological space X is induced by a family of pseudometrics if and only if X is completely regular, see [3, Ch. X, §1.4 Theorem 1 and §1.5 Theorem 2]. A topological space X is called metrizable if its topology is induced by an appropriately chosen metric d on X. Such a metric d on X is then called a *compatible metric.*

From now on we will call a locally compact Hausdorff space simply a "space", for short. Recall that a space is called σ–compact if it can be written as a countable union of compact subsets. A σ–compact space is metrizable if and only if it is second countable, i.e., its topology has a countable base, see [3, Ch. IX, §2.9 Corollary].

A pseudometric d on a space X will be called proper if every ball of finite radius has compact closure. A space X together with a compatible proper metric d will be called a *proper metric space*. It is also called a *Heine–Borel space* by some authors and also a *finitely–compact space* by others. Important examples of proper metric spaces are the Euclidean spaces and the space $\mathbb{Q}_p$ of rational p–adics with their usual metrics.

The topology of a space can be induced by a family of pseudometrics, since a space (understood: locally compact Hausdorff) is completely regular. The topology of a σ–compact space can be induced by a family of proper finite–valued pseudometrics (see corollary 5.3). One of our main results, theorem 1.2, spells out for which actions there is a family of invariant proper finite–valued pseudometrics inducing the topology, namely the proper actions. And theorem 1.3 says that these are essentially the only ones for which such a family exists.

Now let $(X, \mathcal{D})$ be a space X together with a family $\mathcal{D}$ of pseudometrics inducing its topology. A case of particular importance is when $\mathcal{D}$ consists of just one metric, which by assumption induces the topology of X. Let $G = \mathrm{Iso}(X, \mathcal{D})$ be the group of isometries of $(X, \mathcal{D})$, that is the group of all bijections $X \to X$ leaving every $d \in \mathcal{D}$ invariant. Endow G with the topology of pointwise convergence. Then G will be a topological group [3, Ch. X, §3.5 Corollary]. On G there is also the topology of uniform convergence on compact subsets which is the same as the compact–open topology. In our case, these topologies coincide with the topology of pointwise convergence, and the natural action of G on X is continuous [3, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]. We shall prove soon, that if at least one of the pseudometrics d in $\mathcal{D}$ is proper then G is locally compact. In this case the natural action of G on X is even proper. We will discuss this notion now.

Definition 2.2. *A continuous map $f : X \to Y$ between spaces is called proper if one of the following two equivalent conditions holds*

i) $f^{-1}(K)$ is compact for every compact subset K of Y.

ii) f is a closed map and the inverse image of every singleton is compact.

Let G be a topological group. Suppose a continuous action of G on a space X is given.

Proposition 2.3. and Definition *The following conditions are equivalent*

i) The map $G \times X \longrightarrow X \times X$, $(g, x) \longmapsto (gx, x)$, is proper.

ii) For every pair A and B of compact subsets of X the transporter

$$G_{AB} := \{g \in G;\ gA \cap B \neq \emptyset\}$$

from A to B is compact.

iii) Whenever we have two nets $(g_i)_{i\in I}$ in G and $(x_i)_{i\in I}$ in X, for which both $(x_i)_{i\in I}$ and $(g_i x_i)_{i\in I}$ converge, then the net $(g_i)_{i\in I}$ has a convergent subnet.

The action of G on X is called proper if one of these conditions holds.

For a proof see [3, Ch. I, §10.2 Theorem 1 and Ch. III, §4.4 Proposition 7]. For more information on proper group actions see the forthcoming book [1].

Note that if the action of G on X is proper, then G is locally compact, by ii). And if furthermore X is σ–compact, then G is also σ–compact, by ii).

It is useful to rephrase the definition of properness in terms of limit sets. Let $(x_i)_{i\in I}$ be a net in the – not necessarily locally compact – topological space X. We say that the net $(x_i)_{i\in I}$ diverges and write $x_i \underset{i\in I}{\longrightarrow} \infty$, if the net $(x_i)_{i\in I}$ has no convergent subnet. If X is locally compact, a net $(x_i)_{i\in I}$ in X diverges if and only if it converges to the additional point ∞ of the one point (also called Alexandrov–) compactification of X.

Let again the topological group G act on the space X. For $x \in X$ the *limit set* $L(x)$ is defined by

$$\begin{aligned} L(x) := \{y;\ & \textit{there exists a divergent net } (g_i)_{i\in I} \textit{ in } G \\ & \textit{such that } (g_i x)_{i\in I} \textit{ converges to } y\} \end{aligned}$$

and the *extended limit set* $J(x)$ is defined by

$$\begin{aligned}J(x) :=\{y; & \textit{ there exists a divergent net } (g_i)_{i\in I} \textit{ in } G \\ & \textit{and a net } (x_i)_{i\in I} \textit{ in } X \textit{ converging to } x, \\ & \textit{such that } (g_i x_i)_{i\in I} \textit{ converges to } y\}.\end{aligned}$$

Thus, the action of G on X is proper if and only if the following condition holds:

iv) *$J(x) = \emptyset$ for every $x \in X$,*
since iv) is equivalent to iii). If furthermore $\mathcal{D}$ is a family of pseudometrics inducing the topology of X and every $g \in G$ leaves every $d \in \mathcal{D}$ invariant, then it is easy to see that

v) *$L(x) = \emptyset$ implies $J(x) = \emptyset$.*

3 The group of isometries of a proper metric space

Let again X be a locally compact Hausdorff space, let $\mathcal{D}$ be a family of pseudometrics inducing the topology of X and let G be the group of isometries of $(X, \mathcal{D})$ with its natural topology, as above.

Theorem 3.1. *If at least one of the pseudometrics in $\mathcal{D}$ is proper then G is locally compact and the natural action of G on X is proper.*

The special case that $\mathcal{D}$ consists of just one metric is due to Gao and Kechris [6], as follows.

Theorem 3.2. *If (X, d) is a proper metric space then its group G of isometries is locally compact and its natural action of G on X is proper.*

Proof of theorem 3.1. It suffices to show that the natural action of G on X is proper. To prove this we will show that the limit set $L(x)$ is empty for every $x \in X$. Thus let $(g_i)_{i\in I}$ be a net in G for which $(g_i x)_{i\in I}$ converges to a point, say y, in X. We have to show that the net $(g_i)_{i\in I}$ has a convergent subnet. We may assume that $g_i x$ is contained in the relatively compact ball $B_d(y, r)$ for every $i \in I$, where d is a proper pseudometric in $\mathcal{D}$ and $r > 0$. We will use the Arzela–Ascoli theorem. Let $z \in X$. The points $g_i z$, $i \in I$, are contained in the ball $B_d(z, R)$, where $R = r+d(x, z)$. Thus the set $\{g_i z; i \in I\}$ is relatively compact for every $z \in X$. The family of maps $\{g_i; i \in I\}$ is uniformly equicontinuous, being a subset of the uniformly equicontinuous family G of maps from X to X. It follows from the Arzela–Ascoli theorem that the net $(g_i)_{i\in I}$ has a subnet $(g_j)_{j\in J}$ which converges uniformly on compact subsets to a map g. Clearly, g leaves every $d \in \mathcal{D}$ invariant. To see that g is actually invertible look at the net $(g_j^{-1})_{j\in J}$. We have $g_j^{-1} y \in B_d(x, r)$ and hence $g_j^{-1} z \in B_d(z, R')$ where $R' = r + d(x, z)$. Then the net $(g_j^{-1})_{j\in J}$ has a subnet which converges uniformly on compact subsets to a map f. It then follows that f and g are inverse of each other.

Remark 3.3. The sets $K(E) := \{x \in X\,;\, Ex \text{ is relatively compact}\}$, where $E \subset \mathrm{Iso}(X,d)$ played a crucial role in [13] where it is proved that they are open–closed subsets of X. In the case of a proper metric space (X,d) the set $K(E)$ is either the empty set or the whole space X as shown in the proof of Theorem 3.1. Using Bourbaki [3, Ch. X, Exercise 13, p. 323] we may also show that sets $K(E)$ are open-closed subsets of X but we must be careful! Even in the legendary "Topologie Générale" of Bourbaki there is at least one mistake! Precisely in the aforementioned Exercise 13 of Ch. X, p. 323, part d) it is said that if E is a uniformly equicontinuous family of homeomorphisms of a locally compact uniform space X then $K(E)$ is a closed subset of X. This is not true if E is not a subset of a uniformly equicontinuous *group* of homeomorphisms of X as we can easily see by the following counterexample.

Counterexample 3.4. Let

$$X = \bigcup_{k=1}^{\infty}\{(x,y)\,;\, x = \frac{1}{k},\, y \geq 0\} \cup \{(x,y)\,;\, x = 0,\, y > 0\}$$

be endowed with the Euclidean metric. Consider the family $E = \{f_n\}$ of selfmaps of X defined by $f_n(x,y) = (x, \frac{y}{n})$. The family E consists of uniformly equicontinuous homeomorphisms of X and $K(E) = \bigcup_{k=1}^{\infty}\{(x,y)\,;\, x = \frac{1}{k},\, y \geq 0\}$ as can be easily checked. Hence the set $K(E)$ is not closed in X.

4 Proper invariant metrics and pseudometrics, outline of the proof

The main results of our paper are the following converses of theorems 3.1 and 3.2. Again, X is a space, i.e., a locally compact Hausdorff space and G is a Hausdorff topological group. Suppose we are given a continuous action of G on X.

Theorem 4.1. *Suppose X is σ–compact. If the action of G on X is proper then there is a family $\mathcal{D}$ of proper finite–valued G–invariant pseudometrics on X, which induces the topology of X.*

Theorem 4.2. *Suppose X is σ–compact. If the action of G on X is proper and X is metrizable, then there is a compatible G–invariant proper metric d on X.*

Remark 4.3. *If the action is proper, it is easy to see that the kernel of the action $K := \{g \in G\,;\, gx = x \text{ for every } x \in X\}$ is compact and the action map induces an isomorphism of topological groups of G/K onto a closed subgroup of Iso$(X,\mathcal{D})$, resp. Iso(X,d). We thus have a complete correspondence between proper actions and isometry groups of proper metrics or pseudometrics.*

Corollary 4.4. *Suppose X is σ–compact and G acts properly on X. Then the following properties of X are equivalent*

a) X is metrizable.

b) There is a compatible G–invariant proper metric on X.

c) There is a countable family of finite–valued pseudo–metrics on X, which induces the topology of X.

d) There is a countable family of proper finite–valued G–invariant pseudometrics on X, which induces the topology of X.

Proof. a) $\Longrightarrow$ b) by theorem 4.2, b) $\Longrightarrow$ d) and d) $\Longrightarrow$ c) are trivial, c) $\Longrightarrow$ a) is a well known theorem of topology [3, Ch. IX, §2.4 Corollary 1] whose proof is similar to the argument in the last paragraph of the proof of lemma 8.10 a). □

The proof of theorems 4.1 and 4.2 will occupy most of the remainder of the paper. Let us briefly describe the plan of the proof. We describe the plan for the case of a family of pseudometrics, the proof for the metrizable case simplifies at some points.

1. We first construct a family $\mathcal{D}$ of pseudometrics on X, with values in [0,1] which induces the topology of X, see section 5.

2. Next we show how to make every $d \in \mathcal{D}$ G–invariant, see section 6.

3. Then we make every $d \in \mathcal{D}$ orbitwise proper, see section 7.

4. These steps are fairly routine. We then present our main tool, namely the "measuring stick construction". Imagine a family of measuring sticks given by distances of closely neighboring points. We then define a pseudometric d on X by taking for x, y in X as $d(x, y)$ the infimum of all measurements along sequences of points $x = x_0, \ldots, x_n = y$ such that the distance of any two adjacent points is given by measuring sticks. For a precise definition, actually several equivalent ones, see section 8. It turns out that we then get for an appropriate family of measuring sticks a proper pseudometric. The disadvantage of this construction is that there may be points which cannot be connected by sequences as above. Equivalently, there may be points x, y with $d(x, y) = \infty$.

5. We then use our "bridge construction", see section 9. Think of pairs of points with $d(x, y) < \infty$ as lying on the same island. Thus what we call an island is an equivalence class of the equivalence relation defined as $x \sim y$ iff $d(x, y) < \infty$. We connect (some of) these islands by bridges and attribute (high) weights to these bridges. We then define a new pseudometric similarly as above using the already defined pseudometric on the islands and the weights of bridges. We thus obtain a proper pseudometric with finite values and actually a whole family of such, which induces the topology of X. All these constructions are done in a G–invariant way, so that the resulting pseudometrics are G–invariant.

5 A compatible metric and proper pseudometrics

Again, by a space we mean a locally compact Hausdorff space. Recall the following basic metrization result, see [3, Ch. IX, §2.9 Corollary].

Theorem 5.1. *For a space X the following properties are equivalent*

a) X is second countable, i.e., its topology has a countable base.

b) The one–point compactification $\overline{X}$ of X is metrizable.

c) X is metrizable and σ–compact.

If a space is metrizable we may assume that the metric d inducing the topology has values in [0,1]. We just have to replace d by d_1 with $d_1(x, y) := \frac{d(x,y)}{1+d(x,y)}$.

For the general case of a not necessarily metrizable σ–compact space — and for later use — we need the following easy lemma, whose proof is left to the reader.

Lemma 5.2. *A space X is σ–compact if and only if there is a proper continuous function $f : X \longrightarrow [0, \infty)$.*

Corollary 5.3. *On every σ–compact space X there is a family $\mathcal{D}$ of proper finite–valued pseudometrics inducing the topology of X.*

Proof. Let $\mathcal{D}_0$ be the family of pseudometrics on X of the form

$$d_f(x, y) := |f(x) - f(y)|$$

for $x, y \in X$, where $f : X \longrightarrow \mathbb{R}$ is a continuous function. Then $\mathcal{D}_0$ induces the topology of X. Here we do not use that X is σ–compact. But in the next step we do. If X is σ–compact let $\mathcal{D}$ be the family $\mathcal{D} := \{d + d_f \ ; \ d \in \mathcal{D}_0\}$, where $f : X \longrightarrow \mathbb{R}$ is proper and continuous. Then $\mathcal{D}$ induces the topology of X and consists of proper finite–valued pseudometrics. $\square$

The same trick yields the following corollary.

Corollary 5.4. *The following properties of a space X are equivalent.*

a) X has a compatible proper metric.

b) X is metrizable and σ–compact.

c) X is metrizable and separable.

d) X is second countable.

Remark 5.5. Note the if a pseudometric d belongs to a family of pseudometrics inducing the topology of X then d is continuous. Since then $B_d(x, r)$ is a neighborhood of x for every $x \in X$ and every $r > 0$, and hence the function $y \longmapsto d(x, y)$ is continuous at x for every $x \in X$, which easily implies that d is continuous by the triangle inequality.

6 Making the metrics or pseudometrics G–invariant

Now suppose X is a space, G is a Hausdorff topological group and a proper continuous action of G on X is given.

Step 2. *If X is σ–compact, then there is a family of G–invariant continuous finite–valued pseudometrics inducing the topology of X. If X is furthermore metrizable then there is a compatible G–invariant metric on X.*

We present two proofs.

The first one is due to Koszul [12] and uses the concept of a fundamental set, a concept we will need again, later on. The second one uses the notion of an equicontinuous action on the one–point compactification of X. Unfortunately, in the process we loose the property that our (pseudo–)metrics are proper.

Definition 6.1. *A subset F of X is called a* fundamental set *for the action of G on X if the following two conditions hold.*

a) $GF = X$

b) G_{KF} *has compact closure for every compact subset K of X.*

Concerning b), recall the definition of the transporter $G_{AB} = \{g \in G\ ;\ g\,A \cap B \neq \emptyset\}$ from A to B. Note that only proper actions can have a fundamental set, since a) implies that

$$G_{AB} \subset G_{BF}^{-1} \cdot G_{AF}$$

and hence G_{AB} is relatively compact if A and B are compact, by b), and then G_{AB} is actually compact, by continuity of the action. There is the following converse, see [12].

Proposition 6.2. *If X is σ–compact, then there is an open fundamental set for every proper action.*

Step 2, 1$^{\text{st}}$ proof. Let F be an open fundamental set for the action of G on X. Let d be a continuous finite–valued pseudometric on X. Let d' be the supremum of all pseudometrics on X with the property that $d' \mid F \times F \leq d$ and $d' \mid (X \setminus F) \times (X \setminus F) = 0$. Explicitly, let r be the function on X with $r_d(x) = d(x, X \setminus F) := \inf\{d(x,y)\ ;\ y \in X \setminus F\}$. Then

$$d'(x,y) = \min\{d(x,y)\ ,\ r_d(x) + r_d(y)\}.$$

Note that for every $x \in F$ there is a neighborhood of x where d and d' coincide. The function d' is a finite–valued continuous pseudometric and the function $G \longrightarrow \mathbb{R}$, $g \longmapsto d'(gx, gy)$ is continuous and has compact support, namely contained in $G_{\{x,y\},F}$. Define

$$d''(x,y) = \int_G d'(gx, gy)dg$$

where dg is a right invariant Haar measure on G. Then d'' is a G–invariant pseudometric on X. The pseudometric d'' is actually a metric if d is a metric. Furthermore d'' is continuous for every $d \in \mathcal{D}$, by a uniform equicontinuity argument for functions on compact spaces. Thus the family $\mathcal{D}'' = \{d'' \ ; \ d \in \mathcal{D}\}$ induces a weaker topology than $\mathcal{D}$. The two topologies are actually equal since for every neighborhood V of $x \in X$ there are a compact neighborhood V_1 of x in X and a compact neighborhood U_1 of e in G such that $U_1 V_1 \subset V$ and $U_1(X \smallsetminus V) \subset X \smallsetminus V_1$ and hence

$$d''(x,y) \geq d'(x, X \smallsetminus V_1) \cdot \int_{U_1} dg$$

for every $y \in X \smallsetminus V$, which implies our claim for $x \in F$ and hence for every x by G–invariance of the two topologies.

2nd proof. This proof is based on the notion of an equicontinuous group action. Consider the one point compactification $\overline{X} = X \cup \{\infty\}$. The action of G on X extends to an action of G on $\overline{X}$ by defining $g(\infty) = \infty$ for every $g \in G$. The extended action is continuous. Let $\mathcal{D}$ be a family of pseudometrics on $\overline{X}$ which induces the topology of $\overline{X}$. Without further assumptions on X we can take the family $\{d_f ; f : \overline{X} \to [0,1] \text{ continuous}\}$, see the proof of corollary 5.3. If $\overline{X}$ is metrizable, we can take $\mathcal{D}$ to consist of just one element. This is the case if and only if X is metrizable and σ–compact, see theorem 5.1. In any case, define for $d \in \mathcal{D}$ and $x, y \in X$

$$d'(x,y) := \sup_{g \in G} d(gx, gy),$$

and set $\mathcal{D}' = \{d' \ ; \ d \in \mathcal{D}\}$. We claim that $\mathcal{D}'$ induces the topology of X. Obviously, the topology induced by $\mathcal{D}'$ is finer than the topology of X, since $d' \geq d$ and $\mathcal{D}$ induces the topology of X.

Concerning the converse, consider the following property. The action of G on X is called *pointwise equicontinuous with respect* to $\mathcal{D}$ if for every $x \in X$, $d \in \mathcal{D}$ and $\epsilon > 0$ there is a neighborhood U of x such that for $y \in U$ we have $d(gx, gy) < \epsilon$ for every $g \in G$. Clearly, if this holds the topology defined by $\mathcal{D}'$ is weaker than the topology of X and our claim is proved. It thus remains to show

Lemma 6.3. *Let X be a space and let G be a topological group acting properly on X. Let $\mathcal{D}$ be a family of pseudometrics on $\overline{X}$ inducing the topology of $\overline{X}$. Then G acts pointwise equicontinuously on X with respect to $\mathcal{D}$.*

Proof. Arguing by contradiction, assume that there are $d \in \mathcal{D}$, $x \in X$, $\epsilon > 0$ and a net $(x_i)_{i \in I}$ in X converging to x and a net $(g_i)_{i \in I}$ in G such that $d(g_i x, g_i x_i) \geq \epsilon$ for every $i \in I$. It follows that $g_i \longrightarrow \infty$, since otherwise the net $(g_i)_{i \in I}$ has a convergent subnet, say $(g_j)_{j \in J}$ converging to $g \in G$. Then $g_j x \xrightarrow[j \in J]{} gx$ and $g_j x_j \xrightarrow[j \in J]{} gx$ contradicting $d(g_i x, g_i x_i) \geq \epsilon$ for every $i \in I$. It follows next that $g_i x_i \xrightarrow[i \in I]{} \infty$, since otherwise there would be a subnet $(g_j x_j)_{j \in J}$ converging to a point of X, which implies that there would be a convergent subnet of $(g_j)_{j \in J}$, by properness of the action. Thus $g_i x_i \xrightarrow[i \in I]{} \infty$ and $g_i \xrightarrow[i \in I]{} \infty$, which implies $g_i x \underset{i\in I}{\longrightarrow} \infty$, again by properness of the action. But then $d(g_i x, g_i x_i) \underset{i\in I}{\longrightarrow} 0$, since d is continuous on $\overline{X}$. This contradicts our assumption and finishes the proof. $\square$

Remark 6.4. The 2^{nd} proof shows step 2 for the metrizable case only under the additional assumption that $\overline{X}$ is metrizable, i.e., that X is metrizable and σ–compact. This is enough for our main results, though, because there all spaces are σ–compact.

Remark 6.5. The pseudometrics we obtain by these proofs are not proper, in general. This is clear for the second proof. For the first proof, even if we start from a proper (pseudo–) metric d, we obtain in case that the orbit space $G\backslash X$ is compact – so F is relatively compact – that d'' has an upper bound.

Remark 6.6. One could rephrase the notion of pointwise equicontinuity in terms of the unique uniformity on the compact space $\overline{X}$. We chose here to use the language of pseudometrics since proper (pseudo–) metrics are our final goal.

7 Orbitwise proper metrics and pseudometrics

If G acts on X we denote by $\pi : X \longrightarrow G\backslash X$ the natural map to the orbit space. We will call a pseudometric d on X *orbitwise proper* if $\pi\left(B_d(x,r)\right)$ has compact closure for every $x \in X$ and $0 < r < \infty$. Again, we assume the notation and hypotheses of the last section.

Step 3. *If X is σ–compact there is a family of G–invariant orbitwise proper finite–valued pseudometrics on X inducing the topology of X. If X is furthermore metrizable there is a G–invariant orbitwise proper compatible metric on X.*

Proof. If X is a space with a proper action, then the orbit space $G\backslash X$ is Hausdorff as well, see [3]. Clearly, $G\backslash X$ is locally compact. If furthermore X is σ–compact, so is $G\backslash X$. So there is a proper continuous function $f : G\backslash X \to [0,\infty)$, see lemma 5.2. The pseudometric $d' := d_{f\circ\pi}$ on X defined by

$$d'(x,y) = |f\pi(x) - f\pi(y)|$$

for $x, y \in X$ is orbitwise proper, continuous and G–invariant. Hence if $\mathcal{D}$ is a family of finite–valued G–invariant pseudometrics on X inducing the topology of X, so is $\mathcal{D}' = \{d+d' \; ; \; d \in \mathcal{D}\}$ and furthermore every pseudometric of this family is orbitwise proper. $\square$

8 The measuring stick construction

We first present our measuring stick construction in three equivalent ways. We then give a sufficient condition under which the resulting pseudometric is proper. This will be applied to our situation and yields step 4 of our proof.

8.1. Let X be a set, let d be a pseudometric on X and let $\mathcal{U}$ be a covering of X. We then define a new pseudometric $d' = d'(d,\mathcal{U})$ on X depending on d and $\mathcal{U}$ as follows: d' is the supremum of all pseudometrics d'' on X with the property that $d''|U \times U \leq d|U \times U$ for every $U \in \mathcal{U}$.

8.2. We think of pairs (x, y) of points lying in one $U \in \mathcal{U}$ as *measuring sticks* or *sticks,* for short. A sequence $x = x_0,\ x_1, \ldots, x_n = y$ of points in X, such that any two consecutive points form a stick, will be called a *stick path* from x to y of length n and d–length $\sum_{i=1}^{n} d(x_{i-1}, x_i)$. We claim that $d'(x, y)$ is the infimum of d–lengths of all stick paths from x to y. Since on one hand defining d' in this way clearly gives a pseudometric on X and $d'|U \times U \leq d|U \times U$. And, on the other hand, for every pseudometric d'' with the two properties above we have that $d''(x, y)$ is at most equal to the d–length of any stick path from x to y, because for every stick path $x = x_0, x_1, \ldots, x_n = y$ we have

$$d''(x, y) \leq \sum_{i=1}^{n} d''(x_{i-1}, x_i) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i).$$

We thus obtain the following properties of $d' = d'(d,\mathcal{U})$

a) $d' \geq d$

b) $d'|U \times U = d|U \times U$

c) If d is finite–valued on every $U \in \mathcal{U}$ then $d(x, y) < \infty$ if and only if there is a stick path from x to y.

8.3. An alternative way to describe this construction is the following: Let $\Gamma_{\mathcal{U}}$ be the following graph. The vertices of $\Gamma_{\mathcal{U}}$ are the points of X and the edges of $\Gamma_{\mathcal{U}}$ are the sticks, i.e., the pairs (x, y) contained in one $U \in \mathcal{U}$. So the graph $\Gamma_{\mathcal{U}}$ is closely related to the nerve of the covering $\mathcal{U}$. To every edge (x, y) of $\Gamma_{\mathcal{U}}$ we can associate the weight $d(x, y)$. Then for points x, y in X the pseudometric $d'(x, y)$ is the graph distance of the corresponding vertices of this weighted graph.

Let us now return to the case we are interested in. Thus, let X be a σ–compact space with a proper action of a locally compact topological group G. Let F be an open fundamental set for G in X. We consider the covering $\mathcal{U}$ by the translates of F, so $\mathcal{U} = \{gF\ ;\ g \in G\}$. We apply the measuring stick construction for an appropriate pseudometric d and show that the resulting pseudometric d' is proper, but may be infinite–valued. We do this first for the case that the orbit space $G\backslash X$ is compact and then for the general case. We shall need an auxiliary result about Lebesgue numbers of our covering, see below lemma 8.5.The problem of infinite values of d' will be dealt with in the next section. The method will be the "bridge construction".

We start with a well known result, for which we include a proof for the convenience of the reader.

Lemma 8.4. *If the orbit space $G\backslash X$ is compact then every fundamental set is relatively compact. Conversely, if $G\backslash X$ is compact then every relatively compact subset F of X with the property that $GF = X$ is a fundamental set for G in X.*

Proof. The second claim is clear, since property b) of a fundamental set follows immediately from the hypothesis that the action of G on X is proper, see proposition and definition 2.3 ii). To prove the first claim choose a compact neighborhood U_x for every point $x \in X$. A finite number of the $\pi(U_x)$, $x \in X$, cover $G\backslash X$, where π is the natural map $\pi : X \longrightarrow G\backslash X$, which is known to be an open map. Let us say $G\backslash X = \pi(U_{x_1}) \cup \cdots \cup \pi(U_{x_n})$, so $X = GU_{x_1} \cup \cdots \cup GU_{x_n}$. Hence $A \subset G_{U_{x_1},A}U_{x_1} \cup \cdots \cup G_{U_{x_n},A}U_{x_n}$ for every subset A of X. For $A = F$ the subsets $G_{U_{x_i},F}$ of G are relatively compact, by property b) of a fundamental set, see definition 6.1. Hence F is relatively compact. $\square$

A family $\mathcal{D}$ of pseudometrics is called *saturated* if d_1, $d_2 \in \mathcal{D}$ implies $\sup(d_1, d_2) \in \mathcal{D}$.

Lemma 8.5. *Let $\mathcal{D}$ be a saturated family of G–invariant pseudometrics inducing the topology of X. Suppose the orbit space $G\backslash X$ is compact. Then there is a pseudometric $d \in \mathcal{D}$ and a positive number ϵ such that for every $x \in X$ the ball $B_d(x, \epsilon)$ is contained in one translate of F.*

A number ϵ with this property is called a *Lebesgue number* for the covering $\{gF; g \in G\}$ with respect to d.

Proof. By G–invariance, it suffices to show this for points $x \in F$. Since $\overline{F}$ is compact, it is covered by a finite number of gF, say $\overline{F} \subset g_1F \cup \cdots \cup g_nF$. Recall that F is supposed to be open. The set of balls $B_d(x, r)$, $d \in \mathcal{D}$, $x \in X$, $r > 0$, form a base of the topology of X, not only their finite intersections, since $\mathcal{D}$ is saturated. Thus there is for every $x \in \overline{F}$ a pseudometric $d_x \in \mathcal{D}$ and a radius r_x such that $B_{d_x}(x, r_x)$ is contained in one translate of F, since F is open. A finite number of balls $B_{d_x}\left(x, \frac{r_x}{2}\right)$ cover $\overline{F}$, say those for $x = x_1, \ldots, x_n$. Thus for every $y \in \overline{F}$ there is an x_i, $i = 1, \ldots, n$, such that $y \in B_{d_{x_i}}\left(x_i, \frac{r_{x_i}}{2}\right)$ and hence $B_{d_{x_i}}\left(y, \frac{r_{x_i}}{2}\right) \subset B_{d_{x_i}}(x_i, r_{x_i})$ is contained in one translate of F. Hence our claim holds for $d = \sup(d_{x_1}, \ldots, d_{x_n}) \in \mathcal{D}$ and $\epsilon = \inf(r_{x_1}, \ldots, r_{x_n})$. $\square$

Now let again $\mathcal{U} = \{gF \; ; \; g \in G\}$ and for a G–invariant pseudometric d on X let $d' = d'(d, \mathcal{U})$ be the pseudometric obtained by the measuring stick construction.

Proposition 8.6. *Suppose the orbit space $G\backslash X$ is compact. Let d be a continuous G–invariant pseudometric on X, for which there is a Lebesgue number for $\mathcal{U}$. Then d' is a proper pseudometric, i.e., $B_{d'}(x, R)$ is relatively compact for every $x \in X$ and every $R < \infty$.*

Proof. We may assume that $x \in F$, by G–invariance. Then $y \in B_{d'}(x, R)$ if and only if there is a stick path $x = x_0, x_1, \ldots, x_n = y$ with d–length $\sum_{i=1}^{n} d(x_{i-1}, x_i) < R$. We may assume that no three consecutive points x_{i-1}, x_i, x_{i+1} of our stick path are contained in one translate of F, because otherwise we can leave out x_i from our stick path and obtain a stick path of not greater d–length. Let ϵ be the Lebesgue number for $\mathcal{U}$ with respect to d. It follows that $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \geq \epsilon$ for every $i = 1, \ldots, n-1$, because otherwise x_{i-1}, x_i, x_{i+1} are contained in one translate of F. We thus obtain the following upper bound for the length n of our stick path:

$$n < \frac{2R}{\varepsilon} + 1.$$

Thus, let $N \in \mathbb{N} \cup \{0\}$ and let B_N be the set of points $y \in X$ for which there is a stick path of length N starting at a point $x \in F$ and ending at y. We have to show that B_N is relatively compact for every $N \in \mathbb{N} \cup \{0\}$. For $N = 0$ we have $B_N = F$. If $y \in B_{N+1}$ there is a point $y' \in B_N$ such that (y', y) is a stick, say $\{y', y\} \subset g\,F$. Then $y' \in B_N \cap g\,F$ and hence $g \in G_{F,B_N} = G_{B_N^{-1},F}$. This subset of G is relatively compact by induction and property b) of a fundamental set. Thus $y \in g\,F \subset G_{F,B_N}F$, hence $B_{N+1} \subset G_{F,B_N}F$ and so B_{N+1} is relatively compact. □

This yields step 4 of our proof for the case that the orbit space is compact. For the general case we need *one* pseudometric d for which there is a Lebesgue number for every subset of X of the form $\pi^{-1}(K)$ where K is a compact subset of $G\backslash X$. Here we have to suppose that the orbit space is σ–compact.

Before we proceed to do this we need to figure out where d' is finite. Let F and $\mathcal{U}$ be as above. We do not suppose that the orbit space is compact. Let d be a G–invariant pseudometric on X for which $d|F \times F$ has finite values. Let the symbol "$\sim$" denote the smallest G–invariant equivalence relation on X for which F is contained in one equivalence class. Recall that $G_{FF} = \{g \in G;\ gF \cap F \neq \emptyset\}$. Let G_0 be the subgroup of G generated by G_{FF}.

Lemma 8.7. *Let x and y be points of X. The following properties of the pair (x, y) are equivalent*

- *a)* $d'(x, y) < \infty$.
- *b) There is a stick path from x to y.*
- *c)* $x \sim y$.
- *d) The vertices x and y of the graph $\Gamma_{\mathcal{U}}$ belong to the same connected component of $\Gamma_{\mathcal{U}}$.*
- *e) If $x \in g\,F$ and $y \in h\,F$ then $g^{-1}h \in G_0$.*

The equivalence classes will be called *islands* from now on.

Proof. a)$\Longleftrightarrow$b) was noted above, and b)$\Longleftrightarrow$d) and b)$\Longleftrightarrow$c) follow immediately from the definitions.
b) $\Longrightarrow$ e). Let $x \in g\,F$ and $y \in h\,F$ and let (x, y) be a stick, say $\{x, y\} \subset k\,F$ for some $k \in G$. Then $g^{-1}k \in G_{FF}$ and $h^{-1}k \in G_{FF}$ hence $g^{-1}h \in G_0$. The claim b)$\Longleftrightarrow$e) follows now by induction on the length of the stick path.

e) $\Longrightarrow$ c). Let Y be an equivalence class of $\sim$. Thus, if one point of a translate $g\,F$ of F is contained in Y then $g\,F$ is contained in Y. By the same argument applied to $g\,k\,F$ with $k \in G_{FF}$ it then follows that $g\,G_{FF}F \subset Y$, hence $g \cdot G_{FF} \cdot G_{FF}\,F \subset Y$, etc. So $g\,G_0\,F \subset Y$ if $g\,F \cap Y \neq \emptyset$, which proves our claim. □

Corollary 8.8. *The map $g\,G_0 \longmapsto g\,G_0\,F$ establishes a bijection between the set G/G_0 of left cosets of G_0 in G and the set of islands in X.*

Corollary 8.9. *If $G\backslash X$ is σ–compact, so are $\overline{F}$, $G_{\overline{F},\overline{F}}$, G_0 and every island.*

Proof. If K is a compact subset of $G\backslash X$, then so is $F_K := \overline{F} \cap \pi^{-1}(K) = \overline{\pi^{-1}(K) \cap F}$, by lemma 8.4, and hence also G_{F_K,F_K}, since the action of G on X is proper and continuous. It follows that if $G\backslash X$ is σ–compact, so are $\overline{F}$, $G_{\overline{F},\overline{F}}$, the subgroup G_1 of G generated by $G_{\overline{F},\overline{F}}$ and $G_1\overline{F}$. It thus remains to be shown that $G_0 = G_1$ and $G_0\overline{F} = G_0F$. But clearly $G_{\overline{F}F} = G_{FF}$ since F is open, hence $G_{\overline{F}\overline{F}} \subset G^{-1}_{\overline{F}F} \cdot G_{\overline{F}F}$, by the formula following definition 6.1, and thus $G_{\overline{F},\overline{F}} \subset G_0$ and hence $G_1 = G_0$. Furthermore $\overline{F} \subset G^{-1}_{\overline{F}F}F$, by 6.1 a), and hence $G_0\overline{F} = G_0F$. $\square$

We come back to the Lebesgue number and show properness of d' for the case that the orbit space is σ–compact. This accomplishes step 4 of our plan in section 4. Note that at this point we do not need that X is σ–compact, only that the orbit space is σ–compact.

Lemma 8.10. *Suppose the orbit space $G\backslash X$ is σ–compact.*

- *a) Then there is a continuous orbitwise proper G–invariant pseudometric d on X with the following properties: d is finite–valued on every island and for every compact subset K of $G\backslash X$ there is a Lebesgue number for the covering $\mathcal{U}|\pi^{-1}(K)$ of the G–space $\pi^{-1}(K)$ with respect to the restriction of d to $\pi^{-1}(K)$.*
- *b) If d is as in a) then d' is proper, which means that the ball $B_{d'}(x, R)$ has compact closure for every $x \in X$ and every $0 < R < \infty$.*

Proof. a) Let K_n, $n \in \mathbb{N}$, be a sequence of compact subsets of $G\backslash X$ such that $\bigcup_{n=1}^{\infty} K_n = G\backslash X$ and $K_n \subset \overset{\circ}{K}_{n+1}$ for every $n \in \mathbb{N}$. Put $X_n = \pi^{-1}(K_n)$. Then X_n is a closed G–invariant subset of X on which G acts properly with compact orbit space K_n. The set $F_n := F \cap X_n$ is an open fundamental set for G in X_n, hence relatively compact in X_n and in X. So there is a continuous orbitwise proper G–invariant finite–valued pseudometric d_n on X such that there is a Lebesgue number for the covering $\{gF_n; g \in G\}$ of X_n with respect to the pseudometric d_n restricted to X_n. Note that d_n is defined and finite–valued on all of X. To see the existence of such a d_n, we apply lemma 8.5 to the family $d|X_n \times X_n$ where d runs through a saturated family of finite–valued G–invariant pseudometrics on X inducing the topology of X, which we may assume to be orbitwise proper, by Step 3 in section 7.

Let Y be the island G_0F containing F. We use here the notation of lemma 8.7 and its corollaries. Since Y is σ–compact, there is a family L_n, $n \in \mathbb{N}$, of compact subsets of Y such that $\bigcup_{n=1}^{\infty} L_n = Y$ and $L_n \subset \overset{\circ}{L}_{n+1}$. We may assume that $d_n|L_n \times L_n$ has values ≤ 1, by rescaling. Now define

$$d(x, y) = \begin{cases} \Sigma \frac{1}{2^n} d_n(x, y) & \text{if } x \sim y \\ \infty & \text{otherwise.} \end{cases}$$

Then d is G–invariant continuous orbitwise proper pseudometric on X, which is finite–valued on $Y \times Y$ and hence on every island. There is a Lebesgue number for the covering

$\{g\,F_n\,;\,g \in G\}$ of X_n with respect to d, since there is one for d_n and $d \geq \frac{1}{2^n} d_n$. Here we think of d and d_n as restricted to $X_n \times X_n$. This implies our claim under a).

b) Islands are of the form $g\,G_0\,F$, hence open, since F is supposed to be open. It follows that they are also closed. Again, let $Y = G_0 F$ be the island containing F. Let $B_{d'}(x, R)$, $x \in X$, $0 < R < \infty$, be a ball for the pseudometric d' and let B be its closure. We have to show that B is compact. We know that $K := \pi(B)$ is compact, since d is orbitwise proper and hence so is d', since $d' \geq d$ by 8.2 a). We may assume that $x \in F$ and hence $B_{d'}(x, R) \subset Y$ and thus $B \subset Y$.

The subgroup G_0 of G is open since generated by the open subset G_{FF}. It follows that G_0 is a closed subgroup of G. Then the action of G_0 on Y is proper, since the restricted action of G_0 on X is proper and Y is a closed G_0–invariant subset of X. And F is an open fundamental set for G_0 in Y. Let $Z = Y \cap \pi^{-1}(K)$. This is a closed G_0–invariant subset of Y and $F_Z := Z \cap F = F \cap \pi^{-1}(K)$ is an open fundamental set for G_0 in Z. The orbit space $G_0 \backslash Z$ is compact; it can be identified with K. So we can apply proposition 8.6 to the G_0–space Z, the pseudometric $d|Z \times Z$ and the covering $\mathcal{U}_Z := \{gF_Z\,;\,g \in G_0\}$ to obtain that the resulting stick path pseudometric $d'' := d'(d|Z \times Z, \mathcal{U}_Z)$ is proper. It remains to see that $B_{d''}(x, R) = B_{d'}(x, R)$. Clearly $d''(x, y) < R$ implies $d'(x, y) < R$, by looking at the stick paths for $\mathcal{U}_Z$. Conversely, if $d'(x, y) < R$ then there is a stick path $x = x_0, x_1, \ldots, x_n = y$ for $\mathcal{U}$ with $\Sigma d(x_{i-1}, x_i) < R$. Then all the x_i are in $B_{d'}(x, R) \subset Y$ and $\pi(x_i) \in K$, hence $x_i \in Z$ and every pair x_{i-1}, x_i is contained in some translate gF of F. But then $g \in G_0$, by 8.7 e), and so $\{g^{-1}x_{i-1}, g^{-1}x_i\}$ is contained in F and in Z, hence in F_Z. Thus our stick path is also a stick path for $\mathcal{U}_Z$ in Z and thus $d''(x, y) < R$. $\square$

9 Bridges

Again, let X be a σ–compact space and let the locally compact group G act properly on X. Note that then G is σ–compact as well, since if X is the union of countably many compact subsets K_n then G is the union of the countably many sets G_{K_n, K_n} which are compact since the action of G on X is both proper and continuous. Let us again fix an open fundamental set F for G in X. Then, using the notation of the last section, G_0 is an open subgroup of G and hence G/G_0 is a countable discrete space. We can thus choose a finite or infinite sequence of elements g_n, $n = 0, 1, \ldots$, such that G is the union of the disjoint cosets $g_n G_0$. We may assume that g_0 is the identity element. Let S be the set of indices, so $S = \mathbb{N} \cup \{0\}$ or $S = \{0, 1, \ldots, N\}$ for some $N \in \mathbb{N} \cup \{0\}$. Thus $G = \bigcup_{n \in S} g_n G_0$ and hence X is the union of the disjoint subsets $g_n G_0 F$, $n \in S$, by corollary 8.8. Recall that the sets of the form $g\,G_0 F$ are called islands. Consequently we define a *bridge* to be a 2–point subset of X of the form $\{gx, gg_n x\}$ with $g \in G$, $n \in S$, $n \neq 0$, and $x \in F$. Note that gx and $gg_n x$ are always on different islands since $n \neq 0$. But the representation of a bridge in the form above may not be unique. Now suppose a G–invariant pseudometric d on X is given. We then define the *bridge path pseudometric* d_B on X as the supremum of all pseudometrics d'' with the following two properties.

9.1. *a) For every island Y in X we have $d''|Y \times Y \leq d|Y \times Y$.*
b) $d''(gx, gg_n x) \leq n$ for $g \in G$, $n \in S$ and $x \in F$.

There is an alternative description of d_B in terms of paths. Let us define the length of a bridge $\{y,z\}$ as the smallest number $n \in S$ such that $\{y,z\} = \{gx, gg_n x\}$ for some $g \in G$ and $x \in F$. Thus, the length of a bridge is always an integer ≥ 1. Let us call a sequence of points $x = x_0, x_1, \ldots, x_n = y$ a *bridge path* of length n from x to y if any two consecutive points either lie on a common island or form a bridge, i.e., for every $i = 1, \ldots, n$ there is either an island Y such that $\{x_{i-1}, x_i\} \subset Y$ or $\{x_{i-1}, x_i\}$ is a bridge. Define the d–length of such a bridge path as $\sum_{i=1}^n d_i$ where $d_i = d(x_{i-1}, x_i)$ if $\{x_{i-1}, x_i\}$ is on one island or, if $\{x_{i-1}, x_i\}$ is a bridge, then let d_i be the length of this bridge.

9.2. *$d_B(x,y)$ is the infimum of d–lengths of all bridge paths from x to y.*

Proof. The pseudometric d'' defined by the statement of 9.2 has the properties 9.1 a) and b). Conversely, if d'' is a pseudometric with the properties 9.1 a) and b), then $d''(x,y)$ is at most equal to the d–length of any bridge path from x to y, cf. the similar proof in 8.2. ☐

Proposition 9.3. Properties of d_B

a) d_B is G–invariant.

b) d_B is finite–valued if $d|Y \times Y$ is finite–valued for one (equivalently every) island Y.

c) If x is a point of the island Y, then the balls $B_d(x,r) \cap Y$ and $B_{d_B}(x,r)$ coincide for $r < 1$.

d) If d is continuous, so is d_B.

e) Suppose d is continuous, proper and, for every island Y, has finite values on $Y \times Y$. Then d_B is continuous, proper and finite–valued (everywhere).

Proof. a) follows from our construction.

b) follows from the fact that d_B is G–invariant and every island can be reached from F by a bridge.

c) follows from 9.2 and the fact that every bridge has length ≥ 1.

d) A pseudometric is continuous if it is continuous near the diagonal, by the triangle inequality. So d) follows from c).

e) is the main point of these properties. It remains to be shown that d_B is proper if d is proper, continuous and on every island finite–valued. Thus let $x \in X$ and $0 < R < \infty$. We have to show that $B_{d_B}(x,R)$ has compact closure. For a point $y \in X$ we have $d_B(x,y) < R$ if there is a bridge path $x = x_0, \ldots, x_n = y$ with d–length $\Sigma d_i < R$. We may assume that three consecutive points x_{i-1}, x_i, x_{i+1} of our bridge path are not on a common island, since otherwise we could leave out x_i without increasing the d–length of our path, by the triangle inequality for d. So our path has at least $\frac{n+1}{2}$ bridges, all of length ≥ 1. We thus have an upper bound for the length n of our bridge path, namely $n \leq 2R+1$. Furthermore, every bridge in our path has length at most R and every step $d_i = d(x_{i-1}, x_i)$ on one island has length at most R. It thus suffices to prove the following two claims.

a) If K is a compact subset of X, then $B_d(K, R) = \{y \in X\,;\, d(x, y) < R\}$ has compact closure.

b) If K is a compact subset of X, then the set $B(K, R) := \{z \in X;$ there is a bridge $\{y, z\}$ from a point $y \in K$ to z of length $\leq R\}$ has compact closure.

Proof of a). K is contained in a finite union of islands, since K is compact and the islands are open and disjoint and form a cover of X. It thus suffices to prove our claim for the case that K is contained in one island, say Y. Let x be a point of K. Then the function $y \longmapsto d(x, y)$ is continuous and finite–valued on Y, hence has a finite maximum on K, so $K \subset B_d(x, r)$ for some $0 < r < \infty$. Then $B_d(K, R) \subset B_d(x, r + R)$, which has compact closure by hypothesis. This shows our claim.

Proof of b). The bridges $\{y, z\}$ starting from a point of K and having length $\leq R$ are of the form $\{gx, gg_n x\}$ with $x \in F$ and $n \leq R$, and either $gx \in K$ or $gg_n x \in K$. Hence $g \in G_{FK}$ or $g \in G_{g_n F, K} = G_{FK} \cdot g_n^{-1}$ and hence the endpoint z of our bridge is of the form $z = gg_n x \in G_{FK} g_n K$ in the first case or of the form $z = gx \in G_{FK} g_n^{-1} K$ in the second case, thus every endpoint z of such a bridge is contained in the relatively compact set $\bigcup_{n \leq R} G_{FK} g_n^{\pm 1} K$, as was to be shown. □

9.4. We are now ready to finish the proof of our main theorems 1.1 and 1.2. Let X be a σ–compact Hausdorff space and suppose the locally compact topological group G acts properly on X. We have shown that then there is a family of continuous G–invariant pseudometrics on X inducing the topology of X, see step 2 in chapter 6, which we may furthermore assume to be finite–valued and orbitwise proper, by step 3 in chapter 7. Then the stick construction of chapter 8 gave us a pseudometric, which is continuous, proper and on every island finite–valued, namely the pseudometric d' of lemma 8.10. Continuity of d' follows from property 8.2 b) and finiteness on islands from lemma 8.7. If we use this pseudometric in the bridge construction of chapter 9 then the resulting pseudometric d_B is continuous, finite–valued and proper. If now $\mathcal{D}$ is a family of G–invariant pseudometrics inducing the topology of X – we know that such a family exists, by step 2 in chapter 6 – then the family $\{\sup(d, d_B)\,;\, d \in \mathcal{D}\}$ has all the properties we want in theorem 1.2 (theorem 4.1). If X is furthermore metrizable, then there is a compatible G–invariant metric d on X, by step 2 in chapter 6. Again, there is a pseudometric d_B which is continuous, proper, finite–valued and G–invariant. Then the metric $\sup(d, d_B)$ has all these properties, too, and is furthermore a compatible metric. This proves theorem 1.1 (theorem 4.2).

Let us point out the following corollary, due to Haagerup and Przybyszewska [7].

Corollary 9.5. *Every second countable locally compact group has a left invariant compatible proper metric.*

Proof. The underlying space of such a group G is metrizable and σ–compact, by corollary 5.4. The action of G on itself by left translations is obviously proper, so there is a compatible left invariant proper metric on G, by theorem 1.1. □

As a special case we obtain the following old result of Busemann [4].

Corollary 9.6. *The group of isometries of a proper metric space admits a compatible left invariant proper metric.*

Proof. The group G of isometries of a proper metric space is locally compact and Hausdorff, see theorem 3.2, and second countable, see [3, Ch. X, §3.3 Corollary], which implies our claim by the previous corollary.

□

10 Concluding remarks

In this chapter we discuss applications and related work, mention open questions and make other remarks.

10.1. In the non–equivariant context, i.e., if we consider just the topological space X without any group action, it is well known that a σ–compact locally compact metrizable space has a compatible proper metric, see corollary 5.4. More precisely, in [14] it is proved that if d is a complete metric on such a space X then there is a proper metric on X which is locally identical with d, i.e., for every point $x \in X$ there is a neighborhood of x where the two metrics coincide. Note that in our construction the metric is not changed locally in steps 4 and 5 of chapter 4. Thus in the situation of theorem 1.1 if d is a compatible G–invariant metric on X which is orbitwise proper then there is a G–invariant compatible proper metric on X which is locally identical with d. One may thus ask the following question: Suppose, in the situation of theorem 1.1, we are given a G–invariant complete compatible metric on X. Is there a G–invariant proper (compatible) metric on X which is locally identical with d?

10.2. Given an isometric action of a group G on a σ–compact locally compact metric space X with metric d, it is not true in general that there is a compatible proper metric d_p for which the action of G is isometric. For an example let $X = \{(x, y) \in \mathbb{R}^2; x = 0 \text{ or } x = 1\}$ endowed with the metric $d = \min\{d_E, 1\}$ where d_E is the Euclidean metric of $\mathbb{R}^2$ restricted to X. Let G be the group of isometries of (X, d). There is no compatible proper metric d_p on X for which G acts isometrically, for the following reason. The group H of isometries of (X, d_p), endowed with the compact open topology, acts properly, hence the isotropy group $H_{(0,0)}$ of the point $(0, 0)$ is compact and hence has compact orbits. On the other hand, let $G_{(0,0)}$ be the isotropy group of the point $(0, 0)$ in G. The orbit $G_{(0,0)}(1, 0)$ of $(1, 0)$ is $\{1\} \times \mathbb{R}$ and is not relatively compact in X. So G is not contained in H. The point of the example is that the action of G is not proper, no matter which topology we put on G.

10.3. Let us consider the following question. Under which conditions is it true that given a compatible metric d on a locally compact σ–compact space X there is a compatible proper metric d_p with the same group of isometries? A sufficient condition was given by Janos [8], namely if (X, d) is a connected uniformly locally compact metric space.

10.4. Note that if we have a closed subgroup G of the group of isometries of a proper metric space (X, d) then it is not true in general that there is a metric d_1 on X for which G is the precise group of isometries. E.g., the space $X = \mathbb{R}$ of real numbers with the Euclidean metric has the group $G = \mathbb{R}$ as a closed subgroup of its group of isometries. But for every G–invariant metric d_1 on X we have $d_1(x, 0) = d_1(0, -x)$, hence the group of isometries of d_1 contains the reflections of $\mathbb{R}$ and is thus strictly larger than $\mathbb{R}$.

10.5. Given a proper action of a locally compact topological group G on a locally compact metrizable space X, one can ask if there is a G–invariant metric. This is known to be equivalent to $G\backslash X$ being paracompact [12], [1], [2]. The answer is positive in many cases, see [1], [2]. If X is no longer locally compact, the answer is known to be negative if the action is Bourbaki–proper, see [1], but again unknown in general for Palais–proper actions.

10.6. Our theorem 1.1 has potential applications for the Novikov conjecture. Namely, let G be a locally compact second countable group and let μ be a Haar measure on G. Then, using a proper left invariant compatible metric on G, Haagerup and Przybyszewska have proved in [7] that there is a proper affine isometric action of G on some separable strictly convex reflexive Banach space. Kasparov and Yu have recently proved that the Novikov conjecture holds for every discrete countable group which has a uniform embedding into a uniformly convex Banach space, see [10]

References

[1] Abels H. and Strantzalos P., *Proper transformation groups*, Monograph in preparation.

[2] Antonyan, S. and de Neymet, S. *Invariant pseudometrics on Palais proper G-spaces*, Acta Math. Hungar. **98** (2003), no. 1-2, 59–69.

[3] Bourbaki N., *Elements of Mathematics. General Topology*, Parts I and II, Hermann, Paris, 1966.

[4] Busemann, H., *Geometry of Geodesics*, Academic Press, New York, 1955.

[5] van Dantzig D. and van der Waerden B.L., *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.

[6] Gao S. and Kechris A.S., *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766.

[7] Haagerup U. and Przybyszewska A., *Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces*, arXiv:math.OA/0606794 v1 (2006).

[8] Janos L., *A group theoretic property of Heine-Borel metrics*, Geometry and Topology, 215–218, World Sci. Publishing, Singapore, 1989.

[9] Kankaanrinta M., *Proper smooth G-manifolds have complete G-invariant Riemannian metrics*, Topology Appl. **153** (2005), 610–619.

[10] Kasparov, G. and Yu, G., *The coarse geometric Novikov conjecture and uniform convexity*, Adv. Math. **206** (2006), 1–56.

[11] Kasparov G. and Skandalis G., *Groups acting properly on "bolic" spaces and the Novikov conjecture*, Ann. of Math. (2) **158** (2003), 165–206.

[12] Koszul J.L., *Lectures on groups of transformations*, Tata Institute of Fundamental Research, Bombay, 1965.

[13] Manoussos A. and Strantzalos P., *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), 7–12.

[14] Williamson R. and Janos L., *Constructing metrics with the Heine-Borel property*, Proc. Amer. Math. Soc. **100** (1987), 567–573.

Herbert Abels

Universität Bielefeld
Fakultät für Mathematik
Postfach 100 131
33501 Bielefeld
Germany
e–mail: abels@math.uni–bielefeld.de

Antonios Manoussos

Universität Bielefeld
Fakultät für Mathematik, SFB 701
Postfach 100 131
33501 Bielefeld
Germany
e–mail: amanouss@math.uni-bielefeld.de

Gennady Noskov

Sobolev Institute of Mathematics
Pevtsova 13
Omsk 644099
Russia
and
Universität Bielefeld
Fakultät für Mathematik, SFB 701
Postfach 100 131
33501 Bielefeld
Germany
e–mail: g.noskov@gmail.com

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 11, November 2007, Pages 5593–5609
S 0002-9947(07)04377-2
Article electronically published on May 11, 2007

ON EMBEDDINGS OF PROPER AND EQUICONTINUOUS ACTIONS IN ZERO-DIMENSIONAL COMPACTIFICATIONS

ANTONIOS MANOUSSOS AND POLYCHRONIS STRANTZALOS

ABSTRACT. We provide a tool for studying properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces, by embedding such an action in a suitable zero-dimensional compactification of the underlying space with pleasant properties. Precisely, given such an action (G, X) we construct a zero-dimensional compactification μX of X with the properties: (a) there exists an extension of the action on μX, (b) if $\mu L \subseteq \mu X \setminus X$ is the set of the limit points of the orbits of the initial action in μX, then the restricted action $(G, \mu X \setminus \mu L)$ remains properly discontinuous, is indivisible and equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX, and (c) μX is the maximal among the zero-dimensional compactifications of X with these properties. Proper actions are usually embedded in the endpoint compactification εX of X, in order to obtain topological invariants concerning the cardinality of the space of the ends of X, provided that X has an additional "nice" property of rather local character ("property Z", i.e., every compact subset of X is contained in a compact and connected one). If the considered space has this property, our new compactification coincides with the endpoint one. On the other hand, we give an example of a space not having the "property Z" for which our compactification is different from the endpoint compactification. As an application, we show that the invariant concerning the cardinality of the ends of X holds also for a class of actions strictly containing the properly discontinuous ones and for spaces not necessarily having "property Z".

INTRODUCTION

The endpoint compactification of a locally compact space has been proved fruitful for the study of the space in the topological framework, including proper actions. One reason for this is that we have a "clear view" of the embedded space in such a compactification, contrary to the situation when, for example, the Stone-Čech compactification is considered instead. Actually, the endpoint compactification is the quotient space of the Stone-Čech compactification with respect to the equivalence relation whose equivalence classes are the singletons of X and the connected components of $\beta X \setminus X$.

Received by the editors December 9, 2005.
2000 *Mathematics Subject Classification.* Primary 37B05, 54H20; Secondary 54H15.
Key words and phrases. Proper actions, properly discontinuous actions, equicontinuous actions, indivisibility, zero-dimensional compactifications, inverse systems.
This work was partially supported by DFG Forschergruppe "Spektrale Analysis, asymptotische Verteilungen und stochastische Dynamik" and SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany.

Our purpose in this paper is to provide an equivariant and analogously useful notion corresponding to the endpoint compactification in order to have a "clear view" of the embedded proper action. By saying a "clear view" of the embedded action we mean that the embedded action has at least the three properties that follow.

Let (G, X) be the initial proper action, (G, Y) the extended action in a zero-dimensional compactification Y of X and let L be the set of the limit points of the orbits of the initial action in Y (i.e., the cluster points of the nets $\{g_i x\}$, for all nets $\{g_i\}$ divergent in G, and $x \in X$). Then the maximal invariant subspace where the extended action can be proper is, obviously, $Y \setminus L \supseteq X$. So, the required properties are: The action $(G, Y \setminus L)$

(a) remains proper,

(b) is equicontinuous with respect to the uniformity induced on $Y \setminus L$ by that of Y, and

(c) is indivisible (i.e., if $\lim g_i y_0 = e \in L$ for some $y_0 \in Y \setminus L$, then $\lim g_i y = e$ for every $y \in Y \setminus L$).

In this direction the main results of the paper at hand are:

(1) If X is a locally compact, connected and paracompact space and G is a non-compact group acting properly discontinuously on X, there always exists a zero-dimensional compactification μX of X which is the maximal (in the ordering of zero-dimensional compactifications of X) that satisfies the properties: (a) the initial action can be extended on μX, and (b) if μL denotes the set of the limit points of the orbits of the initial action in μX, the restricted action $(G, \mu X \setminus \mu L)$ remains proper, is equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX and indivisible as embedded in the action $(G, \mu X)$ (Theorem 6.2).

(2) μL consists of at most two or infinitely many points (Theorem 6.3).

(3) If X has the *"property Z"*, i.e., every compact subset of X is contained in a compact and connected one (for example if X is locally compact, connected and locally connected), then μX coincides with the endpoint compactification εX of X (Corollary in Section 6).

The proof of the results stated above relies on a *new construction*: The action $(G, \mu X)$ is obtained by taking the initial action as an equivariant inverse limit of properly discontinuous G-actions on polyhedra, which are constructed via G-invariant locally finite open coverings of X, generated by locally finite coverings of (always existing) suitable fundamental sets of the initial action (cf. Section 3).

As an application of these results we prove in Theorem 7.1 that the invariant concerning the cardinality of the ends for proper actions of non-compact groups on locally compact and connected spaces with the "property Z" holds also for proper actions on spaces not necessarily satisfying this property:

If either G_0, the connected component of the neutral element of G, is non-compact, or G_0 is compact and G/G_0 contains an infinite discrete subgroup, then X has at most two or infinitely many ends.

Moreover, in Section 2 we give an example of a properly discontinuous action (G, X), where G is a non-compact group and X is a locally compact, connected and paracompact space not satisfying the "property Z" such that μX does not coincide with the endpoint compactification of X: we show that the sets of the limit points of the actions (G, X) and $(G, \varepsilon X \setminus \varepsilon L)$ in εX coincide, but the action

$(G, \varepsilon X \setminus \varepsilon L)$ is neither proper nor equicontinuous with respect to the uniformity induced on $\varepsilon X \setminus \varepsilon L$ by that of εX.

Properties (a) and (b) in (1) above have already been used, especially concerning embeddings in the endpoint compactification, in order to prove that the existence of a proper action (G, X) of a non-compact group on a locally compact, connected and paracompact space with the "property Z" has implications in the structure and the cardinality of the space of ends of X. The following indications trace the known results in this direction.

The first theorem that relates, although indirectly, equicontinuous actions with structural features of spaces, is formulated by Kerékjártó (1934), who proved that, if the abelian group generated by a homeomorphism of the 2-sphere, S^2, acts equicontinuously on S^2 with respect to the metric uniformity of S^2 except for a finite number of points, then the number of these exceptional points is at most two. These points can be viewed as the set of the endpoints of the maximal subspace of S^2 on which the above group acts equicontinuously. This result is considerably generalized by Lam in [7] for equicontinuous actions of non-compact groups on locally compact, connected metric spaces X with respect to uniformities induced, say, by the uniformities of suitable zero-dimensional compactifications of X, i.e., compactifications with zero-dimensional remainder. Roughly speaking it is shown that, if an action (G, X) can be embedded in an action (G, Y), where Y is a zero-dimensional compactification of X, such that

(a) there exists a subset $R \supseteq X$ of Y such that $Y \setminus R$ is exactly the non-empty set of the points where the action (G, Y) is not equicontinuous, and

(b) the restricted action (G, R) is indivisible (i.e., if $\lim g_i y_0 = e \in Y \setminus R$ for some $y_0 \in R$, then $\lim g_i y = e$ for every $y \in R$),

then $Y \setminus R$ consists of at most two or infinitely many points.

On the other hand, similar results are proved by Abels in [2] for proper actions (G, X), where G is a non-compact topological group and X is a locally compact and connected space with the "property Z" (i.e., every compact subset of X is contained in a compact and connected one). The corresponding property in Lam's work requires X to be a semicontinuum, which ensures the indivisibility of the equicontinuous action on R. In [2] is considered the endpoint compactification, εX, of X, the maximal compactification of it with zero-dimensional remainder, instead of an appropriate zero-dimensional compactification Y of X, and it is proved that such a proper action (G, X) has an extension on εX. To be more precise, let εL denote the set of the limit points of the action (G, X) in εX. Then, it is shown that (a) the action $(G, \varepsilon X \setminus \varepsilon L)$ remains proper and (b) it is indivisible. Using this embedding, it is shown that X has at most two or infinitely many ends, a remarkable invariant of the proper action (G, X) of the non-compact group G.

The interconnection of the main results in [7] and [2] is explained in [11], where it is shown that, for spaces with the "property Z", a group acting equicontinuously in Lam's view may be considered as a dense (not necessarily strict) subgroup of a group acting properly as in Abels' view.

1. Preliminaries

1.1. The *Freudenthal* or *endpoint* compactification εX of a locally compact space X may be defined as the quotient space of the Stone-Čech compactification βX of X with respect to the equivalence relation whose equivalence classes are the singletons of X and the connected components of $\beta X \setminus X$. Recall that the zero-dimensional compactifications of X are ordered with respect to the following ordering: Let Y and Z be two zero-dimensional compactifications of X; then $Y \leq Z$ if there exists a surjection from Z onto Y extending the identity map of X. Therefore, the endpoint compactification is the maximal zero-dimensional compactification of X, i.e., for every zero-dimensional compactification Y of X there is a surjection $p : \varepsilon X \to Y$ extending the identity map of X.

The points of $\varepsilon X \setminus X$ are the *ends* of X.

The following theorem, [9], provides an equivalent definition.

Theorem 1.1.1. *If Y is a compactification of X, it is the endpoint compactification of X iff $Y \setminus X$ is totally disconnected and does not disconnect Y locally, i.e., given an open (in Y) neighborhood V of $y \in Y \setminus X$, then there is no decomposition of $V \cap X$ into two open disjoint subsets U_1, U_2 such that $y \in \overline{U_1} \cap \overline{U_2}$.*

The endpoint compactification has the following useful properties.

Proposition 1.1.2. *Let X and Y be two locally compact topological spaces. Then every proper map $f : X \to Y$ may be extended to a unique map $\varepsilon f : \varepsilon X \to \varepsilon Y$ that maps ends of X to ends of Y.*

Proof. By the characteristic property of the Stone-Čech compactification, the map $f : X \to Y$ has a unique extension $\varepsilon f : \varepsilon X \to \varepsilon Y$. The inclusion $\varepsilon f(\varepsilon X \setminus X) \subseteq \varepsilon Y \setminus Y$ follows from the assumption that f is a proper map. □

Proposition 1.1.3. *Let X be a locally compact and connected space and Y be a zero-dimensional compactification of X. Then, whenever a continuous action (G, X) has an extension (G, Y) this extension is continuous.*

Proof. It suffices to show the continuity of the extended action map at the point (e, z), where e is the neutral element of G and $z \in Y$. Let V and U be two open neighborhoods of z in Y with boundaries in X such that $\overline{V} \subseteq U$. Since the boundaries ∂U and ∂V are compact subsets of X, the set $A = \{g \in G \,|\, g\partial V \subseteq U \text{ and } g^{-1}\partial U \subseteq Y \setminus \overline{V}\}$ is an open neighborhood of e in G. We shall show that $g\overline{V} \subseteq U$ for every $g \in A$: The boundary of the set $g\overline{V} \cap (Y \setminus U)$ is contained in $(g\partial V \cap (Y \setminus U)) \cap (g\overline{V} \cap \partial U)$, which is empty by the definition of A. Since Y is connected, this implies that $g\overline{V} \cap (Y \setminus U)$ is either the empty set or coincides with Y. The latter is impossible since, choosing a point $x \in \partial V$, the definition of A implies that $gx \notin Y \setminus U$. Therefore $g\overline{V} \cap (Y \setminus U) = \emptyset$. □

As an immediate consequence of the above two propositions we state the following:

Corollary 1.1.4. *An action (G, X) of a group G on a locally compact and connected space X may be extended to a unique action on the endpoint compactification εX of X.*

1.2. The notion of a proper action is given in [4, III, 4]. Equivalently, an action (G, X) is *proper* if $J(x)$ is the empty set for every $x \in X$, where

$$J(x) = \{y \in X \,|\, \text{there exist nets } \{x_i\} \text{ in } X \text{ and } \{g_i\} \text{ in } G \text{ with } g_i \to \infty, \\ \lim x_i = x \text{ and } \lim g_i x_i = y\}.$$

Here $g_i \to \infty$ means that the net $\{g_i\}$ does not have any limit point in G.

In the special case where G is locally compact, an action (G, X) is proper iff for every $x, y \in X$ there exist neighborhoods U_x and U_y of x and y, respectively, such that the set

$$G(U_x, U_y) = \{g \in G \mid (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G.

The action is called *properly discontinuous* when $G(U_x, U_y)$ is finite.

Remark. Let (G, X) be a proper action of a non-compact group G and $(G, \varepsilon X)$ its extension on the endpoint compactification of X. Then, the set $J(x)$ with respect to the extended action is a non-empty subset of $\varepsilon X \setminus X$ for every $x \in X$. The study of these sets provides useful information. As an example, we note from [2] the following:

Theorem 1.2.1. *Let (G, X) be a proper action of a non-compact group G on a locally compact and connected space X with the "property Z". Then, X has at most two or infinitely many ends. In particular, if G is connected, then X has at most two ends.*

1.3. A characteristic and very useful feature of a proper action is the fundamental set (cf. [6] and [1]).

Definition. Given an action (G, X), a subset F of X is a *fundamental set* for the action if $GF = X$ and for every compact subset K of X the set $\{g \in G \mid (gK) \cap F \neq \emptyset\}$ is relatively compact in G.

The existence of a fundamental set implies that the action (G, X) is proper, but the converse does not hold, in general. The notion of the fundamental set is relative to the well-known notion of a section but is different in general, in the sense that there are cases where a section is a fundamental set, a fundamental set fails to be a section and cases where a section fails to be a fundamental set.

Theorem 1.3.1. *Let (G, X) be a proper action, where X is a locally compact, connected and paracompact space. Then, there exist open fundamental sets F and S for G in X such that $\overline{F} \subseteq S$.*

This follows immediately by [6, Lemma 2, p. 8], because X is σ-compact; hence the orbit space of the action is paracompact.

1.4. Establishing the notation, we recall

Definition. An inverse system $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ consists of a directed set Λ, a family of topological spaces $\{X_\lambda,\ \lambda \in \Lambda\}$, and continuous mappings $p_{\kappa\lambda} : X_\lambda \to X_\kappa$ with the properties that for every $\kappa, \lambda, \mu \in \Lambda$ with $\kappa \leq \lambda$ and $\lambda \leq \mu$ the map $p_{\lambda\lambda} : X_\lambda \to X_\lambda$ is the identity of X_λ, and $p_{\kappa\lambda} \circ p_{\lambda\mu} = p_{\kappa\mu}$. Let $p_\lambda : \prod_\lambda X_\lambda \to X_\lambda$ be the λ-projection. The (possibly empty) space

$$\{x \in \prod_\lambda X_\lambda \mid p_\kappa(x) = p_{\kappa\lambda} \circ p_\lambda(x) \text{ for every } \kappa \leq \lambda\}$$

is called the *inverse limit* of $\{X_\lambda,\ \lambda \in \Lambda\}$ and is denoted by $\varprojlim X_\lambda$.

Proposition 1.4.1 ([5, Pr. 2.3, p. 428])**.** *The sets $\{p_\lambda^{-1}(U) \mid \lambda \in \Lambda,\ U \text{ open in } X_\lambda\}$ form a basis for $\varprojlim X_\lambda$.*

The following notion provides an alternative way to describe locally compact and paracompact spaces using coverings.

Definition. Let X be a paracompact space, $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ be an inverse system and $\{p_\lambda \,|\, \lambda \in \Lambda\}$ a family of mappings $p_\lambda : X \to X_\lambda$ such that $p_\kappa(x) = p_{\kappa\lambda} \circ p_\lambda(x)$ for every $\kappa \leq \lambda$. We say that the inverse system $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ is a *resolution* of X if the following conditions hold:

(a) For every covering $\mathcal{U}$ of X that admits a subordinated partition of unity, there exist an index $\lambda \in \Lambda$ and a covering $\mathcal{U}_\lambda$ of X_λ that also admits a subordinated partition of unity such that $p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$.

(b) For every $\kappa \in \Lambda$ and every covering $\mathcal{U}_\kappa$ of X_κ, as above, there exists $\lambda \geq \kappa$ such that $p_{\kappa\lambda}(X_\lambda) \subseteq St(p_\kappa(X), \mathcal{U}_\kappa)$, where

$$St(B, \mathcal{U}) = \bigcup \{U_i \,|\, U_i \cap B \neq \emptyset,\, U_i \in \mathcal{U}\}$$

is the *star* of B with respect to the covering $\mathcal{U}$.

Theorem 1.4.2 ([8, Cor. 4, p. 83])**.** *If the spaces X_λ are normal and X is paracompact, then a resolution of X gives X as an inverse limit.*

2. A counterexample

Following the notation in the introduction, we now give an example showing that, if the space X does not have the "property Z", then the action $(G, \varepsilon X \setminus \varepsilon L)$ is not necessarily proper or equicontinuous.

2.1. The *half-open Alexandroff square* Y is the space $[0,1] \times [0,1] \setminus \{(x,y) \,|\, x = 0 \text{ and } y \in (0,1] \text{ or } x = 1 \text{ and } y \in [0,1)\}$ endowed with the topology τ defined as follows: A neighborhood basis of a point $(x,x) \in \Delta = \{(x,x) \,|\, x \in [0,1]\}$ is obtained by the intersection of Y with open (in $Y \subseteq \mathbb{R}^2$) horizontal strips less a finite number of vertical lines; a neighborhood basis for the points $p = (s,t)$ off Δ is obtained by the intersection of $Y \setminus \Delta$ with open vertical segments centered at p (cf. [10, Ex. 101, p. 120]). This space is a compact, connected and not locally connected Hausdorff space. Observe that, if $\{(x_i, y_i)\}$ is a net converging with respect to the Euclidean topology on Y to a point (x,y), then this net converges to (y,y) with respect to τ, unless there is an index i_0 such that $x_i = x$ for all $i \geq i_0$, in which case it converges to (x,y).

2.2. Let X be the subspace $(0,1) \times (0,1)$ of Y. This space is locally compact, connected and paracompact, because the closed horizontal strips are compact subsets of X. It does not have the "property Z", because every closed horizontal strip is not contained in a compact and connected subset of X.

The space Y is the endpoint compactification, εX, of X, and the ends are the points $(x,0)$ for $x \in [0,1)$ and $(x,1)$ for $x \in (0,1]$. In order to prove this, by Theorem 1.1.1, it is sufficient to verify that the set $Y \setminus X$ is totally disconnected and that every point of it does not disconnect Y locally: For the points of the form $(x,0)$ and $(x,1)$ for $x \in (0,1)$ this follows from the fact that a neighborhood basis of every one of these points consists of half-open vertical segments which do not disconnect Y locally. To verify the same for the points $(0,0)$ and $(1,1)$ observe that, if there is a neighborhood V (in Y) for, e.g., $(0,0)$ such that $V \cap X$ is the union of two open sets (in X) having $(0,0)$ as a common point of their closures in Y, then they have common interior points.

2.3. Next we define a properly discontinuous action of the additive group of the integers $\mathbb{Z}$ on X. For convenience, we consider X as $\mathbb{R}^2$ endowed with the topology τ, and we define the action by letting

$$z(x,y) = (x+z, y+z) \text{ for } z \in \mathbb{Z} \text{ and } (x,y) \in \mathbb{R}^2.$$

By Corollary 1.1.4, this action has an extension on $Y = \varepsilon X$. The set εL, of the limit points of this action, consists of the points $(0,0)$ and $(1,1)$. The restricted action $(\mathbb{Z}, \varepsilon X \setminus \varepsilon L)$ is neither proper nor equicontinuous with respect to the uniformity induced on $\varepsilon X \setminus \varepsilon L$ by that of εX. For this, observe that the sequence $\{(x-n,x) \mid n \in \mathbb{N}\}$ converges to the point (x,x), while the sequence $\{n(x-n,x) = (x, x+n)\}$ converges to an end e that corresponds to the vertical line $\{(x,y) \mid y \in \mathbb{R}\}$. Since $e \in J((x,x))$, the action $(\mathbb{Z}, \varepsilon X \setminus \varepsilon L)$ is not proper. On the other hand, $\lim n(x-n,x) = e$ and $\lim n(x,x) = (1,1)$; therefore this action is not equicontinuous at (x,x).

3. The basic construction

In the sequel we shall proceed to answer the question formulated in the introduction. Our answer will be based on an inverse system of properly discontinuous actions on polyhedra, defined from the initial action on X. This is achieved using appropriate invariant locally finite coverings of the given space, in order to have the initial action as an inverse limit of them. To obtain this, it is reasonable to work with invariant coverings of X extending specific coverings of always existing fundamental sets of the initial action. The construction of this inverse system, which follows, is new and will be given in several steps:

3.1. Let (G,X) be a properly discontinuous action of a non-compact group G on a locally compact, connected and paracompact space X. Recall that a covering $\mathcal{V}$ of X is called a barycentric refinement of a given covering if the covering $\{St(x,\mathcal{V}) \mid x \in X\}$ refines it, where $St(x,\mathcal{V})$ has been defined in §1.4. Since X is a locally compact and paracompact space, by [5, Cor. 7.4, p. 242], starting with an open covering of X, we can always find an open locally finite barycentric refinement $\mathcal{V} = \{V_j \mid j \in J\}$ of it consisting of relatively compact open sets.

3.2. Theorem 1.3.1 ensures that there exist open fundamental sets F and S such that $\overline{F} \subset S$. With the previous notation, we can choose $\mathcal{V}$ such that (a) if V_j intersects the boundary of the open fundamental set S, then V_j does not intersect the open fundamental set F, and (b) if V_j does not intersect the boundary of S, then either $V_j \subseteq S$ or $V_j \subseteq X \setminus \overline{S}$. The family $\mathcal{U} = \{St(x,\mathcal{V}) \mid x \in F\}$ is an open locally finite covering of F (in X); it is also a covering of $\overline{F}$, because if some V_j intersects the boundary of F, then it intersects F, hence is a member of a star of some point of F. From (a) and (b) it is easily seen that each member of $\mathcal{U}$ is a subset of S.

3.3. In the sequel we shall use the following modification of the previous construction, aiming to enrich $\mathcal{U}$ with the property: if $U_i \in \mathcal{U}$ and $gU_i \cap F \neq \emptyset$ for some $g \in G$, then $gU_i \in \mathcal{U}$. To this end, let $\mathcal{W} = \{W_k \mid k \in K\}$ be a locally finite refinement of $\mathcal{V}$ with the property that the closures of the stars of it are subsets of corresponding stars of $\mathcal{V}$. Now, we observe that the set $M_i = \{g \in G \mid gU_i \cap \overline{F} \neq \emptyset\}$ is non-empty and finite, because the action is properly discontinuous, U_i is relatively compact and $\overline{F} \subseteq S$ (cf. §§1.2 and 1.3).

If $x \in \overline{W}_k \subseteq U_i$ for some $U_i \in \mathcal{U}$, and $gx \in \overline{F}$, then $g \in M_i$ which is finite. So, for $x \in X$ we can find a neighborhood $N_x \subseteq U_i$ of x such that, if $gN_x \cap F \neq \emptyset$, then gN_x is a subset of some U_j. Since $\overline{W}_k$ is compact, we can replace this W_k by a finite number of neighborhoods such as N_x and the corresponding open sets gN_x for $g \in M_i$. In this way, we obtain a refinement of $\mathcal{U}$, which will be denoted again with $\mathcal{U}$ and shall be used in the sequel.

This refinement remains locally finite and, in addition, has the required property, because if $gN_x \cap F \neq \emptyset$, then $gU_i \cap F \neq \emptyset$, from which follows that $g \in M_i$; hence gN_x is a member of our refinement. It is easily seen that this property passes to the family $\{St(x,\mathcal{U}) \,|\, x \in F\}$, because if $gx \in \overline{F}$, then $gSt(x,\mathcal{U}) = St(gx,\mathcal{U})$.

3.4. Next, using the covering $\mathcal{U}$ of $\overline{F}$ defined in §3.3, we consider the invariant covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ of X. We show that it is locally finite: For $x \in X$ there exists $h \in G$ such that $hx \in F$. Since F is open, there exists an open and relatively compact neighborhood $N \subseteq F$ of hx that intersects finitely many members of $\mathcal{U}$. Then, the neighborhood $h^{-1}N$ of x intersects finitely many members of $\mathcal{C}$, because by §3.3, if $gU_i \cap N \neq \emptyset$, then $gU_i \in \mathcal{U}$.

3.5. To each covering $\mathcal{C}$ corresponds a polyhedron $X_{\mathcal{C}}$, namely the nerve of the covering $\mathcal{C}$ with the CW-topology. A subordinated partition of unity $\Phi_{\mathcal{C}} = \{\varphi_U \,|\, U \in \mathcal{C}\}$ determines a *canonical map* $p_{\mathcal{C}} : X \to X_{\mathcal{C}}$ with the property that $p_{\mathcal{C}}$ maps a point $x \in X$ to the point of $X_{\mathcal{C}}$ whose barycentric coordinate corresponding to the vertex U equals $\varphi_U(x)$.

Since $\mathcal{C}$ is invariant, the initial properly discontinuous action (G, X) induces a natural action $(G, X_{\mathcal{C}})$ defined as follows: For $g \in G$ and U a vertex of $\mathcal{C}$ we let $(g, U) \mapsto gU$ and we extend the action map by linearity. This action is properly discontinuous as is easily verified.

3.6. The construction of the desired inverse system of properly discontinuous actions on polyhedra will be based in the proof of the following theorem (cf. [3]; see also [8, Th. 7 and Cor. 5, pp. 84-85]).

Theorem. *Every connected, locally compact and paracompact space is the inverse limit of polyhedra.*

For the convenience of the reader, we outline the proof: Let X be a connected, locally compact paracompact space and $\mathcal{F}$ be the family of all coverings of X admitting a subordinated partition of unity. For every $\mathcal{D} \in \mathcal{F}$ we choose a locally finite partition of unity $\Phi_{\mathcal{D}}$ subordinated to $\mathcal{D}$. Let $X_{\mathcal{D}}$ be the nerve of $\mathcal{D}$ with the CW-topology. Let Λ be the set of all finite subsets $\lambda = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ of $\mathcal{F}$ ordered by inclusion. We denote by X_λ the nerve of the covering

$$\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n = \{V_1 \cap \ldots \cap V_n \,|\, (V_1, \ldots, V_n) \in \mathcal{D}_1 \times \ldots \times \mathcal{D}_n\}.$$

If $\lambda \leq \mu = \{\mathcal{D}_1, \ldots, \mathcal{D}_n, \ldots, \mathcal{D}_l\}$, let $p_{\lambda\mu} : X_\mu \to X_\lambda$ be the simplicial map which maps the vertex $(V_1, \ldots, V_n, \ldots, V_l)$ of the nerve of $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n \wedge \ldots \wedge \mathcal{D}_l$ to the vertex $(V_1, \ldots, V_n)$ of the nerve of $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n$.

As is shown in [3], the family

$$\Phi_{\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n} = \{\varphi_{(V_1, \ldots, V_n)} \,|\, (V_1, \ldots, V_n) \in \mathcal{D}_1 \times \ldots \times \mathcal{D}_n\},$$

where $\varphi_{(V_1,\ldots,V_n)} = \varphi_{V_1} \cdot \ldots \cdot \varphi_{V_n}$, is a partition of unity subordinated to the covering $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n$. Using this, for $\lambda = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ we define the canonical map $p_\lambda : X \to X_\lambda$ as in §3.5.

In order to obtain a polyhedral resolution of X (cf. §1.4), a slight modification of the above construction is needed:

We replace the previous inverse system $(X_\lambda, p_{\lambda\mu}, \Lambda)$ by a larger system (Y_r, q_{rs}, S) defined as follows: For $\lambda \in \Lambda$ let $\mathcal{V}_\lambda$ be a neighborhood basis of the closure of $p_\lambda(X)$ in X_λ, and let

$$S = \{r = (\lambda, V) \,|\, \lambda \in \Lambda \text{ and } V \in \mathcal{V}_\lambda\}.$$

Let $r \leq s = (\mu, W)$ if $\lambda \leq \mu$ and $p_{\lambda\mu}(W) \subseteq V$. Moreover, letting $Y_r = V$, for $r \leq s$ we define the map $q_{rs} : Y_s \to Y_r$ as the restriction of $p_{\lambda\mu}$ on W.

Taking into account the fact that $\mathcal{F}$ consists of *all* coverings of X admitting subordinated partitions of unity, it is proved that $X = \varprojlim Y_r = \varprojlim X_\lambda$.

3.7. If we replace $\mathcal{F}$ by $\mathcal{P}$, the family of the coverings of X of the form $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ defined in §3.4, and we repeat the previous steps, we obtain an inverse system, denoted (for simplicity) again by $(X_\lambda, p_{\lambda\mu}, \Lambda)$. Since we use star coverings, we note that $St(x, \mathcal{D}_1) \cap St(x, \mathcal{D}_2) = St(x, \mathcal{D}_1 \wedge \mathcal{D}_2)$.

3.8. If we restrict ourselves to the fundamental set $\overline{F} \subseteq X$, the coverings from $\mathcal{P}$ induce a family of coverings on $\overline{F}$ defined by intersections of each one covering with $\overline{F}$. This family is cofinal to the corresponding one defined analogously via $\mathcal{F}$ on $\overline{F}$. Since $\mathcal{P}$ is not cofinal to $\mathcal{F}$, regarded as families of coverings of X, we shall focus on the induced coverings of the fundamental set $\overline{F}$, where we may assume that the members of both families are the same. Note that, by [4, I, Cor., p. 49], $\overline{F} = \varprojlim p_\lambda(\overline{F})$ holds, with respect to both $\mathcal{F}$ and $\mathcal{P}$. Moreover, with respect to $\mathcal{F}$, we have $\varprojlim p_\lambda(\overline{F}) = \overline{F} \subseteq X$, by the theorem in §3.6, while, with respect to $\mathcal{P}$ and the notation from §3.7, $\overline{F} \subseteq \varprojlim X_\lambda$.

4. The initial action as inverse limit of actions on polyhedra

Lemma 4.1. *Let $\mathcal{C}_i \in \mathcal{P}$. For the covering $\mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_n$, there exists a subordinated partition of unity $\Phi_{\mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_n} = \{\varphi_{(V_1,\ldots,V_n)} \,|\, (V_1, \ldots, V_n) \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_n\}$ such that $\varphi_{(V_1,\ldots,V_n)} = \varphi_{(gV_1,\ldots,gV_n)} \circ g$, for every $g \in G$.*

Proof. If the assertion is true for every single covering $\mathcal{C}$, then

$$\varphi_{(gV_1,gV_2,\ldots,gV_n)} \circ g = [(\varphi_{V_1} \circ g^{-1}) \cdot \ldots \cdot (\varphi_{V_n} \circ g^{-1})] \circ g = \varphi_{V_1} \cdot \ldots \cdot \varphi_{V_n} = \varphi_{(V_1,\ldots,V_n)}.$$

So, it suffices to prove the assertion for a covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ as in §3.4. We follow the usual construction (cf. [5, Th. 4.2, p. 170]): We choose locally finite coverings $\{V_i \,|\, i \in I\}$ and $\{W_i \,|\, i \in I\}$ of the open fundamental set F such that $\overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i$ for every $i \in I$. We can apply Urysohn's Theorem in order to find continuous maps $f_{U_i} : X \to [0, 1]$ which are identically 1 on $\overline{W_i}$ and vanish on $X \setminus V_i$. We set $f_{gU_i} = f_{U_i} \circ g^{-1}$ for every $g \in G$. Since the covering $\{g\overline{W_i} \,|\, i \in I,\, g \in G\}$ is locally finite, it follows that for each $x \in X$ at least one and at most finitely many f_{gU_i} are not zero; therefore $\sum f_{gU_i}$ is a well-defined continuous real-valued map on X and is never zero. So, we can define the required partition of unity by setting

$$\varphi_{gU_i}(y) = \frac{f_{gU_i}(y)}{\sum f_{gU_i}(y)}.$$

Since $x \in U_i$ iff $gx \in gU_i$, we have

$$\varphi_{gU_i}(gx) = \frac{f_{gU_i}(gx)}{\sum f_{gU_i}(gx)} = \frac{f_{U_i} \circ g^{-1}(gx)}{\sum f_{U_i} \circ g^{-1}(gx)} = \frac{f_{U_i}(x)}{\sum f_{U_i}(x)} = \varphi_{U_i}(x).$$

□

Theorem 4.2. *With the notation from §3.7, X is equivariantly homeomorphic to $\varprojlim X_\lambda$.*

Proof. We recall that the actions (G, X_λ) defined in §3.5 induce an action on $\varprojlim X_\lambda$ as follows: Let $g \in G$ and $\mathbf{x} \in \varprojlim X_\lambda$ with coordinates $p_\lambda(x)$. The coordinates of $g\mathbf{x}$ are $gp_\lambda(x)$, i.e., $p_\lambda(gx) = gp_\lambda(x)$. This action is well defined since the maps $p_{\lambda\mu} : X_\mu \to X_\lambda$ have the property $p_{\lambda\mu}(gx_\mu) = gp_{\lambda\mu}(x_\mu)$ for every $g \in G$ and $x_\mu \in X_\mu$, by the definition of the action on each X_λ.

An equivariant homeomorphism $f : X \to \varprojlim X_\lambda$ may be defined in the following way: For $x \in X$ there exists some $g \in G$ such that $gx \in \overline{F}$ (cf. §1.3). We let $f(x)$ be the point with coordinates $p_\lambda(f(x)) = g^{-1}(p_\lambda(gx))$. We will prove that f is well defined: It suffices to prove that, if $x \in X$ and $g \in G$ with $gx \in \overline{F}$, then $g^{-1}(p_\lambda(gx))$ is independent of the choice of g. Indeed, with the notation from §§3.5 and 3.6 let $x \in V_1 \cap \ldots \cap V_n$. Then, by the definition of the actions on the polyhedra and the previous lemma, we have:

$$\varphi_{(g^{-1}gV_1,\ldots,g^{-1}gV_n)}(g^{-1}(p_\lambda(gx))) = \varphi_{(gV_1,\ldots,gV_n)}(p_\lambda(gx)) = \varphi_{(V_1,\ldots,V_n)}(p_\lambda(x)).$$

Using this and the fact that f is the identity map on the open fundamental set F, we can first verify that f is equivariant and then a homeomorphism. □

5. The embedding of the action in $\varprojlim \varepsilon X_\lambda$ and its basic properties

Theorem 5.1. *The space $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of X. Moreover, G acts on $\varprojlim \varepsilon X_\lambda$ and (G, X) is equivariantly embedded in $(G, \varprojlim \varepsilon X_\lambda)$.*

Proof. The simplicial maps $p_{\lambda\mu} : X_\mu \to X_\lambda$ are proper surjections. Hence, by Proposition 1.1.2, they have unique extensions $\varepsilon p_{\lambda\mu} : \varepsilon X_\mu \to \varepsilon X_\lambda$ that map the space of the ends of X_μ onto that of X_λ. Furthermore, $\varepsilon p_{\lambda\lambda} : \varepsilon X_\lambda \to \varepsilon X_\lambda$ is the identity map of εX_λ, and for $\kappa \leq \lambda$ and $\lambda \leq \mu$ we have $\varepsilon p_{\kappa\lambda} \circ \varepsilon p_{\lambda\mu} = \varepsilon p_{\kappa\mu}$. Hence, they define an inverse limit, $\varprojlim \varepsilon X_\lambda$.

Using the fact that each εX_λ is a zero-dimensional compactification of X_λ and applying Proposition 1.4.1, we see that $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of $\varprojlim X_\lambda$. By Corollary 1.1.4, the action of G on X_λ is extended to an action on εX_λ such that the following equivariant diagram commutes:

$$\begin{tikzcd}
(G, \varprojlim X_\lambda) \arrow[r, "\mathrm{id_G \times h}"'] \arrow[d, "\mathrm{id_G \times p_\lambda}"'] & (G, \varprojlim \varepsilon X_\lambda) \arrow[d, "\mathrm{id_G \times \varepsilon p_\lambda}"] \\
(G, X_\lambda) \arrow[r, "\mathrm{id_G \times i_\lambda}"'] & (G, \varepsilon X_\lambda)
\end{tikzcd}$$

where $i_\lambda : X_\lambda \to \varepsilon X_\lambda$ are the inclusion maps, $p_\lambda : \varprojlim X_\lambda \to X_\lambda$ and $\varepsilon p_\lambda : \varprojlim \varepsilon X_\lambda \to \varepsilon X_\lambda$ are projections, id_G is the identity map of G and $h : \varprojlim X_\lambda \to \varprojlim \varepsilon X_\lambda$ is defined by setting $h_\lambda = i_\lambda$.

That (G, X) embeds equivariantly in $(G, \varprojlim \varepsilon X_\lambda)$ is an immediate consequence of Theorem 4.2 and the above diagram. □

Remark. The example in Section 2 shows that $\varprojlim \varepsilon X_\lambda$ does not necessarily coincide with the endpoint compactification εX of X. However,

Proposition 5.2. *If X has finitely many ends, then $\varepsilon X = \varprojlim \varepsilon X_\lambda$.*

Proof. Let e_i for $i = 1, 2, \ldots, n$ be the ends of X and $V_1, V_2, \ldots, V_n$ be open neighborhoods of them in εX, respectively, with disjoint closures and boundaries lying in X. Then, the set $K = \varepsilon X \setminus \bigcup_{i=1}^n V_i$ is a compact subset of X. Let e_1, e_2 be two distinct ends in εX with the same image in $\varprojlim \varepsilon X_\lambda$ via the projection map $p : \varepsilon X \to \varprojlim \varepsilon X_\lambda$. Such a projection exists since, by Theorem 5.1, $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of X and εX is the maximal one. Therefore, e_1 and e_2 should have the same image under the composition map $\varepsilon p_\lambda \circ p$. With the notation from §3.4, this means that there is a subfamily of $\mathcal{C}$ with infinitely many members gU_i with the property $g_i U_i \cap K \neq \emptyset$. Then, we can find a sequence $\{x_k\}$ with $x_k \in g_k U_k \cap K$. Since K is compact, we may assume that $\lim x_k = x \in K$, from which follows that the covering $\mathcal{C}$ fails to be locally finite at x, a contradiction. $\square$

The following proposition shows that, especially for equicontinuous actions, the sets $J(x)$, defined in §1.2, can be replaced by the *limit sets*

$$L(x) = \{y \in X \mid \text{there exists a net } \{g_i\} \text{ in } G \text{ with } g_i \to \infty \text{ and } \lim g_i x = y\},$$

which are simpler to handle. The points of the sets $L(x)$ are the *limit points* of the action.

Proposition 5.3. *Let (G, X) be an equicontinuous action of a locally compact group G on a locally compact space X. Then $J(x) = L(x)$ holds for every $x \in X$. Moreover, if the nets $\{x_i\}$ in X and $\{g_i\}$ in G are such that $\lim x_i = x$, $g_i \to \infty$ and $\lim g_i x_i = y \in J(x)$, then $\lim g_i x = y \in L(x)$.*

Proof. Since (G, X) is equicontinuous, for every entourage U there exists an entourage V such that for $y \in X$,

$$(x, y) \in V \text{ implies } (gx, gy) \in U \text{ for every } g \in G.$$

But $\lim x_i = x$, so we may assume that $(x, x_i) \in V$; therefore $(g_i x, g_i x_i) \in U$ and $(g_i x_i, y) \in U$. So $(g_i x, y) \in U \circ U$; hence $\lim g_i x = y$. $\square$

A kind of converse of the previous proposition is the following:

Proposition 5.4. *Let Y be a zero-dimensional compactification of the locally compact and connected space Z. Let (G, Y) be an action such that Z is an invariant subspace of Y and the action (G, Z) is proper. The restricted action (G, Z) is equicontinuous with respect to the uniformity induced on Z by that of Y iff the following condition is satisfied: If $z \in Z$ is such that there exist a net $\{z_i\}$ in Z with $\lim z_i = z$, and a net $\{g_i\}$ in G with $g_i \to \infty$ and $\lim g_i z_i = e \in Y \setminus Z$, then $\lim g_i z = e$.*

Proof. The necessity may be proved by arguments analogous to those applied in the proof of the previous proposition.

For the sufficiency, note that if the action (G, Z) is not equicontinuous at the point z, then there exists an entourage U such that for every entourage V there exist a point $z_V \in Z$ and some $g_V \in G$ such that

$$(z, z_V) \in V \text{ and } (g_V z, g_V z_V) \notin U.$$

Since the entourages of the uniformity may be directed by setting $V_1 \leq V_2$ if $V_2 \subseteq V_1$, we may assume that $g_V \to \infty$ and $\lim z_V = z$. By the compactness of Y, we may assume that the nets $\{g_V z\}$ and $\{g_V z_V\}$ converge to different points of $Y \setminus Z$, a contradiction to our hypothesis. □

For the formulation of the next basic theorem, we recall that an action (G, X) embedded in an action (G, Y), where Y is a zero-dimensional compactification of X, is *indivisible* if whenever $\lim g_i y_0 = e \in Y \setminus X$ for some $y_0 \in X$, then $\lim g_i y = e$ for every $y \in X$.

Theorem 5.5. *Let $(G, \varprojlim \varepsilon X_\lambda)$ be the action defined in Theorem 5.1 and L be the set of the limit points of the action $(G, X) = (G, \varprojlim X_\lambda)$ in $\varprojlim \varepsilon X_\lambda$. Then, the action $(G, \varprojlim \varepsilon X_\lambda \setminus L)$ is*

(a) *proper,*

(b) *equicontinuous with respect to the uniformity induced on $\varprojlim \varepsilon X_\lambda \setminus L$ by that of $\varprojlim \varepsilon X_\lambda$, and*

(c) *indivisible as embedded in the action $(G, \varprojlim \varepsilon X_\lambda)$.*

For the proof we need the following:

Lemma 5.6. *Let L_λ be the set of the limit points of the action (G, X_λ) in εX_λ. Then $L = \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$.*

Proof. Let $w \in L$, $\lim g_i x = w$ for $x \in X = \varprojlim X_\lambda$, and $g_i \to \infty$. Then $\lim g_i \varepsilon p_\lambda(x) = \lim \varepsilon p_\lambda(g_i x) = \varepsilon p_\lambda(w)$, from which follows that $\varepsilon p_\lambda(w) \in L_\lambda$; therefore $L \subseteq \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$.

For the inverse inclusion, let $v \in \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$, that is, $\varepsilon p_\lambda(v) \in L_\lambda$ for every $\lambda \in \Lambda$. This means that for each $\lambda \in \Lambda$ there exist a net $\{g_i^\lambda\}$ in G with $g_i^\lambda \to \infty$ and $x_\lambda \in X_\lambda$ with $\lim g_i^\lambda x_\lambda = \varepsilon p_\lambda(v)$. Since the polyhedron X_λ is a connected, locally compact and locally connected space, it has the "property Z"; therefore, by [2, 3.4], the action (G, X_λ) is indivisible as a restriction of $(G, \varepsilon X_\lambda)$. So, we may assume that $x_\lambda = \varepsilon p_\lambda(x)$ for a fixed $x \in X$ and every $\lambda \in \Lambda$. By the compactness of $\varprojlim \varepsilon X_\lambda$, we may assume that $\lim g_i^\lambda x = v^\lambda \in L$. So, we have

$$\varepsilon p_\lambda(v) = \lim g_i^\lambda x_\lambda = \lim g_i^\lambda \varepsilon p_\lambda(x) = \lim \varepsilon p_\lambda(g_i^\lambda x) = \varepsilon p_\lambda(v^\lambda).$$

Let $\lim v^\lambda = u \in \varprojlim \varepsilon X_\lambda$. This u is contained in $\varprojlim \varepsilon X_\lambda \setminus X$, because $v^\lambda \in \varprojlim \varepsilon X_\lambda \setminus X$, which, by Theorem 5.1, is a compact set. But, for each $\kappa \in \Lambda$ and every $\lambda \in \Lambda$ with $\kappa \leq \lambda$, by §1.4, we have

$$\varepsilon p_\kappa(u) = \lim \varepsilon p_\kappa(v^\lambda) = \lim \varepsilon p_{\kappa\lambda} \circ \varepsilon p_\lambda(v^\lambda) = \lim \varepsilon p_{\kappa\lambda} \circ \varepsilon p_\lambda(v) = \varepsilon p_\kappa(v),$$

from which it follows that $u = v$. Taking into account that $\lim g_i^\lambda x = v^\lambda$ and applying a diagonal procedure, we may find a net $\{g_j\}$ in G such that $\lim g_j x = v \in \varprojlim \varepsilon X_\lambda \setminus X$. The properness of the action (G, X) implies that this net is divergent, and therefore $v \in L$, as required. □

Proof of Theorem 5.5. (a) Assume that $\{g_i\}$ is a net in G and x, x_i and y are points in $\varprojlim \varepsilon X_\lambda \setminus L$ such that $\lim x_i = x$ and $\lim g_i x_i = y$. By the previous lemma, $\varprojlim \varepsilon X_\lambda \setminus L = \bigcup_\lambda \varepsilon p_\lambda^{-1}(\varepsilon X_\lambda \setminus L_\lambda)$. So, there exist κ and λ such that $x \in \varepsilon p_\kappa^{-1}(\varepsilon X_\kappa \setminus L_\kappa)$ and $y \in \varepsilon p_\lambda^{-1}(\varepsilon X_\lambda \setminus L_\lambda)$.

For an index μ with $\kappa \leq \mu$ and $\lambda \leq \mu$, we may assume that

$$\varepsilon p_{\kappa\mu}^{-1}(\varepsilon X_\kappa \setminus L_\kappa) \cup \varepsilon p_{\lambda\mu}^{-1}(\varepsilon X_\lambda \setminus L_\lambda) \subseteq \varepsilon X_\mu \setminus L_\mu.$$

Indeed, note that if, e.g., $z \in \varepsilon p_{\kappa\mu}^{-1}(\varepsilon X_\kappa \setminus L_\kappa)$ and $z \in L_\mu$, there exist a net $\{h_j\}$ in G with $h_j \to \infty$ and some $x_\mu \in X_\mu$ with $\lim h_j x_\mu = z$; hence $\lim h_j \varepsilon p_{\kappa\mu}(x_\mu) = \varepsilon p_{\kappa\mu}(z) \in L_\kappa$, a contradiction.

By this, we may assume that the points x, x_i and y are contained in the open and invariant set $\varepsilon p_\mu^{-1}(\varepsilon X_\mu \setminus L_\mu)$. Since X_μ is connected, locally compact and locally connected, it has the "property Z"; therefore the action $(G, \varepsilon X_\mu \setminus L_\mu)$ is proper [2, 4.7], and hence $J(\varepsilon p_\mu(x)) = \emptyset$. From this and the fact that $\lim g_i \varepsilon p_\mu(x_i) = \varepsilon p_\mu(y)$, it follows that the net $\{g_i\}$ cannot be divergent. Hence, by §1.2, the action $(G, \varprojlim \varepsilon X_\lambda \setminus L)$ is proper.

(b) We shall use Proposition 5.4 for $Z = \varprojlim \varepsilon X_\lambda \setminus L$ and the notation there. Let $\lim g_i z = e_1$. For every $\lambda \in \Lambda$ we have

$$\lim \varepsilon p_\lambda(z_i) = \varepsilon p_\lambda(z), \ \lim g_i \varepsilon p_\lambda(z_i) = \varepsilon p_\lambda(e), \ \text{and} \ \lim g_i \varepsilon p_\lambda(z) = \varepsilon p_\lambda(e_1).$$

By (a), the action (G, Z) is proper; hence $e, e_1 \in L$. Therefore, by Lemma 5.6, $\varepsilon p_\lambda(e), \varepsilon p_\lambda(e_1) \in L_\lambda$. From this and the indivisibility of the action $(G, \varepsilon X_\lambda \setminus L_\lambda)$ (cf. [2, 3.4]), it follows that $\varepsilon p_\lambda(e) = \varepsilon p_\lambda(e_1)$ for every $\lambda \in \Lambda$, i.e., $e = e_1$, and the assertion follows.

(c) The proof follows by repeating the arguments in the proof of (b). $\square$

Remark. We note that $X \subseteq \varprojlim \varepsilon X_\lambda \setminus L$, because (G, X) is proper and $X = \varprojlim X_\lambda \subseteq \varprojlim \varepsilon X_\lambda$.

6. The maximality of $\varprojlim \varepsilon X_\lambda = \mu X$ and the cardinality of $L = \mu L$

In this paragraph we prove the main results of the paper.

Lemma 6.1. *Let $(X, \mathbf{D})$ be a uniform space, and (G, X) be an equicontinuous action. Then, there exists a finer uniformity $\mathbf{D}^*$ compatible with the topology of X such that G acts on X by pseudoisometries with respect to the pseudometrics generating $\mathbf{D}^*$.*

Proof. Let $\{d_i,\, i \in I\}$ be a saturated family of bounded pseudometrics on X which generates $\mathbf{D}$ [4, II, Th. 1, p. 142]. We obtain a pseudometric d_i^* on X such that every $h \in G$ acts on X as a d_i^*-pseudoisometry, by letting $d_i^*(x, y) = \sup_{g \in G} d_i(gx, gy)$. Let $\mathbf{D}^*$ denote the uniformity generated by the family $\{d_i^* \,|\, i \in I\}$. The topologies τ, τ^* induced on X by $\mathbf{D}$ and $\mathbf{D}^*$, respectively, coincide: Since $d_i^*(x, y) \geq d_i(x, y)$, we have $\mathbf{D} \subseteq \mathbf{D}^*$ and $\tau \subseteq \tau^*$. Conversely, if $U_x^* = \bigcap_{k=1}^n S_k(x, \epsilon)$ is a neighborhood of x in τ^*, where $S_k(x, \epsilon)$ denotes a $d_{i_k}^*$-ball of radius ϵ, centered at x, then the equicontinuity of G implies the existence of a neighborhood U_x of x in τ, such that $U_x \subseteq U_x^*$. $\square$

Theorem 6.2. *The compactification $\varprojlim \varepsilon X_\lambda = \mu X$ is maximal among the zero-dimensional compactifications of X satisfying simultaneously the following properties:*

(a) *The initial action (G, X) is extended to an action $(G, \mu X)$.*

(b) *The action $(G, \mu X \setminus \mu L)$, where μL is the set of the limit points of the orbits of the initial action (G, X) in μX, is proper, equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX, and indivisible.*

Proof. By Theorem 5.5, the zero-dimensional compactification μX of X satisfies the properties (a) and (b). So, it remains to prove the maximality of μX: Suppose that Y is a zero-dimensional compactification of X also satisfying these properties, such that $q : Y \to \varprojlim \varepsilon X_\lambda$ is a surjection extending the identity map of X (cf. §1.1). We have to show that q is bijective.

Claim 1. The restriction of q on the set L_Y of the limit points of the action (G, X) in Y is a bijection.

Let L_Y be the set of the limit points of the action (G, X) in Y, and c_1, $c_2 \in L_Y$ be two distinct points such that $q(c_1) = q(c_2)$. Then, there are open neighborhoods V_1 and V_2 of c_1 and c_2, respectively, in Y with disjoint closures.

Due to the indivisibility of the action $(G, Y \setminus L_Y)$, we may assume that $\lim g_j x = c_1$ and $\lim h_j x = c_2$ with $g_j x \in V_1$ and $h_j x \in V_2$. If there exists a covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$, as in §3.7, such that the members of it containing $g_j x$, respectively $h_j x$, are pairwise disjoint, then it is easily seen that $\lim \varepsilon p_\lambda(q(g_j x)) = \lim \varepsilon p_\lambda(g_j x) \neq \lim \varepsilon p_\lambda(q(h_j x))$, a contradiction to the assumption that $q(c_1) = q(c_2)$. Therefore, there exist cofinal families $\{A_j\}$ and $\{B_j\}$ of members of $\mathcal{C}$ such that $g_j x \in A_j$, $h_j x \in B_j$ and $A_j \cap B_j \neq \emptyset$.

Since we consider star coverings, using refinements if necessary, we may assume that $A_j \cup B_j = f_j U_j$ is a member of our covering intersecting both V_1 and V_2. Then $g_j x \in f_j U_j$; hence $x \in g_j^{-1} f_j U_j \in \mathcal{C}$. Since $\mathcal{C}$ is a locally finite covering, passing if necessary to a subnet, we may assume that $x \in g_j^{-1} f_j U_j = gU_r$ for suitable g and r. It follows that $h_j x = g_j g x_j$, where $x_j \in U_r$. Since U_r is relatively compact in X, we may assume that $\lim x_j = y \in X$. Thus, $c_2 \in J(y)$ with respect to the action (G, Y), because $\lim h_j x = c_2$. From this and the assumption that the action $(G, Y \setminus L_Y)$ is equicontinuous, taking into account Proposition 5.4, we conclude that $\lim g_j g y = c_2$. This contradicts the fact that $gy \in X$, $\lim g_j x = c_1$ and the action $(G, Y \setminus L_Y)$ is indivisible.

Claim 2. The restriction of q on the set $Y \setminus L_Y$ is also a bijection.

Since q is, by definition, the identity map on X, we have to show that it is bijective on $Y \setminus (L_Y \cup X)$. The action $(G, Y \setminus L_Y)$ is equicontinuous; hence, by Lemma 6.1, we may assume that G acts by pseudoisometries. So, we are allowed to assume that the invariant covering $\mathcal{C}$ consists of open sets leading to invariant entourages.

Let $b_1, b_2 \in Y \setminus (L_Y \cup X)$ be two distinct points such that $q(b_1) = q(b_2)$, and $V = \{(x, y) \in (Y \setminus L_Y) \times (Y \setminus L_Y) \,|\, d_k(x, y) < \epsilon,\, k = 1, 2, \ldots, n\}$ be an entourage such that $(b_1, b_2) \notin V$. Moreover, we may assume that $\mathcal{C}$ consists of open sets leading to invariant entourages of the form

$$W = \{(x, y) \in (Y \setminus L_Y) \times (Y \setminus L_Y) \,|\, d_k(x, y) < \epsilon/2,\, k = 1, 2, \ldots, n, n+1, \ldots, m\}.$$

As in the proof of Claim 1 and the notation there, since $b_1, b_2 \notin X$, we can find families $\{A_j\}$ and $\{B_j\}$ of members of $\mathcal{C}$ and $x_j \in A_j$, $y_j \in B_j$ such that $\lim x_j = b_1$, $\lim y_j = b_2$ and $A_j \cup B_j$ is a member of our covering. From this and the specific choice of the entourages V and W, it follows that $(b_1, b_2) \in V$, a contradiction. □

Corollary. *If X has the "property Z", then $\mu X = \varepsilon X$.*

Proof. We have to show that if (G, X) is a properly discontinuous action, then the properties (a) and (b) of the previous theorem are satisfied for εX, the maximal zero-dimensional compactification of X, instead of μX. This follows from Corollary 1.1.4, the already mentioned results of [2] in the introduction (cf. [2, 4.7 and 3.7]), and from Proposition 5.4 for $Y = \varepsilon X$ and $Z = \varepsilon X \setminus \varepsilon L$. □

Example. In our counterexample the set μL consists of the two endpoints of the diagonal and the zero-dimensional compactification μX may obtained as a quotient space of the half-open Alexandroff square by identifying on the one hand the points $\{(x, 1) \mid x \in (0, 1]\}$, and on the other hand the points $\{(x, 0) \mid x \in [0, 1)\}$.

Theorem 6.3. *The space μL of the limit points of the action (G, X) in μX either consists of at most two points or it is a perfect compact set. In the case where the group G is abelian, μL has at most two points.*

Proof. Let μL have infinitely many points. We have to show that for every point e of it and every neighborhood $V = \varepsilon p_\lambda^{-1}(U)$ of e (cf. Proposition 1.4.1), we can find a point $e_1 \in \mu L$ with $e_1 \in \overline{V}$ and $e_1 \neq e$. By [2, 4.2], the action $(G, \varepsilon X_\lambda \setminus L_\lambda)$ is proper. In the case under consideration, L_λ is a perfect compact set, by [2, 4.11, Satz D, 4]. Since, by Lemma 5.6, $\varepsilon p_\lambda(e) \in L_\lambda$, we can find $e_\lambda \in L_\lambda$ with $e_\lambda \in U$ and $e_\lambda \neq \varepsilon p_\lambda(e)$. By the indivisibility of the action (G, X_λ) (cf. [2, 3.4]), there is a net $\{g_i\}$ in G with $g_i \to \infty$ and $\lim g_i \varepsilon p_\lambda(x) = e_\lambda$ for every $x \in X$. Since $\lim \varepsilon p_\lambda(g_i x) = e_\lambda$ and μX is compact, fixing an $x \in X$, we may assume that $g_i x \in V$ and $\lim g_i x = e_1 \in \mu L$. Therefore $e_1 \in \overline{V}$. Since $\varepsilon p_\lambda(e_1) = \lim \varepsilon p_\lambda(g_i x) = e_\lambda \neq \varepsilon p_\lambda(e)$, we have $e_1 \neq e$.

Now, assume that μL has finitely many points. Since, by Lemma 5.6, $L = \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$, the set μL is the inverse limit of the inverse system $(L_\lambda, \varepsilon p_{\lambda\mu}, \Lambda)$. By Theorem 1.2.1, every L_λ consists of at most two points. From this and the fact that every simplicial map $p_{\lambda\mu} : X_\mu \to X_\lambda$ is defined by deleting the last coordinates (cf. §3.6), we conclude that μX also has at most two points.

If the acting group is abelian, then the action (G, X_λ) fulfills the assumptions of Theorem 1.17 of [7]; therefore every L_λ consists of one or two points. From this and using the same arguments as before, we see that μL consists of at most two points. □

7. An application

In this section we apply our main results to show that the already known necessary condition for the existence of a proper action of a non-compact group on a locally compact and connected space with the "property Z" (cf. Theorem 1.2.1) remains also necessary in a broad class of actions, containing the properly discontinuous ones, on spaces that do not have the "property Z".

Theorem 7.1. *Let X be a locally compact, connected and paracompact space, and G be a non-compact group acting properly on X such that either G_0, the connected component of the neutral element of G, is non-compact, or G_0 is compact and G/G_0 contains an infinite discrete subgroup. Then X has*

(a) *at most two or infinitely many ends, and*
(b) *at most two ends, if G_0 is not compact.*

Proof. We begin with the proof of (b) and we shall restrict ourselves in the proof of (a) only in the case where G_0 is compact.

(b) If G_0 is non-compact, we consider the restricted action (G_0, X). By Iwasawa's Decomposition Theorem, G_0 contains a closed subgroup isomorphic to $\mathbb{R}$; therefore it contains a closed subgroup isomorphic to $\mathbb{Z}$, the additive group of the integers. The restricted action $(\mathbb{R}, X)$ is proper; therefore the action $(\mathbb{Z}, X)$ is properly discontinuous.

Since the space of the ends of X is totally disconnected, every end is a fixed point for the action $(G_0, \varepsilon X)$, therefore for the restricted action $(\mathbb{Z}, \varepsilon X)$ too. Since the projection $p : \varepsilon X \to \mu X$ is equivariant, every point of $\mu X \setminus X$ is a fixed point for the action $(\mathbb{Z}, \mu X)$, where μX is the zero-dimensional compactification of X that corresponds to the action $(\mathbb{Z}, X)$ by Theorem 6.2. From this and Theorem 6.3, there exist at most two limit points for the action $(\mathbb{Z}, \mu X \setminus \mu L)$. The set $\mu X \setminus X$ cannot have any other point except these limit points, because by Theorem 5.5, the action $(\mathbb{Z}, \mu X \setminus \mu L)$ is properly discontinuous, therefore has compact isotropy groups.

We claim that in this case $\varepsilon X = \mu X$ holds, which implies (b). To this end, we have to prove that p is injective. In order to be able to repeat the arguments in the proof of Theorem 6.2, Claim 2, replacing Y by εX, we need the following:

Claim. The action $(\mathbb{R}, X)$ is equicontinuous with respect to the uniformity induced on X by that of εX.

We shall use Proposition 5.4. Let $x \in X$ and $\lim x_i = x$ for $x_i \in X$. To arrive at a contradiction, assume that there exists a net $\{t_i\}$ in $\mathbb{R}$ with $t_i \to +\infty$ and $\lim t_i x = e_1 \in \varepsilon X \setminus X$, while $\lim t_i x_i = e_2 \in \varepsilon X \setminus X$, where $e_1 \neq e_2$. Let U and V_1 be disjoint neighborhoods in εX of x and e_1, respectively, with boundaries in X. Then, there exists t_0 such that $tx \in V_1$ for every $t \geq t_0$, because otherwise, by the connectedness of the orbits, we can find a net $\{r_i x\}$ in the boundary of V_1 with $\lim r_i x = y \in X$ and $r_i \to +\infty$; this is not possible, because the action $(\mathbb{R}, X)$ is proper; hence $L(x) \subseteq J(x) = \emptyset$ for every $x \in X$ (cf. §1.2 and Section 5). So, we can find a neighborhood V_2 in εX of e_2 with boundary in X, disjoint from U such that $tx \notin \overline{V_2}$ for every $t \geq 0$. Since $\lim x_i = x \in U$ and $\lim t_i x_i = e_2 \in V_2$ there exists a net $\{s_i x_i\}$ in the boundary of V_2 such that $\lim s_i x_i = z \in X$. As before, the net $\{s_i\}$ cannot be divergent; therefore we may assume that $\lim s_i = s \geq 0$. Hence $z = sx \in \overline{V_2}$, a contradiction.

(a) We have to consider only the case where G_0 is compact and G/G_0 contains an infinite discrete subgroup. Since X is connected and σ-compact, the orbit space $X \backslash G_0$ of the action (G_0, X) is connected and σ-compact, therefore paracompact. The group G/G_0 acts on $X \backslash G_0$, by letting

$$(gG_0, G_0(x)) \mapsto G_0(gx), \text{ for every } g \in G \text{ and } x \in X.$$

This action is well defined, since G_0 is a normal subgroup of G. Moreover, it is proper: Since the initial action is proper, G is locally compact. Therefore, by §1.2, there exist compact neighborhoods U_x, U_y in X of x and y, respectively, such that the set

$$G(U_x, U_y) = \{g \in G \mid (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G. Then $W_1 = \{G_0(z) \mid z \in U_x\}$ and $W_2 = \{G_0(z) \mid z \in U_y\}$ are compact neighborhoods of the points $G_0(x)$, $G_0(y)$ in $X \backslash G_0$, respectively. The set

$$(G/G_0)(W_1, W_2) = \{gG_0 \in G/G_0 \mid (gG_0 W_1) \cap W_2 \neq \emptyset\}$$

is relatively compact in G/G_0. Indeed, let $\{g_i G_0\}$ be a net in $(G/G_0)(W_1, W_2)$. Then, there exist h_i, $q_i \in G_0$ and $x_i \in U_x$, $y_i \in U_y$ such that $g_i h_i x_i = q_i y_i$, i.e., $q_i^{-1} g_i h_i \in G(U_x, U_y)$. Therefore $g_i \in G_0 \cdot G(U_x, U_y) \cdot G_0$, which is a relatively compact subset of G. This means that $\{g_i G_0\} \to \infty$ is not possible. Hence

$(G/G_0)(W_1, W_2)$ is relatively compact. Therefore, every non-compact discrete subgroup F of G/G_0 acts properly discontinuously on $X\backslash G_0$, which is a locally compact, connected and paracompact space.

So, we can apply our results for the action $(F, X\backslash G_0)$. The surjective map $q : X \to X\backslash G_0$ with $q(x) = G_0(x)$ is proper, because G_0 is compact. Therefore, by Proposition 1.1.2, it has a unique extension $\varepsilon q : \varepsilon X \to \varepsilon(X\backslash G_0)$ that maps the ends of X onto those of $X\backslash G_0$. So, this map relates the ends of X with those of $X\backslash G_0$.

Claim. The restriction of the map εq on the set of the ends of X is a bijection.

Since G_0 is connected, as before, the ends of X are fixed points for the action $(G_0, \varepsilon X)$. The map εq is equivariant; therefore the ends of $X\backslash G_0$ are also fixed points with respect to the action $(G/G_0, \varepsilon(X\backslash G_0))$. Since every end of X is a G_0-orbit, the assertion follows.

If X has infinitely many ends there is nothing to prove. If X has finitely many ends, let $\mu(X\backslash G_0)$ be the zero-dimensional compactification of $X\backslash G_0$ that corresponds to the action $(F, X\backslash G_0)$ by Theorem 6.2. According to Proposition 5.2, we have that $\mu(X\backslash G_0) = \varepsilon(X\backslash G_0)$, and by Theorem 6.3, the set L^* of the limit points of the action $(F, X\backslash G_0)$ consists of at most two points. There are no other ends except those of L^*, because by Theorem 6.2(b), the non-compact group F acts properly on $\varepsilon(X\backslash G_0) \setminus L^*$ which has finitely many points. This and the previous claim prove the theorem. □

References

1. H. Abels, *Über die erzeugung von eigentlichen transformationgruppen*, Math. Z. **103** (1968), 333–357. MR0224738 (37:337)
2. H. Abels, *Enden von räumen mit eigentlichen transformationsgruppen*, Comment. Math. Helv. **47** (1972), 457–473. MR0317305 (47:5852)
3. M.D. Alder, *Inverse limits of simplicial complexes*, Compositio Math. **29** (1974), 1–7. MR0356070 (50:8541)
4. N. Bourbaki, *General Topology*, Parts I and II, Hermann, Paris, 1966.
5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966. MR0193606 (33:1824)
6. J.L. Koszul, *Lectures on groups of transformations*, Tata Institute of Fundamental Research, Bombay, 1965. MR0218485 (36:1571)
7. P.F. Lam, *Almost equicontinuous transformation groups*, Trans. Amer. Math. Soc. **195** (1974), 165–200. MR0377957 (51:14126)
8. S. Mardešić, J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982. MR676973 (84b:55020)
9. F. Raymond, *The end point compactification of manifolds*, Pacific J. Math. **10** (1960), 947–963. MR0120637 (22:11387)
10. L.A. Steen, J.A. Seebach Jr., *Counterexamples in Topology*, Springer-Verlag, New York, 1978. MR507446 (80a:54001)
11. P. Strantzalos, *Strikt fast gleihgradig-stetige und eigentliche aktionen*, Pacific J. Math. **70** (1977), 253–267. MR0482702 (58:2758)

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: amanouss@math.uni-bielefeld.de

Department of Mathematics, University of Athens, Panepistimioupolis, GR-157 84, Athens, Greece

E-mail address: pstrantz@math.uoa.gr

Journal of Lie Theory
Volume **13** (2003) 7–12

On the Group of Isometries on a Locally Compact Metric Space

Antonios Manoussos and Polychronis Strantzalos*

Communicated by W. A. F. Ruppert

Abstract. In the present paper we study conditions under which the group of isometries on a locally compact metric space is locally compact, or acts properly.

Keywords: Ellis semigroup, isometry, pointwise convergence, proper action.
Mathematics Subject Classification: Primary: 54H20, 37B; Secondary: 54E15, 54H15.

1. Introduction

It is long known from the work of van Dantzig and van der Waerden ([1], cf. also [2, Ch.I, Th.4.7]) that if (X, d) is a connected locally compact metric space then its group of isometries $I(X, d)$, when endowed with the topology of pointwise convergence, is always locally compact and acts properly on X. More recently it was shown by one of the authors ([6]) that the pointwise closure of $I(X, d)$ is locally compact if the space $\Sigma(X)$ of the connected components of X is quasicompact (compact but not necessarily Hausdorff) with respect to the quotient topology. The question whether $I(X, d)$ is closed in $C(X, X)$ (the space of all continuous selfmaps of X endowed with the topology of pointwise convergence) remained open. In this note we fill this gap (cf. also [4]), i.e., we show that if $\Sigma(X)$ is quasicompact then $I(X, d)$ coincides with its Ellis' semigroup, completing the proof of the following:

Theorem. *Let (X, d) be a locally compact metric space. Denote by $I(X, d)$ its group of isometries, with the topology of pointwise convergence, and by $\Sigma(X)$ the space of the connected components of X, endowed with the quotient topology. Then*

1. *If $\Sigma(X)$ is not quasicompact, then $I(X, d)$ need not be locally compact, nor act properly on X.*
2. *If $\Sigma(X)$ is quasicompact then*

* The authors want to thank the referee for his useful and constructive comments.

ISSN 0949–5932 / \$2.50

(a) *$I(X,d)$ is locally compact,*

(b) *the action $(I(X,d),X)$ is not always proper, and*

(c) *the action $(I(X,d),X)$ is proper if X is connected.*

For the sake of completeness, we give short and slightly improved proofs of some of the previously published partial results of the authors, these are crucial for a unified proof of the above theorem. Our treatment is based on the sets $(x,V_x)=\{g\in I(X,d): g(x)\in V_x\}$, where V_x is a neighborhood of $x\in X$. These sets form a neighborhood subbasis at the identity with respect to the topology of pointwise convergence, the natural topology of $I(X,d)$.

2. Generalities

2.1. The following simple examples establish 1 and 2(b) of the above theorem.

Example. Let $X=\mathbb{Z}$ with the discrete metric. Obviously $\Sigma(X)$ is not quasi-compact. It can be easily seen that $I(X,d)$ is the group of all bijections of $\mathbb{Z}$, which is not locally compact with respect to the topology of pointwise convergence, therefore it cannot act properly on a locally compact space.

Example. Let $X=Y\cup\{(1,0)\}\subset\mathbb{R}^2$ where $Y=\{(0,y): y\in\mathbb{R}\}$, and $d=\min\{1,\delta\}$, where δ denotes the Euclidean metric. As we shall see in §3, by Theorem 3.7, $I(X,d)$ is locally compact; however the action $(I(X,d),X)$ is not proper, because the isotropy group of $(1,0)$ is not compact, since it contains the translations of Y. So, the action of $I(X,d)$ on X is not proper, even if X has two components.

Since the sets (x,V_x) as above form a neighborhood subbasis at the identity in $I(X,d)$, the following condition is necessary for the local compactness of $I(X,d)$:

(a) There exist $x_i\in X$, $i=1,\dots,m$ such that $\bigcap_{i=1}^m(x_i,V_{x_i})$ is relatively compact in $C(X,X)$.

This condition becomes also sufficient if, in addition, the following condition is satisfied:

(b) $I(X,d)$ is closed in $C(X,X)$.

So, to prove that $I(X,d)$ is locally compact, we have to ensure that both of the above conditions are satisfied.

3. The local compactness of $I(X,d)$

The following is crucial for the investigation of the conditions 2.1(a) and (b):

3.1. Lemma. *Let (X,d) be a locally compact metric space, $F\subseteq I(X,d)$, and*

$$K(F)=\{x\in X: F(x)=\{f(x): f\in F\} \text{ is relatively compact}\}.$$

Then $K(F)$ is an open and closed subset of X.

Proof. Since F is an equicontinuous family of selfmaps of X we see that $K(F)$ is open. It remains to prove that $K(F)$ is closed.

We write $S(x,\eta) = \{y \in X \mid d(x,y) < \eta\}$ for any $x \in X$ and $\eta > 0$, and $S(M,\eta) = \bigcup\{S(x,\eta) \mid x \in M\}$ for subsets $M \subseteq X$. Let x be a cluster point of $K(F)$ and let η be a positive real such that $\overline{S(x,5\eta)}$ is relatively compact. Choose a point $k \in K(F) \cap S(x,\eta)$. Then $\overline{F(k)} \subseteq S(F(S(x,\eta)),\eta) = F(S(x,2\eta))$, and by the compactness of $\overline{F(k)}$ we can find a finite subset $L \subseteq F$ such that $\overline{F(k)} \subseteq L(S(x,2\eta))$. We show that $F(x)$ is contained in the relatively compact set $L(S(x,5\eta))$. To see this, pick $f \in F$ and let $g \in L$ such that $f(k) \in g(S(x,2\eta))$. Then

$$\begin{aligned} d(f(x),g(k)) &\leq d(f(x),f(k)) + d(f(k),g(x)) + d(g(x),g(k)) \\ &= d(x,k) + d(f(k),g(x)) + d(x,k) \leq 4\eta \end{aligned}$$

and therefore

$$f(x) \in S(g(k),4\eta) = g(S(k,4\eta)) \subseteq g(S(x,5\eta)) \subseteq L(S(x,5\eta)).$$

Thus $x \in K(F)$ and the proof is finished.

3.2. Remark. In the sequel we assume that $\Sigma(X)$ *is quasicompact* in the quotient topology via the natural map $q : X \to \Sigma(X)$. Note that $\Sigma(X)$ is a T_1–space, and need not be Hausdorff. Nevertheless

X is separable, hence second countable; so sequences are adequate in C(X,X).

The proof is similar to the lengthy one in [5] (see also [2, Appendix 2]).

3.3. Lemma. *Let (X,d) be a locally compact metric space with a quasicompact space of connected components $\Sigma(X)$. Then condition 2.1(a) is satisfied.*

Proof. Let V_x be a relatively compact neighborhood of $x \in X$. Then

$$(x,V_x) = \{g \in I(X,d) : g(x) \in V_x\}$$

is a neighborhood of the identity in $I(X,d)$. Since $x \in K((x,V_x))$, $K((x,V_x))$ is not empty, and by Lemma 3.1 is open and contains entire components of X. Therefore $q(K((x,V_x)))$ is an open subset of $\Sigma(X)$. Since $\Sigma(X)$ is quasicompact, there are x_i, $i = 1,\ldots,m$, such that the corresponding $q(K((x_i,V_{x_i})))$' s cover $\Sigma(X)$. This means that $X = \bigcup_{i=1}^{m} K((x_i,V_{x_i}))$, i.e., the neighborhood $F = \bigcap_{i=1}^{m}(x_i,V_{x_i})$ of the identity has the property: for every $x \in X$ the set $F(x)$ is relatively compact in X. Therefore, by Ascoli' s theorem, F is relatively compact in $C(X,X)$.

3.4. Now we prove that if $\Sigma(X)$ is quasicompact then $I(X,d)$ is a closed subspace of $C(X,X)$. Because of Remark 3.2, the elements f of the boundary of $I(X,d)$ in $C(X,X)$ are limits of sequences $\{f_n \in I(X,d), n \in \mathbb{N}\}$. Obviously, such an f preserves d; so the question is whether f is surjective. If $\Sigma(X)$ is not quasicompact then this is not always true:

Example. Let $X = \mathbb{Z}$ with the discrete metric. If $f_n(z) = z$ for $-n < z < 0$, $f_n(-n) = 0$, and $f_n(z) = z+1$ otherwise, then $f_n \to f$, where $f(z) = z$ for $z < 0$, and $f(z) = z+1$ for $z \geq 0$. Hence each f_n is an isometry, but f is not surjective since $0 \notin f(\mathbb{Z})$.

3.5. Lemma. *If $\Sigma(X)$ is quasicompact and $\{(f_n) : f_n \in I(X,d)\}$ is a sequence such that $f_n \to f$ for some selfmap f of X with respect to the topology of pointwise convergence, then $f(X)$ is open and closed in X.*

Proof. By Lemma 3.1, it suffices to show that $f(X) = K(F)$, where $F = \{f_n^{-1}, n \in \mathbb{N}\}$. Indeed, since $d(f_n(x), f(x)) = d(x, f_n^{-1}(f(x)))$, we have $f_n^{-1}(f(x)) \to x$, so (since X is locally compact) $f(x) \in K(F)$, for every $x \in X$. Now, if $y \in K(F)$, we may assume $f_{n_k}^{-1}(y) \to x$ for some $x \in X$, because $F(y)$ is relatively compact in X, hence $f(x) = y$.

3.6. Proposition. *If (X,d) is a locally compact metric space, and $\Sigma(X)$ is quasicompact, then $I(X,d)$ is closed in $C(X,X)$.*

Proof. Let $\{(f_n) : f_n \in I(X,d)\}$ be a sequence such that $f_n \to f$ for some selfmap f of X with respect to the topology of pointwise convergence. We prove that f is surjective. Let $y \in X$. We denote by S_x the connected component containing $x \in X$, and by S_n the component of $f_n^{-1}(y)$. If $\{S_n, n \in \mathbb{N}\}$ has a constant subnet $\{S_{n_i}, i \in I\}$, then $S_{n_i} = S_0$, for some $S_0 \in \Sigma(X)$. Hence $S_{f_{n_i}^{-1}(y)} = S_0$, so $f_{n_i}(S_0) = S_y$, for every $i \in I$. Pick an $x \in S_0$, then $f_{n_i}(x) \in S_y$. But $f_{n_i}(x) \to f(x)$, so $f(x) \in S_y$. By Lemma 3.5 $S_y \subseteq f(X)$, hence $y \in f(X)$.

Suppose that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet. By the quasicompactness of $\Sigma(X)$, there exists a subnet $\{S_{n_i}, i \in I\}$ of $\{S_n, n \in \mathbb{N}\}$ such that $S_{n_i} \to S$, for some $S \in \Sigma(X)$. With the above notation, the following is true:

Claim. *There exists a subsequence $\{S_k, k \in \mathbb{N}\}$ of $\{S_n, n \in \mathbb{N}\}$ such that there are $x_k \in S_k$ with $x_k \to x_0$, for some $x_0 \in X$.*

Proof. If not, $R = (\bigcup_{n=1}^{\infty} S_n) \setminus S$ is closed in X. Indeed, let $\{(y_m) : y_m \in R\}$ be a sequence such that $y_m \to y \in X$. If $y_m \in (\bigcup_{n=1}^{n_0} S_n) \setminus S$ for $m > m_0$, then a subsequence of $\{y_m, m \in \mathbb{N}\}$ is contained in some S_i for some $i \in \{1, \dots, n_0\}$, therefore $y \in S_i \subseteq R$, as required. If this is not the case, we construct a subsequence $\{y_{m_p}, p \in \mathbb{N}\}$ of $\{y_m, m \in \mathbb{N}\}$ in the following way: For S_1 we choose a point $y_{m_1} \in S_{n_1}$ with $n_1 > 1$ and $d(y_{m_1}, y) < 1$, for $(\bigcup_{n=1}^{n_1} S_n) \setminus S$ a point $y_{m_2} \in S_{n_2}$ with $n_2 > n_1$ and $d(y_{m_2}, y) < \frac{1}{2}$, and so on. Obviously, $y_{m_p} \in S_{n_p}$ and $y_{m_p} \to y$, a contradiction.

Since S does not meet R, then $S \subseteq X \setminus R$. On the other hand $X \setminus R$ is open (since R is closed in X) and contains entire components (recall that R is a union of components), so $S_{n_i} \subseteq X \setminus R$, eventually. Therefore $S_{n_i} = S$, a contradiction, since we have assumed that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet.

According to the Claim, there exists a sequence $\{(x_k) : x_k \in S_k\}$ such that $x_k \to x_0 \in X$, where $S_k = S_{f_k^{-1}(y)} = f_k^{-1}(S_y)$, from which follows $x_k = f_k^{-1}(y_k)$ for some $y_k \in S_y$. Then

$$\begin{aligned} d(y_k, f(x_0)) &\leq d(y_k, f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &= d(f_k^{-1}(y_k), x_0) + d(f_k(x_0), f(x_0)) \to 0, \end{aligned}$$

therefore $f(x_0) \in S_y$, which means that $S_y \cap f(X) \neq \emptyset$ and, by Lemma 3.5, $S_y \subseteq f(X)$, hence $y \in f(X)$, and f is surjective.

3.7. Theorem. *If $\Sigma(X)$ is quasicompact, then $I(X,d)$ is locally compact.*

Proof. This assertion follows from Lemma 3.3 and Proposition 3.6, since both conditions 2.1(a) and (b) are satisfied.

4. The properness of the action $(I(X,d), X)$

In this short section, applying the methods used previously, we give a complete proof of the following:

Proposition. *If (X,d) is locally compact and connected, then $I(X,d)$ is locally compact and the action $(I(X,d), X)$ is proper.*

Proof. Since X is connected $G = I(X,d)$ is locally compact by Theorem 3.7. So, we have to show that, for every x, $y \in X$, there are neighborhoods U_x, U_y of x and y respectively such that

$$(U_x, U_y) := \{g \in G : (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G. Let $U_x = S(x,\varepsilon)$ and $U_y = S(y,\varepsilon)$ be such that $S(y,2\varepsilon)$ is relatively compact. Then, for $g \in (U_x, U_y)$ and $z \in U_x$ with $g(z) \in U_y$, we have

$$d(g(x),(y)) \leq d(g(x), g(z)) + d(g(z), y) = d(x,z) + d(g(z), y) < 2\varepsilon,$$

therefore $g \in F = \{g \in G : g(x) \in S(y, 2\varepsilon)\}$. Then $x \in K(F)$, and, according to Lemma 3.1, $K(F)$ coincides with the connected space X. From this and Ascoli's theorem it follows that F is relatively compact in $C(X,X)$. So $(U_x, U_y) \subseteq F$ is relatively compact in $C(X,X)$, hence in G, because G is closed (cf. Proposition 3.6).

This proves the Proposition and completes the proof of the Theorem in the Introduction.

5. Final Remark

Using the same arguments we can prove that if X is a locally compact metrizable space, then $I(X,d)$ is locally compact for all admissible metrics d, provided that the space $Q(X)$ of the quasicomponents of X is compact with respect to the quotient topology (note that $Q(X)$ is always Hausdorff) (cf. [3]). Recall that the quasicomponent of a point is the intersection of all open and closed sets which contain it. Our exposition is given via $\Sigma(X)$ because we regard the condition "$\Sigma(X)$ is quasicompact" as a topologically more natural condition than "$Q(X)$ is compact", although it is more restrictive: There are locally compact metric spaces with compact $Q(X)$ and non quasicompact $\Sigma(X)$ as the following example shows:

Example. The space of all connected components of the locally compact space

$$X = \left(\bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : y \in [-1,1]\}\right) \cup \{(0,y) : y \in [-1,0)\} \cup \left(\bigcup_{k=1}^{\infty} I_k\right) \subseteq \mathbb{R}^2,$$

where

$$I_k = \{(0,y) : y \in (\frac{1}{k+1}, \frac{1}{k})\}, \quad k \in \mathbb{N}^*,$$

is not quasicompact, because the sequence $\{I_k\} \subseteq \Sigma(X)$ does not have a convergent subsequence in $\Sigma(X)$. On the contrary, $Q(X)$ is compact, because the quasicomponent of the point $(0,-1)$ consists of the set $\{(0,y) : y \in [-1,0)\}$ and the intervals I_k, $k \in \mathbb{N}^*$.

So the quasicompactness of $\Sigma(X)$ is not necessary for the local compactness of $I(X,d)$.

References

[1] van Dantzig, D., and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.

[2] Kobayashi, S., and K. Nomizu, "Foundations of Differential Geometry, Vol. I," Interscience, New York, 1963.

[3] Manoussos, A., *Contribution to the study of D–stable actions*, Ph.D. Thesis, University of Athens, Greece (1993) (in Greek).

[4] Manoussos, A., *The Ellis semigroup of the full group of isometries of a locally compact metric space*, Preprint.

[5] Sierpinsky, W., *Sur les espaces métriques localement séparables*, Fund. Math. **21** (1933), 107–113.

[6] Strantzalos, P., *Actions by Isometries*, Lecture Notes in Math. **1375** (1989), 319– 325.

Antonios Manoussos
123 Sapfous Street
176 75 Kallithea
Athens
Greece
amanou@cc.uoa.gr

Polychronis Strantzalos
Department of Mathematics
University of Athens
Athens, GR – 157 84
Greece
pstrantz@math.uoa.gr

Received August 25, 2000
and in final form March 9, 2002

ON THE ACTION OF THE GROUP OF ISOMETRIES ON A LOCALLY COMPACT METRIC SPACE: CLOSED-OPEN PARTITIONS AND CLOSED ORBITS

ANTONIOS MANOUSSOS

Abstract. In the present work we study the dynamic behavior of the orbits of the natural action of the group G of isometries on a locally compact metric space X using suitable closed-open subsets of X. Precisely, we study the dynamic behavior of an orbit even in cases where G is not locally compact with respect to the compact-open topology. In case G is locally compact we decompose the space X into closed-open invariant disjoint sets that are related to various limit behaviors of the orbits. We also provide a simple example of a locally compact separable and complete metric space X with discrete group of isometries G such that the natural action of G on X has closed and non-closed orbits.

1. Introduction

The group of isometries and their actions play an important role in many branches of Mathematics (especially in Geometry). This class of actions is rich, as a recent result of Abels, Noskov and the author in [2] shows. In [2] it is shown that if Y is a locally compact σ-compact metrizable space then a locally compact group Γ acts properly on Y if and only if there exists a Γ-invariant proper compatible metric on Y (recall that a metric on Y is called proper or Heine-Borel if every ball has compact closure in Y). So, in this case, we can consider such a group as a closed subgroup of the group of isometries of a proper metric space (modulo the kernel of the action). The first result concerning the local compactness of the group of isometries of a locally compact metric space is the van Dantzig - van der Waerden theorem in 1928 (see [7] and [10, Theorem 4.7]) which says that the group G of isometries of

2000 *Mathematics Subject Classification.* Primary 37B05; Secondary 54H20.

Key words and phrases. Isometries, closed-open partitions, closed orbits, pseudo-components.

During this research the author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

a connected, locally compact metric space X is locally compact (with respect to the compact-open topology) and acts properly on X. In [12] (see also [14]) this result is generalized for the case of a locally compact metric space which has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components). In particular it is shown that the group of isometries of X is locally compact but may fail to act properly on X even for the case that X has only two connected components. A crucial point in obtaining this result is making use of suitable closed-open subsets of X (for more details see the next section). At the same time Gao and Kechris in [8, Theorem 5.4 and Corollary 6.2] (see also [6]) showed a stronger result: that the group of isometries G of a locally compact separable metric space X with finitely many pseudo-components (which are also closed-open subsets of X, see [8, Proposition 5.3]) is locally compact and in case X is locally compact, separable and pseudo-connected then G acts properly on X (for definitions and more details see [8, p. 32] and Section 3 below). Important examples of locally compact, separable and pseudo-connected spaces are the proper (Heine-Borel) spaces. Comparing the results of [12] and [8] we would like to mention that the assumption about the quasi-compactness of the space of connected components of X in [12] is purely topological hence the result in [12] applies to any metric that induces the topology of X. Obviously the assumption in [8] about finitely many pseudo-components depends on the choice of the metric on X but the result is stronger since a locally compact metric space with quasi-compact space of connected components has finitely many pseudo-components.

The purpose of this paper is to show that the closed-open subsets of X used in [12] and [8] also give information for the space X and the dynamic behavior of the orbits of the natural action of G on X, even for the case that G is not a locally compact group. In what follows, X will denote a locally compact metric space with a fixed metric d and $G := Iso(X, d)$ will denote the group of (surjective) isometries of X endowed with the compact-open topology. The natural action of G on X is the action with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$. The main results in this work are stated below:

In Section 3 (see Propositions 3.1 and Corollary 3.4 below) we show the following:

Proposition. *Let $x, y \in X$ and a net $\{g_i\}$ in G with $g_i x \to y$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$, a closed-open subset A of X that contains x and a map $f : A \to X$ which preserves the distance such that $g_j \to f$ pointwise on A, $f(x) = y$ and $f(A)$ is an open subset of X.*

The same result also holds if we replace A with the pseudo-component C_x that contains x. In this case $f(C_x) = C_{f(x)}$.

The previous proposition gives as corollaries the van Dantzig - van der Waerden theorem (see Corollary 3.2 below) and the results of Gao and Kechris in [8, Theorem 5.4 and Corollary 6.2] (see Corollary 3.5 below). In Section 4 we give some applications in case G is locally compact and there exist closed orbits for the action of G on X. We also give a simple example of a locally compact separable and complete metric space with a discrete group of isometries such that the natural action of G on X has closed and non-closed orbits (see Example 4.4 below). In Section 5 we show that the closed-open subsets of X used in [12] leads to a decomposition of the space X into closed-open G-invariant disjoint sets that are related to the limit behavior of the orbits (see Theorem 5.1 below): Let

$$\begin{aligned} L(x) = \{y \in X \,|\, &\text{there exists a net } \{g_i\} \text{ in } G \\ &\text{with } g_i \to \infty \text{ and } \lim g_i x = y\}, \end{aligned}$$

denote the limit set of $x \in X$, where $g_i \to \infty$ means that the net $\{g_i\}$ has no cluster point in G.

Theorem. *Assume that G is locally compact and not compact and let*

$$\begin{aligned} CL &= \{x \in X \,|\, L(x) \textit{ is not empty and compact}\}, \\ NCL &= \{x \in X \,|\, L(x) \textit{ is not compact}\} \textit{ and} \\ P &= \{x \in X \,|\, L(x) \textit{ is the empty set}\}. \end{aligned}$$

Then the sets CL, NCL and P are closed-open G-invariant disjoint, their union is X and each one of them is a union of pseudo-components.

2. Preliminaries

A continuous action of a topological group Γ on a topological space Y is a continuous map $\Gamma \times Y \to Y$ with $(g, x) \mapsto gx$, $g \in \Gamma, x \in Y$ such that $(1, x) \mapsto x$, for every $x \in Y$ where 1 denotes the unit element of Γ, and $h(gx) = (hg)x$ for every $h, g \in \Gamma$ and $x \in Y$. For $U \subseteq Y$ let ΓU denote the set $\{gx \,|\, g \in \Gamma, x \in U\}$. Especially, if $U = \{x\}$ then the set $\Gamma x := \Gamma\{x\}$ is called the orbit of $x \in Y$ under Γ. If $\Gamma U = U$ we say that U is Γ-invariant. The subgroup $\Gamma_x := \{g \in \Gamma \,|\, gx = x\}$ of Γ is called the isotropy group of $x \in Y$.

In what follows, X will denote a locally compact metric space with a fixed metric d and $G := Iso(X, d)$ will denote the group of (surjective) isometries of X endowed with the compact-open topology. The natural action of G on X is the action with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$.

If we endow G with the topology of pointwise convergence then G is a topological group (see [5, Ch. X, §3.5 Corollary]). On G there is also the topology of uniform convergence on compact subsets which is the same as the compact–open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural action of G on X with $(g, x) \mapsto g(x)$ is continuous (see [5, Ch. X, §2.4 Theorem 1 and §3.4 Corollary 1]).

We recall that in [4, Ch. III, §4.1 Definition 1] a continuous action of a topological group Γ on a topological space Y is said to be *proper* (or *Bourbaki proper*) if the map

$$\Gamma \times Y \to Y \times Y \text{ with } (g, x) \mapsto (x, gx), \text{ for } g \in \Gamma \text{ and } x \in Y$$

is proper, i.e. it is continuous, closed and the inverse image of a singleton is a compact set.

To simplify the proofs we shall use the following equivalent definition for properness: a continuous action is proper if the extended limit sets $J(x)$ are empty for every $x \in Y$, where

$$J(x) = \{y \in Y \,|\, \text{there exist nets } \{x_i\} \text{ in } Y \text{ and } \{g_i\} \text{ in } \Gamma \\ \text{with } g_i \to \infty, \lim x_i = x \text{ and } \lim g_i x_i = y\},$$

where $g_i \to \infty$ means that the net $\{g_i\}$ has no cluster point in G. It is easy to see that in the special case of actions by isometries $J(x) = L(x)$ holds for every $x \in Y$, where

$$L(x) = \{y \in Y \,|\, \text{there exists a net } \{g_i\} \text{ in } \Gamma \\ \text{with } g_i \to \infty \text{ and } \lim g_i x = y\},$$

denotes the limit set of $x \in Y$ under the action of Γ on Y. Hence an action by isometries is proper if and only if $L(x)$ is the empty set for every $x \in Y$. The limit and the generalized limit sets for locally compact spaces and groups are closed and Γ-invariant (see [3]). The following example shows that even in case that X has two connected components the action of G on X may not be proper (see also [14]).

Example 2.1. Let $X = L_1 \cup L_2 \subset \mathbb{R}^2$ where $L_1 = \{(0, t) \,|\, t \in \mathbb{R}\}$ and $L_2 = \{(2, t) \,|\, t \in \mathbb{R}\}$. We consider the metric $d = \min\{d_E, 1\}$ where d_E is the usual Euclidean metric on $\mathbb{R}^2$. With this metric X is a locally compact separable space. Since for a point $x \in X$ (actually for every $x \in X$) the isotropy group G_x contains an isomorphic copy of the reals the action of G on X is not proper.

Let F be a subset of G. We define $K(F)$ to be the set

$$K(F) := \{x \in X \,|\, \text{the set } Fx \text{ has compact closure in } X\}.$$

These sets played a crucial role in [12] where it is shown that they are closed-open subsets of X (see [12, Lemma 3.1], [14]). Actually we can find the same definition and result in Bourbaki (see [5, Ch. X, Exercise 13, p. 323]) but as we mentioned in [2] there is a mistake in part d) of this exercise (for a counterexample see [2] or [11]).

3. The general case

In this section *the only assumption is that X is locally compact without assuming the same for G*. We study the following problem: Assume that there is a pair of points $x, y \in X$ and a net $\{g_i \,|\, i \in I\}$ in G such that $g_i x \to y$. What can we say about the convergence of $\{g_i\}$?

Proposition 3.1. *Let $x, y \in X$ and a net $\{g_i \,|\, i \in I\}$ in G with $g_i x \to y$. Then there exist a subnet $\{g_j \,|\, j \in J\}$ of $\{g_i \,|\, i \in I\}$, a closed-open subset A of X that contains x and a map $f : A \to X$ which preserves the distance such that $g_j \to f$ pointwise on A, $f(x) = y$ and $f(A)$ is an open subset of X. Moreover, if $\{g_i\}$ is a sequence and X is a second countable space (in which case the limit sets can be described using sequences) then $f(A)$ is closed.*

Proof. Since $d(x, g_i^{-1}y) = d(g_i x, y) \to 0$ it follows that $g_i^{-1}y \to x$. By the local compactness of X there exists an index $i_0 \in I$ such that, if $F := \{g_i \,|\, i \geq i_0\}$ then $x \in K(F)$ and $y \in K(F^{-1})$, where $F^{-1} := \{g_i^{-1} \,|\, i \geq i_0\}$. Set $A := K(F)$. By [12, Lemma 3.1] A is a closed-open subset of X. If $g_i|_A$ denotes the restriction of each g_i on A, then the Arzela-Ascoli theorem implies that the set $\{g_i|_A : A \to X \,|\, i \geq i_0\}$ has compact closure in $C(A, X)$ (this the set of all continuous maps from A to X). Thus, there exists a subnet $\{g_j \,|\, j \in J\}$ of $\{g_i \,|\, i \in I\}$ and a map $f : A \to X$ with $f(x) = y$ which preserves the distance such that $g_j \to f$ pointwise on A. We show that $f(A)$ is open: Let $z \in f(A)$. That is, there is $w \in A$ such that $f(w) = z$. It is enough to show that if $\{z_l\} \subset X$ is a net such that $z_l \to z$ then $z_l \in A$ eventually for every l. Since $w \in A$ then $g_j w \to f(w) = z$. Hence $g_j^{-1}z \to w$. As before there exists an index j_0 such that, if $F_1 := \{g_j \,|\, j \geq j_0\}$ then $w \in K(F_1)$ and $z \in K(F_1^{-1})$. Again by the Arzela-Ascoli theorem there exist a subnet $\{g_k\}$ of $\{g_j\}$ and a map $h : K(F_1^{-1}) \to X$ which preserves the distance such that $g_k^{-1} \to h$ pointwise on $K(F_1^{-1})$ and $h(z) = w$. Since $K(F_1^{-1})$ is open and $z \in K(F_1^{-1})$ we may assume that $z_l \in K(F_1^{-1})$ eventually for every l. Hence $h(z_l) \to h(z) = w \in A$ as $l \to \infty$ and for each l, $g_k^{-1}z_l \to h(z_l)$ as $k \to \infty$. Therefore $h(z_l) \in A$ eventually for every l. Fix a point $h(z_l) \in A$. Then $g_k(h(z_l)) \to f(h(z_l))$. Thus $d(z_l, f(h(z_l))) \leq d(g_k g_k^{-1} z_l, g_k h(z_l)) +$

$d(g_k h(z_l), f(h(z_l))) = d(g_k^{-1} z_l, h(z_l)) + d(g_k h(z_l), f(h(z_l))) \to 0$ as $k \to \infty$. Hence $z_l = f(h(z_l)) \in A$ eventually for every l.

Note that up to this point we have only used the property that the sets $K(F)$ are open. If $\{g_i \,|\, i \in \mathbb{N}\}$ is a sequence we can set $F_2 := \{g_i \,|\, i \in \mathbb{N}\}$ and $A := K(F_2)$. Then A is a non-empty closed-open subset of X and, as before, there exists a subsequence $\{g_{i_n} \,|\, n \in \mathbb{N}\}$ of $\{g_i\}$ (here we use that X is second countable) and a map $f : A \to X$ with $f(x) = y$ which preserves the distance such that $g_{i_n} \to f$ pointwise on A. We will use now the property that the sets $K(F)$ are closed to show that $f(A)$ is also a closed subset of X. If we set $F_3 := \{g_{i_n} \,|\, n \in \mathbb{N}\}$ then it is easy to verify that $f(A) \subseteq K(F_3^{-1})$ and there exist a subsequence $\{g_{i_{n_l}} \,|\, l \in \mathbb{N}\}$ of $\{g_{i_n}\}$ and a map $h : K(F_3^{-1}) \to X$ which preserves the distance such that $g_{i_{n_l}}^{-1} \to h$ pointwise on $K(F_3^{-1})$. Take a sequence $\{f(a_k) \,|\, k \in \mathbb{N}\}$, $a_k \in A$ such that $f(a_k) \to b$ for some $b \in X$. We will show that $b \in f(A)$. Fix $k \in \mathbb{N}$. Since $f(A) \subseteq K(F_3^{-1})$ and $K(F_3^{-1})$ is closed then $f(a_k) \in K(F_3^{-1})$, $g_{i_{n_l}}^{-1} f(a_k) \to h(f(a_k))$ as $l \to \infty$ and $b \in K(F_3^{-1})$. The latter implies that $g_{i_{n_l}}^{-1} b \to h(b)$. Note that $d(g_{i_{n_l}}^{-1} b, h(b)) = d(b, g_{i_{n_l}} h(b)) \to 0$ so $g_{i_{n_l}} h(b) \to b$. We will show that $h(b) \in A$ and $g_{i_{n_l}} h(b) \to f(h(b))$, hence $b = f(h(b)) \in f(A)$ and the proof is finished. Indeed, observe that $d(g_{i_{n_l}}^{-1} f(a_k), a_k) = d(f(a_k), g_{i_{n_l}} a_k) \to 0$ as $l \to \infty$. Therefore $h(f(a_k)) = a_k$. Thus $a_k = h(f(a_k)) \to h(b)$ as $k \to \infty$. But $a_k \in A$ and A is a closed subset of X hence $h(b) \in A$. So $g_{i_{n_l}} h(b) \to f(h(b))$. □

Note that $f(A)$ may not be G-invariant, see for instance Example 2.1. As an application of Proposition 3.1 we can prove the van Dantzig - van der Waerden theorem in a short and elegant way comparing to the proof in the original work of van Dantzig and van der Waerden [7] or to the lengthy one in [10, Theorem 4.7, pp. 46–49]:

Corollary 3.2. *(The van Dantzig - van der Waerden theorem) The group G of isometries of a connected, locally compact metric space X is locally compact (with respect to the compact-open topology) and G acts properly on X.*

Proof. It is enough to show that G acts properly on X (i.e. $L(x) = \emptyset$ for every $x \in X$, see Section 2) because in this case for every pair of points $x, y \in X$ there exist open neighborhoods U_x, U_y of x, y respectively such that the set $\{g \in G \,|\, gU_x \cap U_y \neq \emptyset\}$ has compact closure in G (see e.g. [3]). Let $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$. Proposition 3.1 implies that there exist a subnet $\{g_j\}$ of $\{g_i\}$, a closed-open subset A of X and a map $f : A \to X$ which preserves the distance such that $g_j \to f$ pointwise on A. Since X is connected it follows that $A = X$. Note that $d(x, g_i^{-1}y) = d(g_i x, y) \to 0$ hence we can repeat the same procedure as before and find a subnet $\{g_k\}$ of $\{g_i\}$ and maps $f, h : X \to X$ which preserve the distance such that $g_k \to f$ and $g_k^{-1} \to h$ pointwise on X. Obviously h is the inverse of f. This shows that $f \in G$. Hence, $L(x) = \emptyset$ and since $x \in X$ was arbitrary the action is proper. $\square$

A question which arises from Proposition 3.1 is whether there is any difference if one replaces A with the pseudo-component that contains the point $x \in X$. We answer this question in the affirmative in Corollary 3.4. Before we present these result we need some formulation that we can also find in [8, p. 32]:

An important notion in the definition of the pseudo-component of a point $x \in X$ is the *radius of compactness* $\rho(x)$ *of* x:

$$\rho(x) := \sup\{\, r > 0 \mid \text{the open ball } B(x,r) \text{ has compact closure}\}$$

where $B(x, r)$ denotes the open ball centered at $x \in X$ with radius $r > 0$. It is easy to see that if $g \in G$ then $\rho(gx) = \rho(x)$. We define an equivalence relation $\mathcal{E}$ on X as follows: Firstly we define a directed graph $\mathcal{R}$ on X by

$$x\mathcal{R}y \quad \text{if and only if} \quad d(x,y) < \rho(x).$$

Let $\mathcal{R}^*$ be the transitive closure of $\mathcal{R}$, i.e.

$$x\mathcal{R}^*y \quad \text{if and only if for some} \quad u_0 = x, u_1, \ldots, u_n = y$$

we have $u_i \mathcal{R} u_{i+1}$ for every $i < n$. Finally, define the following equivalence relation $\mathcal{E}$ on X

$$x\mathcal{E}y \quad \text{if and only if} \quad x = y \text{ or } (x\mathcal{R}^*y \text{ and } y\mathcal{R}^*x).$$

We call the $\mathcal{E}$-equivalence class of $x \in X$ the *pseudo-component* of x, and we denote it by C_x. We call X *pseudo-connected* if it has only one pseudo-component. It follows that pseudo-components are closed-open subsets of X (see [8, Proposition 5.3]). An immediate consequence of the definitions is that $gC_x = C_{gx}$ for every $g \in G$.

The following example shows that in many cases the closed-open set A in Proposition 3.1 may contain strictly the pseudo-component that contains the point $x \in X$.

Example 3.3. Let $X = L_1 \cup L_2 \cup L_3 \subset \mathbb{R}^2$ where $L_1 = \{(0,t) \mid t \in \mathbb{R}\}$, $L_2 = \{(2,t) \mid t \in \mathbb{R}\}$ and $L_3 = \{(4,t) \mid t \in \mathbb{R}\}$ endowed with the metric $d = \min\{d_E, 1\}$ where d_E is the usual Euclidean metric on $\mathbb{R}^2$. With this metric X is a locally compact separable and complete metric space with finitely many pseudo-components. Let $x := (0, 0)$ and let

$g_n : X \to X$, $n \in \mathbb{N}$ with $g_n(t,a) = (t,a)$ if $a = 0$ or 2 and $t \in \mathbb{R}$ and $g_n(4,t) = (4, t+n)$, $t \in \mathbb{R}$. Obviously $g_n x = x$ for every $n \in \mathbb{N}$, the map g_n restricted to $L_1 \cup L_2$ is the identity and the pseudo-component C_x of x is the set L_1. Hence, if we take as $A := L_1 \cup L_2$ then A contains strictly C_x.

Corollary 3.4. *Assume that X is locally compact (perhaps with infinitely many pseudo-components). Let $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x, $f(x) = y$ and $f(C_x) = C_{f(x)}$.*

Proof. With a slight modification of the technical Lemma 5.5 in [8] in order to use nets instead of sequences we have the following: Let $x, y \in X$ and $\{g_i\}$ be a net in G with $g_i x \to y$. Then for $F := \{g_i\}$ the set $F(z)$ has compact closure in X for every $z \in C_x$. For $A := K(F)$ Proposition 3.1 implies that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x and $f(x) = y$. Since $g_j^{-1} f(x) \to x$ then there exist a subnet $\{g_k\}$ of $\{g_j\}$ and a map $h : C_{f(x)} \to X$ which preserves the distance such that $g_k^{-1} \to h$ pointwise on $C_{f(x)}$ and $h(y) = x$. Take a point $z \in C_x$. Since $g_k z \to f(z)$ and the pseudo-component $C_{f(x)}$ is a closed-open subset of X then $f(z) \in C_{f(x)}$ (so $f(C_x) \subseteq C_{f(x)}$) and $g_k z \in C_{f(x)}$ eventually for every k. Hence $z = g_k^{-1}(g_k z) \to h(f(z))$. In the same way we can deduce that $f(h(w)) = w$ for every $w \in C_{f(x)}$, thus $f(C_x) = C_{f(x)}$. □

As an implication of the previous corollary we can take the results of Gao and Kechris [8, Theorem 5.4 and Corollary 6.2]:

Corollary 3.5. *(The Gao - Kechris theorem) The isometry group of a locally compact metric space with only finitely many pseudo-components is locally compact. In case X is locally compact and pseudo-connected then G acts properly on X.*

Proof. Let $C_1, C_2, \ldots, C_n$ denote the pseudo-components of X and take points $x_1 \in C_1, x_2 \in C_2, \ldots, x_n \in C_n$ and open balls $B(x_m, r) \subseteq C_m$, $m = 1, 2, \ldots, n$, $r > 0$ such that all $B(x_m, r)$ have compact closures. We will show that the set $V := \displaystyle\bigcap_{m=1}^{n} \{g \in G \,|\, g x_m \in B(x_m, r)\}$ is an (open) neighborhood of the identity in G with compact closure. Indeed take a net $\{g_i\} \subseteq V$. Since each $B(x_m, r)$ has compact closure there exist a subnet $\{g_j\}$ of $\{g_i\}$ and points $y_1 \in C_1, y_2 \in C_2, \ldots, y_n \in C_n$ such that $g_j x_m \to y_m$, as $j \to \infty$, for every $m = 1, 2, \ldots, n$. Corollary 3.4 implies that there exist a subnet $\{g_l\}$ of $\{g_j\}$ and maps $f_m : C_m \to C_m$ which preserve the distance such that $g_l \to f_m$ on C_m and $f_m(C_m) = C_m$ for all m. This shows that $\{g_k\}$ converges to a *surjection* of X which actually gives that $\{g_k\}$ converges to an isometry of X.

Assume that X is pseudo-connected. In order to show that G acts properly on X it is enough to show that the limit set $L(x)$ is empty for every $x \in X$ (see Section 2). Let $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$. Corollary 3.4 implies that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x, $f(x) = y$ and $f(C_x) = C_{f(x)}$. Since X is pseudo-connected then $C_x = C_{f(x)} = X$. Hence $f \in G$ thus $L(x)$ is empty. ☐

Remark 3.6. Note that in Corollary 3.5 we do not require that X is separable like in [8, Theorem 5.4 and Corollary 6.2]. This is *not* a real improvement since we can show that *a locally compact metric space X with countably many pseudo-components is separable*. For a proof we can imitate the proof of Lemma 3 in [10, Appendix 2] (actually this is a result of Sierpinski, see [13]): We define a relation $\mathcal{S}$ on X by $x\mathcal{S}y$ if and only if there exist separable open balls $B(x, r_1)$ and $B(y, r_2)$ with $y \in B(x, r_1)$ and $x \in B(y, r_2)$. For every $A \subseteq X$ we denote by $\mathcal{S}A := \{y \in X \mid y\mathcal{S}x \text{ for some } x \in A\}$. If $A = \{x\}$ is a singleton we write $\mathcal{S}x$ instead of $\mathcal{S}\{x\}$. Set $\mathcal{S}^{n+1}x := \mathcal{S}\mathcal{S}^n x$ for every $n \in \mathbb{N}$ and $U(x) := \displaystyle\bigcup_{n=1}^{+\infty} \mathcal{S}^n x$. Then by [10, Lemma 3 in Appendix 2] each $U(x)$ is a separable closed-open subset of X and if $U(x) \cap U(y) \neq \emptyset$ then $U(x) = U(y)$. By construction every $U(x)$ contains the pseudo-component of $x \in X$. Therefore X is separable.

Remark 3.7. Proposition 3.1 and Corollary 3.4 point out a natural generalization of the notion of properness for locally compact metric spaces with groups of isometries which are not closed in the space of all continuous selfmaps of X endowed with the compact-open topology: In particular, it will be interesting to study actions with the property "if $x, y \in X$ and there is a net $\{g_i\}$ in G such that $g_i x \to y$ then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : X \to X$ *not necessarily surjective*, which preserves the distance and such that $g_i \to f$ pointwise on X". That is, let's say, if $g_i \to \infty$ then this happens in a "strong" way.

4. Closed orbits

In this section we assume that *both* X and G are *locally compact* and we will discuss some implications of the existence of closed orbits. In the previous section we saw that if there is a pair of points $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$ then there exist a subnet $\{g_j\}$ of $\{g_i\}$, a closed-open subset A of X that contains x and a map $f : A \to X$ which preserves the distance such that $g_j \to f$ pointwise on A, $f(x) = y$ and $f(A)$ is an open subset of X. A question which arises naturally is: When is the map f a restriction of some element of G on A? An answer can be given using the following general proposition but before we see that we need again some formulation: Given a continuous action of a locally compact group Γ on a locally compact space Y we can define a homomorphism $\phi : \Gamma \to H(Y)$ with $\phi(g)(x) := gx$, $g \in \Gamma$, $x \in Y$ where $H(Y)$ denotes the group of homeomorphisms of Y endowed with the compact-open topology. This homomorphism is always continuous (see e.g. [3] or [15, Lemma 10.4 (c)]). If $g|_A$ denotes the restriction of $g \in \Gamma$ on a subset A of Y we define $\phi : \Gamma \to C(A, Y)$ with $\phi(g)(x) := gx$, $g \in \Gamma$, $x \in A$ where $C(A, Y)$ denotes the space of all continuous maps from A to Y endowed with the compact-open topology. Note that ϕ is a continuous map.

Proposition 4.1. *Let Y be a locally compact space, A be an open or closed subset of Y and Γ be a locally compact σ-compact group which acts continuously on Y. If there exists a point $x \in A$ with closed orbit such that $\phi(\Gamma_x)$ is closed in $C(A, Y)$ then $\phi(\Gamma)$ is closed in $C(A, Y)$.*

Proof. Since Γ is locally compact and σ-compact and $\Gamma(x)$ is closed in Y, the map $\varphi : \Gamma/\Gamma_x \to \Gamma(x)$ with $\varphi(g\Gamma_x) := \Gamma x$, $g \in \Gamma$ is a homeomorphism (see [15, Theorem 10.10 (c)]). Let $\{g_i \,|\, i \in I\}$ be a net in Γ such that $\phi(g_i) \to h$ for some $h \in C(A, Y)$. Since the orbit $\Gamma(x)$ is closed, there exists $\gamma \in \Gamma$ such that $\gamma x = h(x)$ so $g_i\Gamma_x \to \gamma\Gamma_x$. The quotient map $\Gamma \to \Gamma/\Gamma_x$ is open and Γ is locally compact hence there exist an open neighborhood V of v with compact closure and nets $\{f_i\}$ in V, $\{v_i\}$ in Γ_x such that $g_i = f_i v_i$ eventually for every $i \in I$. Thus, there exist a subnet $\{f_j\}$ of the net $\{f_i\}$ and $f \in \Gamma$ such that $f_j \to f$. The set A is locally compact, hence the composition map $T : C(Y, Y) \times C(A, Y) \to C(A, Y)$ with $T(f_1, f_2) = f_1 \circ f_2$, $f_1 \in C(Y, Y)$, $f_2 \in C(A, Y)$ is continuous (see [15, Lemma 9.4 (c)]). Thus, $\phi(v_j) = \phi(f_j^{-1}) \circ \phi(g_j) \to \phi(f^{-1}) \circ h$. Since $\phi(\Gamma_x)$ is closed in $C(A, Y)$ there exists $g \in \Gamma_x$ such that $\phi(f^{-1}) \circ h = \phi(g)$ from which follows that $h = \phi(fg)$. Hence $\phi(\Gamma)$ is closed in $C(A, Y)$. □

Note that in our case, if X is a second countable locally compact metric space with locally compact group of isometries G then G is σ-compact (see [5, Ch. X, §3.3 Corollary]). Following the proof of the previous proposition, if $x, y \in Y$ and $\{g_i\}$ is a net in Γ such that

$g_i x \to y$ then $g_i = f_i v_i$ for some nets $\{f_i\}$ in a compact subset of Γ and $\{v_i\}$ in Γ_x, so we have the following:

Corollary 4.2. *Let Y be a locally compact space and Γ be a locally compact σ-compact group which acts continuously on Y. If x is a point of Y with closed orbit then the limit set $L(x)$ is not empty if and only if the isotropy group Γ_x of x is not compact.*

If we assume that X is second countable (hence separable) and complete and its group of isometries G is locally compact then G is second countable (see [5, Ch. X, §3.3 Corollary]). So, in this case, both X and G are *Polish* spaces. In [9] Glimm showed that for the case of an action of a locally compact separable group on a locally compact separable space the existence of a *Borel section* (or selection) is actually equivalent to the fact that each orbit is locally closed. Recall that a subset S of a topological space Y is called a *section* (or selection) for a continuous action of a topological group Γ on Y if S meets every orbit in exactly one point. In our case if x, y is a pair of points of X and $\{g_i\}$ is a net in G such that $g_i x \to y$ then $g_i^{-1} y \to x$, since g_i preserves the metric. Thus, for isometric actions, locally closed orbits are closed and vice versa. A question which arises naturally is the following:

Question 4.3. If X and G are locally compact do there exist always closed orbits?

Note that we are considering the full group of isometries of X because if we ask the same question for the action of a closed subgroup of G on X then the answer is negative (see [11]). The following simple example shows that the answer is also negative for the action of G on X:

Example 4.4. Let $X = R \cup Q \subset \mathbb{R}^2$ where $R = \{(t,0) \,|\, t \in \mathbb{R}\}$ and $Q = \{(q,1) \,|\, q \in \mathbb{Q}\}$. For every pair of points $w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2) \in X$ define

$$d(w_1, w_2) := \begin{cases} |x_1 - x_2|, & \text{if } w_1, w_2 \in R \\ |x_1 - x_2| + 1, & \text{if only one of } w_1, w_2 \text{ is not in } R \\ |x_1 - x_2| + 2, & \text{if } w_1, w_2 \in Q. \end{cases}$$

It is easy to verify that d is a metric on X and X with this metric is a locally compact, separable and complete space. The group of isometries G is generated by the horizontal translations by rationals and by the horizontal reflections with centers of the form $(x, y) \in X$ with $x \in \mathbb{Q}$ (if we want to have only translations we may take $Q := \{(q,1) \,|\, q \in \mathbb{Q} + \sqrt{2}\,\mathbb{N}\}$). Hence G is a discrete group (so it is locally compact). If $w = (x,1) \in Q$ then the orbit $G(w) = Q$ and if $w = (x,0) \in R$ then the orbit $G(w) = \{(x,0) \in R \,|\, x \in \mathbb{Q}\}$ so it is not closed in X. Moreover if $w \in Q$ then $L(x) = \emptyset$ and if $w \in R$ then $x \in L(x) = R$.

In this direction Gao and Kechris in [8, p. 35] asked the following question which still remains open:

Question 4.5. (Gao - Kechris) Let (X,d) be a locally compact complete metric space with finitely many pseudo-components (or connected components). Is it true that the action of $G := Iso(X,d)$ on X has closed orbits?

Based on this we ask the following question.

Question 4.6. Let (X,d) be a locally compact and complete metric with only two connected components, one compact and one not compact. If the action of $G := Iso(X,d)$ on the non-compact component is proper is it true that the orbits of points in the compact component are closed?

The last question is of great interest in case of a metric space having only one end in its Freudenthal (end-point) compactification.

As we saw in Example 4.4 the set of points of X with closed orbits may not be the whole space X, so it is natural to ask the following:

Question 4.7. Let X be a locally compact metric space. Is the set of points of X with closed orbits closed or open? Does it contain entire pseudo-components?

In the following example we give (a partial) negative answer to this question. Namely the set of points of X with closed orbits may not be open or may not contains entire pseudo-components:

Example 4.8. This example is based on the same idea as Example 4.4. Let $X = D \cup S \subset \mathbb{R}^2$ where D is the closed unit disk and S is an orbit of a point on a circle with center the origin and radius 2 under an irrational rotation $2e^{2\pi a}$, $a \notin \mathbb{Q}$. The distance of two points in the unit disk is the usual Euclidean one. To measure the distance from a point of $x \in S$ to a point of $y \in D$ we firstly move on the radius connecting x with the origin until we meet the circle with center the origin and radius the distance from y to the origin. Then we move on this circle in the shortest way until we meet the point y. In a similar way we measure the distance of two points $x, y \in S$: Firstly we move on the radius connecting x with the origin until we meet the unit circle and then we move on this circle and follow back the radius connecting y with the origin in the shortest way until we meet the point y. The space X endowed with this metric is locally compact, separable and complete and its group of isometries G is discrete (hence locally compact) like in Example 4.4. Moreover, all the points of the closed unit disk except the origin do not have closed orbits and the origin is a fixed point. The action of G on S is proper and the orbits coincide with S. Note that the pseudo-component (and the connected component) that contains the origin is the closed unit disk hence the set of points of X with closed orbit is not closed and does not contain entire pseudo-components.

The only thing that remains to be clarified is whether the set of points of a space X with closed orbits is a closed subset of X. In this direction we know that if there is a pair of points $x, y \in X$ and a net $\{g_i\}$ in G such that $g_i x \to y$ then, by Corollary 3.4, there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x. So, if X is separable then by Proposition 4.1, we know that if there exists some point $z \in C_x$ such that $\phi(\Gamma_z)$ is closed in $C(C_x, X)$ then $\phi(\Gamma)$ is closed in $C(C_x, X)$. From this we can deduce that the set of points of X with closed orbits is closed. The general question remains open as well as the following generalized one:

Question 4.9. Is there any locally compact, separable and complete metric space such that G is locally compact and *every* orbit is not closed?

5. Closed-open G-invariant partitions

In [12, Theorem] we showed that in case X has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components) the group G is locally compact. This is an application of the fact that the sets $K(F)$, defined in Section 2, are closed and open. In this section we will see another application of this property of $K(F)$ concerning the structure of X. We will show that if G is locally compact then there is a decomposition of X into closed-open G-invariant sets that are related to various limit behaviors of the orbits: To be more precise, let

$$\begin{aligned} CL &= \{x \in X \mid L(x) \text{ is not empty and compact}\}, \\ NCL &= \{x \in X \mid L(x) \text{ is not compact}\} \text{ and} \\ P &= \{x \in X \mid L(x) \text{ is the empty set}\}. \end{aligned}$$

Theorem 5.1. *Let (X, d) be a locally compact metric space and $G := Iso(X, d)$ the group of isometries of X. Assume that G is locally compact and not compact. Then*

(i) *The closure of each orbit is a minimal set.*

(ii) *If the closure of an orbit is compact then it is stable. Hence each $x \in CL$ has a stable orbit closure.*

(iii) *The sets CL, NCL and P are closed-open G-invariant disjoint, their union is X and each one of them is a union of pseudo-components.*

Proof. (i) Recall that a non-empty G-invariant closed subset A of X is called *minimal* if it has no G-invariant closed proper subsets. Equivalently, $A = \overline{G(x)}$ for every $x \in A$. Let $y \in X$ and $x \in \overline{G(y)}$. Obviously $\overline{G(x)} \subseteq \overline{G(y)}$ and there exists a sequence $\{g_n\}$ in G such that $g_n y \to x$. Since $d(g_n y, x) = d(y, g_n^{-1}x)$ it follows that $y \in \overline{G(x)}$, thus $\overline{G(y)} \subseteq \overline{G(x)}$. Note also that since $L(y)$ is a G-invariant closed subset of X then whenever $L(y) \neq \emptyset$ we have that $L(y) = \overline{G(y)}$. So, if G is not compact then $CL = \{x \in X \mid \overline{G(x)} \text{ is compact}\}$ and $NCL = \{x \in X \mid \overline{G(x)} \text{ is not compact and } L(x) \neq \emptyset\}$.

(ii) Assume that $\overline{G(x)}$ is compact for some $x \in X$. We will show that $\overline{G(x)}$ is stable, that is, for every open set $U \subseteq X$ with $\overline{G(x)} \subseteq U$ there exists a G-invariant open set V such that $\overline{G(x)} \subseteq V \subseteq U$. Since X is locally compact and $\overline{G(x)}$ is compact then $\overline{G(x)}$ has a neighborhood base consisting of compact sets, let's say $\mathcal{W}$ (see [4, Ch. I, §9.7 Proposition 10]). There is a natural direction defined on $\mathcal{W}$: $W_1 \leq W_2$ if and only if $W_2 \subseteq W_1$ for $W_1, W_2 \in \mathcal{W}$. We argue by contradiction: Assume that for every $W \in \mathcal{W}$ there exist $x_w \in W$ and a point $g_w \in G$ such that $g_w x_w \notin U$. It is not hard to see that there exist a point $y \in \overline{G(x)}$ and a subnet $\{x_i\}$ of the net $\{x_w \mid w \in \mathcal{W}\}$ such that $x_i \to y$. Since $g_i y \in \overline{G(x)}$ and $\overline{G(x)}$ is compact there exist a subnet $\{g_j y\}$ of $\{g_i y\}$ and a point $z \in \overline{G(x)}$ such that $g_j y \to z$. Note that $d(g_j x_j, z) \leq d(g_j x_j, g_j y) + d(g_j y, z) = d(x_j, y) + d(g_j y, z) \to 0$ which is a contradiction since we have assumed that $g_i x_i \notin U$ for every index i.

(iii) Obviously the sets CL, NCL and P are G-invariant, disjoint and their union is X. Item (i) implies that $CL = K(G)$ hence, by [12, Lemma 3.1], CL is closed and open. Since $J(x) = L(x)$ for every $x \in X$ (because we have an action by isometries, see Section 2), $P = \{x \in X \mid J(x) = \emptyset\}$. Take a point x in the complement of P. Then $J(x) \neq \emptyset$ and since $J(x) = L(x)$ we have that $J(x) = L(x) = \overline{G(x)}$. Hence $X \setminus P = \{x \in X \mid x \in J(x)\}$ and it is well known (see e.g. [3]) that this is a closed subset of X, so P is open. We claim that P is also closed. Let $\{x_n \mid n \in \mathbb{N}\}$ be a sequence of points of P such that $x_n \to x$ for some $x \in X$. We argue by contradiction: If $x \notin P$ then $x \in J(x) = L(x)$ hence there exists a net $\{g_i \mid i \in I\}$ in G with $g_i \to \infty$ and $g_i x \to x$. Fix a positive real number $r > 0$ such that the ball centered at x with radius

$2r$ has compact closure and fix also a point x_{n_0} such that $d(x_{n_0}, x) < r$. Note that $d(g_i x_{n_0}, x) \leq d(g_i x_{n_0}, g_i x) + d(g_i x, x) = d(x_{n_0}, x) + d(g_i x, x)$. Since $g_i x \to x$ the net $\{g_i x_{n_0} \mid i \in I\}$ is eventually in the open ball $B(x, 2r)$ hence it has a convergent subnet. This implies that $L(x_{n_0}) \neq \emptyset$ (since $g_i \to \infty$) thus $x_{n_0} \notin P$ which is a contradiction.

In Corollary 3.4 we saw that if $x, y \in X$ and $\{g_i\}$ is a net in G such that $g_i x \to y$ then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and a map $f : C_x \to X$ which preserves the distance such that $g_j \to f$ pointwise on C_x. This shows that the set P is a union of pseudo-components. If we work as in the proof of Corollary 3.4 and take $F := G$ then it easy to see that the set CL has also the same property. □

A question which arises from the previous theorem is the following: Can the sets CL, NCL and P may coexist in any combination? We answer this question in the affirmative using the following simple examples. Note that the Arzela-Ascoli theorem implies that $CL = X$ if and only if the group G is compact. If X is connected the van Dantzig - van der Waerden theorem implies that $P = X$ and in Example 2.1 we have that $NCL = X$.

Example 5.2. ($CL \neq \emptyset$, $NCL \neq \emptyset$ and $P = \emptyset$). Let $X = \{(0,0)\} \cup L_1 \cup L_2 \subset \mathbb{R}^2$ where $L_1 = \{(2,t) \mid t \in \mathbb{R}\}$ and $L_2 = \{(4,t) \mid t \in \mathbb{R}\}$. We consider the metric $d = \min\{d_E, 1\}$ where d_E is the usual Euclidean metric on $\mathbb{R}^2$. As in Example 2.1 it is easy to see that $CL = \{(0,0)\}$, $NCL = L_1 \cup L_2$ and $P = \emptyset$.

Example 5.3. ($CL \neq \emptyset$, $NCL = \emptyset$ and $P \neq \emptyset$). If we take $X = \{(0,0)\} \cup L_1$, where $L_1 = \{(2,t) \mid t \in \mathbb{R}\}$, and the metric as in the previous example then $CL = \{(0,0)\}$, $NCL = \emptyset$ and $P = L_1$.

Example 5.4. ($CL = \emptyset$, $NCL \neq \emptyset$ and $P \neq \emptyset$). In Example 4.4 we have that $CL = \emptyset$, $NCL = R$ and $P = Q$.

Example 5.5. ($CL \neq \emptyset$, $NCL \neq \emptyset$ and $P \neq \emptyset$). This example is a modification of the Example 4.4. Firstly we replace the metric d in Example 4.4 by the bounded metric $d' = \frac{d}{1+d}$ (note that d' and d give the same group of isometries). Then we add the point $(3,0)$ to X and finally we endow the set $Y := X \cup \{(3,0)\}$ with a new metric d^* requiring that $d^*|_{X\times X} = d'|_{X\times X}$ and $d^*((3,0), w) = 1$ for every $w \in X$. If G denotes the group of isometries of Y with respect to d^* it is easy to see that $CL = \{(3,0)\}$, $NCL = R$ and $P = Q$.

In case P is not empty we have a very interesting result concerning its structure. This result is an application of a theorem of Abels in [1]. Namely, in [1], Abels proved that if a non-compact locally compact group G with compact space of connected components acts properly on a locally compact space Y such that the orbit space $G\backslash Y$ is paracompact then Y is homeomorphic to a product of the form $\mathbb{R}^n \times M$ for some $n \in \mathbb{N}$ where M is a closed subset of X. Actually n is the same n if we write the group G as a homeomorphic image of the product $\mathbb{R}^n \times K$ where K is a maximal compact subgroup of G in the Malcev-Iwasawa's decomposition theorem for G (see [15, Theorem 32.5]). If we apply this theorem to our case we have the following:

Proposition 5.6. *Let (X, d) be a locally compact metric space and $G := Iso(X, d)$ the group of isometries of X. Assume that G is locally compact, not compact with compact space of connected components (or the connected component of the identity of G is not compact). Then P, if it is not empty, it is homeomorphic to a product of the form $\mathbb{R}^n \times M$ for some $n \in \mathbb{N}$ where M is a closed subset of P.*

Proof. The proof is an immediate consequence of the previous mentioned theorem of Abels in [1] taking into account that if G_1 denotes the connected component of the identity of G then G_1 is a closed subgroup of G. Hence G_1 acts properly on P and the orbit space $G_1\backslash X$ is metrizable (see [3]). □

Remark 5.7. As a final remark we would like to point out that the results of this paper also hold for the natural action of a locally compact, pointwise equicontinuous group of homeomorphisms Γ on a locally compact uniform space Y.

References

[1] H. Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Ann. **212** (1974/75), 1–19.

[2] H. Abels, G. Noskov and A. Manoussos, *Proper actions and proper invariant metrics*, preprint.

[3] H. Abels and P. Strantzalos, *Proper Transformation Groups*, preprint.

[4] N. Bourbaki, *Elements of Mathematics. General topology*, Part 1. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[5] N. Bourbaki, *Elements of Mathematics. General topology*, Part 2. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[6] J. D. Clemmens, S. Gao and A. S. Kechris, *Polish Metric Spaces: Their classification and isometry groups*, Bull. Symbolic Logic **7** (2001), 361–375.

[7] D. van Dantzig and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Seminar Hamburg **6** (1928), 367–376.

[8] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766.

[9] J. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124–138.

[10] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol I, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[11] A. Manoussos, *Contribution to the study of D–stable actions*, Ph.D. Thesis, University of Athens, Greece (1993) (in Greek).

[12] A. Manoussos and P. Strantzalos, *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), 7–12.

[13] W. Sierpinski, *Sur les espaces métriques localement séparables*, Fund. Math. **21** (1933), 107–113.

[14] P. Strantzalos, *Actions by isometries*, Transformation groups (Osaka, 1987), 319–325, Lecture Notes in Math., **1375**, Springer, Berlin, 1989.

[15] M. Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006.

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: `amanouss@math.uni-bielefeld.de`

A GROUP OF ISOMETRIES WITH NON-CLOSED ORBITS

H. ABELS AND A. MANOUSSOS

ABSTRACT. In this note we give an example of a one-dimensional manifold with two connected components and a complete metric whose group of isometries has an orbit which is not closed. This answers a question of S. Gao and A. S. Kechris.

1. PRELIMINARIES AND THE CONSTRUCTION OF THE EXAMPLE

In [3, p. 35] S. Gao and A. S. Kechris asked the following question. Let (X, d) be a locally compact complete metric space with finitely many pseudo-components or connected components. Does its group of isometries have closed orbits? This is the case if X is connected since then the group of isometries acts properly by an old result of van Dantzig and van der Waerden [1] and hence all of its orbits are closed. The above question arose in the following context. Suppose a locally compact group with a countable base acts on a locally compact space with a countable base. Then the action has locally closed orbits (i.e. orbits which are open in their closures) if and only if there exists a Borel section for the action (see [4], [2]) or, in other terminology, the corresponding orbit equivalence relation is smooth. For isometric actions it is easy to see that an orbit is locally closed if and only if it is closed. In this note we give a negative answer to the question of Gao and Kechris. Our space is a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. In the course of the construction we give an example of a 2-dimensional

2000 *Mathematics Subject Classification.* Primary 37B05, 54H15; Secondary 54H20.

Key words and phrases. Proper action, group of isometries, smooth orbit equivalence relation.

During this research the second author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He is grateful for its generosity and hospitality.

manifold with two connected components one compact and one non-compact and a complete metric whose group G of isometries also has non-closed dense orbits on the compact component. The difference is that G contains a subgroup of index 2 which is isomorphic to $\mathbb{R}$.

Let (Y, d_1) be a metric space. Later on Y will be a torus with a flat Riemannian metric. Let $Z = Y \cup (Y \times \mathbb{R})$. We fix two positive real numbers R and M. We endow Z with the following metric d depending on R and M.

$$\begin{aligned} &d(y_1, y_2) = d_1(y_1, y_2) \\ &d((y_1, t_1), (y_2, t_2)) = d_1(y_1, y_2) + \min(|t_1 - t_2|, M) \\ &d(y_1, (y_2, t_2)) = d((y_2, t_2), y_1) = d(y_1, y_2) + R, \end{aligned}$$

for $y_1, y_2 \in Y$ and $t_1, t_2 \in \mathbb{R}$. It is easy to check that d is a metric on Z if $2R \geq M$. The metric space Z has the following properties

1.1. a) For a given point $(y, r) \in Y \times \mathbb{R}$ there is a unique point in Y which is closest to (y, r), namely y.

b) Given a point $y \in Y$ the set of points in $Y \times \mathbb{R}$ which are closest to y is the line $\{y\} \times \mathbb{R}$.

c) For every point $(y, r) \in Y \times \mathbb{R}$ and every $y' \in Y$ there is a unique point on the line $\{y'\} \times \mathbb{R}$ which is closest to (y, r), namely (y', r).

d) Let g_Y be an isometry of Y and let $g_{\mathbb{R}}$ be an isometry of the Euclidean line $\mathbb{R}$. Define a map $g = g(g_Y, g_{\mathbb{R}}) : Z \to Z$ by $g|Y := g_Y$ and $g(y, r) = (g_Y(y), g_{\mathbb{R}}(r))$ for $(y, r) \in Y \times \mathbb{R}$. Then g is an isometry of Z.

e) Every isometry of Z is of the form given in d) if Y is compact.

Proof. a) through d) are easily checked. To prove e) let g be an isometry of Z. Then $g(Y) = Y$ and $g(Y \times \mathbb{R}) = Y \times \mathbb{R}$, since Y is compact and $Y \times \mathbb{R}$ consists of non-compact components. Then $g_Y := g|Y$ is an isometry of Y. The map $g(g_Y, id)^{-1} \circ g$, where id denotes the identity map, is an isometry of Z which fixes Y, hence maps every line $\{y\} \times \mathbb{R}$ to itself, by b). Let $h_y : \mathbb{R} \to \mathbb{R}$ be defined by $g(y, t) = (y, h_y(t))$. Then h_y is an isometry of the Euclidean line $\mathbb{R}$ for every $y \in Y$ and all the h_y's are the same, by c), say $h_y = g_{\mathbb{R}}$. Thus $g = (g_Y, g_{\mathbb{R}})$. □

1.2. Let now Y be a 2-dimensional torus with a flat Riemannian metric. Y is also an abelian Lie group whose composition we write as multiplication. Every translation L_x of Y, $L_x(y) = x \cdot y$, is an isometry. Let $g(t)$, $t \in \mathbb{R}$, be a dense one parameter subgroup of Y. Let $H \subset Y \times \mathbb{R}$ be its graph, $H = \{(g(t), t)\,;\, t \in \mathbb{R}\}$. Our example is $X = Y \cup H$ with the metric induced from $Z = Y \cup (Y \times \mathbb{R})$.

1.3. a) If $g_{\mathbb{R}}$ is an isometry of the Euclidean line $\mathbb{R}$ then there is a unique isometry g of X such that $g(y,t) \in Y \times \{g_{\mathbb{R}}(t)\}$. If $g_{\mathbb{R}}$ is the translation by a, so $g_{\mathbb{R}} = L_a$ with $L_a(t) = t+a$, then g is the restriction of $g(L_{g(a)}, L_a)$ to X. If $g_{\mathbb{R}}$ is the reflection at O, $g_{\mathbb{R}} = -\mathbf{1}$, then g is the restriction of $g(inv, -\mathbf{1})$ to X, where $inv : Y \to Y$, $inv(y) = y^{-1}$. The reflection in $a \in \mathbb{R}$ is the composition $L_{-2a} \circ (-\mathbf{1}) = -\mathbf{1} \circ L_{2a}$.

b) Every isometry of X is of the form in a). It follows that the group of isometries of X has dense non-closed orbits on Y and the other component H is one orbit.

c) H is locally isometric to the real line with the Euclidean metric, actually $d((g(t),t),(g(s),s)) = (1+\|\dot{g}(0)\|)\,|t-s|$ for small $|t-s|$, where $\dot{g}(0)$ is the tangent of the one-parameter group $g(t)$, $t \in \mathbb{R}$, and $\|\cdot\|$ is the norm on the tangent space of Y at the identity element derived from the Riemannian tensor.

Proof. c) follows from the definition of the metric d on $Y \times \mathbb{R}$. The maps given in a) are isometries of Z and map X to X, hence are isometries of X. To prove the uniqueness claim in a) it suffices to prove it for $g_{\mathbb{R}} = id$. But then g is the identity on the image of the one-parameter group $g(t)$, $t \in \mathbb{R}$, by 1.1 a) and hence on all of Y. Hence g has the form given by 1.1 d). To show b) it suffices to show that every isometry h of H is of the form given in a). This follows from c). $\square$

1.4 Remark. In our example the space has dimension 2 and the group of orientation preserving isometries is of index 2 in the group of all isometries and is isomorphic to $\mathbb{R}$. We can reduce the dimension of our space to 1 to obtain a group of isometries with closed orbits on the non-compact component, which is diffeomorphic and locally isometric to $\mathbb{R}$, and non-closed dense orbits on the compact component, which isometric to S^1. The example is as follows. Take a one-dimensional subtorus Y_1 of Y containing the identity element of Y. Define $X_1 = Y_1 \cup H \subset Y \cup H$. Then the group of isometries of Y_1 consists of those maps $g_a = g(L_{g(a)}, L_a)$ restricted to Y_1 with $g(a) \in Y_1$, and of the maps $g(inv \circ L_{g(2a)}, -\mathbf{1} \circ L_a)$ restricted to Y_1 with $g(2a) \in Y_1$. The proof follows from the proof of 1.3.

References

[1] D. van Dantzig and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Seminar Hamburg **6** (1928), 367-376.

[2] E. G. Effros. *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38-55.

[3] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766.

[4] J. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124-138.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
E-mail address: abels@math.uni-bielefeld.de

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
E-mail address: amanouss@math.uni-bielefeld.de

THE GROUP OF ISOMETRIES OF A LOCALLY COMPACT METRIC SPACE WITH ONE END

ANTONIOS MANOUSSOS

Abstract. In this note we study the dynamics of the action of the group of isometries G of a locally compact metric space (X, d) with one end. Using the notion of pseudo-components introduced by S. Gao and A. S. Kechris we show that X has only finitely many pseudo-components exactly one of which is not compact and G acts properly on this pseudo-component. The complement of the non-compact component is a compact subset of X and G may fail to act properly on it.

1. Preliminaries and the main result

The idea to study the dynamics of the action of the group of isometries G of a locally compact metric space (X, d) with one end, using the notion of pseudo-components introduced by S. Gao and A. S. Kechris in [2], came from a paper of E. Michael [6]. In this paper he introduced the notion of a J-space, i.e. a topological space with the property that whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact. In terms of compactifications locally compact non-compact J-spaces are characterized by the property that their end-point compactification coincides with their one-point compactification (see [6, Proposition 6.2], [7, Theorem 6]). Recall that the Freudenthal or end-point compactification of a locally compact non-compact space X is the maximal zero-dimensional compactification of X. By zero-dimensional compactification of X we here mean a compactification Y of X such that $Y \setminus X$ has a base of closed-open sets (see [5], [7]). From the topological point of view locally compact spaces with one end are something very general since the product of two non-compact locally compact connected spaces is a space with one end (see [7, Proposition

2000 *Mathematics Subject Classification.* Primary 37B05; Secondary 54H20.

Key words and phrases. Proper action, pseudo-component, end-point compactification, J-space.

During this research the author was fully supported by SFB 701 “Spektrale Strukturen und Topologische Methoden in der Mathematik” at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

8], [6, Proposition 2.5]), so it is rather surprising that the dynamics of the action of the group of isometries G of a locally compact metric space (X, d) with one end has a certain structure as our main result shows.

Theorem 1.1. *Let (X, d) be a locally compact metric space with one end and let G be its group of isometries. Then*

(i) *X has finitely many pseudo-components exactly one of which is not compact and G is locally compact.*
(ii) *Let P be the non-compact pseudo-component. Then G acts properly on P, $X \setminus P$ is a compact subset of X and G may fail to act properly on it.*

Recall that the action of G on X is the map $G \times X \to X$ with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$ and it is proper if and only if the limit sets $L(x) = \{y \in X \mid \text{there exists a net } \{g_i\} \text{ in } G \text{ with } g_i \to \infty \text{ and } \lim g_i x = y\}$ are empty for every $x \in X$, where $g_i \to \infty$ means that the net $\{g_i\}$ has no cluster point in G (see [4]). A few words about pseudo-components. They were introduced by S. Gao and A. S. Kechris in [2] and we used them in [4] to study the dynamics of the action of the group of isometries of a locally compact metric space. For the convenience of the reader we repeat what a pseudo-component is. For each point $x \in X$ we define the radius of compactness $\rho(x)$ of x as $\rho(x) := \sup\{ r > 0 \mid B(x, r) \text{ has compact closure}\}$ where $B(x, r)$ denotes the open ball centered at $x \in X$ with radius $r > 0$. We define next an equivalence relation $\mathcal{E}$ on X as follows: Firstly we define a directed graph $\mathcal{R}$ on X by $x\mathcal{R}y$ if and only if $d(x, y) < \rho(x)$. Let $\mathcal{R}^*$ be the transitive closure of $\mathcal{R}$, i.e. $x\mathcal{R}^*y$ if and only if for some $u_0 = x, u_1, \ldots, u_n = y$ we have $u_i\mathcal{R}u_{i+1}$ for every $i < n$. Finally, we define the following equivalence relation $\mathcal{E}$ on X: $x\mathcal{E}y$ if and only if $x = y$ or ($x\mathcal{R}^*y$ and $y\mathcal{R}^*x$). We call the $\mathcal{E}$-equivalence class of $x \in X$ the pseudo-component of x, and we denote it by C_x. It follows that pseudo-components are closed-open subsets of X, see [2, Proposition 5.3] and $gC_x = C_{gx}$ for every $g \in G$.

Before we give the proof of Theorem 1.1 we need some results that may be of independent interest.

Lemma 1.2. *Let X be non-compact J-space and let $\mathcal{A} = \{A_i,\, i \in I\}$ be a partition of X with closed-open non-empty sets. Then $\mathcal{A}$ contains only finitely many sets exactly one of which is not compact; its complement is a compact subset of X.*

Proof. We show firstly that there exists a set in $\mathcal{A}$ which is not compact. We argue by contradiction. Assume that every set $B \in \mathcal{A}$ is compact.

Then $\mathcal{A}$ contains infinitely many distinct sets because otherwise X must be a compact space. Let $\{B_n,\, n \in \mathbb{N}\} \subset \mathcal{A}$ with $B_n \neq B_k$ for $n \neq k$ (i.e. $B_n \cap B_k = \emptyset$) and choose $x_n \in B_n$. Obviously the sequences $\{x_{2n-1}\}$, $\{x_{2n}\}$ have no limit points in X since $\mathcal{A}$ is an open partition of X. The sets $D =: \bigcup_{n=1}^{+\infty} B_{2n-1}$ and $X \setminus D$ are open (since $X \setminus D$ is a union of elements of $\mathcal{A}$) and disjoint so they form a closed partition of X. Hence, one of them must be compact. Therefore, at least one of the sequences $\{x_{2n-1}\}$, $\{x_{2n}\}$ has a limit point which is a contradiction.

Fix a non-compact $P \in \mathcal{A}$. Since P is a closed-open subset of X then $\{P, X \setminus P\}$ is a closed partition of X. Hence P or $X \setminus P$ must be compact. But P is non-compact so $X \setminus P$ is compact. If $K \in \mathcal{A}$ with $K \neq P$ then $K \subset X \setminus P$. Therefore, K is compact. Moreover $\mathcal{A}$ contains finitely many sets, since $X \setminus P$ is compact and $\mathcal{A}$ is a partition of X with closed-open non-empty sets. □

The previous lemma makes X a second countable space (i.e. X has a countable base):

Proposition 1.3. *A metrizable locally compact J-space has a countable base.*

Proof. We follow the proof of Lemma 3 in [3, Appendix 2] (actually this is a result of Sierpinski, see [8]). We define a relation $\mathcal{S}$ on X by $x\mathcal{S}y$ if an only if there exist separable open balls $B(x, r_1)$ and $B(y, r_2)$ with $y \in B(x, r_1)$ and $x \in B(y, r_2)$. For every $A \subseteq X$ we denote by $\mathcal{S}A := \{y \in X \,|\, y\mathcal{S}x \text{ for some } x \in A\}$. If $A = \{x\}$ is a singleton we write $\mathcal{S}x$ instead of $\mathcal{S}\{x\}$. Set $\mathcal{S}^{n+1}x := \mathcal{S}\mathcal{S}^n x$ for every $n \in \mathbb{N}$ and $U(x) := \bigcup_{n=1}^{+\infty} \mathcal{S}^n x$. Then, by [3, Lemma 3 in Appendix 2], each $U(x)$ is a separable closed-open subset of X and if $U(x) \cap U(y) \neq \emptyset$ then $U(x) = U(y)$. Lemma 1.2 implies that we have finitely many of these sets, hence X is separable so it is second countable. □

Proof of Theorem 1.1. Since every pseudo-component is a closed-open subset of X we can apply Lemma 1.2 for the family of the pseudo-components of X. Hence, X has finitely many pseudo-components exactly one of which, say P, is not compact and its complement $X \setminus P$ is a compact subset of X. Take any $g \in G$. Then gP is a non-compact pseudo-component hence $gP = P$. This shows that P is G-invariant. Then G is locally compact, since X has finitely many pseudo-components (see [2, Corollary 6.2]). We shall show that G acts properly on P. Assume that there are points $x, y \in P$ and a sequence

$\{g_n\}$ in G with $g_n x \to y$. We can use sequences in the definition of limit sets because X has a countable base. Let $\{P, C_1, C_2, \dots, C_k\}$ be an enumeration of the pseudo-components of X. Each pseudo-component C_i, $i = 1, \dots, k$ is compact. Choose points $x_i \in C_i$, $i = 1, \dots, k$. Since $X \setminus P$ is compact we may assume that there exist points $y_i \in X \setminus P$, $i = 1, \dots, k$ and a subsequence $\{g_{n_l}\}$ of $\{g_n\}$ such that $g_{n_l} x_i \to y_i$ for every $i = 1, \dots, k$. By Corollary 3.4 in [4] there is a subsequence of $\{g_{n_m}\}$ of $\{g_n\}$ and a map $f : X \to X$ which preserves the distance such that $g_{n_m} x \to f$ pointwise on X (we may find a subsequence instead of a subnet because X has a countable base). Then $g_n^{-1} y \to x \in P$, since $d(g_n^{-1} y, x) = d(y, g_n x)$. Repeating the previous arguments we conclude that there exists a map $h : X \to X$ such that $g_n^{-1} \to h$ pointwise on X and h preserves the distance. Obviously h is the inverse map of f, hence $f \in G$ and G acts properly on P. The group G may fail to act properly on $X \setminus P$. As an example we may take as $X = P \cup S \subset \mathbb{R}^3$, where P is the plane $\{(x, y, 0)) \mid x, y \in \mathbb{R}\}$ and S is the circle $\{(x, y, 2) \mid x^2 + y^2 = 1\}$. We endow X with the metric $d = \min\{d_E, 1\}$, where d_E is the usual Euclidean metric on $\mathbb{R}^3$. Then the action of G on S is not proper, since for a point $x \in S$ the isotropy group $G_x := \{g \in G \mid gx = x\}$ is not compact. □

Remark 1.4. If G does not act properly on $X \setminus P$ one may ask if the orbits on $X \setminus P$ are closed or if the isotropy groups of points $x \in X \setminus P$ are non-compact (see also Question 4.6 in [4]). The answer is negative in general. As an example we may consider the example in [1]. In this paper we constructed a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. We can regard the real line as a distorted helix with a locally Euclidean metric. The problem is that this manifold has two ends. But this is not really a problem. Following the same arguments as in [1] we can replace the distorted helix by a small distorted helix-like stripe and have a space with one end and two connected components, one compact isometric to S^1, and one non-compact with a locally Euclidean metric so that the group of isometries has non-closed dense orbits on the compact component.

References

[1] H. Abels and A. Manoussos, *A group of isometries with non-closed orbits*, preprint arXiv:0910.4717.

[2] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766.

[3] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol I, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[4] A. Manoussos, *On the action of the group of isometries on a locally compact metric space: closed-open partitions and closed orbits*, preprint arXiv: 0902.0319.

[5] J. R. McCartney, *Maximum zero-dimensional compactifications*, Proc. Cambridge Philos. Soc. **68** (1970), 653-661.

[6] E. Michael, *J-spaces*, Topology Appl. **102** (2000), 315-339.

[7] K. Nowiński, *Closed mappings and the Freudenthal compactification*, Fund. Math. **76** (1972), 71-83.

[8] W. Sierpinski, *Sur les espaces métriques localement séparables*, Fund. Math. **21** (1933), 107-113.

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: amanouss@math.uni-bielefeld.de

Papers on Operator Theory

J-CLASS OPERATORS AND HYPERCYCLICITY

GEORGE COSTAKIS AND ANTONIOS MANOUSSOS

Abstract. The purpose of the present work is to treat a new notion related to linear dynamics, which can be viewed as a "localization" of the notion of hypercyclicity. In particular, let T be a bounded linear operator acting on a Banach space X and let x be a non-zero vector in X such that for every open neighborhood $U \subset X$ of x and every non-empty open set $V \subset X$ there exists a positive integer n such that $T^n U \cap V \neq \emptyset$. In this case T will be called a J-class operator. We investigate the class of operators satisfying the above property and provide various examples. It is worthwhile to mention that many results from the theory of hypercyclic operators have their analogues in this setting. For example we establish results related to the Bourdon-Feldman theorem and we characterize the J-class weighted shifts. We would also like to stress that even some non-separable Banach spaces which do not support topologically transitive operators, as for example $l^\infty(\mathbb{N})$, do admit J-class operators.

1. INTRODUCTION

Let X be a complex (or real) Banach space. In the rest of the paper the symbol T stands for a bounded linear operator acting on X. We first fix some notation. Consider any subset C of X. The symbols C^o, $\overline{C}$ and ∂C denote the interior, the closure and the boundary of C respectively. The symbol $Orb(T, C)$ denotes the orbit of C under T, i.e. $Orb(T, C) = \{T^n x : x \in C,\ n = 0, 1, 2, \ldots\}$. If $C = \{x\}$ is a singleton and the orbit $Orb(T, x)$ is dense in X, the operator T is called hypercyclic and the vector x is a hypercyclic vector for T. If $C = \{\lambda x : \lambda \in \mathbb{C}\} = \mathbb{C}x$ and the set $Orb(T, C)$ is dense in X, the operator T is called supercyclic and the vector x is a supercyclic vector for T. A

2000 *Mathematics Subject Classification.* Primary 47A16; Secondary 37B99, 54H20.

Key words and phrases. Hypercyclic operators, J-class operators, J^{mix}-class operators, unilateral and bilateral weighted shifts, cyclic vectors.

During this research the second author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

nice source of examples and properties of hypercyclic and supercyclic operators is the survey article [18], see also some recent survey articles [30], [19], [24], [8], [15], [20] and the recent book [2]. Observe that in case the operator T is hypercyclic the underlying Banach space X should be separable. Then it is well known and easy to show that *an operator $T : X \to X$ is hypercyclic if and only if for every pair of non-empty open sets U, V of X there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$.* The purpose of this paper is twofold. Firstly we somehow "localize" the notion of hypercyclicity by introducing certain sets, which we call J-sets. The notion of J-sets is well known in the theory of topological dynamics, see [6]. Roughly speaking, if x is a vector in X and T an operator, then the corresponding J-set of x under T describes the asymptotic behavior of all vectors nearby x. To be precise for a given vector $x \in X$ we define

$$\begin{aligned} J(x) = \{y \in X : \ & \text{there exist a strictly increasing sequence of positive} \\ & \text{integers}\, \{k_n\} \,\text{and a sequence}\ \ \{x_n\} \subset X \,\text{such that}\, x_n \to x \,\text{and} \\ & T^{k_n} x_n \to y\}. \end{aligned}$$

Secondly we try to develop a systematic study of operators whose J-set under some vector is the whole space. As it turns out this new class of operators although different from the class of hypercyclic operators, shares some similarities with the behavior of hypercyclic operators. In fact it is not difficult to see that if T is hypercyclic then $J(x) = X$ for every $x \in X$. On the other hand we provide examples of operators T such that $J(x) = X$ for some vector $x \in X$ but T fails to be hypercyclic and in general T need not be even multi-cyclic. This should be compared with the results of Feldman in [16] where he shows that a countably hypercyclic operator need not be multi-cyclic. We would like to stress that some non-separable Banach spaces, such as the space $l^\infty(\mathbb{N})$ of bounded sequences, support J-class operators, (see Proposition 5.2), while it is known that the space $l^\infty(\mathbb{N})$ does not support topologically transitive operators, see [3].

The paper is organized as follows. In section 2 we define the J-sets and we examine some basic properties of these sets. In section 3 we investigate the relation between hypercyclicity and J-sets. In particular we show that $T : X \to X$ is hypercyclic if and only if there exists a cyclic vector $x \in X$ such that $J(x) = X$. Recall that a vector x is cyclic for T if the linear span of the orbit $Orb(T, x)$ is dense in X. The main result of section 4 is a generalization of a theorem due to Bourdon and Feldman, see [11]. Namely, we show that if x is a cyclic vector for an operator $T : X \to X$ and the set $J(x)$ has non-empty interior then $J(x) = X$ and, in addition, T is hypercyclic. In section 5 we introduce the notion of J-class operator and we establish some of its properties. We also present examples of J-class operators which are not hypercyclic. On the other hand, we show that if T is a bilateral or a unilateral weighted shift on the space of square summable sequences then T is hypercyclic if and only if T is a J-class operator. Finally, in section 6 we give a list of open problems.

2. PRELIMINARIES AND BASIC NOTIONS

If one wants to work on general non-separable Banach spaces and in order to investigate the dynamical behavior of the iterates of T, the suitable substitute of hypercyclicity is the following well known notion of topological transitivity which is frequently used in dynamical systems.

Definition 2.1. An operator $T : X \to X$ is called *topologically transitive* if for every pair of open sets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \emptyset$.

Definition 2.2. Let $T : X \to X$ be an operator. For every $x \in X$ the sets

$$\begin{aligned} L(x) = \{y \in X :\ & \text{there exists a strictly increasing sequence} \\ & \text{of positive integers } \{k_n\} \text{ such that } T^{k_n}x \to y\} \end{aligned}$$

and

$$\begin{aligned} J(x) = \{y \in X :\ & \text{there exist a strictly increasing sequence of positive} \\ & \text{integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \to x \text{ and} \\ & T^{k_n}x_n \to y\} \end{aligned}$$

denote the *limit set* and the *extended (prolongational) limit set* of x under T respectively. In case T is invertible and for every $x \in X$ the sets $L^+(x)$, $J^+(x)$ ($L^-(x)$, $J^-(x)$) denote the limit set and the extended limit set of x under T (T^{-1}).

Remark 2.3. An equivalent definition of $J(x)$ is the following.

$$\begin{aligned} J(x) = \{y \in X :\ & \text{for every pair of neighborhoods } U, V \text{ of } x, y \\ & \text{respectively, there exists a positive integer } n, \\ & \text{such that } T^n U \cap V \neq \emptyset\}. \end{aligned}$$

Observe now that T is topologically transitive if and only if $J(x) = X$ for every $x \in X$.

Definition 2.4. Let $T : X \to X$ be an operator. A vector x is called *periodic* for T if there exists a positive integer n such that $T^n x = x$.

The proof of the following lemma can be found in [12].

Lemma 2.5. *Let $T : X \to X$ be an operator and $\{x_n\}$, $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$ for some $x, y \in X$. If $y_n \in J(x_n)$ for every $n = 1, 2, \ldots$, then $y \in J(x)$.*

Proposition 2.6. *For all $x \in X$ the sets $L(x)$, $J(x)$ are closed and T-invariant.*

Proof. It is an immediate consequence of the previous lemma. □

Remark 2.7. Note that the set $J(x)$ is not always invariant under the operation T^{-1} even in the case T is surjective. For example consider the operator $T = \frac{1}{2}B$ where B is the backward shift operator on $l^2(\mathbb{N})$, the space of square summable sequences. Since $\|T\| = \frac{1}{2}$ it follows that $L(x) = J(x) = \{0\}$ for every $x \in l^2(\mathbb{N})$. For any non-zero vector $y \in KerT$ we have $Ty = 0 \in J(x)$ and $y \in X \setminus J(x)$. However, if T is invertible it is easy to verify the following.

Proposition 2.8. *Let $T : X \to X$ be an invertible operator. Then $T^{-1}J(x) = J(x)$ for every $x \in X$.*

Proof. By Proposition 2.6 it follows that $J(x) \subset T^{-1}J(x)$. Take $y \in T^{-1}J(x)$. There are a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{x_n\} \subset X$ so that $x_n \to x$ and $T^{k_n}x_n \to Ty$, hence $T^{k_n-1}x_n \to y$. □

Proposition 2.9. *Let $T : X \to X$ be an invertible operator and $x, y \in X$. Then $y \in J^+(x)$ if and only if $x \in J^-(y)$.*

Proof. If $y \in J^+(x)$ there exist a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{x_n\} \subset X$ such that $x_n \to x$ and $T^{k_n}x_n \to y$. Then $T^{-k_n}(T^{k_n}x_n) = x_n \to x$, hence $x \in J^-(y)$. □

Proposition 2.10. *Let $T : X \to X$ be an operator. If T is power bounded then $J(x) = L(x)$ for every $x \in X$.*

Proof. Since T is power bounded there exists a positive number M such that $\|T^n\| \leq M$ for every positive integer n. Fix a vector $x \in X$. If $J(x) = \emptyset$ there is nothing to prove. Therefore assume that $J(x) \neq \emptyset$. Since the inclusion $L(x) \subset J(x)$ is always true, it suffices to show that $J(x) \subset L(x)$. Take $y \in J(x)$. There exist a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{x_n\} \subset X$ such that $x_n \to x$ and $T^{k_n}x_n \to y$. Then we have $\|T^{k_n}x - y\| \leq \|T^{k_n}x - T^{k_n}x_n\| + \|T^{k_n}x_n - y\| \leq M\|x - x_n\| + \|T^{k_n}x_n - y\|$ and letting n goes to infinity to the above inequality, we get that $y \in L(x)$. □

Lemma 2.11. *Let $T : X \to X$ be an operator. If $J(x) = X$ for some non-zero vector $x \in X$ then $J(\lambda x) = X$ for every $\lambda \in \mathbb{C}$.*

Proof. For $\lambda \in \mathbb{C} \setminus \{0\}$ it is easy to see that $J(\lambda x) = X$. It remains to show that $J(0) = X$. Fix a sequence of non-zero complex numbers $\{\lambda_n\}$ converging to 0 and take $y \in J(x)$. Then $y \in J(\lambda_n x)$ for every n and since $\lambda_n \to 0$, Lemma 2.5 implies that $y \in J(0)$. Hence $J(0) = X$. $\square$

Proposition 2.12. *Let $T : X \to X$ be an operator. Define the set $A = \{x \in X : J(x) = X\}$. Then A is a closed, connected and $T(A) \subset A$.*

Proof. The T-invariance follows immediately from the T-invariance of $J(x)$. By Lemma 2.5 we conclude that A is closed. Let $x \in A$. Lemma 2.11 implies that for every $\lambda \in \mathbb{C}$, $J(0) = J(\lambda x) = X$, hence A is connected. $\square$

3. A CHARACTERIZATION OF HYPERCYCLIC OPERATORS THROUGH J-SETS

The following characterization of hypercyclic operators appears more or less in [18]. However we sketch the proof for the purpose of completeness.

Theorem 3.1. *Let $T : X \to X$ be an operator acting on a separable Banach space X. The following are equivalent.*

(i) *T is hypercyclic;*
(ii) *For every $x \in X$ it holds that $J(x) = X$;*
(iii) *The set $A = \{x \in X : J(x) = X\}$ is dense in X;*
(iv) *The set $A = \{x \in X : J(x) = X\}$ has non-empty interior.*

Proof. We first prove that (i) implies (ii). Let $x, y \in X$. Since the set of hypercyclic vectors is G_δ and dense in X there exist a sequence $\{x_n\}$ of hypercyclic vectors and a strictly increasing sequence $\{k_n\}$ of positive integers such that $x_n \to x$ and $T^{k_n} x_n \to y$ as $n \to \infty$. Hence $y \in J(x)$. That (ii) implies (iii) is trivial. A consequence of Lemma 2.5 is that (iii) gives (ii). Next we show that (iv) implies (ii). Fix $x \in A^o$ and consider $y \in X$ arbitrary. Then $y \in J(x) = X$, hence there exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence $\{k_n\}$ of positive integers such that $x_n \to x$ and $T^{k_n} x_n \to y$. Since $x \in A^o$ without loss of generality we may assume that $x_n \in A$ for every n. Moreover A is T-invariant, hence $T^{k_n} x_n \in A$ for every n. Since $T^{k_n} x_n \to y$ and A is closed we conclude that $y \in A$. Let us now prove that (ii) implies (i). Fix $\{x_j\}$ a countable dense set of X. Define the sets $E(j, s, n) = \{x \in X : \|T^n x - x_j\| < \frac{1}{s}\}$ for every $j, s = 1, 2, \dots$ and every $n = 0, 1, 2, \dots$. In view of Baire's Category Theorem and the well known set theoretical description of hypercyclic vectors through the sets $E(j, s, n)$, it suffices to show that the set $\bigcup_{n=0}^{\infty} E(j, s, n)$ is dense in X for every j, s. Indeed, let $y \in X$, $\epsilon > 0$, j, s be given. Since $J(y) = X$, there exist $x \in X$ and $n \in \mathbb{N}$ such that $\|x - y\| < \epsilon$ and $\|T^n x - x_j\| < 1/s$. □

The following lemma -see also Corollary 3.4- which is of great importance in the present paper, gives information about the spectrum of the adjoint T^* of an operator $T : X \to X$ provided there is a vector $x \in X$ whose extended limit set $J(x)$ has non-empty interior. The corresponding result for hypercyclic operators has been proven by P. Bourdon in [9].

Lemma 3.2. *Let $T : X \to X$ be an operator acting on a complex or real Banach space X. Suppose there exists a vector $x \in X$ such that $J(x)$ has non-empty interior and x is cyclic for T. Then for every non-zero polynomial P the operator $P(T)$ has dense range. In particular the point spectrum $\sigma_p(T^*)$ of T^* (the adjoint operator of T) is empty, i.e. $\sigma_p(T^*) = \emptyset$.*

Proof. Assume first that X is a complex Banach space. Since $P(T)$ can be decomposed in the form $P(T) = \alpha(T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_k I)$ for some $\alpha, \lambda_i \in \mathbb{C}$, $i = 1, \dots, k$, where I stands for the identity operator, it suffices to show that $T - \lambda I$ has dense range for any $\lambda \in \mathbb{C}$. If not, there exists a non-zero linear functional x^* such that $x^*((T - \lambda I)(x)) = 0$ for every $x \in X$. The last implies that $x^*(T^n x) = \lambda^n x^*(x)$ for every $x \in X$ and every n non-negative integer. Take y in the interior of $J(x)$. Then there exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence $\{k_n\}$ of positive integers such that $x_n \to x$ and $T^{k_n} x_n \to y$ as $n \to +\infty$. Suppose first that $|\lambda| < 1$. Observe that $x^*(T^{k_n} x_n) = \lambda^{k_n} x^*(x_n)$ and letting $n \to +\infty$ we arrive at $x^*(y) = 0$. Since the functional x^* is zero on an open subset of X must be identically zero on X, which is a contradiction. Working for $|\lambda| = 1$ as before, it is easy to show that for every y in the interior of $J(x)$, $x^*(y) = \mu x^*(x)$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$, which is again a contradiction since x^* is surjective. Finally we deal with the case $|\lambda| > 1$. *At this part of the proof we shall use the hypothesis that x is cyclic.* Letting n tend to infinity in the relation $x^*(x_n) = \frac{1}{\lambda^{k_n}} x^*(T^{k_n} x_n)$, it is plain that $x^*(x) = 0$ and therefore $x^*(T^n x) = 0$ for every n non-negative integer. The last implies that $x^*(P(T)x) = 0$ for every P non-zero polynomial and since x is cyclic the linear functional x^* vanishes everywhere, which gives a contradiction. It remains to handle the real case. For that it suffices to consider the case where P is an irreducible and monic polynomial of the form $P(t) = t^2 - 2Re(w)t + |w|^2$ for some non-real complex number w. Assume that $P(T)$ does not have dense range. Then there exists a non-zero $x^* \in Ker(P(T)^*)$. Following the proof of the main result in [5], there exists a real 2×2 matrix A such that $J_{A^t}((x^*(Tx), x^*(x))^t) = \mathbb{R}^2$, where the symbol A^t stands for the transpose of A. By Proposition 5.5 (which holds in the real case as well) we get $x^*(Tx) = x^*(x) = 0$. The last implies that $x^*(Q(T)x) = 0$ for every real polynomial Q. Since x is cyclic we conclude that $x^* = 0$ which is a contradiction. This completes the proof of the lemma. □

Theorem 3.3. *Let $T : X \to X$ be an operator acting on a separable Banach space X. Then T is hypercyclic if and only if there exists a cyclic vector $x \in X$ for T such that $J(x) = X$.*

Proof. We need only to prove that if $x \in X$ is a cyclic vector for T and $J(x) = X$ then T is hypercyclic. Take any non-zero polynomial P. It is easy to check that $P(T)(J(x)) \subset J(P(T)x)$. By the previous lemma it follows that $P(T)$ has dense range and since $J(x) = X$ we conclude that $X = \overline{P(T)(X)} \subset J(P(T)x)$. Therefore $J(P(T)x) = X$ for every non-zero polynomial P. The fact that x is a cyclic vector it now implies that there exists a dense set D in X so that $J(y) = X$ for every $y \in D$. Hence, in view of Theorem 3.1, T is hypercyclic. □

Corollary 3.4. *Let $T : X \to X$ be an operator. Suppose there exists a vector $x \in X$ such that $J(x)$ has non-empty interior. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ the operator $T - \lambda I$ has dense range.*

Proof. See the proof of Lemma 3.2. □

Remark 3.5. At this point we would like to comment on Theorem 3.3. First of all under the hypothesis that x is a cyclic vector for T and $J(x) = X$ one cannot get a stronger conclusion than T is hypercyclic. In particular it is not true in general that x is a hypercyclic vector. To see this, take $T = 2B$ where B is the backward shift operator acting on the space of square summable sequences $l^2(\mathbb{N})$ over $\mathbb{C}$. In [14] Feldman showed that for a given positive number ϵ there exists a vector $x \in l^2(\mathbb{N})$ such that the set $Orb(2B, x)$ is ϵ-dense in $l^2(\mathbb{N})$ (this means that for every $y \in l^2(\mathbb{N})$ there exists a positive integer n such that $T^n x$ is ϵ-close to y), but x is not hypercyclic for $2B$. It is straightforward to check that x is supercyclic for $2B$ and hence it is cyclic. In addition $J(x) = l^2(\mathbb{N})$ since $2B$ is hypercyclic (see Theorem 3.1).

Remark 3.6. Let us now show that the hypothesis x is cyclic in Theorem 3.3 cannot be omitted. Let $B : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the backward shift operator. Consider the operator $T = 2I \oplus 2B : \mathbb{C} \oplus l^2(\mathbb{N}) \to \mathbb{C} \oplus l^2(\mathbb{N})$, where I is the identity operator acting on $\mathbb{C}$. It is obvious that $2I \oplus 2B$ is not a hypercyclic operator. However we shall show that for every hypercyclic vector $y \in l^2(\mathbb{N})$ for $2B$ it holds that $J(0 \oplus y) = \mathbb{C} \oplus l^2(\mathbb{N})$. Therefore there exist (non-cyclic) non-zero vectors $x \in \mathbb{C} \oplus l^2(\mathbb{N})$ with $J(x) = \mathbb{C} \oplus l^2(\mathbb{N})$ and T is not hypercyclic. Indeed, fix a hypercyclic vector $y \in l^2(\mathbb{N})$ for $2B$ and let $\lambda \in \mathbb{C}$, $w \in l^2(\mathbb{N})$. There exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $T^{k_n} y \to w$. Define $x_n = \frac{\lambda}{2^{k_n}} \oplus y$. Then $x_n \to 0 \oplus y$ and $T^{k_n} x_n \to \lambda \oplus w$. Hence, $J(0 \oplus y) = \mathbb{C} \oplus l^2(\mathbb{N})$.

4. AN EXTENSION OF BOURDON-FELDMAN'S THEOREM

In this section we establish an extension of the following striking result due to Bourdon and Feldman [11]: *if X is a separable Banach space, $T : X \to X$ an operator and for some vector $x \in X$ the orbit $Orb(T, x)$ is somewhere dense then $\overline{Orb(T, x)} = X$.* This theorem was an answer to a question raised by Peris in [26]. We shall prove the following theorem.

Theorem 4.1. *Let x be a cyclic vector for T. If $J(x)^o \neq \emptyset$ then $J(x) = X$.*

In order to prove Theorem 4.1 we follow the steps of the proof of Bourdon-Feldman's theorem. Of course there are some extra technicalities which have to be taken care since the orbit $Orb(T, x)$ of x under T is replaced by the extended limit set $J(x)$ of x.

Lemma 4.2. *If for some non-zero polynomial P the operator $P(T)$ has dense range and x is a cyclic vector for T then $P(T)x$ is cyclic for T.*

Proof. Take $P(T)y$ for some $y \in X$. Since x is cyclic there is a sequence of polynomials $\{Q_n\}$ such that $Q_n(T)x \to y$. Therefore, $Q_n(T)(P(T)x) \to P(T)y$. $\square$

Lemma 4.3. *Assume that x is a cyclic vector for T and $J(x)$ has non-empty interior. Then the set $X \setminus J(x)^o$ is T-invariant.*

Proof. We argue by contradiction. Let $y \in X \setminus J(x)^o$ be such that $Ty \in J(x)^o$. By the continuity of T we may assume that $y \notin J(x)$. Moreover, since x is cyclic we may find a non-zero polynomial $P(T)$ such that $P(T)x \in X \setminus J(x)^o$ and $TP(T)x \in J(x)^o$. Hence, there exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $x_n \to x$ and $T^{k_n}x_n \to TP(T)x$. Taking any polynomial Q we get $Q(T)x_n \to Q(T)x$ and $T^{k_n}Q(T)x_n = Q(T)(T^{k_n}x_n) \to Q(T)TP(T)x$. So it follows that $P(T)TQ(T)x \in J(Q(T)x)$ for every polynomial Q. But $J(Q(T)x) \subset J(TQ(T)x)$, hence we get $P(T)TQ(T)x \in J(TQ(T)x)$ for every polynomial Q. By Lemmata 3.2 and 4.2, Tx is a cyclic vector for T, hence there exists a sequence of the form $\{Q_n(T)x\}$, for some non-zero polynomials Q_n, such that $TQ_n(T)x \to x$. Therefore it follows that $P(T)TQ_n(T)x \to P(T)x$. Observe that $P(T)TQ_n(T)x \in J(TQ_n(T)x)$ and using Lemma 2.5 it follows that $P(T)x \in J(x)$ which is a contradiction. □

Lemma 4.4. *Assume that x is a cyclic vector for T and $J(x)$ has non-empty interior. Suppose that $Q(T)x \in X \setminus J(x)$ for some non-zero polynomial Q. Then $Q(T)(J(x)) \subset X \setminus J(x)^o$.*

Proof. Let $y \in J(x)$. There exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $x_n \to x$ and $T^{k_n}x_n \to y$. Since $X \setminus J(x)$ is an open set we may assume that $Q(T)x_n \in X \setminus J(x)$ for every n and thus $Q(T)x_n \in X \setminus J(x)^o$. By Lemma 4.3 the set $X \setminus J(x)^o$ is T-invariant, therefore $T^{k_n}Q(T)x = Q(T)T^{k_n}x_n \in X \setminus J(x)^o$. Now it is plain that $Q(T)y \in X \setminus J(x)^o$. □

Lemma 4.5. *Assume that x is a cyclic vector for T, $J(x)$ has non-empty interior and let P be any non zero polynomial. Then $P(T)x \notin \partial(J(x)^o)$.*

Proof. In view of Lemma 4.4 let us define the set

$$\mathcal{A} = \{Q : Q \,\text{is a polynomial and}\, Q(T)x \in X \setminus J(x)\}.$$

Note that the set $\{Qx : Q \in \mathcal{A}\}$ is dense in $X \setminus J(x)^o$. We argue by contradiction. Suppose there exists a non-zero polynomial P so that $P(T)x \in \partial(J(x)^o)$. The inclusion $\partial(J(x)^o) \subset \partial J(x)$ gives that $P(T)x \in \partial(X \setminus J(x))$. We will prove that $P(T)(J(x)^o) \subset X \setminus J(x)^o$. Since x is a cyclic vector and $J(x)^o$ is open, it is enough to show that: if $S(T)x \in J(x)^o$ for some non-zero polynomial S then $P(T)S(T)x \in X \setminus J(x)^o$. We have $P(T)x \in \partial(X \setminus J(x))$. Therefore there exists a sequence $\{Q_n(T)x\}$ such that $Q_n \in \mathcal{A}$ and $Q_n(T)x \to P(T)x$. Hence Lemma 4.4 yields that $Q_n(T)S(T)x \in X \setminus J(x)^o$. So, we get $Q_n(T)S(T)x \to P(T)S(T)x$ and $P(T)S(T)x \in X \setminus J(x)^o$. Consider the set $D := J(x)^o \bigcup \{Q(T)x : Q \in \mathcal{A}\}$ which is dense in X. By Lemma 3.2, $P(T)D$ is dense in X. Since $P(T)x \in J(x)$, Lemma 4.4 implies that $Q(T)P(T)x \in X \setminus J(x)^o$ for every $Q \in \mathcal{A}$. Hence

$$P(T)D = P(T)(J(x)^o) \bigcup \{P(T)Q(T)x : Q \in \mathcal{A}\} \subset X \setminus J(x)^o,$$

which is a contradiction. □

Proof of Theorem 4.1 The set $\{P(T)x : P \text{ is a non-zero polynomial}\}$ is dense and connected. Assume that $J(x) \neq X$. So we can find a non-zero polynomial P such that $P(T)x \in \partial(J(x)^o)$. This contradicts Lemma 4.5. □

Corollary 4.6. *Let $T : X \to X$ be an operator. If there exists a cyclic vector $x \in X$ for T such that $J(x)$ has non-empty interior then T is hypercyclic.*

Proof. The proof follows by combining Theorems 3.3 and 4.1. □

Corollary 4.7 (Bourdon-Feldman's theorem)**.** *Let $T : X \to X$ be an operator. If for some vector $x \in X$ the orbit $Orb(T, x)$ is somewhere dense then it is everywhere dense.*

Proof. It is easy to see that x is a cyclic vector for T. Since $Orb(T, x)$ is somewhere dense, it follows that $L(x)^o \neq \emptyset$. Note that $\overline{L(x)} \subset J(x)$. Hence Theorem 4.1 implies that $J(x) = X$. The set $\overline{Orb(T, x)}$ has non-empty interior so we can find a positive integer l such that $T^l x \in \overline{Orb(T, x)}^o$. Since $J(x) = X$ and $J(x) \subset J(T^l x)$ we arrive at $J(T^l x) = X$. So it is enough to prove that $\overline{Orb(T, x)} = J(T^l x)$. Let $y \in J(T^l x)$. There exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $x_n \to T^l x$ and $T^{k_n} x_n \to y$. Observing that $T^l x \in \overline{Orb(T, x)}^o$, without loss of generality we may assume that $x_n \in \overline{Orb(T, x)}^o$ for every n. Moreover $\overline{Orb(T, x)}$ is T-invariant, hence $T^{k_n} x_n \in \overline{Orb(T, x)}$ for every n. Since $T^{k_n} x_n \to y$ we conclude that $y \in \overline{Orb(T, x)}$. □

Corollary 4.8. *Let $T : X \to X$ be an operator. Suppose there exist a vector $x \in X$ and a polynomial P such that $P(T)x$ is a cyclic vector for T. If the set $J(x)$ has non-empty interior then T is hypercyclic.*

Proof. Since $P(T)x$ is a cyclic vector for T it is obvious that x is a cyclic vector for T. Using the hypothesis that the set $J(x)$ has non-empty interior, Corollary 4.6 implies the desired result. □

Remark 4.9. The conclusion of Corollary 4.6 does not hold in general if x is a cyclic vector for T and $J(P(T)x) = X$ for some polynomial P. To see that, consider the space $X = \mathbb{C} \oplus l^2(\mathbb{N})$ and let $B : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the backward shift operator. Define the operator $T = 2I \oplus 3B : X \to X$, where I denotes the identity operator acting on $\mathbb{C}$. Take any hypercyclic vector y for $3B$ and define $x = 1 \oplus y$. Then x is cyclic for T (in fact x is supercyclic for T) and obviously T is not hypercyclic.

In fact it holds that $J(x) = \emptyset$. Consider the polynomial $P(z) = z - 2$. Then $P(T)x = 0 \oplus P(3B)y$. Since y is hypercyclic for $3B$, by a classical result due to Bourdon [9], the vector $P(3B)x$ is hypercyclic for $3B$ as well. Then using a similar argument as in Remark 3.6 we conclude that $J(P(T)x) = J(0 \oplus P(3B)y) = X$. In particular, the above shows that, *if T is cyclic and $J(x) = X$ for some vector $x \in X$ then T is not hypercyclic in general.* On the other hand, we have the following.

Corollary 4.10. *Let $T : X \to X$ be an operator. Suppose P is a non-zero polynomial such that $P(T)$ has dense range. If x is a cyclic vector for T, $P(T)x \neq 0$ and $J(P(T)x)^o \neq \emptyset$ then T is hypercyclic.*

Proof. Lemma 4.2 implies that $P(T)x$ is a cyclic vector for T. Since $J(P(T)x)^o \neq \emptyset$, Corollary 4.6 implies that T is hypercyclic. □

5. J-CLASS OPERATORS

Definition 5.1. An operator $T : X \to X$ will be called a *J-class operator* provided there exists a non-zero vector $x \in X$ so that the extended limit set of x under T (see Definition 2.2) is the whole space, i.e. $J(x) = X$. In this case x will be called a *J-class vector* for T.

The reason we exclude the extended limit set of the zero vector is to avoid certain trivialities, as for example the multiples of the identity operator acting on finite or infinite dimensional spaces. To explain briefly, for any positive integer n consider the operator $\lambda I : \mathbb{C}^n \to \mathbb{C}^n$, where λ is a complex number of modulus greater than 1 and I is the identity operator. It is then easy to check that $J_{\lambda I}(0) = X$ and $J_{\lambda I}(x) \neq \mathbb{C}^n$ for every $x \in \mathbb{C}^n \setminus \{0\}$. However, the extended limit set of the zero vector plays an important role in checking whether an operator $T : X \to X$ -acting on a Banach space X- supports non-zero vectors x with $J_T(x) = X$, see Proposition 5.9. Let us also point out that from the examples we presented in section 3, see Remark 3.6, it clearly follows that this new class of operators does not coincide with the class of hypercyclic operators.

Let us turn our attention to non-separable Banach spaces. Obviously a non-separable Banach space cannot support hypercyclic operators. However, it is known that topologically transitive operators may exist in non-separable Banach spaces, see for instance [7]. On the other hand in [3], Bermúdez and Kalton showed that the non-separable Banach space $l^\infty(\mathbb{N})$ of bounded sequences over $\mathbb{C}$ does not support topologically transitive operators. Below we prove that the Banach space $l^\infty(\mathbb{N})$ supports J-class operators.

Proposition 5.2. *Let $B : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ be the backward shift where $l^\infty(\mathbb{N})$ is the Banach space of bounded sequences over $\mathbb{C}$, endowed with the usual supremum norm. Then for every $|\lambda| > 1$, λB is a J-class operator. In fact we have the following complete characterization of the set of J-class vectors. For every $|\lambda| > 1$ it holds that*

$$\{x \in l^\infty(\mathbb{N}) : J_{\lambda B}(x) = l^\infty(\mathbb{N})\} = c_0(\mathbb{N}),$$

where $c_0(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}) : \lim_{n\to+\infty} x_n = 0\}$.

Proof. Fix $|\lambda| > 1$. Let us first show that if x is a vector in $l^\infty(\mathbb{N})$ with finite support then $J_{\lambda B}(x) = l^\infty(\mathbb{N})$. For simplicity let us assume that $x = e_1 = (1, 0, 0, \ldots)$. Take any $y = (y_1, y_2, \ldots) \in l^\infty(\mathbb{N})$. Define $x_n = (1, 0, \ldots, 0, \frac{y_1}{\lambda^n}, \frac{y_2}{\lambda^n}, \ldots)$ where 0's are taken up to the n-th coordinate. Obviously $x_n \in l^\infty(\mathbb{N})$ and it is straightforward to check that $x_n \to e_1$ and $(\lambda B)^n x_n = y$ for all n. Hence, $J_{\lambda B}(e_1) = l^\infty(\mathbb{N})$. Since the closure of the set consisting of all the vectors with finite support is $c_0(\mathbb{N})$, an application of Lemma 2.5 gives that $c_0(\mathbb{N})$ is contained in $\{x \in l^\infty(\mathbb{N}) : J_{\lambda B}(x) = l^\infty(\mathbb{N})\}$. It remains to show the converse implication. Suppose that $J_{\lambda B}(x) = l^\infty(\mathbb{N})$ for some non-zero vector $x = (x_1, x_2, \ldots) \in l^\infty(\mathbb{N})$. Then there exist a sequence $y_n = (y_{n1}, y_{n2}, \ldots)$, $n = 1, 2, \ldots$ in $l^\infty(\mathbb{N})$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $y_n \to x$ and $(\lambda B)^{k_n} y_n \to 0$. Consider $\epsilon > 0$. There exists a positive integer n_0 such that $\|y_n - x\| < \epsilon$ and $\|(\lambda B)^{k_n} y_n\| = |\lambda|^{k_n} \sup_{m\geq k_n+1} |y_{nm}| < \epsilon$ for every $n \geq n_0$. Hence for every $m \geq k_{n_0} + 1$ and since $|\lambda| > 1$ it holds that $|x_m| \leq \|y_{n_0} - x\| + |y_{n_0 m}| < 2\epsilon$. The last implies that $x \in c_0(\mathbb{N})$ and this completes the proof. □

Remark 5.3. The previous proof actually yields that for every $|\lambda| > 1$, $J_{\lambda B}(x) = l^\infty(\mathbb{N})$ if and only if $0 \in J_{\lambda B}(x)$.

Next we show that certain operators, such as positive, compact, hyponormal and operators acting on finite dimensional spaces cannot be J-class operators. It is well known that the above mentioned classes of operators are disjoint from the class of hypercyclic operators, see [23], [10].

Proposition 5.4. (i) *Let X be an infinite dimensional separable Banach space and $T : X \to X$ be an operator. If T is compact then it is not a J-class operator.*

(ii) *Let H be an infinite dimensional separable Hilbert space and $T : H \to H$ be an operator. If T is positive or hyponormal then it is not a J-class operator.*

Proof. Let us prove assertion (i). Suppose first that T is compact. If T is a J-class operator, there exists a non-zero vector $x \in X$ so that $J(x) = X$. It is clear that there exists a bounded set $C \subset X$ such that the set $Orb(T, C)$ is dense in X. Then according to Proposition 4.4 in [16] no component of the spectrum, $\sigma(T)$, of T can be contained in the open unit disk. However, for compact operators the singleton $\{0\}$ is always a component of the spectrum and this gives a contradiction.

We proceed with the proof of the second statement. Suppose now that T is hyponormal. If T is a J-class operator, there exists a non-zero vector $h \in H$ so that $J(h) = H$. Therefore there exists a bounded set $C \subset H$ which is bounded away from zero (since $h \neq 0$) such that the set $Orb(T, C)$ is dense in X. The last contradicts Theorem 5.10 in [16]. The case of a positive operator is an easy exercise and is left to the reader. □

Below we prove that any operator acting on a finite dimensional space cannot be J-class operator.

Proposition 5.5. *Fix any positive integer l and let $A : \mathbb{C}^l \to \mathbb{C}^l$ be a linear map. Then A is not a J-class operator. In fact $J(x)^o = \emptyset$ for every $x \in \mathbb{C}^l \setminus \{0\}$.*

Proof. By the Jordan's canonical form theorem for A we may assume that A is a Jordan block with eigenvalue $\lambda \in \mathbb{C}$. Assume on the contrary that there exists a non-zero vector $x \in \mathbb{C}^l$ with coordinates $z_1, \dots, z_l$ such that $J(x)^o = \emptyset$. If $\{x_n\} \in \mathbb{C}^l$ is such that $x_n \to x$ and $z_{n1}, \dots, z_{nl}$ be the corresponding coordinates to x_n then the m-th coordinate of $A^n x_n$ equals to

$$\sum_{k=0}^{l-m} \left(\begin{array}{c} n \\ k \end{array} \right) \lambda^{n-k} z_{n(m+k)}.$$

If $|\lambda| < 1$ then $J(x) = \{0\}$. It remains to consider the case $|\lambda| \geq 1$. Suppose $z_l \neq 0$. Then, for every strictly increasing sequence of positive integers $\{k_n\}$ the possible limit points of the sequence $\{\lambda^{k_n} z_{nl}\}$ are: either ∞ in case $|\lambda| > 1$ or a subset of the circumference $\{z \in \mathbb{C} : |z| = |z_l|\}$ in case $|\lambda| = 1$. This leads to a contradiction since $J(x)^o \neq \emptyset$. Therefore, the last coordinate z_l of the non-zero vector $x \in \mathbb{C}^l$ should be 0. In case $|\lambda| = 1$ and since $z_l = 0$ the only limit point of $\{\lambda^{k_n} z_{nl}\}$ is 0 for every strictly increasing sequence of positive integers $\{k_n\}$. So $J(x)^o \subset \mathbb{C}^{l-1} \times \{0\}$, a contradiction. Assume now that $|\lambda| > 1$. For the convenience of the reader we give the proof in the case $l = 3$. Take $y = (y_1, y_2, y_3) \in J(x)$. There exist a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{x_n\} \subset \mathbb{C}^3$

such that $x_n = (x_{n1}, x_{n2}, x_{n3}) \to (z_1, z_2, 0) = x$ and $A^{k_n} x_n \to y$. Let $y_n = (y_{n1}, y_{n2}, y_{n3}) = A^{k_n} x_n$. Hence we have

$$\begin{aligned} y_{n3} &= \lambda^{k_n} x_{n1} + k_n \lambda^{k_n - 1} x_{n2} + \tfrac{k_n(k_n-1)}{2} \lambda^{k_n-2} x_{n3} \\ y_{n2} &= \lambda^{k_n} x_{n2} + k_n \lambda^{k_n-1} x_{n3} \\ y_{n1} &= \lambda^{k_n} x_{n3}. \end{aligned}$$

Since $y_{n3} = \lambda^{k_n} x_{n3} \to y_3$ then $k_n(k_n - 1)x_{n3} \to 0$. From $y_{n2} \to y_2$ we get $\frac{y_{n2}}{k_n} = \frac{\lambda^{k_n}}{k_n^2} k_n x_{n2} + \lambda^{k_n-1} x_{n3} \to 0$. Using the fact that $\lambda^{k_n} x_{n3} \to y_3$ it follows that the sequence $\{\frac{\lambda^{k_n}}{k_n^2} k_n x_{n2}\}$ converges to a finite complex number, hence $k_n x_{n2} \to 0$. The last implies $x_{n2} \to 0$, therefore $z_2 = 0$. We have $x_{n1} = \frac{y_{n3}}{\lambda^{k_n}} - \frac{1}{\lambda} k_n x_{n2} - \frac{1}{2}\lambda^2 k_n(k_n - 1)x_{n3}$. Observing that each one term on the right hand side in the previous equality goes to 0, since $y_{n3} \to y_3$, we arrive at $z_1 = 0$. Therefore $x = 0$ which is a contradiction. □

Remark 5.6. The previous result does not hold in general if we remove the hypothesis that A is linear even if the dimension of the space is 1. It is well known that the function $f : (0, 1) \to (0, 1)$ with $f(x) = 4x(1-x)$ is chaotic, see [13]. Consider any homeomorphism $g : (0, 1) \to \mathbb{R}$. Take $h = gfg^{-1} : \mathbb{R} \to \mathbb{R}$. Then it is obvious that there is a G_δ and dense set of points with dense orbits in $\mathbb{R}$. Applying Theorem 3.1 (observe that this corollary holds without the assumption of linearity for T) we get that $J(x) = \mathbb{R}$, for every $x \in \mathbb{R}$.

It is well known, see [22], that if T is a hypercyclic and invertible operator, its inverse T^{-1} is hypercyclic. On the other hand, as we show below, the previously mentioned result fails for J-class operators.

Proposition 5.7. *There exists an invertible J-class operator T acting on a Banach space X so that its inverse T^{-1} is not a J-class operator.*

Proof. Take any hypercyclic invertible operator S acting on a Banach space Y and consider the operator $T = \lambda I_{\mathbb{C}} \oplus S : \mathbb{C} \oplus Y \to \mathbb{C} \oplus Y$, for any fixed complex number λ with $|\lambda| > 1$. Then, arguing as in Remark 3.6 it is easy to show that T is a J-class operator. However its inverse $T^{-1} = \lambda^{-1} I_{\mathbb{C}} \oplus S^{-1}$ is not a J-class operator since $|\lambda^{-1}| < 1$. □

Salas in [28] answering a question of D. Herrero constructed a hypercyclic operator T on a Hilbert space such that its adjoint T^* is also hypercyclic but $T \oplus T^*$ is not hypercyclic. In fact the following (unpublished) result of Deddens holds: *suppose T is an operator, acting on a complex Hilbert space, whose matrix with respect to some orthonormal basis, consists entirely of real entries. Then $T \oplus T^*$ is not cyclic.* A proof of Deddens result can be found in the expository paper [30]. Recently, Montes and Shkarin, see [25], extended Deddens' result to the general setting of Banach space operators. Hence it is natural to ask if there exists an operator T such that $T \oplus T^*$ is a J-class operator. Below we show that this is not the case.

Proposition 5.8. *Let T be an operator acting on a Hilbert space H. Then $T \oplus T^*$ is not a J-class operator.*

Proof. We argue by contradiction, so assume that $T \oplus T^*$ is a J-class operator. Hence there exist vectors $x, y \in H$ such that $J(x \oplus y) = H \oplus H$ and $x \oplus y \neq 0$.

Case I: suppose that one of the vectors x, y is zero. Without loss of generality assume $x = 0$. Then there exist sequences $\{x_n\}, \{y_n\} \subset H$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $x_n \to x = 0$, $y_n \to y$, $T^{k_n} x_n \to y$ and $T^{*k_n} y_n \to x = 0$. Taking limits to the following equality $< T^{k_n} x_n, y_n >=< x_n, T^{*k_n} y_n >$ we get that $\|y\| = \|x\| = 0$ and hence $y = 0$. Therefore $x \oplus y = 0$, which yields a contradiction.

Case II: suppose that $x \neq 0$ and $y \neq 0$. Let us show first that $J(\lambda x \oplus \mu y) = H \oplus H$ for every $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Indeed, fix $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Take any $z, w \in H$. Since $J(x \oplus y) = H \oplus H$, there exist sequences $\{x_n\}, \{y_n\} \subset H$ and a strictly increasing sequence of positive integers $\{k_n\}$ such that $x_n \to x$, $y_n \to y$, $T^{k_n} x_n \to \lambda^{-1} z$ and $T^{*k_n} y_n \to \mu^{-1} w$. The last implies that $z \oplus w \in J(\lambda x \oplus \mu y)$, hence $J(\lambda x \oplus \mu y) = H \oplus H$. With no loss of generality we may assume that $\|x\| \neq \|y\|$ (because if $\|x\| = \|y\|$, by multiplying with a suitable $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\|\lambda x\| \neq \|y\|$ and $J(\lambda x \oplus y) = H \oplus H$). Then we proceed as in Case I and arrive at a contradiction. The details are left to the reader. □

Below we establish that, for a quite large class of operators, an operator T is a J-class operator if and only if $J(0) = X$. What we need to assume is that there exists at least one non-zero vector having "regular" orbit under T.

Proposition 5.9. *Let $T : X \to X$ be an operator on a Banach space X.*

(i) *For every positive integer m it holds that $J_T(0) = J_{T^m}(0)$.*
(ii) *Suppose that z is a non-zero periodic point for T. Then the following are equivalent.*
   (1) *T is a J-class operator;*
   (2) *$J(0) = X$;*
   (3) *$J(z) = X$.*

(iii) *Suppose there exist a non-zero vector $z \in X$, a vector $w \in X$ and a sequence $\{z_n\} \subset X$ such that $z_n \to z$ and $T^n z_n \to w$. Then the following are equivalent.*
(1) *T is a J-class operator;*
(2) *$J(0) = X$;*
(3) *$J(z) = X$.*

In particular, this statement holds for operators with non trivial kernel or for operators having at least one non-zero fixed point.

Proof. Let us first show item (i). Fix any positive integer m and let $y \in J_T(0)$. There exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\}$ in X such that $x_n \to 0$ and $T^{k_n} x_n \to y$. Then for every n there exist non-negative integers l_n, ρ_n with $\rho_n \in \{0, 1, \ldots, m-1\}$ such that $k_n = l_n m + \rho_n$. Hence without loss of generality we may assume that there is $\rho \in \{0, 1, \ldots, m-1\}$ such that $k_n = l_n m + \rho$ for every n. The last implies that $T^{ml_n}(T^\rho x_n) \to y$ and $T^\rho x_n \to 0$ as $n \to \infty$. Hence $J_T(0) \subset J_{T^m}(0)$. The converse inclusion is obvious. Let us show assertion (ii). That (1) implies (2) is an immediate consequence of Lemma 2.11. We shall prove that (2) gives (3). Suppose that N is the period of the periodic point z. Fix $w \in X$. Assertion (i) yields that $J_{T^N}(0) = X$. Hence there exist a strictly increasing sequence of positive integers $\{m_n\}$ and a sequence $\{y_n\}$ in X such that $y_n \to 0$ and $T^{Nm_n} y_n \to w - z$. It follows that $y_n + z \to z$ and $T^{Nm_n}(y_n + z) \to w$, from which we conclude that $J_T(z) = X$. This proves assertion (ii). We proceed with the proof of assertion (iii). It only remains to show that (2) implies (3). Take any $y \in X$. There exist a sequence $\{x_n\} \subset X$ and a strictly increasing sequence $\{k_n\}$ of positive integers such that $x_n \to 0$ and $T^{k_n} x_n \to y - w$. Our hypothesis implies that $x_n + z_{k_n} \to z$ and $T^{k_n}(x_n + z_{k_n}) \to y$. Hence $y \in J(z)$. $\square$

In the following proposition we provide a construction of J-class operators which are not hypercyclic.

Proposition 5.10. *Let X be a Banach space and let Y be a separable Banach space. Consider an operator $S : X \to X$ so that $\sigma(S) \subset \{\lambda : |\lambda| > 1\}$. Let also $T : Y \to Y$ be a hypercyclic operator. Then*

(i) *$S \oplus T : X \oplus Y \to X \oplus Y$ is a J-class operator but not a hypercyclic operator and*
(ii) *the set $\{x \oplus y : x \in X, y \in Y$ such that $J(x \oplus y) = X \oplus Y\}$ forms an infinite dimensional closed subspace of $X \oplus Y$ and in particular*

$$\{x \oplus y : x \in X, y \in Y \text{ such that } J(x \oplus y) = X \oplus Y\} = \{0\} \oplus Y.$$

Proof. We first prove assertion (i). That $S \oplus T$ is not a hypercyclic operator is an immediate consequence of the fact that $\sigma(S) \subset \{\lambda : |\lambda| > 1\}$. Let us now prove that $S \oplus T$ is a J-class operator. Fix any hypercyclic vector $y \in Y$ for T. We shall show that $J(0 \oplus y) = X \oplus Y$. Take $x \in X$ and $w \in Y$. Since $\sigma(S) \subset \{\lambda : |\lambda| > 1\}$ it follows that S is invertible and $\sigma(S^{-1}) \subset \{\lambda : |\lambda| < 1\}$. Hence the spectral radius formula implies that $\|S^{-n}\| \to 0$. Therefore $S^{-n}x \to 0$. Since y is hypercyclic for T there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $T^{k_n}y \to w$. Observe now that $(S \oplus T)^{k_n}(S^{-k_n}x \oplus y) = x \oplus T^{k_n}y \to x \oplus w$ and $S^{-k_n}x \oplus y \to 0 \oplus y$. We proceed with the proof of (ii). Fix any hypercyclic vector $y \in Y$ for T. From the proof of (i) we get $J(0 \oplus y) = X \oplus Y$. Since for every positive integer n the vector $T^n y$ is hypercyclic for T, by the same reasoning as above we have that $J(0 \oplus T^n y) = X \oplus Y$. Using Lemma 2.5 and that y is hypercyclic for T we conclude that $J(0 \oplus w) = X \oplus Y$ for every $w \in Y$. To finish the proof, it suffices to show that if $x \in X \setminus \{0\}$ then for every $w \in Y$, $J(x \oplus w) \neq X$. In particular we will show that $J(x \oplus w) = \emptyset$. Suppose there exists $h \in J^+(x) = J(x)$ (see Definition 2.2). Propositions 2.9 and 2.10 imply that $x \in J^-(h) = L^-(h)$ (since S^{-1} is power bounded). On the other hand $\|S^{-n}\| \to 0$ and therefore $x \in L^-(h) = \{0\}$, which is a contradiction. $\square$

Let us point out that Proposition 5.10 shows that the cyclicity assumption is indeed necessary in Corollary 4.8 and Lemma 3.2. We next provide some information on the spectrum of a J-class operator. Recall that if T is hypercyclic then every component of the spectrum $\sigma(T)$ intersects the unit circle, see [23]. Although the spectrum of a J-class operator intersects the unit circle ∂D, see Proposition 5.12 below, it may admits components not intersecting ∂D. For instance consider the J-class operator $2B \oplus 3I$, where B is the backward shift on $l^2(\mathbb{N})$ and I is the identity operator on $\mathbb{C}$.

Proposition 5.11. *Let $T : X \to X$ be an operator on a complex Banach space X. If $r(T) < 1$, where $r(T)$ denotes the spectral radius of T, or $\sigma(T) \subset \{\lambda : |\lambda| > 1\}$ then T is not a J-class operator.*

Proof. If $r(T) < 1$ then we have $\|T^n\| \to 0$. Hence T is not a J-class operator. If $\sigma(T) \subset \{\lambda : |\lambda| > 1\}$ the conclusion follows by the proof of Proposition 5.10. $\square$

Proposition 5.12. *Let X be a complex Banach space. If $T : X \to X$ is a J-class operator, it holds that $\sigma(T) \cap \partial D \neq \emptyset$.*

Proof. Assume, on the contrary, that $\sigma(T) \cap \partial D = \emptyset$. Then we have $\sigma(T) = \sigma_1 \cup \sigma_2$ where $\sigma_1 = \{\lambda \in \sigma(T) : |\lambda| < 1\}$ and $\sigma_2 = \{\lambda \in$

$\sigma(T) : |\lambda| > 1\}$. If at least one of the sets σ_1, σ_2 is empty, we reach a contradiction because of Proposition 5.11. Assume now that both σ_1, σ_2 are non-empty. Applying Riesz decomposition theorem, see [27], there exist invariant subspaces X_1, X_2 of X under T such that $X = X_1 \oplus X_2$ and $\sigma(T_i) = \sigma_i$, $i = 1, 2$, where T_i denotes the restriction of T to X_i, $i = 1, 2$. It follows that $T = T_1 \oplus T_2$ and since T is J-class it is easy to show that at least one of T_1, T_2 is a J-class operator. By Proposition 5.11 we arrive again at a contradiction. □

Proposition 5.13. *Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be a unilateral backward weighted shift with positive weight sequence $\{\alpha_n\}$ and consider a vector $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. The following are equivalent.*

(i) *T is hypercyclic;*
(ii) *$J(x) = l^2(\mathbb{N})$;*
(iii) *$J(x)^o \neq \emptyset$.*

Proof. It only remains to prove that (iii) implies (i). Suppose $J(x)^o \neq \emptyset$. Then there exists a vector $y = (y_1, y_2, \ldots) \in J(x)$ such that $y_1 \neq 0$. Hence we may find a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{z_n\}$ in $l^2(\mathbb{N})$, $z_n = (z_{n1}, z_{n2}, \ldots)$, such that $z_n \to x$ and $T^{k_n} z_n \to y$. We have

$$|(T^{k_n} z_n)_1 - y_1| = \left| \left(\prod_{i=1}^{k_n} \alpha_i \right) z_{n(k_n+1)} - y_1 \right| \to 0.$$

Observe that $|z_{n(k_n+1)}| \leq |z_{n(k_n+1)} - x_{k_n+1}| + |x_{k_n+1}| \leq \|z_n - x\| + |x_{k_n+1}|$. The above inequality implies $z_{n(k_n+1)} \to 0$ and since $y_1 \neq 0$ we arrive at $\prod_{i=1}^{k_n} \alpha_i \to +\infty$. By Salas' characterization of hypercyclic unilateral weighted shifts, see [29], it follows that T is hypercyclic. □

Remark 5.14. We would also like to mention that (ii) implies (i) in the previous proposition, is an immediate consequence of Proposition 5.3 in [16]. Let us stress that in case T is a unilateral backward weighted shift on $l^2(\mathbb{N})$, the condition $J(0) = l^2(\mathbb{N})$ implies that T is hypercyclic. For a characterization of J-class unilateral weighted shifts on $l^\infty(\mathbb{N})$ in terms of their weight sequence see [12].

Proposition 5.15. *Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a bilateral backward weighted shift with positive weight sequence $\{\alpha_n\}$ and consider a non-zero vector $x = (x_n)_{n\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$. The following are equivalent.*

(i) *T is hypercyclic;*
(ii) *$J(x) = l^2(\mathbb{Z})$;*
(iii) *$J(x)^o \neq \emptyset$.*

Proof. It suffices to show that (iii) implies (i). In view of Salas' Theorem 2.1 in [29], we shall prove that there exists a strictly increasing sequence $\{k_n\}$ of positive integers such that for any integer q, $\prod_{i=1}^{k_n} \alpha_{i+q} \to +\infty$ and $\prod_{i=0}^{k_n-1} \alpha_{q-i} \to 0$. Since x is a non-zero vector, there exists an integer m such that $x_m \neq 0$. Without loss of generality we may assume that m is positive. Suppose $J(x)^o \neq \emptyset$. Then there exists a vector $y = (y_n)_{n\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$ such that $y_1 \neq 0$. Hence we may find a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{z_n\}$ in $l^2(\mathbb{Z})$, $z_n = (z_{nl})_{l\in\mathbb{Z}}$, such that $z_n \to x$ and $T^{k_n} z_n \to y$. For simplicity reasons we assume that $q = 0$. Arguing as in the proof of Proposition 5.13 we get that $\prod_{i=1}^{k_n} \alpha_i \to +\infty$. On the other hand observe that

$$|(T^{k_n} z_n)_{m-k_n} - y_{m-k_n}| = \left| \left(\prod_{i=0}^{m} \alpha_i \right) \left(\prod_{i=1}^{k_n-m+1} \alpha_{-i} \right) z_{nm} - y_{m-k_n} \right| \to 0.$$

Since $x_m \neq 0$ there exists a positive integer n_0 such that $|z_{nm}| \geq \frac{|x_m|}{2}$ for every $n \geq n_0$. We also have $(T^{k_n} z_n)_{m-k_n} \to 0$. The above imply that $\prod_{i=0}^{k_n-1} \alpha_{-i} \to 0$. $\square$

6. OPEN PROBLEMS

Below we give a list of open problems.

Problem 1.
Let $T : X \to X$ be an operator on an infinite dimensional Banach space X. Suppose there exists a vector $x \in X$ such that $J(x)^o \neq \emptyset$. Is it true that $J(x) = X$?

Ansari [1] and Bernal [4] gave a positive answer to Rolewicz' question if every separable and infinite dimensional Banach space supports a hypercyclic operator. Observe that we showed that the non-separable Banach space $l^\infty(\mathbb{N})$ admits a J-class operator, while on the other hand Bermúdez and Kalton [3] showed that $l^\infty(\mathbb{N})$ does not support topologically transitive operators. Hence it is natural to raise the following question.

Problem 2.
Does every non-separable and infinite dimensional Banach space support a J-class operator?

D. Herrero in [21] established a spectral description of the closure of the set of hypercyclic operators acting on an infinite dimensional and separable Hilbert space. Below we ask a similar question for J-class operators.

Problem 3.
Is there a spectral description of the closure of the set of J-class operators acting on a separable and infinite dimensional Hilbert space?

Problem 4.
Let X be a separable and infinite dimensional Banach space and $T : X \to X$ be an operator. Suppose that $J(x)^o \neq \emptyset$ for every $x \in X$. Does it follow that T is hypercyclic?

Grivaux in [17] showed that every operator on a complex infinite dimensional Hilbert space can be written as a sum of two hypercyclic operators. We consider the following

Problem 5.
Is it true that any operator on $l^\infty(\mathbb{N})$ can be written as a sum of two J-class operators?

Acknowledgements. We are grateful to the referees for providing us useful remarks and suggestions concerning the presentation of the present paper.

References

[1] S. I. Ansari, Hypercyclic and cyclic vectors, *J. Funct. Anal.* **128** (1995), 374-383.
[2] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, 179. Cambridge University Press, Cambridge, 2009. xiv+337 pp.
[3] T. Bermúdez and N. J. Kalton, The range of operators on von Neumann algebras, *Proc. Amer. Math. Soc.* **130** (2002), 1447-1455.
[4] L. Bernal-González, On hypercyclic operators on Banach spaces, *Proc. Amer. Math. Soc.* **127** (1999), 1003-1010.
[5] J. P. Bès, Invariant manifolds of hypercyclic vectors for the real scalar case, *Proc. Amer. Math. Soc.* **127** (1999), 1801-1804.
[6] N. P. Bhatia and G. P. Szego, *Stability theory of dynamical systems.* Die Grundlehren der mathematischen Wissenschaften, Band 161 Springer-Verlag, New York-Berlin 1970 xi+225 pp.
[7] J. Bonet, L. Frerick, A. Peris and J. Wengenroth, Transitive and hypercyclic operators on locally convex spaces, *Bull. London Math. Soc.* **37** (2005), 254-264.
[8] J. Bonet, F. Martínez-Giménez and A. Peris, Linear chaos on Frechet spaces, Dynamical systems and functional equations (Murcia, 2000), *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), 1649-1655.
[9] P. S. Bourdon, Invariant Manifolds of Hypercyclic Vectors, *Proc. Amer. Math. Soc.* **118** (1993), 845-847.
[10] P. S. Bourdon, Orbits of hyponormal operators, *Michigan Math. J.* **44** (1997), 345-353.
[11] P. S. Bourdon and N. S. Feldman, Somewhere dense orbits are everywhere dense, *Indiana Univ. Math. J.* **52** (2003), 811-819.

[12] G. Costakis and A. Manoussos, J-class weighted shifts on the space of bounded sequences of complex numbers, *Integral Equations Operator Theory* **62** (2008), 149-158.

[13] R. L. Devaney, *An introduction to chaotic dynamical systems.* Second edition. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989, xviii+336 pp.

[14] N. S. Feldman, Perturbations of hypercyclic vectors, *J. Math. Anal. Appl.* **273** (2002), 67-74.

[15] N. S. Feldman, (A Survey of) The Dynamics of Cohyponormal Operators, *Contemp. Math. Amer. Math. Soc.* **321** (2003), 71-85.

[16] N. S. Feldman, Countably hypercyclic operators, *J. Operator Theory* **50** (2003), 107-117.

[17] S. Grivaux, Sums of hypercyclic operators, *J. Funct. Anal.* **202** (2003), 486-503.

[18] K. -G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* **36** (1999), 345-381.

[19] K. -G. Grosse-Erdmann, Recent developments in hypercyclicity, *RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* (2003), 273-286.

[20] K. -G. Grosse-Erdmann, *Dynamics of linear operators*, Topics in complex analysis and operator theory, 41-84, Univ. Malaga, Malaga, 2007.

[21] D. Herrero, Limits of hypercyclic and supercyclic operators, *J. Funct. Anal.* **99** (1991), 179-190.

[22] D. A. Herrero and C. Kitai, On invertible hypercyclic operators, *Proc. Amer. Math. Soc.* **116** (1992), 873-875.

[23] C. Kitai, *Invariant closed sets for linear operators*, PhD Dissertation, University of Toronto, (1982).

[24] A. Montes-Rodríguez and H. N. Salas, Supercyclic subspaces, *Bull. London Math. Soc.* **35** (2003), 721-737.

[25] A. Montes-Rodríguez and S. A. Shkarin, Non-weakly supercyclic operators, *J. Operator Theory* **58** (2007), 39-62.

[26] A. Peris, Multihypercyclic operators are hypercyclic, *Math. Z.* **236** (2001), 778-786.

[27] H. Radjavi and P. Rosenthal, *Invariant subspaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973 xi+219 pp.

[28] H. N. Salas, A hypercyclic operator whose adjoint is also hypercyclic, *Proc. Amer. Math. Soc.* **112** (1991), 765-770.

[29] H. N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.* **347** (1995), 993-1004.

[30] J. H. Shapiro, *Notes on the Dynamics of Linear Operators*, Unpublished Lecture Notes, (available at www.math.msu.edu/~shapiro).

Department of Mathematics, University of Crete, Knossos Avenue, GR-714 09 Heraklion, Crete, Greece
E-mail address: `costakis@math.uoc.gr`

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
E-mail address: `amanouss@math.uni-bielefeld.de`

Integr. equ. oper. theory 62 (2008), 149–158
0378-620X/020149-10, DOI 10.1007/s00020-008-1621-6

Integral Equations and Operator Theory

J-Class Weighted Shifts on the Space of Bounded Sequences of Complex Numbers

George Costakis and Antonios Manoussos

Abstract. We provide a characterization of J-class and J^{mix}-class unilateral weighted shifts on $l^{\infty}(\mathbb{N})$ in terms of their weight sequences. In contrast to the previously mentioned result we show that a bilateral weighted shift on $l^{\infty}(\mathbb{Z})$ cannot be a J-class operator.

Mathematics Subject Classification (2000). Primary 47A16; Secondary 37B99, 54H20.

Keywords. Hypercyclic operators, J-class operators, J^{mix}-class operators, unilateral and bilateral weighted shifts.

1. Introduction

During the last years the dynamics of linear operators on infinite dimensional spaces has been extensively studied, see the survey articles [4], [7], [8], [9], [10], [12] and the recent book [1]. Let us recall the notion of *hypercyclicity*. Let X be a separable Banach space and $T : X \to X$ be a bounded linear operator. The operator T is said to be hypercyclic provided there exists a vector $x \in X$ such that its orbit under T, $Orb(T, x) = \{T^n x : n = 0, 1, 2, \dots\}$, is dense in X. If X is Banach space (possibly non-separable) and $T : X \to X$ is a bounded linear operator then T is called *topologically transitive* (*topologically mixing*) if for every pair of non-empty open subsets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \emptyset$ ($T^m U \cap V \neq \emptyset$ for every $m \geq n$ respectively). It is well known, and easy to prove, that if T is a bounded linear operator acting on separable Banach space X then T is hypercyclic if and only if T is topologically transitive.

A first step to understand the dynamics of linear operators is to look at particular operators as for example the weighted shifts. Salas [11] was the first who

During this research the second author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

characterized the hypercyclic weighted shifts in terms of their weight sequences. We would like to point out that $l^\infty(\mathbb{N})$ and $l^\infty(\mathbb{Z})$ do not support hypercyclic operators since they are not separable Banach spaces. In fact they do not support topologically transitive operators as it was shown by Bermúdez and Kalton in [2]. Recently Bès, Chan and Sanders [3] showed that there exists a weak* hypercyclic weighted shift T on $l^\infty(\mathbb{N})$, i.e there exists a vector $x \in l^\infty(\mathbb{N})$ whose orbit $Orb(T,x)$ is dense in the weak* topology of $l^\infty(\mathbb{N})$. In fact they give a characterization of the weak* hypercyclic weighted shifts in terms of their weight sequences. In [5] we studied the dynamics of operators by replacing the orbit of a vector with its extended limit set. To be precise, let $T : X \to X$ be a bounded linear operator on a Banach space X (not necessarily separable) and $x \in X$. A vector y belongs to the extended limit set $J(x)$ of x if there exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\} \subset X$ such that $x_n \to x$ and $T^{k_n}x_n \to y$. If $J(x) = X$ for some non-zero vector $x \in X$ then T is called J-class operator. Roughly speaking, the use of the extended limit set "localizes" the notion of hypercyclicity. The last can be justified by the following: $J(x) = X$ if and only if for every open neighborhood U of x and every non-empty open set $V \subset X$ there exists a positive integer n such that $T^nU \cap V \neq \emptyset$.

The purpose of this paper is to study the dynamical behavior of weighted shifts on the spaces of bounded sequences of complex numbers $l^\infty(\mathbb{N})$ and $l^\infty(\mathbb{Z})$ through the use of the extended limit sets. Our main result is the following (see Theorem 3.1).

Theorem. *Let $T : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ be a backward unilateral weighted shift with positive weights $(\alpha_n)_{n\in\mathbb{N}}$. The following are equivalent.*

(i) *T is a J-class operator.*

(ii) $\displaystyle\lim_{n\to+\infty}\left(\inf_{j\geq 0}\prod_{i=1}^{n}\alpha_{i+j}\right) = +\infty.$

In particular, if T is a J-class operator then the sequence of weights $(\alpha_n)_{n\in\mathbb{N}}$ is bounded from below by a positive number and we have the following complete description of the set of J-vectors.

$$\{x \in l^\infty(\mathbb{N}) : J(x) = l^\infty(\mathbb{N})\} = c_0(\mathbb{N}),$$

where $c_0(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}) : \lim_{n\to+\infty} x_n = 0\}$.

Observe that if T is a J-class backward unilateral weighted shift on $l^\infty(\mathbb{N})$ then in view of the above theorem and Salas' characterization of hypercyclic weighted shifts, see [11], we conclude that T is hypercyclic on $l^p(\mathbb{N})$ for every $1 \leq p < +\infty$. However, as we show in section 3, the converse is not always true.

On the other hand the situation is completely different in the case of bilateral weighted shifts. In particular we show that a bilateral weighted shift on $l^\infty(\mathbb{Z})$ cannot be a J-class operator, see Theorem 3.3. In addition, we prove similar results for J^{mix}-class weighted shifts (see Definitions 2.1 and 2.2).

2. Preliminaries

Definition 2.1. Let $T : X \to X$ be a bounded linear operator on a Banach space X. For every $x \in X$ the sets

$$\begin{aligned} J(x) = \{y \in X : \ & \text{there exist a strictly increasing sequence of positive} \\ & \text{integers}\, \{k_n\} \text{ and a sequence } \{x_n\} \subset X \,\text{such that}\, x_n \to x \,\text{and} \\ & T^{k_n} x_n \to y\}, \end{aligned}$$

$$\begin{aligned} J^{\text{mix}}(x) = \{y \in X : \ & \text{there exists a sequence}\, \{x_n\} \subset X \,\text{such that} \\ & x_n \to x \text{ and } T^n x_n \to y\} \end{aligned}$$

will be called the extended limit set of x under T and the extended mixing limit set of x under T respectively.

Definition 2.2. A bounded linear operator $T : X \to X$ acting on a Banach space X will be called a J-class (J^{mix}-class) operator if there exists a non-zero vector $x \in X$ such that $J(x) = X$ ($J^{\text{mix}}(x) = X$ respectively).

Definition 2.3. Let T be a bounded linear operator acting on a Banach space X. A vector $x \in X$ will be called a J-vector (J^{mix}-vector) if $J(x) = X$ ($J^{\text{mix}}(x) = X$ respectively).

Remark 2.4. Observe that

(i) an operator $T : X \to X$ is topologically transitive if and only if $J(x) = X$ for every $x \in X$,
(ii) an operator $T : X \to X$ is topologically mixing if and only if $J^{\text{mix}}(x) = X$ for every $x \in X$,

see [5]. Hence every hypercyclic operator (topologically mixing) is a J-class operator (J^{mix}-class operator). However the converse is not true. To see that consider the operator $3I \oplus 2B : \mathbb{C} \oplus l^2(\mathbb{N}) \to \mathbb{C} \oplus l^2(\mathbb{N})$ where I is the identity map on $\mathbb{C}$ and B is the backward shift on the space of square summable sequences $l^2(\mathbb{N})$. Consider any non-zero vector $x \in l^2(\mathbb{N})$. We shall prove that $J^{\text{mix}}_{3I \oplus 2B}(0 \oplus x) = \mathbb{C} \oplus l^2(\mathbb{N})$. Let $y \in l^2(\mathbb{N})$ and $\lambda \in \mathbb{C}$. There exists a sequence $\{x_n\}$ in $l^2(\mathbb{N})$ such that $(2B)^n x_n \to y$. Define the vectors $\frac{\lambda}{3^n} \oplus x_n$. Then we have $\frac{\lambda}{3^n} \oplus x_n \to 0 \oplus x$ and $(3I \oplus 2B)^n(\frac{\lambda}{3^n} \oplus x_n) \to \lambda \oplus y$. Hence $3I \oplus 2B$ is a J^{mix}-class operator which is not hypercyclic. In fact it is not even supercyclic, see [6].

Let us also give an example of a backward weighted shift, acting on a non-separable space, which is a J-class operator but not topologically transitive. Consider the operator $2B : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ where B is the backward shift and $l^\infty(\mathbb{N})$ is the space of bounded sequences. Theorem 3.6 implies that $2B$ is a J^{mix}-class operator. On the other hand the space $l^\infty(\mathbb{N})$ does not support topologically transitive operators, see [2].

The next lemma, which will be of use to us, also appears in [5]. For the convenience of the reader we give its proof.

Lemma 2.5. *Let $T : X \to X$ be a bounded linear operator on a Banach space X and $\{x_n\}$, $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$ for some $x, y \in X$.*

(i) *If $y_n \in J(x_n)$ for every $n = 1, 2, \ldots$, then $y \in J(x)$.*
(ii) *If $y_n \in J^{mix}(x_n)$ for every $n = 1, 2, \ldots$, then $y \in J^{mix}(x)$.*

Proof. (i) For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{k_1} - x\| < \frac{1}{2} \text{ and } \|y_{k_1} - y\| < \frac{1}{2}.$$

Since $y_{k_1} \in J(x_{k_1})$ we may find a positive integer l_1 and $z_1 \in X$ such that

$$\|z_1 - x_{k_1}\| < \frac{1}{2} \text{ and } \|T^{l_1} z_1 - y_{k_1}\| < \frac{1}{2}.$$

Therefore,

$$\|z_1 - x\| < 1 \text{ and } \|T^{l_1} z_1 - y\| < 1.$$

Proceeding inductively we find a strictly increasing sequence of positive integers $\{l_n\}$ and a sequence $\{z_n\}$ in X such that

$$\|z_n - x\| < \frac{1}{n} \text{ and } \|T^{l_n} z_n - y\| < \frac{1}{n}.$$

This completes the proof of assertion (i).

(ii) For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{k_1} - x\| < \frac{1}{2} \text{ and } \|y_{k_1} - y\| < \frac{1}{2}.$$

Since $y_{k_1} \in J^{\text{mix}}(x_{k_1})$ we may find a positive integer l_1 and a sequence $\{z_n\} \subset X$ such that

$$\|z_n - x_{k_1}\| < \frac{1}{2} \text{ and } \|T^n z_n - y_{k_1}\| < \frac{1}{2}$$

for every $n \geq l_1$. Therefore,

$$\|z_n - x\| < 1 \text{ and } \|T^n z_n - y\| < 1$$

for every $n \geq l_1$. Proceeding in the same way we may find a positive integer $l_2 > l_1$ and a sequence $\{w_n\} \subset X$ such that

$$\|w_n - x\| < \frac{1}{2} \text{ and } \|T^n w_n - y\| < \frac{1}{2}$$

for every $n \geq l_2$. Set $v_n = z_n$ for every $l_1 \leq n < l_2$, hence

$$\|v_n - x\| < 1 \text{ and } \|T^n v_n - y\| < 1.$$

Proceeding inductively we find a strictly increasing sequence of positive integers $\{n_k\}$ and a sequence $\{v_n\}$ in X such that if $n \geq n_k$ then

$$\|v_n - x\| < \frac{1}{k} \text{ and } \|T^n v_n - y\| < \frac{1}{k}.$$

Take any $\epsilon > 0$. There exists a positive integer k_0 such that $\frac{1}{k_0} < \epsilon$. Hence for every $n \geq n_{k_0}$ we get

$$\|v_n - x\| < \frac{1}{k_0} < \epsilon \text{ and } \|T^n v_n - y\| < \frac{1}{k_0} < \epsilon.$$

This completes the proof of assertion (ii). $\square$

3. Main results

Theorem 3.1. *Let $T : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ be a backward unilateral weighted shift with positive weights $(\alpha_n)_{n\in\mathbb{N}}$. The following are equivalent.*

(i) *T is a J-class operator.*

(ii) $\displaystyle\lim_{n\to+\infty}\left(\inf_{j\geq 0}\prod_{i=1}^{n}\alpha_{i+j}\right) = +\infty.$

In particular, if T is a J-class operator then the sequence of weights $(\alpha_n)_{n\in\mathbb{N}}$ is bounded from below by a positive number and we have the following complete description of the set of J-vectors.

$$\{x \in l^\infty(\mathbb{N}) : J(x) = l^\infty(\mathbb{N})\} = c_0(\mathbb{N}),$$

where $c_0(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}) : \lim_{n\to+\infty} x_n = 0\}$.

Proof. Let us prove that (i) implies (ii). There exists a non-zero vector $x \in l^\infty(\mathbb{N})$ such that $J(x) = l^\infty(\mathbb{N})$. Consider the vector $y = (1, 1, \ldots)$. Then there exists a strictly increasing sequence $\{k_n\}$ of positive integers and a sequence $\{y_n\} \in l^\infty(\mathbb{N})$, $y_n = (y_{nm})_{m=1}^\infty$, such that

$$\|y_n - x\|_\infty \to 0 \quad \text{and} \quad \|T^{k_n} y_n - (1, 1, \ldots)\|_\infty \to 0.$$

Observe that

$$\|T^{k_n} y_n - (1, 1, \ldots)\|_\infty = \sup_{j\geq 0}\left|\left(\prod_{i=1}^{k_n}\alpha_{i+j}\right) y_{n(k_n+j+1)} - 1\right| \to 0$$

as $n \to \infty$. Fix $0 < \epsilon < 1$. There exists a positive integer n_1 such that

$$\|y_n - x\|_\infty < \epsilon \quad \text{for every} \quad n \geq n_1 \tag{3.1}$$

and

$$\sup_{j\geq 0}\left|\left(\prod_{i=1}^{k_{n_1}}\alpha_{i+j}\right) y_{n_1(k_{n_1}+j+1)} - 1\right| < \epsilon.$$

Therefore

$$\left|\left(\prod_{i=1}^{k_{n_1}}\alpha_{i+j}\right) y_{n_1(k_{n_1}+j+1)}\right| > 1 - \epsilon \quad \text{for every} \quad j \geq 0. \tag{3.2}$$

On the other hand, using (3.1), we have

$$\begin{aligned}\left|\left(\prod_{i=1}^{k_{n_1}}\alpha_{i+j}\right)y_{n_1(k_{n_1}+j+1)}\right| &\leq \left(\prod_{i=1}^{k_{n_1}}\alpha_{i+j}\right)\|y_{n_1}\|_\infty \\ &< \left(\prod_{i=1}^{k_{n_1}}\alpha_{i+j}\right)(\epsilon+\|x\|_\infty)\end{aligned} \tag{3.3}$$

for every $j \geq 0$. By (3.2) and (3.3) it follows that

$$\prod_{i=1}^{m_1}\alpha_{i+j} > \frac{1-\epsilon}{\epsilon+\|x\|_\infty} \quad \text{for every} \quad j \geq 0,$$

where $m_1 := k_{n_1}$. For every $l = 2, 3, \ldots$ consider the vector $(l, l, \ldots)$. Since $(l, l, \ldots) \in J(x)$ and working as before we inductively construct a strictly increasing sequence $\{m_l\}$ of positive integers such that

$$\prod_{i=1}^{m_l}\alpha_{i+j} > \frac{l-\epsilon}{\epsilon+\|x\|_\infty} \quad \text{for every} \quad j \geq 0 \quad \text{and every} \quad l \geq 1.$$

The last implies that

$$\lim_{l\to+\infty}\left(\inf_{j\geq 0}\prod_{i=1}^{m_l}\alpha_{i+j}\right) = +\infty$$

which in turn yields

$$\limsup_{n\to+\infty}\left(\inf_{j\geq 0}\prod_{i=1}^{n}\alpha_{i+j}\right) = +\infty.$$

It remains to show that

$$\lim_{n\to+\infty}\left(\inf_{j\geq 0}\prod_{i=1}^{n}\alpha_{i+j}\right) = +\infty.$$

Let us first show that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is bounded from below by a positive number. Fix a positive number $M > 1$. There exists a positive integer N such that

$$\prod_{i=1}^{N}\alpha_{i+j} > M \quad \text{for every } j \geq 0.$$

If $N = 1$ there is nothing to prove. Assume that $N > 1$. For every $j \geq 0$ and since $\|T\| = \sup_n \alpha_n$, we have

$$\alpha_{j+1}\|T\|^{N-1} \geq \alpha_{j+1}\left(\prod_{i=2}^{N}\alpha_{i+j}\right) > M.$$

Proceeding inductively we conclude that

$$\alpha_n \geq \frac{M}{\|T\|^{N-1}}$$

for every $n \in \mathbb{N}$. Take any positive integer $n > N$. There exist positive integers p_n, v_n such that $n = Np_n + v_n$ and $0 \le v_n \le N-1$. Since $(\alpha_n)_{n\in\mathbb{N}}$ is bounded from below by $\frac{M}{\|T\|^{N-1}}$ it follows that

$$\prod_{i=1}^{n} \alpha_{i+j} > M^{p_n} C \quad \text{for every } j \ge 0,$$

where

$$C = \min\left\{ \left(\frac{M}{\|T\|^{N-1}} \right)^{N-1}, 1 \right\}.$$

From the last and the fact that $M > 1$ it clearly follows that

$$\lim_{n\to+\infty} \left(\inf_{j\ge 0} \prod_{i=1}^{n} \alpha_{i+j} \right) = +\infty.$$

We shall now prove that (ii) implies (i). Fix a vector $x = (x_1, x_2, \ldots)$ in $l^\infty(\mathbb{N})$ with finite support. There exists a positive integer n_0 such that $x_n = 0$ for every $n \ge n_0$ and $\inf_{j\ge 0} \prod_{i=1}^{n} \alpha_{i+j} > 0$ for every $n \ge n_0$. Consider any vector $y = (y_1, y_2, \ldots) \in l^\infty(\mathbb{N})$. We set

$$y_n = \left(x_1, x_2, \ldots, x_{n_0-1}, 0, \ldots, 0, \frac{y_1}{\prod_{i=1}^{n} \alpha_i}, \frac{y_2}{\prod_{i=1}^{n} \alpha_{i+1}}, \frac{y_3}{\prod_{i=1}^{n} \alpha_{i+2}}, \ldots \right)$$

for every $n \ge n_0$, where the 0's fill all the coordinates from the n_0-th up to n-th position. Then for every $n \ge n_0$ we have

$$\|y_n - x\|_\infty = \sup_{j\ge 0} \left| \frac{y_{j+1}}{\prod_{i=1}^{n} \alpha_{i+j}} \right| \le \frac{\|y\|_\infty}{\inf_{j\ge 0} \prod_{i=1}^{n} \alpha_{i+j}},$$

hence $y_n \to x$. Observe also that $T^n y_n = y$, so $y \in J(x)$. Thus T is a J-class operator and this completes the proof that (ii) implies (i).

It remains to show that the set of J-vectors is $c_0(\mathbb{N})$. From the proof that (ii) implies (i) we have that if x is a vector with finite support then $J(x) = l^\infty(\mathbb{N})$. Since the closure of the set of all vectors with finite support is $c_0(\mathbb{N})$, by Lemma 2.5, we conclude that

$$c_0(\mathbb{N}) \subset \{x \in l^\infty(\mathbb{N}) : J(x) = l^\infty(\mathbb{N})\}.$$

To prove the converse inclusion, take a vector x such that $J(x) = l^\infty(\mathbb{N})$. Consider the zero vector and let ϵ be a positive number. There exist positive integers n_0, n_1 and a vector $y_{n_0} = (y_{n_0 k})_{k\in\mathbb{N}}$ such that

$$\|y_{n_0} - x\|_\infty < \epsilon, \ \|T^{n_1} y_{n_0}\|_\infty < \epsilon \quad \text{and} \quad \prod_{i=1}^{n_1} \alpha_{i+j} > 1 \ \text{ for every } j \ge 0.$$

Hence we have

$$\left| \left(\prod_{i=1}^{n_1} \alpha_{i+j} \right) y_{n_0(n_1+j+1)} \right| < \epsilon$$

for every $j \geq 0$. The last and the previous bound on the weights imply that

$$|y_{n_0(n_1+j+1)}| < \frac{\epsilon}{\prod_{i=1}^{n_1} \alpha_{i+j}} < \epsilon$$

for every $j \geq 0$. Hence it follows that

$$|x_{n_1+j+1}| \leq \|y_{n_0} - x\|_\infty + |y_{n_0(n_1+j+1)}| < 2\epsilon$$

for every $j \geq 0$. Thus x belongs to $c_0(\mathbb{N})$. This completes the proof of the theorem. □

Remark 3.2. As we promised in the introduction, we provide below an example of a hypercyclic backward unilateral weighted shift on the space of square summable sequences $l^2(\mathbb{N})$, which is not a J-class operator on $l^\infty(\mathbb{N})$. Consider the backward unilateral weighted shift T with weight sequence

$$(\alpha_1, \alpha_2, \dots) = (\frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots).$$

It is easy to check that T is hypercyclic on $l^2(\mathbb{N})$. On the other hand we have that

$$\inf_{j\geq 0} \prod_{i=1}^{n} \alpha_{i+j} \leq \frac{1}{2^n} \quad \text{for every} \quad n = 1, 2, \dots.$$

Hence,

$$\lim_{n\to+\infty} \left(\inf_{j\geq 0} \prod_{i=1}^{n} \alpha_{i+j} \right) = 0.$$

Theorem 3.1 implies that T is not a J-class operator on $l^\infty(\mathbb{N})$.

To complete our study on J-class backward unilateral weighted shifts we would like to mention the following result from [5]: *a backward unilateral weighted shift T is a J-class operator on $l^p(\mathbb{N})$ if and only if T is hypercyclic on $l^p(\mathbb{N})$, for* $1 \leq p < +\infty$. A similar result holds for bilateral shifts, see [5].

Theorem 3.3. *Let $T : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ be a backward bilateral weighted shift with positive weights $(\alpha_n)_{n\in\mathbb{Z}}$. Then T is not a J-class operator.*

Proof. Following a similar line of reasoning as in the proof that (i) implies (ii) in Theorem 3.1 and using the vectors $(\dots, l, l, l, \dots) \in l^\infty(\mathbb{Z})$ for $l = 1, 2, \dots$ we conclude that the sequence $(\alpha_n)_{n\in\mathbb{Z}}$ is bounded from below by a positive number and

$$\lim_{n\to+\infty} \left(\inf_{j\in\mathbb{Z}} \prod_{i=1}^{n} \alpha_{i+j} \right) = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \left(\inf_{j\in\mathbb{Z}} \prod_{i=1}^{n} \alpha_{j-i} \right) = +\infty.$$

Assume that there exists a non-zero vector $x = (x_j)_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ such that $J(x) = l^\infty(\mathbb{Z})$. Since $x \neq 0$ there is some $j \in \mathbb{Z}$ such that $x_j \neq 0$. By our assumption $0 \in J(x)$ hence there exist a sequence of positive integers k_n and vectors $y_n = (y_{nm})_{m\in\mathbb{Z}}$ such that

$$\|y_n - x\|_\infty \to 0 \quad \text{and} \quad \|T^{k_n} y_n\|_\infty \to 0.$$

Therefore, taking the $-k_n+1+j$-th coordinate of the vector $T^{k_n}y_n$ we conclude that

$$\left|\left(\prod_{i=1}^{k_n}\alpha_{j-i}\right)y_{nj}\right|\to 0.$$

Since $\prod_{i=1}^{k_n}\alpha_{j-i}\to+\infty$ then $x_j=\lim\limits_{n\to+\infty}y_{nj}=0$, a contradiction. $\square$

Corollary 3.4. *Let T be a backward unilateral (bilateral) weighted shift with weight sequence $(\alpha_n)_{n\in\mathbb{N}}$ ($(\alpha_n)_{n\in\mathbb{Z}}$ respectively). The following are equivalent*

(i) $J(0)=l^\infty(\mathbb{N})$ $(J(0)=l^\infty(\mathbb{Z}))$.

(ii) $\lim\limits_{n\to+\infty}\left(\inf\limits_{j\geq 0}\prod\limits_{i=1}^{n}\alpha_{i+j}\right)=+\infty$, $\left(\lim\limits_{n\to+\infty}\left(\inf\limits_{j\in\mathbb{Z}}\prod\limits_{i=1}^{n}\alpha_{i+j}\right)=+\infty\right)$.

Remark 3.5. By the previous corollary and Theorem 3.1 it follows that if T is a backward unilateral weighted shift and $J(0)=l^\infty(\mathbb{N})$ then T is J-class operator. However, for backward bilateral weighted shifts this is no longer true. For example consider the backward bilateral weighted shift $T:l^\infty(\mathbb{Z})\to l^\infty(\mathbb{Z})$ with weight sequence $(\alpha_n)_{n\in\mathbb{Z}}$, $\alpha_n=2$ for $n\geq 1$ and $\alpha_n=1$ for $n\leq 0$. Corollary 3.4 gives that $J(0)=l^\infty(\mathbb{Z})$ and Theorem 3.3 implies that T is not a J-class operator.

Using similar arguments as in the proof of Theorem 3.1 we obtain the following.

Theorem 3.6. *Let $T:l^\infty(\mathbb{N})\to l^\infty(\mathbb{N})$ be a backward unilateral weighted shift with positive weights $(\alpha_n)_{n\in\mathbb{N}}$. The following are equivalent.*

(i) *T is a J^{mix}-class operator.*

(ii) $\lim\limits_{n\to+\infty}\left(\inf\limits_{j\geq 0}\prod\limits_{i=1}^{n}\alpha_{i+j}\right)=+\infty$.

In addition, if T is a J^{mix}-class operator we have the following complete description of the set of J^{mix}-vectors.

$$\{x\in l^\infty(\mathbb{N}):J^{mix}(x)=l^\infty(\mathbb{N})\}=c_0(\mathbb{N}).$$

Combining Theorems 3.1 and 3.6 we obtain the following.

Corollary 3.7. *Let $T:l^\infty(\mathbb{N})\to l^\infty(\mathbb{N})$ be a backward unilateral weighted shift with positive weights $(\alpha_n)_{n\in\mathbb{N}}$. The following are equivalent.*

(i) *T is a J^{mix}-class operator.*

(ii) *T is a J-class operator.*

Acknowledgment

We are indebted to the referee for an extremely careful reading of the manuscript and for sending us a list of very helpful comments concerning the presentation of the paper. He/she pointed out a gap in the proof of Theorem 3.1 and he/she also corrected the statement of Theorem 3.3.

References

[1] F. Bayart and É. Matheron, *Topics in linear dynamics*, Cambridge Tracts in Mathematics (to appear).

[2] T. Bermúdez and N.J. Kalton, *The range of operators on von Neumann algebras*, Proc. Amer. Math. Soc. **130** (2002), 1447–1455.

[3] J. Bès, K.C. Chan and R. Sanders, *Weak* Hypercyclicity and Supercyclicity of Shifts on l^∞*, Integral Equations and Operator Theory **55** (2006), 363–376.

[4] J. Bonet, F. Martínez-Giménez and A. Peris, *Linear chaos on Frechet spaces*, Dynamical systems and functional equations (Murcia, 2000). Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 1649–1655.

[5] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, preprint (arXiv:0704.3354).

[6] M. González, F. León-Saavedra and A. Montes-Rodríguez, *Semi-Fredholm theory: hypercyclic and supercyclic subspaces*, Proc. London Math. Soc. **81** (2000), 169–189.

[7] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. **36** (1999), 345–381.

[8] K.-G. Grosse-Erdmann, *Recent developments in hypercyclicity*, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. (2003), 273–286.

[9] K.-G. Grosse-Erdmann, *Dynamics of linear operators*, preprint.

[10] A. Montes-Rodríguez and H.N. Salas, *Supercyclic subspaces*, Bull. London Math. Soc. **35** (2003), 721–737.

[11] H.N. Salas, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc. **347** (1995), 993–1004.

[12] J.H. Shapiro, *Notes on the Dynamics of Linear Operators*, Unpublished Lecture Notes, (available at www.math.msu.edu/∼shapiro).

George Costakis
Department of Mathematics
University of Crete
Knossos Avenue
GR-714 09 Heraklion, Crete
Greece
e-mail: `costakis@math.uoc.gr`

Antonios Manoussos
Fakultät für Mathematik, SFB 701
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany
e-mail: `amanouss@math.uni-bielefeld.de`

Submitted: September 25, 2007.
Revised: June 24, 2008.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 3, March 2009, Pages 1025–1034
S 0002-9939(08)09717-7
Article electronically published on October 17, 2008

DYNAMICS OF TUPLES OF MATRICES

G. COSTAKIS, D. HADJILOUCAS, AND A. MANOUSSOS

(Communicated by Nigel J. Kalton)

ABSTRACT. In this article we answer a question raised by N. Feldman in 2008 concerning the dynamics of tuples of operators on $\mathbb{R}^n$. In particular, we prove that for every positive integer $n \geq 2$ there exist n-tuples $(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices over $\mathbb{R}$ such that $(A_1, A_2, \ldots, A_n)$ is hypercyclic. We also establish related results for tuples of 2×2 matrices over $\mathbb{R}$ or $\mathbb{C}$ being in Jordan form.

1. INTRODUCTION

Following the recent work of Feldman in [4] an n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \ldots, T_n$ acting on a locally convex space X. The tuple $(T_1, T_2, \ldots, T_n)$ is hypercyclic if there exists a vector $x \in X$ such that the set

$$\{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} x : k_1, k_2, \ldots, k_n \geq 0\}$$

is dense in X. Such a vector x is called hypercyclic for $(T_1, T_2, \ldots, T_n)$ and the set of hypercyclic vectors for $(T_1, T_2, \ldots, T_n)$ will be denoted by $HC((T_1, T_2, \ldots, T_n))$. The above definition generalizes the notion of hypercyclicity to tuples of operators. For an account of results, comments and an extensive bibliography on hypercyclicity we refer to [1], [5], [6] and [7]. For results concerning the dynamics of tuples of operators see [2], [3], [4] and [9].

In [4] Feldman showed, among other things, that in $\mathbb{C}^n$ there exist diagonalizable $(n+1)$-tuples of matrices having dense orbits. In addition he proved that there is no n-tuple of diagonalizable matrices on $\mathbb{R}^n$ or $\mathbb{C}^n$ that has a somewhere dense orbit. Therefore the following question arose naturally.

Question (Feldman [4]). *Are there non-diagonalizable n-tuples on $\mathbb{R}^k$ that have somewhere dense orbits?*

We give a positive answer to this question in a very strong form, as the next theorem shows.

Received by the editors March 24, 2008.

2000 *Mathematics Subject Classification.* Primary 47A16.

Key words and phrases. Hypercyclic operators, tuples of matrices.

During this research the third author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

Theorem 1.1. *For every positive integer $n \geq 2$ there exist n-tuples $(A_1, \ldots, A_n)$ of $n \times n$ non-(simultaneously) diagonalizable matrices over $\mathbb{R}$ such that $(A_1, \ldots, A_n)$ is hypercyclic.*

Restricting ourselves to tuples of 2×2 matrices in Jordan form either on $\mathbb{R}^2$ or $\mathbb{C}^2$, we prove the following.

Theorem 1.2. *There exist 2×2 matrices $A_j, j = 1, 2, 3, 4$, in Jordan form over $\mathbb{R}$ such that (A_1, A_2, A_3, A_4) is hypercyclic. In particular*

$$HC((A_1, A_2, A_3, A_4)) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\}.$$

Theorem 1.3. *There exist 2×2 matrices $A_j, j = 1, 2, \ldots, 8$, in Jordan form over $\mathbb{C}$ such that $(A_1, A_2, \ldots, A_8)$ is hypercyclic.*

2. Products of 2×2 matrices

Lemma 2.1. *Let m be a positive integer and for each $j = 1, 2, \ldots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, i.e. $A_j = \begin{pmatrix} a_j & 1 \\ 0 & a_j \end{pmatrix}$ for $a_1, a_2, \ldots, a_m \in \mathbb{F}$. Then $(A_1, A_2, \ldots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic if and only if the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$ (respectively $\mathbb{R}^2$).

Proof. We prove the above in the case $\mathbb{F} = \mathbb{C}$, since the other case is similar. Observe that

$${A_j}^l = \begin{pmatrix} {a_j}^l & l{a_j}^{l-1} \\ 0 & {a_j}^l \end{pmatrix}$$

for $l \in \mathbb{N}$. As a result we have

$${A_1}^{k_1} {A_2}^{k_2} \ldots {A_m}^{k_m} = \begin{pmatrix} \prod\limits_{j=1}^{m} {a_j}^{k_j} & \prod\limits_{j=1}^{m} {a_j}^{k_j} \sum\limits_{s=1}^{m} \frac{k_s}{a_s} \\ 0 & \prod\limits_{j=1}^{m} {a_j}^{k_j} \end{pmatrix}.$$

Assume that $(A_1, A_2, \ldots, A_m)$ is hypercyclic and let $\binom{z_1}{z_2} \in \mathbb{C}^2$ be a hypercyclic vector for $(A_1, A_2, \ldots, A_m)$. Then the sequence

$$\left\{ {A_1}^{k_1} {A_2}^{k_2} \ldots {A_m}^{k_m} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$
$$= \left\{ \begin{pmatrix} z_1 \prod\limits_{j=1}^{m} {a_j}^{k_j} + z_2 \prod\limits_{j=1}^{m} {a_j}^{k_j} \sum\limits_{s=1}^{m} \frac{k_s}{a_s} \\ z_2 \prod\limits_{j=1}^{m} {a_j}^{k_j} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. This implies that $z_2 \neq 0$. Dividing the element in the first row by that in the second, it can easily be shown that the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. The converse can easily be shown. $\square$

Remark 2.2. Let m be a positive integer and for each $j = 1, 2, \ldots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. By the proof of Lemma 2.1 it is immediate that whenever $(A_1, A_2, \ldots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic, one can completely describe the set of hypercyclic vectors as

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 : z_2 \neq 0 \right\} \quad \left(\text{respectively } \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\} \right).$$

2.1. **The real case.** The one-dimensional version of Kronecker's theorem stated below (see for example [8, Theorem 438, p. 375]) will be used repeatedly throughout this work.

Theorem 2.3. *If x is a positive irrational number, then the sequence $\{kx - s : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Remark 2.4. If x is a positive irrational number, then the sequence $\{s - kx : k, s \in \mathbb{N}\}$ is also dense in $\mathbb{R}$. Likewise, if x is a negative irrational number, then the sequence $\{s + kx : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

We shall need the following well-known result; see for example [4].

Theorem 2.5. *If $a, b > 1$ and $\frac{\ln a}{\ln b}$ is irrational, then the sequence $\{\frac{a^n}{b^m} : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$.*

Lemma 2.6. *Let $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $b > 1$ and $\frac{\ln |a|}{\ln b}$ is irrational. Then the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Proof. Since $\frac{\ln |a|}{\ln b}$ is irrational it follows that $\ln b / \ln \frac{1}{a^2}$ is irrational as well. Applying Theorem 2.5 we conclude that the sequence $\{a^{2n} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$. On the other hand the fact that a is negative implies that the sequence $\{a^{2n+1} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^-$. This completes the proof of the lemma. $\square$

Proposition 2.7. *There exist $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \frac{k_3}{a_3} + \frac{k_4}{a_4} \\ {a_1}^{k_1} {a_2}^{k_2} {a_3}^{k_3} {a_4}^{k_4} \end{pmatrix} : k_1, k_2, k_3, k_4 \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$.

Proof. By the lemma above fix $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $a + \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - x_2| < \epsilon$. Note that $a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s$ for every $k, s \in \mathbb{N}$. Note also that $a + \frac{1}{a} < 0$. Hence, by Remark 2.4, the sequence

$$\left\{ s + k \left(a + \frac{1}{a} \right) : k, s \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$; i.e. there exist $k, s \in \mathbb{N}$ such that

$$\left| s + k \left(a + \frac{1}{a} \right) - \left(x_1 - \frac{n}{a} - \frac{m}{b} \right) \right| < \epsilon,$$

i.e.

$$\left| \frac{n}{a} + \frac{m}{b} + k \left(a + \frac{1}{a} \right) + s - x_1 \right| < \epsilon.$$

Hence, setting $a_1 = a, a_2 = b, a_3 = \frac{1}{a}, a_4 = 1$ we prove the result. $\square$

Proof of Theorem 1.2. This is an immediate consequence of Lemma 2.1, Proposition 2.7 and Remark 2.2.

Example 2.8. One may construct many concrete examples of four 2×2 matrices, in Jordan form over $\mathbb{R}$, being hypercyclic. For example, fix $a, b\in\mathbb{R}$ such that $-1<a<0$, $b>1$ and both $a+\frac{1}{a}$, $\frac{\ln|a|}{\ln b}$ are irrational. From the above we conclude that

$$\left(\begin{pmatrix} a & 1\\ 0 & a\end{pmatrix}, \begin{pmatrix} b & 1\\ 0 & b\end{pmatrix}, \begin{pmatrix} \frac{1}{a} & 1\\ 0 & \frac{1}{a}\end{pmatrix}, \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\right)$$

is hypercyclic.

We shall now prove Theorem 1.1 for $n=2$; see Proposition 2.10 (ii). For this we need the following result due to Feldman; see Corollary 3.2 in [4].

Proposition 2.9 (Feldman). *Let $\mathbb{D}$ denote the open unit disk centered at 0 in the complex plane. If $b\in\mathbb{D}\setminus\{0\}$, then there exists a dense set $\Delta\subset\mathbb{C}\setminus\mathbb{D}$ such that for every $a\in\Delta$ the sequence $\{a^nb^m : n,m\in\mathbb{N}\}$ is dense in $\mathbb{C}$.*

Proposition 2.10. *(i) Every pair (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ with A_j, $j=1,2$, being either diagonal or in Jordan form is not hypercyclic.*

(ii) There exist pairs (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ such that A_1 is diagonal, A_2 is antisymmetric (rotation matrix) and (A_1,A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1,A_2); i.e.

$$HC((A_1,A_2))=\mathbb{R}^2\setminus\{(0,0)\}.$$

(iii) There exist pairs (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ such that both A_1 and A_2 are antisymmetric and (A_1,A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1,A_2), i.e.

$$HC((A_1,A_2))=\mathbb{R}^2\setminus\{(0,0)\}.$$

Proof. Let us prove assertion (i). The case of A_1, A_2 both diagonal is covered by Feldman; see [4].

Assume that A_1 is diagonal and A_2 is in Jordan form; i.e.

$$A_1=\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix},\quad A_2=\begin{pmatrix} b & 1\\ 0 & b\end{pmatrix}\ \text{ for } a,b\in\mathbb{R}.$$

Suppose that (A_1,A_2) is hypercyclic and let $\binom{x_1}{x_2}\in\mathbb{R}^2$ be a hypercyclic vector for (A_1,A_2). Then the sequence

$$\left\{{A_1}^n{A_2}^m\begin{pmatrix} x_1\\ x_2\end{pmatrix} : n,m\in\mathbb{N}\right\}=\left\{\begin{pmatrix} a^nb^mx_1+ma^nb^{m-1}x_2\\ a^nb^mx_2\end{pmatrix} : n,m\in\mathbb{N}\right\}$$

is dense in $\mathbb{R}^2$. Therefore b cannot be zero. Observe that x_2 cannot be zero either. Take any $y_1\in\mathbb{R}$ and $y_2\in\mathbb{R}\setminus\{0\}$. Then there exist sequences of positive integers $\{n_k\}$, $\{m_k\}$ such that $m_k\to+\infty$ and

$$\begin{aligned} &a^{n_k}b^{m_k}x_1+m_ka^{n_k}b^{m_k-1}x_2\to y_1,\\ &a^{n_k}b^{m_k}x_2\to y_2\end{aligned}$$

as $k\to+\infty$. Since $b\neq 0$, $y_2\neq 0$ and $x_2\neq 0$ we get that

$$a^{n_k}b^{m_k}x_1\to\frac{y_2x_1}{x_2}\quad\text{and}\quad |m_ka^{n_k}b^{m_k-1}x_2|=\frac{m_k}{|b|}|a^{n_k}b^{m_k}x_2|\to+\infty$$

as $k \to +\infty$. From the last, it clearly follows that

$$|a^{n_k} b^{m_k} x_1 + m_k a^{n_k} b^{m_k - 1} x_2| \to +\infty,$$

which is a contradiction.

Assume now that both A_1, A_2 are in Jordan form; i.e.

$$A_1 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},$$

for $a, b \in \mathbb{R}$ and (A_1, A_2) is hypercyclic. Lemma 2.1 implies that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ a^n b^m \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Observe that neither $|a|$ nor $|b|$ is equal to 1. By taking the absolute value in the second coordinate and then applying the logarithmic function, we find that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ n \ln |a| + m \ln |b| \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence the sequence

$$\left\{ \begin{pmatrix} n\frac{\ln |a|}{a} + m\frac{\ln |a|}{b} \\ n\frac{\ln |a|}{a} + m\frac{\ln |b|}{a} \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Subtracting the second coordinate from the first one, we conclude that the sequence

$$\left\{ m \left(\frac{\ln |a|}{b} - \frac{\ln |b|}{a} \right) : m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$, which is absurd. We proceed with the proof of assertion (ii). By Proposition 2.9 there exist $a \in \mathbb{R} \setminus \mathbb{Q}$ and $b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Write $b = |b| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b| \cos\theta & -|b| \sin\theta \\ |b| \sin\theta & |b| \cos\theta \end{pmatrix}.$$

Then we have

$$A_1{}^n A_2{}^m = \begin{pmatrix} a^n |b|^m \cos m\theta & -a^n |b|^m \sin m\theta \\ a^n |b|^m \sin m\theta & a^n |b|^m \cos m\theta \end{pmatrix}.$$

Applying in the above relation the vector $\binom{1}{0}$ and taking into account that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$, we conclude that the sequence

$$\left\{ A_1{}^n A_2{}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n, m \in \mathbb{N} \right\} = \left\{ \begin{pmatrix} a^n |b|^m \cos m\theta \\ a^n |b|^m \sin m\theta \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence (A_1, A_2) is hypercyclic. It is now easy to show that every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1, A_2).

In order to prove the last assertion we follow a similar line of reasoning as above. That is, by Proposition 2.9 there exist $a, b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Write $a = |a| e^{i\phi}$, $b = |b| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a| \cos\phi & -|a| \sin\phi \\ |a| \sin\phi & |a| \cos\phi \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b| \cos\theta & -|b| \sin\theta \\ |b| \sin\theta & |b| \cos\theta \end{pmatrix}.$$

A direct computation gives that $\left\{ {A_1}^n {A_2}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n, m \in \mathbb{N} \right\}$ is equal to

$$\left\{ \begin{pmatrix} |a|^n |b|^m \cos(n\phi + m\theta) \\ |a|^n |b|^m \sin(n\phi + m\theta) \end{pmatrix} : n, m \in \mathbb{N} \right\},$$

and by the choice of a, b we conclude that the vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is hypercyclic for (A_1, A_2). This completes the proof of the proposition. □

Question 2.11. What is the minimum number of 2×2 matrices over $\mathbb{R}$ in Jordan form so that their tuple forms a hypercyclic operator?

2.2. **The complex case.** In what follows we will be writing $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of a complex number z respectively.

Proposition 2.12. *There exist $a_j \in \mathbb{C}$, $j = 1, 2, \dots, 8$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \dots + \frac{k_8}{a_8} \\ {a_1}^{k_1} {a_2}^{k_2} \dots {a_8}^{k_8} \end{pmatrix} : k_1, k_2, \dots, k_8 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

Proof. The proof is in the same spirit as the proof of Proposition 2.7. Fix $a, b \in \mathbb{C}$ such that $-1 < a < 0$, $a + \frac{1}{a}, a - \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$ (see Proposition 2.9). Let $z_1, z_2 \in \mathbb{C}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - z_2| < \epsilon$. Note that

$$a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s (ia)^\xi \left(\frac{1}{ia}\right)^\xi (4i)^\rho \left(-\frac{1}{4}\right)^\rho$$

for every $k, s, \xi \in \mathbb{N}$ and $\rho \in 4\mathbb{N}$. Note that $a + \frac{1}{a} < 0$ and $a - \frac{1}{a} > 0$. Hence, by Theorem 2.3, the sequence

$$\left\{ \xi \left(a - \frac{1}{a}\right) - \left(\frac{\rho}{4}\right) : \xi \in \mathbb{N}, \rho \in 4\mathbb{N} \right\}$$

is dense in $\mathbb{R}$. As a result, there exist $\xi \in \mathbb{N}$ and $\rho \in 4\mathbb{N}$ such that

$$\left| \Im\left(i\xi \left(a - \frac{1}{a}\right) - i\left(\frac{\rho}{4}\right)\right) - \Im\left(z_1 - \frac{n}{a} - \frac{m}{b}\right) \right| < \epsilon;$$

i.e. we have that

$$\left| \Im\left(\frac{n}{a} + \frac{m}{b} + i\xi \left(a - \frac{1}{a}\right) - i\left(\frac{\rho}{4}\right)\right) - \Im(z_1) \right| < \epsilon.$$

By Remark 2.4, the sequence

$$\left\{ k\left(a + \frac{1}{a}\right) + s : k, s \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$. Hence, there exist $k, s \in \mathbb{N}$ such that

$$\left| k\left(a + \frac{1}{a}\right) + s - \left(4\rho + \Re\left(z_1 - \frac{n}{a} - \frac{m}{b}\right)\right) \right| < \epsilon;$$

i.e. we have that

$$\left| \Re\left(\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) - 4\rho + s\right) - \Re(z_1) \right| < \epsilon.$$

But this means that the real and imaginary parts of the complex number

$$\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) + s + i\xi\left(a - \frac{1}{a}\right) - i\frac{\rho}{4} - 4\rho$$

are within ϵ of the real and imaginary parts of z_1. Hence, setting $a_1 = a, a_2 = b, a_3 = \frac{1}{a}, a_4 = 1, a_5 = ia, a_6 = \frac{1}{ia}, a_7 = 4i, a_8 = -\frac{1}{4}$, we prove the result. □

Proof of Theorem 1.3. By Proposition 2.12, Lemma 2.1 and Remark 2.2 the assertion follows.

Example 2.13. Fix $a, b \in \mathbb{C}$ such that $-1 < a < 0$, $a + \frac{1}{a}, a - \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. From the above it is evident that the 8-tuple of 2×2 matrices in Jordan form over $\mathbb{C}$ given by

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \begin{pmatrix} \frac{1}{a} & 1 \\ 0 & \frac{1}{a} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} ia & 1 \\ 0 & ia \end{pmatrix}, \begin{pmatrix} \frac{1}{ia} & 1 \\ 0 & \frac{1}{ia} \end{pmatrix}, \begin{pmatrix} 4i & 1 \\ 0 & 4i \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 1 \\ 0 & -\frac{1}{4} \end{pmatrix}$$

is hypercyclic.

Question 2.14. What is the minimum number of 2×2 matrices over $\mathbb{C}$ in Jordan form so that their tuple forms a hypercyclic operator?

3. Products of 3×3 matrices

In this section we start with the following special case of Corollary 3.5 in [4], due to Feldman, which will be of use to us in the following.

Proposition 3.1 (Feldman). *If $b_1, b_2 \in \mathbb{D} \setminus \{0\}$, then there exists a dense set $\Delta \subset \mathbb{C} \setminus \mathbb{D}$ such that for every $a_1, a_2 \in \Delta$ the sequence*

$$\left\{ \begin{pmatrix} {a_1}^n {b_1}^m \\ {a_2}^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

In order to handle products of 3×3 matrices, we establish the following:

Corollary 3.2. *There exist $a \in \mathbb{C}$ and $b, c, d \in \mathbb{R}$ such that the sequence*

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times \mathbb{R}$.

Proof. Fix two real numbers b_1, b_2 with $b_1, b_2 \in (0, 1)$. By Proposition 3.1 there exist $a_1, a_2 \in \mathbb{C} \setminus \mathbb{D}$ such that the sequence

$$\left\{ \begin{pmatrix} {a_1}^n {b_1}^m \\ {a_2}^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. Define $a = a_1$, $b = b_1$, $c = |a_2|$ and $d = -\sqrt{b_2}$. Observe that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times [0, +\infty)$. Take $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

Case I. $x \geq 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}, \{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to x.$$

Since ${b_2}^{l_k} = d^{2l_k}$ we get $c^{n_k} d^{2l_k} \to x$.

Case II. $x < 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}, \{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to \frac{x}{d}.$$

The last implies that $c^{n_k} d^{2l_k+1} \to x$. This completes the proof of the corollary. $\square$

The main result of this section is to prove Theorem 1.1 for $n = 3$. This is stated and proved below.

Proposition 3.3. *There exist* 3 *tuples* (A_1, A_2, A_3) *of* 3×3 *matrices over* $\mathbb{R}$ *such that* (A_1, A_2, A_3) *is hypercyclic.*

Proof. By Corollary 3.2 there exist $a \in \mathbb{C}$ and $b, c, d \in \mathbb{R}$ such that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times \mathbb{R}$. Write $a = |a| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a|\cos\theta & -|a|\sin\theta & 0 \\ |a|\sin\theta & |a|\cos\theta & 0 \\ 0 & 0 & c \end{pmatrix}, \; A_2 = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Then we have

$${A_1}^n {A_2}^m {A_3}^l = \begin{pmatrix} |a|^n b^m \cos n\theta & -|a|^n b^m \sin n\theta & 0 \\ |a|^n b^m \sin n\theta & |a|^n b^m \cos n\theta & 0 \\ 0 & 0 & c^n d^l \end{pmatrix},$$

which in turn gives

$${A_1}^n {A_2}^m {A_3}^l \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} |a|^n b^m \cos n\theta \\ |a|^n b^m \sin n\theta \\ c^n d^l \end{pmatrix}.$$

The last and the choice of a, b, c, d imply that (A_1, A_2, A_3) is hypercyclic with $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ being a hypercyclic vector for (A_1, A_2, A_3). $\square$

4. Proof of Theorem 1.1

By Proposition 2.10, there exist 2×2 matrices B_1 and B_2 such that (B_1, B_2) is hypercyclic.

Case I. $n = 2k$ for some positive integer k. For $k = 1$ the result follows by Proposition 2.10. Assume that $k > 1$. Each A_j will be constructed by blocks of 2×2 matrices. Let I_2 be the 2×2 identity matrix. We will be using the notation $diag(D_1, D_2, \ldots, D_n)$ to denote the diagonal matrix with diagonal entries the block matrices $D_1, D_2, \ldots, D_n$. Define $A_1 = diag(B_1, I_2, \ldots, I_2)$, $A_2 = diag(B_2, I_2, \ldots, I_2)$, $A_3 = diag(I_2, B_1, I_2, \ldots, I_2)$, $A_4 = diag(I_2, B_2, I_2, \ldots, I_2)$ and so on up to $A_{n-1} = diag(I_2, \ldots, I_2, B_1)$, $A_n = diag(I_2, \ldots, I_2, B_2)$.

It is now easy to check that $(A_1, A_2, \ldots, A_n)$ is hypercyclic and furthermore that the set $HC((A_1, A_2, \ldots, A_n))$ is

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{2j-1}^2 + x_{2j}^2 \neq 0, \forall j = 1, 2, \ldots, k\}.$$

Case II. $n = 2k + 1$ for some positive integer k. If $k = 1$ the result follows by Proposition 3.3. Suppose $k > 1$. For simplicity we treat the case $k = 2$, since the general case follows by similar arguments. By Proposition 3.3 there exist C_1, C_2, C_3, 3×3 matrices such that (C_1, C_2, C_3) is hypercyclic. Let I_3 be the 3×3 identity matrix. Define $A_1 = diag(B_1, I_3)$, $A_2 = diag(B_2, I_3)$, $A_3 = diag(I_2, C_1)$, $A_4 = diag(I_2, C_2)$ and $A_5 = diag(I_2, C_3)$.

It can easily be shown that $(A_1, A_2, \ldots, A_5)$ is hypercyclic. The details are left to the reader.

Acknowledgement

The authors would like to thank the anonymous referee for helpful and constructive comments.

References

[1] F. Bayart, É. Matheron, *Topics in linear dynamics*, Cambridge Tracts in Math. (to appear).
[2] N. S. Feldman, *Hypercyclic Tuples of Operators*. Mini-workshop: Hypercyclicity and Linear Chaos. Abstracts from the mini-workshop held August 13-19, 2006. Organized by Teresa Bermúdez, Gilles Godefroy, Karl-G. Grosse-Erdmann and Alfredo Peris. Oberwolfach Reports, Vol. 3, no. 3. Oberwolfach Rep. **3** (2006), no. 3, 2227-2276. MR2332177
[3] N. S. Feldman, *Hypercyclic pairs of coanalytic Toeplitz operators*, Integral Equations Operator Theory **58** (2007), 153-173. MR2324885 (2008d:47057)
[4] N. S. Feldman, *Hypercyclic tuples of operators and somewhere dense orbits*, J. Math. Anal. Appl. **346** (2008), 82-98.
[5] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. **36** (1999), 345-381. MR1685272 (2000c:47001)
[6] K.-G. Grosse-Erdmann, *Recent developments in hypercyclicity*, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **97** (2003), 273-286. MR2068180 (2005c:47010)
[7] K.-G. Grosse-Erdmann, *Dynamics of linear operators*, Topics in complex analysis and operator theory, 41-84, Univ. Malaga, Malaga, 2007. MR2394656

[8] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, The Clarendon Press, Oxford University Press, New York, 1979. MR0568909 (81i:10002)

[9] L. Kérchy, *Cyclic properties and stability of commuting power bounded operators*, Acta Sci. Math. (Szeged) **71** (2005), 299-312. MR2160368 (2006e:47014)

Department of Mathematics, University of Crete, Knossos Avenue, GR-714 09 Heraklion, Crete, Greece

E-mail address: costakis@math.uoc.gr

Department of Computer Science and Engineering, European University Cyprus, 6 Diogenes Street, Engomi, P.O. Box 22006, 1516 Nicosia, Cyprus

E-mail address: d.hadjiloucas@euc.ac.cy

Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: amanouss@math.uni-bielefeld.de

J. Math. Anal. Appl. 365 (2010) 229–237

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the minimal number of matrices which form a locally hypercyclic, non-hypercyclic tuple

G. Costakis [a,*], D. Hadjiloucas [b], A. Manoussos [c,1]

[a] Department of Mathematics, University of Crete, Knossos Avenue, GR-714 09 Heraklion, Crete, Greece
[b] Department of Computer Science and Engineering, The School of Sciences, European University Cyprus, 6 Diogenes Street, Engomi, PO Box 22006, 1516 Nicosia, Cyprus
[c] Fakultät für Mathematik, SFB 701, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

ARTICLE INFO

Article history:
Received 21 April 2009
Available online 15 October 2009
Submitted by J.H. Shapiro

Keywords:
Hypercyclic operators
Locally hypercyclic operators
J-class operators
Tuples of matrices

ABSTRACT

In this paper we extend the notion of a locally hypercyclic operator to that of a locally hypercyclic tuple of operators. We then show that the class of hypercyclic tuples of operators forms a proper subclass to that of locally hypercyclic tuples of operators. What is rather remarkable is that in every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, a pair of commuting matrices exists which forms a locally hypercyclic, non-hypercyclic tuple. This comes in direct contrast to the case of hypercyclic tuples where the minimal number of matrices required for hypercyclicity is related to the dimension of the vector space. In this direction we prove that the minimal number of diagonal matrices required to form a hypercyclic tuple on $\mathbb{R}^n$ is $n+1$, thus complementing a recent result due to Feldman.

1. Introduction

Locally hypercyclic (or J-class) operators form a class of linear operators which possess certain dynamic properties. These were introduced and studied in [5]. The notion of a locally hypercyclic operator can be viewed as a "localization" of the notion of hypercyclic operator. For a comprehensive study and account of results on hypercyclic operators we refer to the book [1] by Bayart and Matheron.

Hypercyclic tuples of operators were introduced and studied by Feldman in [6–8], see also [12]. An n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \ldots, T_n$ acting on a locally convex topological vector space X. The tuple $(T_1, T_2, \ldots, T_n)$ is hypercyclic if there exists a vector $x \in X$ such that the set

$$\left\{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x\colon\ k_1, k_2, \ldots, k_n \in \mathbb{N} \cup \{0\}\right\}$$

is dense in X. The tuple $(T_1, T_2, \ldots, T_n)$ is topologically transitive if for every pair (U, V) of non-empty open sets in X there exist $k_1, k_2, \ldots, k_n \in \mathbb{N} \cup \{0\}$ such that $T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(U) \cap V \neq \emptyset$. If X is separable it is easy to show that $(T_1, T_2, \ldots, T_n)$ is topologically transitive if and only if $(T_1, T_2, \ldots, T_n)$ is hypercyclic. Following Feldman [8], we denote the semigroup generated by the tuple $T = (T_1, T_2, \ldots, T_n)$ by $\mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}\colon\ k_i \in \mathbb{N} \cup \{0\}\}$ and the orbit of x under the tuple T by $Orb(T, x) = \{Sx\colon\ S \in \mathcal{F}_T\}$. Furthermore, we denote by $HC((T_1, T_2, \ldots, T_n))$ the set of hypercyclic vectors for the tuple $(T_1, T_2, \ldots, T_n)$.

* Corresponding author.
E-mail addresses: costakis@math.uoc.gr (G. Costakis), d.hadjiloucas@euc.ac.cy (D. Hadjiloucas), amanouss@math.uni-bielefeld.de (A. Manoussos).

[1] During this research the author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.020

In this article we extend the notion of a locally hypercyclic operator (locally topologically transitive) to that of a locally hypercyclic tuple (locally topologically transitive tuple) of operators as follows. For $x \in X$ we define the extended limit set $J_{(T_1,T_2,\dots,T_n)}(x)$ to be the set of $y \in X$ for which there exist a sequence of vectors $\{x_m\}$ with $x_m \to x$ and sequences of non-negative integers $\{k_m^{(j)}\colon m \in \mathbb{N}\}$ for $j = 1, 2, \dots, n$ with

$$k_m^{(1)} + k_m^{(2)} + \cdots + k_m^{(n)} \to +\infty \tag{1.1}$$

such that

$$T_1^{k_m^{(1)}} T_2^{k_m^{(2)}} \dots T_n^{k_m^{(n)}} x_m \to y.$$

Note that condition (1.1) is equivalent to having at least one of the sequences $\{k_m^{(j)}\colon m \in \mathbb{N}\}$ for $j = 1, 2, \dots, n$ containing a strictly increasing subsequence tending to $+\infty$. This is in accordance with the well-known definition of J-sets in topological dynamics, see [9]. In Section 2 we provide an explanation as to why condition (1.1) is reasonable. The tuple $(T_1, T_2, \dots, T_n)$ is *locally topologically transitive* if there exists $x \in X \setminus \{0\}$ such that $J_{(T_1,T_2,\dots,T_n)}(x) = X$. Using simple arguments it is easy to show the following equivalence. $J_{(T_1,T_2,\dots,T_n)}(x) = X$ if and only if for every open neighborhood U_x of x and every non-empty open set V there exist $k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{0\}$ such that $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(U_x) \cap V \neq \emptyset$. In the case when X is separable and there exists $x \in X \setminus \{0\}$ such that $J_{(T_1,T_2,\dots,T_n)}(x) = X$, the tuple $(T_1, T_2, \dots, T_n)$ will be called *locally hypercyclic*.

In a finite dimensional space over $\mathbb{R}$ or $\mathbb{C}$, no linear operator can be hypercyclic (see [13]) or locally hypercyclic (see [5]). However, it was shown recently by Feldman in [8] that the situation for tuples of linear operators in finite dimensional spaces over $\mathbb{R}$ or $\mathbb{C}$ is quite different. There, it was shown that there exist hypercyclic $(n+1)$-tuples of diagonal matrices on $\mathbb{C}^n$ and that no n-tuple of diagonal matrices is hypercyclic. We complement this result by showing that the minimal number of diagonal matrices required to form a hypercyclic tuple in $\mathbb{R}^n$ is $n+1$. We also mention at this point that in [3] it is proved that non-diagonal hypercyclic n-tuples exist on $\mathbb{R}^n$, answering a question of Feldman.

In the present work we make a first attempt towards studying locally hypercyclic tuples of linear operators on finite dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. We show that if a tuple of linear operators is hypercyclic then it is locally hypercyclic (see Section 2). We then proceed to show that in the finite dimensional setting, the class of hypercyclic tuples of operators forms a proper subclass of the class of locally hypercyclic tuples of operators. What is rather surprising is the fact that the minimal number of matrices required to construct a locally hypercyclic tuple in any finite dimensional space over $\mathbb{R}$ or $\mathbb{C}$ is 2. This comes in direct contrast to the class of hypercyclic tuples where the minimal number of matrices required depends on the dimension of the vector space. Examples of diagonal pairs of matrices as well as pairs of upper triangular non-diagonal matrices and matrices in Jordan form which are locally hypercyclic but not hypercyclic are constructed. We mention that some of our constructions can be directly generalized to the infinite dimensional case, see Section 4.

2. Basic properties of locally hypercyclic tuples of operators

Let us first comment on the condition (1.1) in the definition of a locally hypercyclic tuple. This comes as an extension to the definition of a locally hypercyclic operator given in [5]. Recall that a hypercyclic operator $T : X \to X$ is locally hypercyclic and furthermore $J_T(x) = X$ for every $x \in X$. In the definition of a locally hypercyclic tuple, one may have been inclined to demand that $k_m^{(j)} \to +\infty$ for every $j = 1, 2, \dots, n$. However this would lead to a situation where the class of hypercyclic tuples would not form a subclass of the locally hypercyclic tuples. To clarify this issue, we give an example. Take any hypercyclic operator $T : X \to X$ and consider the tuple $(T, 0)$ where $0 : X \to X$ is the zero operator defined by $0(x) = 0$ for every $x \in X$. Obviously, this is a hypercyclic tuple ($Orb(0, x) = \{x, 0\}$). On the other hand, for every pair of sequences of integers $\{n_k\}$, $\{m_k\}$ with $n_k, m_k \to +\infty$ and for every sequence of vectors x_k tending to some vector x we have $T^{n_k} 0^{m_k} x_k \to 0$ and so $(T, 0)$ would not be a locally hypercyclic pair.

Let us now proceed by stating some basic facts which will be used in showing that the class of hypercyclic tuples is contained in the class of locally hypercyclic tuples.

Lemma 2.1. *If $x \in HC((T_1, T_2, \dots, T_n))$ then $J_{(T_1,T_2,\dots,T_n)}(x) = X$.*

Proof. Let $y \in X$, $\epsilon > 0$ and $m \in \mathbb{N}$. Since the set $Orb(T, x)$ is dense in X it follows that the set

$$\left\{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\colon k_1 + k_2 + \cdots + k_n > m\right\}$$

is dense in X (only a finite number of vectors is omitted from the orbit $Orb(T, x)$). Hence, there exist $(k_1, k_2, \dots, k_n) \in \mathbb{N}^n$ with $k_1 + k_2 + \cdots + k_n > m$ such that

$$\left\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x - y\right\| < \epsilon. \qquad \square$$

The proof of the following lemma is an immediate variation of the proof of Lemma 2.5 in [4].

Lemma 2.2. *If $\{x_m\}$, $\{y_m\}$ are two sequences in X such that $x_m \to x$ and $y_n \to y$ for some $x, y \in X$ and $y_m \in J_{(T_1,T_2,\dots,T_n)}(x_m)$ for every $m \in \mathbb{N}$ then $y \in J_{(T_1,T_2,\dots,T_n)}(x)$.*

Lemma 2.3. *For all $x \in X$ the set $J_{(T_1,T_2,\dots,T_n)}(x)$ is closed and T_j invariant for every $j = 1, 2, \dots, n$.*

Proof. This is an easy consequence of Lemma 2.2. $\square$

Proposition 2.4. *$(T_1, T_2, \dots, T_n)$ is hypercyclic if and only if it is locally hypercyclic and $J_{(T_1,T_2,\dots,T_n)}(x) = X$ for every $x \in X$.*

Proof. Assume first that $(T_1, T_2, \dots, T_n)$ is hypercyclic. By Lemma 2.1 it follows that $(T_1, T_2, \dots, T_n)$ is locally hypercyclic. Denote by A the set of vectors $\{x \in X\colon\ J_{(T_1,T_2,\dots,T_n)}(x) = X\}$. By Lemma 2.1 we have $HC((T_1, T_2, \dots, T_n)) \subset A$. Since $HC((T_1, T_2, \dots, T_n))$ is dense (see [8]) and A is closed by Lemma 2.2, it is plain that $A = X$. For the converse implication let us consider $x \in X$. Since $J_{(T_1,T_2,\dots,T_n)}(x) = X$ then for every open neighborhood U_x of x and every non-empty open set V there exist $k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{0\}$ such that $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(U_x) \cap V \neq \emptyset$. Therefore $(T_1, T_2, \dots, T_n)$ is topologically transitive and since X is separable it follows that $(T_1, T_2, \dots, T_n)$ is hypercyclic. $\square$

3. Locally hypercyclic pairs of diagonal matrices which are not hypercyclic

In [8], Feldman showed that there exist $(n+1)$-tuples of diagonal matrices on $\mathbb{C}^n$ and that there are no hypercyclic n-tuples of diagonalizable matrices on $\mathbb{C}^n$. In the same paper, Feldman went a step further to show that no n-tuple of diagonal matrices on $\mathbb{R}^n$ is hypercyclic while, on the other hand, there exists an $(n+1)$-tuple of diagonal matrices on $\mathbb{R}^n$ that has a dense orbit in $(\mathbb{R}^+)^n$. We complement the last result by showing that there is an $(n+1)$-tuple of diagonal matrices on $\mathbb{R}^n$ which is hypercyclic. Throughout the rest of the paper for a vector u in $\mathbb{R}^n$ or $\mathbb{C}^n$ we will be denoting by u^t the transpose of u.

Theorem 3.1. *For every $n \in \mathbb{N}$ there exists an $(n+1)$-tuple of diagonal matrices on $\mathbb{R}^n$ which is hypercyclic.*

Proof. Choose negative real numbers $a_1, a_2, \dots, a_n$ such that the numbers

$$1, a_1, a_2, \dots, a_n$$

are linearly independent over $\mathbb{Q}$. By Kronecker's theorem (see Theorem 442 in [10]) the set

$$\left\{(ka_1 + s_1, ka_2 + s_2, \dots, ka_n + s_n)^t\colon\ k, s_1, \dots, s_n \in \mathbb{N} \cup \{0\}\right\}$$

is dense in $\mathbb{R}^n$. The continuity of the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x_1, x_2, \dots, x_n) = (e^{x_1}, e^{x_2}, \dots, e^{x_n})$ implies that the set

$$\left\{\left(\left(e^{a_1}\right)^k e^{s_1}, \left(e^{a_2}\right)^k e^{s_2}, \dots, \left(e^{a_n}\right)^k e^{s_n}\right)^t\colon\ k, s_1, \dots, s_n \in \mathbb{N} \cup \{0\}\right\}$$

is dense in $(\mathbb{R}^+)^n$. An easy argument (see for example the proof of Lemma 2.6 in [3]) shows that the set

$$\left\{\begin{pmatrix} (e^{a_1})^k(-\sqrt{e}\,)^{s_1} \\ (e^{a_2})^k(-\sqrt{e}\,)^{s_2} \\ \vdots \\ (e^{a_n})^k(-\sqrt{e}\,)^{s_n} \end{pmatrix}\colon\ k, s_1, \dots, s_n \in \mathbb{N} \cup \{0\}\right\}$$

is dense in $\mathbb{R}^n$. Let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} e^{a_1} & & & \\ & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\sqrt{e} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \dots, \quad B_n = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -\sqrt{e} \end{pmatrix}.$$

Then the set

$$\left\{A^k B_1^{s_1} \dots B_n^{s_n} \mathbf{1}\colon\ k, s_1, \dots, s_n \in \mathbb{N} \cup \{0\}\right\}$$

is dense in $\mathbb{R}^n$, which implies that the $(n+1)$-tuple $(A, B_1, \dots, B_n)$ of diagonal matrices is hypercyclic. $\square$

All of the results mentioned at the beginning of this section as well as the one proved above show that the length of a hypercyclic tuple of diagonal matrices depends on the dimension of the space. It comes as a surprise that this is not the case for locally hypercyclic tuples of diagonal matrices. In fact, we show that on a vector space of any finite dimension $n \geqslant 2$ one may construct a pair of diagonal matrices which is locally hypercyclic.

Theorem 3.2. *Let $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $b > 1$ and $\frac{\ln|a|}{\ln b}$ is irrational. Let n be a positive integer with $n \geqslant 2$ and consider the $n \times n$ matrices*

$$A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{pmatrix}$$

where $a_1 = a$, $b_1 = b$, a_j, b_j are real numbers with $|a_j| > 1$ and $|b_j| > 1$ for $j = 2, \dots, n$. Then (A, B) is a locally hypercyclic pair on $\mathbb{R}^n$ which is not hypercyclic. In particular, we have

$$\{x \in \mathbb{R}^n\colon\ J_{(A,B)}(x) = \mathbb{R}^n\} = \{(x_1, 0, \dots, 0)^t \in \mathbb{R}^n\colon\ x_1 \in \mathbb{R}\}.$$

Proof. Note that

$$A^k B^l = \begin{pmatrix} a^k b^l & 0 & 0 & \dots & 0 \\ 0 & a_2^k b_2^l & 0 & \dots & 0 \\ 0 & 0 & a_3^k b_3^l & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_n^k b_n^l \end{pmatrix}.$$

Let $x = (1, 0, 0, \dots, 0)^t \in \mathbb{R}^n$. We will show that $J_{(A,B)}(x) = \mathbb{R}^n$. Fix a vector $y = (y_1, \dots, y_n)^t$. By [3, Lemma 2.6], the sequence $\{a^k b^l\colon\ k, l \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence there exist sequences of positive integers $\{k_i\}$ and $\{l_i\}$ with $k_i, l_i \to +\infty$ such that $a^{k_i} b^{l_i} \to y_1$. Let

$$x_i = \left(1, \frac{y_2}{a_2^{k_i} b_2^{l_i}}, \dots, \frac{y_n}{a_n^{k_i} b_n^{l_i}}\right)^t.$$

Obviously $x_i \to x$ and

$$A^{k_i} B^{l_i} x_i = \left(a^{k_i} b^{l_i}, y_2, \dots, y_n\right)^t \to y.$$

In [8, Theorems 3.4 and 3.6] Feldman showed that there exists a hypercyclic $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$, for every $n \in \mathbb{N}$ but there is no hypercyclic n-tuple of diagonal matrices on $\mathbb{C}^n$ or on $\mathbb{R}^n$. Feldman actually showed that there is no n-tuple of diagonal matrices on $\mathbb{C}^n$ or $\mathbb{R}^n$ that has a somewhere dense orbit [8, Theorem 4.4]. So the pair (A, B) is not hypercyclic. To finish, note that for every $\lambda \in \mathbb{R} \setminus \{0\}$ it holds that $J_{(A,B)}(\lambda x) = \lambda J_{(A,B)}(x) = \mathbb{R}^n$. In view of Lemma 2.2 it follows that $J_{(A,B)}(0) = \mathbb{R}^n$. On the other hand, by the choice of a_j, b_j for $j = 2, \dots, n$ it is clear that for any vector $u = (u_1, u_2, \dots, u_n)^t$ with $u_j \neq 0$ for some $j \in \{2, 3, \dots, n\}$ we have $J_{(A,B)}(u) \neq \mathbb{R}^n$. This completes the proof of the theorem. □

A direct analogue to the previous theorem also holds in the complex setting. We will make use of the following result in [8] due to Feldman.

Proposition 3.3.

(i) *If $b \in \mathbb{C} \setminus \{0\}$ with $|b| < 1$ then there is a dense set $\Delta_b \subset \{z \in \mathbb{C}\colon\ |z| > 1\}$ such that for any $a \in \Delta_b$, we have that $\{a^k b^l\colon\ k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$.*

(ii) *If $a \in \mathbb{C}$ with $|a| > 1$, then there is a dense set $\Delta_a \subset \{z \in \mathbb{C}\colon\ |z| < 1\}$ such that for any $b \in \Delta_a$, we have that $\{a^k b^l\colon\ k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$.*

Theorem 3.4. *Let $a, b \in \mathbb{C}$ such that $\{a^k b^l\colon\ k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Let n be a positive integer with $n \geqslant 2$ and consider the diagonal matrices A and B as in Theorem 3.2 where $a_1 = a$, $b_1 = b$, $a_j, b_j \in \mathbb{C}$ with $|a_j| > 1$ and $|b_j| > 1$ for $j = 2, \dots, n$. Then (A, B) is a locally hypercyclic pair on $\mathbb{C}^n$ which is not hypercyclic. In particular, we have*

$$\{z \in \mathbb{C}^n\colon\ J_{(A,B)}(z) = \mathbb{C}^n\} = \{(z_1, 0, \dots, 0)^t \in \mathbb{C}^n\colon\ z_1 \in \mathbb{C}\}.$$

Proof. The proof follows along the same lines as that of Theorem 3.2. □

4. Locally hypercyclic pairs of diagonal operators which are not hypercyclic in infinite dimensional spaces

In this section we slightly modify the construction in Theorem 3.2 in order to obtain similar results in infinite dimensional spaces. As usual the symbol $l^p(\mathbb{N})$ stands for the Banach space of p-summable sequences, where $1 \leqslant p < \infty$ and by $l^\infty(\mathbb{N})$ we denote the Banach space of bounded sequences (either over $\mathbb{R}$ or $\mathbb{C}$).

Theorem 4.1. *Let $a, b \in \mathbb{C}$ such that $\{a^k b^l\colon k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$ and let $c \in \mathbb{C}$ with $|c| > 1$. Consider the diagonal operators $T_j : l^p(\mathbb{N}) \to l^p(\mathbb{N})$, $1 \leqslant p \leqslant \infty$, $j = 1, 2$, defined by*

$$T_1(x_1, x_2, x_3, \ldots) = (ax_1, cx_2, cx_3, \ldots),$$

$$T_2(x_1, x_2, x_3, \ldots) = (bx_1, cx_2, cx_3, \ldots),$$

for $x = (x_1, x_2, x_3, \ldots) \in l^p(\mathbb{N})$, $1 \leqslant p \leqslant \infty$. Then (T_1, T_2) is a locally hypercyclic, non-hypercyclic pair in $l^p(\mathbb{N})$ for every $1 \leqslant p < \infty$ and (T_1, T_2) is a locally topologically transitive, non-topologically transitive pair in $l^\infty(\mathbb{N})$. In particular we have

$$\left\{x \in l^p(\mathbb{N})\colon\ J_{(T_1,T_2)}(x) = l^p(\mathbb{N})\right\} = \left\{(x_1, 0, 0, \ldots)\colon\ x_1 \in \mathbb{C}\right\}$$

for every $1 \leqslant p \leqslant \infty$.

Proof. Fix $1 \leqslant p \leqslant \infty$ and consider a vector $y = (y_1, y_2, \ldots) \in l^p(\mathbb{N})$. There exist sequences of positive integers $\{k_i\}$ and $\{l_i\}$ with $k_i, l_i \to +\infty$ such that $a^{k_i} b^{l_i} \to y_1$. Let

$$x_i = \left(1, \frac{y_2}{c^{k_i+l_i}}, \frac{y_3}{c^{k_i+l_i}}, \ldots\right).$$

Obviously $x_i \to x = (1, 0, 0, \ldots)$ and

$$T_1^{k_i} T_2^{l_i} x_i = \left(a^{k_i} b^{l_i}, y_2, y_3, \ldots\right) \to y.$$

Therefore $J_{(T_1,T_2)}(x) = l^p(\mathbb{N})$. For $p = 2$ the pair (T_1, T_2) is not hypercyclic by Feldman's result which says that there are no hypercyclic tuples of normal operators in infinite dimensions, see [8]. However, one can show directly that for every $1 \leqslant p < \infty$ the pair (T_1, T_2) is not hypercyclic and (T_1, T_2) is not topologically transitive in $l^\infty(\mathbb{N})$. Indeed, suppose that $x = (x_1, x_2, \ldots) \in l^p(\mathbb{N})$ is hypercyclic for the pair (T_1, T_2), where $1 \leqslant p < \infty$. Then necessarily $x_2 \neq 0$ and the sequence $\{c^n\}$ should be dense in $\mathbb{C}$ which is a contradiction. For the case $p = \infty$, assuming that the pair (T_1, T_2) is topologically transitive we conclude that the pair (A, B) is topologically transitive in $\mathbb{C}^2$, where $A(x_1, x_2) = (ax_1, cx_2)$, $B(x_1, x_2) = (bx_1, cx_2)$, $(x_1, x_2) \in \mathbb{C}^2$. The latter implies that (A, B) is hypercyclic. Since no pair of diagonal matrices is hypercyclic in $\mathbb{C}^2$, see [8], we arrive at a contradiction. It is also easy to check that $\{x \in l^p(\mathbb{N})\colon\ J_{(T_1,T_2)}(x) = l^p(\mathbb{N})\} = \{(x_1, 0, 0, \ldots)\colon\ x_1 \in \mathbb{C}\}$ for every $1 \leqslant p \leqslant \infty$. □

Remark 4.2. Theorem 4.1 is valid for the $l^p(\mathbb{N})$ spaces over the reals as well. Concerning the non-separable Banach space $l^\infty(\mathbb{N})$ we stress that this space does not support topologically transitive operators, see [2]. On the other hand there exist operators acting on $l^\infty(\mathbb{N})$ which are locally topologically transitive, see [5].

5. Locally hypercyclic pairs of upper triangular non-diagonal matrices which are not hypercyclic

We first show that it is possible for numbers $a_1, a_2 \in \mathbb{R}$ to exist with the property that the set

$$\left\{\frac{a_1^k a_2^l}{\frac{k}{a_1} + \frac{l}{a_2}}\colon k, l \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}$ and at the same time the sequences on both the numerator and denominator stay unbounded. For our purposes we will show that the set above with $a_2 = -1$ and $a_1 = a$ is dense in $\mathbb{R}$ for any $a \in \mathbb{R}$ with $a > 1$. Actually we shall prove that the set

$$\left\{\frac{\frac{k}{a} - l}{a^k(-1)^l}\colon k, l \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}$ for any $a \in \mathbb{R}$ with $a > 1$. From this it should be obvious that the result above follows since the image of a dense set in $\mathbb{R} \setminus \{0\}$ under the map $f(x) = 1/x$ is also dense in $\mathbb{R}$.

Lemma 5.1. *The set*

$$\left\{\frac{\frac{k}{a} - l}{a^k(-1)^l}\colon k, l \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}$ for any $a > 1$.

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be given. We want to find $k, l \in \mathbb{N}$ such that

$$\left| \frac{\frac{k}{a} - l}{a^k (-1)^l} - x \right| < \epsilon.$$

There are two cases to consider, namely the cases $x > 0$ and $x < 0$, and we consider them separately (the case $x = 0$ is trivial since keeping l fixed we can find k big enough which does the job).

Case I ($x > 0$): There exists $k \in \mathbb{N}$ such that $1/a^k < \epsilon/2$. We will show that there exists a positive odd integer $l = 2s - 1$ for some $s \in \mathbb{N}$ for which

$$\left| \frac{\frac{k}{a} - l}{a^k (-1)^l} - x \right| = \left| \frac{2s}{a^k} - \frac{1}{a^k} - \frac{k}{a^{k+1}} - x \right| < \epsilon.$$

But note that this is true since consecutive terms in the sequence $\{2s/a^k\colon s \in \mathbb{N}\}$ are at distance $2/a^k < \epsilon$ units apart and so, for some $s \in \mathbb{N}$ it holds that $\frac{2s}{a^k} - \frac{1}{a^k} - \frac{k}{a^{k+1}} \in (x - \epsilon, x + \epsilon)$.

Case II ($x < 0$): There exists $k \in \mathbb{N}$ such that $1/a^k < \epsilon/2$. We will show that there exists a positive even integer $l = 2s$ for some $s \in \mathbb{N}$ for which

$$\left| \frac{\frac{k}{a} - l}{a^k (-1)^l} - x \right| = \left| \frac{k}{a^{k+1}} - \frac{2s}{a^k} - x \right| < \epsilon.$$

But note that this is true since consecutive terms in the sequence $\{2s/a^k\colon s \in \mathbb{N}\}$ are at distance $2/a^k < \epsilon$ units apart and so, for some $s \in \mathbb{N}$ it holds that $\frac{k}{a^{k+1}} - \frac{2s}{a^k} \in (x - \epsilon, x + \epsilon)$. $\square$

Lemma 5.2. *Let $x \in \mathbb{R} \setminus \{0\}$, $a > 1$ and consider sequences $\{k_i\}$, $\{l_i\}$ of natural numbers with $k_i, l_i \to +\infty$ such that*

$$\frac{\frac{k_i}{a} - l_i}{a^{k_i} (-1)^{l_i}} \to x.$$

Then both the numerator and denominator stay unbounded.

Proof. This is trivial since the denominator grows unbounded and so it forces the numerator to keep up. $\square$

Remark 5.3. The case where $x = 0$ is the only one for which one has the freedom of having the denominator grow unbounded and keep the numerator bounded. However, if one requires both numerator and denominator to stay unbounded then the numerator can also be made to grow unbounded (growing at a slower rate than the denominator).

Let us now proceed with the construction of a locally hypercyclic pair of upper triangular non-diagonal matrices on $\mathbb{R}^n$ which is not hypercyclic.

Theorem 5.4. *Let n be a positive integer with $n \geqslant 2$ and consider the $n \times n$ matrices*

$$A_j = \begin{pmatrix} a_j & 0 & 0 & \dots & 1 \\ 0 & a_j & 0 & \dots & 0 \\ 0 & 0 & a_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_j \end{pmatrix}$$

for $j = 1, 2$ where $a_1 > 1$ and $a_2 = -1$. Then (A_1, A_2) is a locally hypercyclic pair on $\mathbb{R}^n$ which is not hypercyclic. In particular, we have

$$\{x \in \mathbb{R}^n\colon J_{(A_1, A_2)}(x) = \mathbb{R}^n\} = \{(x_1, 0, \dots, 0)^t \in \mathbb{R}^n\colon x_1 \in \mathbb{R}\}.$$

Proof. It easily follows that

$$A_1^k A_2^l = \begin{pmatrix} a_1^k a_2^l & 0 & 0 & \dots & a_1^k a_2^l (\frac{k}{a_1} + \frac{l}{a_2}) \\ 0 & a_1^k a_2^l & 0 & \dots & 0 \\ 0 & 0 & a_1^k a_2^l & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_1^k a_2^l \end{pmatrix}.$$

Let $x \neq 0$. We want to find a sequence $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^t$, $i \in \mathbb{N}$ which converges to the vector $(x, 0, \ldots, 0)^t$ and such that for any vector $w = (w_1, w_2, \ldots, w_n)^t$ there exist strictly increasing sequences $\{k_i\}$, $\{l_i\}$ of positive integers for which $A_1^{k_i} A_2^{l_i} x_i \to w$. Without loss of generality we may assume that $w_n \neq 0$. This is equivalent to having

$$a_1^{k_i} a_2^{l_i} x_{i1} + a_1^{k_i} a_2^{l_i} \left(\frac{k_i}{a_1} + \frac{l_i}{a_2}\right) x_{in} \to w_1$$

and

$$a_1^{k_i} a_2^{l_i} x_{ij} \to w_j$$

for $j = 2, \ldots, n$. By Lemma 5.1 there exist sequences $\{k_i\}$ and $\{l_i\}$ of positive integers such that $k_i, l_i \to +\infty$ and

$$\frac{a_1^{k_i} a_2^{l_i}}{\frac{k_i}{a_1} + \frac{l_i}{a_2}} \to -\frac{w_n}{x}.$$

We set

$$x_{i1} = x - \frac{w_1 x}{w_n(\frac{k_i}{a_1} + \frac{l_i}{a_2})}, \qquad x_{ij} = -\frac{w_j x}{w_n(\frac{k_i}{a_1} + \frac{l_i}{a_2})}$$

for $j = 2, \ldots, n-1$, and

$$x_{in} = -\frac{x}{\frac{k_i}{a_1} + \frac{l_i}{a_2}}.$$

Note that, because of Lemma 5.2, $x_{i1} \to x$ and $x_{ij} \to 0$ for $j = 2, \ldots, n$. Substituting into the equations above we find

$$a_1^{k_i} a_2^{l_i} x_{i1} + a_1^{k_i} a_2^{l_i} \left(\frac{k_i}{a_1} + \frac{l_i}{a_2}\right) x_{in} = a_1^{k_i} a_2^{l_i} \left(-\frac{w_1 x}{w_n(\frac{k_i}{a_1} + \frac{l_i}{a_2})}\right) \to w_1$$

and

$$a_1^{k_i} a_2^{l_i} x_{ij} = a_1^{k_i} a_2^{l_i} \left(-\frac{w_j x}{w_n(\frac{k_i}{a_1} + \frac{l_i}{a_2})}\right) \to w_j$$

for $j = 2, \ldots, n-1$ as well as

$$a_1^{k_i} a_2^{l_i} x_{in} = a_1^{k_i} a_2^{l_i} \left(-\frac{x}{\frac{k_i}{a_1} + \frac{l_i}{a_2}}\right) \to w_n.$$

The pair (A_1, A_2) is not hypercyclic. The reason is that if it is hypercyclic then there is a vector $y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$ such that the set $\{A_1^k A_2^l y\colon\ k, l \in \mathbb{N} \cup \{0\}\}$ is dense in $\mathbb{R}^n$. Hence the set of vectors

$$\left\{ \begin{pmatrix} a_1^k a_2^l y_1 + a_1^k a_2^l (\frac{k}{a_1} + \frac{l}{a_2}) y_n \\ a_1^k a_2^l y_2 \\ \vdots \\ a_1^k a_2^l y_n \end{pmatrix} : k, l \in \mathbb{N} \cup \{0\} \right\}$$

is dense in $\mathbb{R}^n$. If $y_n = 0$ then it is clear that the last coordinate cannot approximate anything but 0. If $y_n \neq 0$ then, since $a_1 > 1$ and $a_2 = -1$ the sequence $\{|a_1^k a_2^l y_n|\colon\ k, l \in \mathbb{N} \cup \{0\}\} = \{|a_1|^k |y_n|\colon\ k \in \mathbb{N} \cup \{0\}\}$ is geometric and so cannot be dense in $\mathbb{R}^+$. It is left to the reader to check that

$$\{x \in \mathbb{R}^n\colon\ J_{(A_1, A_2)}(x) = \mathbb{R}^n\} = \{(x_1, 0, \ldots, 0)^t \in \mathbb{R}^n\colon\ x_1 \in \mathbb{R}\}. \qquad \square$$

In what follows we establish an analogue of Theorem 5.4 in the complex setting.

Lemma 5.5. *Let a, θ be real numbers such that $a > 1$ and θ an irrational multiple of π. Then the set*

$$\left\{ \frac{\frac{k}{ae^{i\theta}} - l}{a^k e^{ik\theta} (-1)^l} : k, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}$.

Proof. Let $w = |w|e^{i\phi} \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$. By the denseness of the irrational rotation on the unit circle and by the choice of a, there exists a positive integer k such that

$$\left|e^{-ik\theta} - e^{i\phi}\right| < \frac{\epsilon}{4|w|} \quad \text{and} \quad \frac{k}{a^{k-1}} < \frac{\epsilon}{4}.$$

By the proof of Lemma 5.1 there exists a non-negative odd integer $l = 2s - 1$ for some $s \in \mathbb{N}$ such that

$$\left|\frac{-l}{a^k(-1)^l} - |w|\right| = \left|\frac{2s}{a^k} - \frac{1}{a^k} - |w|\right| < \frac{\epsilon}{2}.$$

Using the above estimates it follows that

$$\begin{aligned}\left|\frac{\frac{k}{ae^{i\theta}} - l}{a^k e^{ik\theta}(-1)^l} - |w|e^{i\phi}\right| &\leqslant \left|\frac{\frac{k}{ae^{i\theta}}}{a^k e^{ik\theta}(-1)^l}\right| + \left|\frac{-l}{a^k e^{ik\theta}(-1)^l} - |w|e^{i\phi}\right| \\ &\leqslant \frac{k}{a^{k-1}} + \left|\frac{-l}{a^k(-1)^l} - |w|\right| + |w|\left|e^{-ik\theta} - e^{i\phi}\right| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon. \qquad \square\end{aligned}$$

We now construct a pair of upper triangular non-diagonal matrices which is locally hypercyclic on $\mathbb{C}^n$ and not hypercyclic.

Theorem 5.6. *Let n be a positive integer with $n \geqslant 2$ and consider the $n \times n$ matrices*

$$A_j = \begin{pmatrix} a_j & 0 & 0 & \dots & 1 \\ 0 & a_j & 0 & \dots & 0 \\ 0 & 0 & a_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_j \end{pmatrix}$$

for $j = 1, 2$ where $a_1 = ae^{i\theta}$ for $a > 1$, θ an irrational multiple of π and $a_2 = -1$. Then (A_1, A_2) is a locally hypercyclic pair on $\mathbb{C}^n$ which is not hypercyclic. In particular, we have

$$\left\{z \in \mathbb{C}^n \colon J_{(A_1,A_2)}(z) = \mathbb{C}^n\right\} = \left\{(z_1, 0, \dots, 0)^t \in \mathbb{C}^n \colon z_1 \in \mathbb{C}\right\}.$$

Proof. The proof follows along the same lines as the proof of Theorem 5.4 where use is made of Lemma 5.5 instead of Lemma 5.1. $\square$

Remark 5.7. Note that for $n = 2$ the upper triangular matrices we obtain in Theorems 5.4 and 5.6 are in Jordan form. This gives an example of a locally hypercyclic pair of matrices in Jordan form which is not hypercyclic.

6. Concluding remarks and questions

We stress that all the tuples considered in this work consist of commuting matrices/operators. Recently, in [11] Javaheri deals with the non-commutative case. In particular, he shows that for every positive integer $n \geqslant 2$ there exist non-commuting linear maps $A, B : \mathbb{R}^n \to \mathbb{R}^n$ so that for every vector $x = (x_1, x_2, \dots, x_n)$ with $x_1 \neq 0$ the set

$$\left\{B^{k_1} A^{l_1} \dots B^{k_n} A^{l_n} x \colon k_j, l_j \in \mathbb{N} \cup \{0\},\ 1 \leqslant j \leqslant n\right\}$$

is dense in $\mathbb{R}^n$. In other words the $2n$-tuple $(B, A, \dots, B, A)$ is hypercyclic.

The following open question was kindly posed by the referee.

Question. Suppose $(T_1, T_2, \dots, T_m)$ is a locally hypercyclic tuple of (commuting) matrices such that $J_{(T_1,T_2,\dots,T_m)}(x) = \mathbb{R}^n$ for a finite set of vectors x in $\mathbb{R}^n$ whose linear span is equal to $\mathbb{R}^n$. Is it true that the tuple $(T_1, T_2, \dots, T_m)$ is hypercyclic? Similarly for $\mathbb{C}^n$.

Acknowledgment

We would like to thank the referee for an extremely careful reading of the manuscript. Her/his remarks and comments helped us to improve considerably the presentation of the paper. Finally we mention that Theorem 4.1 is due to the referee.

References

[1] F. Bayart, É. Matheron, Dynamics of Linear Operators, Cambridge Tracts in Math., vol. 179, Cambridge University Press, 2009.
[2] T. Bermúdez, N.J. Kalton, The range of operators on von Neumann algebras, Proc. Amer. Math. Soc. 130 (2002) 1447–1455.
[3] G. Costakis, D. Hadjiloucas, A. Manoussos, Dynamics of tuples of matrices, Proc. Amer. Math. Soc. 137 (2009) 1025–1034.
[4] G. Costakis, A. Manoussos, J-class weighted shifts on the space of bounded sequences of complex numbers, Integral Equations Operator Theory 62 (2008) 149–158.
[5] G. Costakis, A. Manoussos, J-class operators and hypercyclicity, preprint, arXiv:0704.3354.
[6] N.S. Feldman, Hypercyclic tuples of operators, Oberwolfach Rep. 3 (3) (2006) 2254–2256. Mini-workshop: Hypercyclicity and Linear Chaos. Abstracts from the mini-workshop held August 13–19, 2006. Organized by Teresa Bermudez, Gilles Godefroy, Karl-G. Grosse-Erdmann and Alfredo Peris. Oberwolfach Rep. 3 (3) (2006) 2227–2276.
[7] N.S. Feldman, Hypercyclic pairs of coanalytic Toeplitz operators, Integral Equations Operator Theory 58 (2007) 153–173.
[8] N.S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, J. Math. Anal. Appl. 346 (2008) 82–98.
[9] O. Hájek, Prolongations in topological dynamics, in: Seminar on Differential Equations and Dynamical Systems, II, Univ. Maryland, College Park, MD, 1969, in: Lecture Notes in Math., vol. 144, Springer, Berlin, 1970, pp. 79–89.
[10] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford Sci. Publ., Oxford University Press, 1979.
[11] M. Javaheri, Semigroups of matrices with dense orbits, preprint, arXiv:0905.1311.
[12] L. Kérchy, Cyclic properties and stability of commuting power bounded operators, Acta Sci. Math. (Szeged) 71 (2005) 299–312.
[13] C. Kitai, Invariant closed sets for linear operators, Dissertation, University of Toronto, 1982.

Papers on Dynamical Systems

Journal of Functional Analysis **187**, 129–145 (2001)
doi:10.1006/jfan.2001.3819, available online at http://www.idealibrary.com on IDEAL®

The Jacobson Radical for Analytic Crossed Products

Allan P. Donsig

Mathematics and Statistics Department, University of Nebraska—Lincoln, Lincoln, Nebraska 68588
E-mail: adonsig@math.unl.edu

Aristides Katavolos

Department of Mathematics, University of Athens, Panepistimioupolis, GR-157 84 Athens, Greece
E-mail: akatavol@eudoxos.math.uoa.gr

and

Antonios Manoussos

123, Sapfous Street, 176 75 Kallithea, Athens, Greece
E-mail: amanou@cc.uoa.gr

Communicated by D. Sarason

Received November 8, 2000; accepted April 20, 2001

We characterise the Jacobson radical of an analytic crossed product $C_0(X) \times_\phi \mathbb{Z}_+$, answering a question first raised by Arveson and Josephson in 1969. In fact, we characterise the Jacobson radical of analytic crossed products $C_0(X) \times_\phi \mathbb{Z}_+^d$. This consists of all elements whose "Fourier coefficients" vanish on the recurrent points of the dynamical system (and the first one is zero). The multidimensional version requires a variation of the notion of recurrence, taking into account the various degrees of freedom.

Key Words: semicrossed products; analytic crossed products; Jacobson radical; recurrence; wandering sets.

There is a rich interplay between operator algebras and dynamical systems, going back to the founding work of Murray and von Neumann in the 1930's. Crossed product constructions continue to provide fundamental examples of von Neumann algebras and C*-algebras. Comparatively recently, Arveson [1] in 1967 introduced a nonselfadjoint crossed product construction, called the analytic crossed product or the semicrossed product, which has the remarkable property of capturing all of the information about the dynamical system.

0022-1236/01 $35.00

The construction starts with a dynamical system (X, ϕ), i.e., a locally compact Hausdorff space X and a continuous, proper surjection $\phi: X \to X$. Regarding the elements of $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $\sum_{n \geqslant 0} U^n f_n$, define a multiplication by requiring $fU = U(f \circ \phi)$. The analytic crossed product, $C_0(X) \times_\phi \mathbb{Z}_+$, is a suitable completion of $\ell^1(\mathbb{Z}_+, C_0(X))$; we give a detailed discussion below. Then the property mentioned above is that, subject to a mild condition on periodic points, two analytic crossed product algebras are isomorphic as complex algebras if, and only if, the underlying dynamical systems are topologically conjugate; i.e., there is a homeomorphism between the spaces that intertwines the two actions. In this generality, the result is due to Hadwin and Hoover [9, 10]—see also [20], which gives an elegant direct proof of this if the maps ϕ are homeomorphisms and extends the result to analytic crossed products by finitely many distinct commuting homeomorphisms on X, i.e., by $\mathbb{Z}_+^d$.

Arveson's original work [1] was for weakly-closed operator algebras and Arveson and Josephson in [2] gave an extension to norm closed operator algebras, including a structure theorem for bounded isomorphisms between two such algebras. Motivated by this, they asked if the analytic crossed product algebras were always semisimple (which would imply that all isomorphisms are bounded), noting that the evidence suggested a negative answer. This question stimulated considerable work on the ideal structure of analytic crossed products.

Another stimulus is the close connections between the ideal structure of C*-crossed products and dynamical systems, such as the characterisation of primitive ideals of C*-crossed products in terms of orbit closures by Effros and Hahn [5]. In this connection, we should mention Lamoureux's development of a generalisation of the primitive ideal space for various nonselfadjoint operator algebras, including analytic crossed products [12, 13].

We state our main result for the case $d = 1$. Recall that a point $x \in X$ is *recurrent* for the dynamical system (X, ϕ) if for every neighbourhood V of x, there is $n \geqslant 1$ so that $\phi^n(x) \in V$. If X is a metric space, then this is equivalent to having a sequence (n_k) tending to infinity so that $\phi^{n_k}(x)$ converges to x. Let X_r denote the recurrent points of (X, ϕ). Denoting elements of the analytic crossed product by formal series $\sum_{n \geqslant 0} U^n f_n$ we prove:

THEOREM 1. *If X is a locally compact metrisable space, then*

$$\operatorname{Rad}(C_0(X) \times_\phi \mathbb{Z}_+) = \left\{ \sum_{n \geqslant 1} U^n f_n \in C_0(X) \times_\phi \mathbb{Z}_+ : f_n|_{X_r} = 0 \textit{ for all } n \right\}.$$

Important progress towards a characterisation has been made by a number of authors. In [16], Muhly gave two sufficient conditions, one for an analytic crossed product to be semisimple and another for the Jacobson radical to be nonzero. The sufficient condition for a nonzero Jacobson radical is that the dynamical system (X, ϕ) possess a *wandering set*, i.e., an open set $V \subset X$ so that $V, \phi^{-1}(V), \phi^{-2}(V), \ldots$ are pairwise disjoint. If there are no wandering open sets, then the recurrent points are dense, so it turns out that this sufficient condition is also necessary.

Peters in [18, 19] characterised the strong radical (namely, the intersection of the maximal (modular, two-sided) ideals) and the closure of the prime radical and described much of the ideal structure for analytic crossed products arising from *free* actions of $\mathbb{Z}^+$. He also gave a sufficient condition for semisimplicity and showed that this condition is necessary and sufficient for semisimplicity of the norm dense subalgebra of polynomials in the analytic crossed product.

Most recently, Mastrangelo *et al.* [15], using powerful coordinate methods and the crucial idea from [4], characterised the Jacobson radical for analytic subalgebras of groupoid C*-algebras. For those analytic crossed products that can be coordinatised in this way (those with a free action), their characterisation is the same as ours. The asymptotic centre of the dynamical system that is used in [15] is also important to our approach.

However, we are able to dispose of the assumption of freeness (and thus our dynamical systems can have fixed points or periodic points); in fact, our methods are applicable to irreversible dynamical systems having several degrees of freedom (that is, actions of $\mathbb{Z}_+^d$). In the multidimensional case the usual notions of recurrence and centre are not sufficient to describe the Jacobson radical, as we show by an example. Accordingly, we introduce appropriate modifications.

After discussing the basic properties of analytic crossed products and some of the radicals of Banach algebras, we develop the key lemma in Section 1. This lemma, which is based on the idea of [4, Lemma 1], relates (multi-) recurrent points in the dynamical system with elements not in the Jacobson radical. In Section 2, we give a characterisation of semisimplicity. The proof has three ingredients: the key lemma, a sufficient condition for an element to belong to the prime radical (a descendant of Muhly's condition mentioned earlier), and a basic fact from dynamical systems theory which is known in the one-dimensional case. Our main result, Theorem 18, is proved in the last section using a modification of the centre of a dynamical system.

0.1. *Definition of analytic crossed products.* Analytic crossed products or semicrossed products have been defined in various degrees of generality by several authors (see for example [9, 13, 18, 19, 20]), generalising the concept of the crossed product of a C*-algebra by a group of *-automorphisms. To fix our conventions, we present the definition in the form that we will use it. Let X be a locally compact Hausdorff space and $\Phi = \{\phi_{\mathbf{n}}: \mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d\}$ be a semigroup of continuous and proper surjections isomorphic (as a semigroup) to $\mathbb{Z}_+^d$.

An action of $\Sigma = \mathbb{Z}_+^d$ on $C_0(X)$ by isometric *-endomorphisms $\alpha_{\mathbf{n}}$ ($\mathbf{n} \in \Sigma$) is obtained by defining $\alpha_{\mathbf{n}}(f) = f \circ \phi_{\mathbf{n}}$.

We write elements of the Banach space $\ell^1(\Sigma, C_0(X))$ as formal multi-series $A = \sum_{\mathbf{n} \in \Sigma} U_{\mathbf{n}} f_{\mathbf{n}}$ with the norm given by $\|A\|_1 = \sum \|f_{\mathbf{n}}\|_{C_0(X)}$. The multiplication on $\ell^1(\Sigma, C_0(X))$ is defined by setting

$$U_{\mathbf{n}} f U_{\mathbf{m}} g = U_{\mathbf{n}+\mathbf{m}}(\alpha_{\mathbf{m}}(f)\, g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\Sigma, C_0(X))$ is a Banach algebra.

We will represent $\ell^1(\Sigma, C_0(X))$ faithfully as a (concrete) operator algebra on Hilbert space, and define the analytic crossed product as the closure of the image.

Assuming we have a faithful action of $C_0(X)$ on a Hilbert space H_o, we can define a faithful contractive representation π of $\ell^1(\Sigma, C_0(X))$ on the Hilbert space $\mathscr{H} = H_o \otimes \ell^2(\Sigma)$ by defining $\pi(U_{\mathbf{n}} f)$ as

$$\pi(U_{\mathbf{n}} f)(\xi \otimes e_{\mathbf{k}}) = \alpha_{\mathbf{k}}(f)\, \xi \otimes e_{\mathbf{k}+\mathbf{n}}$$

To show that π is faithful, let $A = \sum_{\mathbf{n} \in \Sigma} U_{\mathbf{n}} f_{\mathbf{n}}$ be in $\ell^1(\mathbb{Z}_+^d, C_0(X))$ and $x, y \in H_o$ be unit vectors. Since π is clearly contractive, the series $\pi(A) = \sum_{\mathbf{n} \in \Sigma} \pi(U_{\mathbf{n}} f_{\mathbf{n}})$ converges absolutely. For $\mathbf{m} \in \Sigma$, we have

$$\begin{aligned} \langle \pi(A)(x \otimes e_0), y \otimes e_{\mathbf{m}} \rangle &= \sum_{\mathbf{n}} \langle \pi(U_{\mathbf{n}} f_{\mathbf{n}})(x \otimes e_0), y \otimes e_{\mathbf{m}} \rangle \\ &= \sum_{\mathbf{n}} \langle f_{\mathbf{n}} x \otimes e_{\mathbf{n}}, y \otimes e_{\mathbf{m}} \rangle \\ &= \langle f_{\mathbf{m}} x \otimes e_{\mathbf{m}}, y \otimes e_{\mathbf{m}} \rangle = \langle f_{\mathbf{m}} x, y \rangle \end{aligned}$$

as $x \otimes e_{\mathbf{n}}$ and $y \otimes e_{\mathbf{m}}$ are orthogonal for $\mathbf{n} \neq \mathbf{m}$. It follows that

$$\|\pi(A)\| \geqslant \|f_{\mathbf{m}}\|.$$

Hence if $\pi(A) = 0$ then $f_{\mathbf{m}} = 0$ for all $\mathbf{m}$, showing $A = 0$. Thus π is a monomorphism.

Definition 2. The *analytic crossed product* $\mathscr{A} = C_0(X) \times_\phi \mathbb{Z}_+^d$ is the closure of the image of $\ell^1(\mathbb{Z}_+^d, C_0(X))$ in $\mathscr{B}(\mathscr{H})$ in the representation just defined.

This is a generalisation of the definition given in [19]. Note that $\mathscr{A}$ is in fact independent of the faithful action of $C_0(X)$ on H_o (up to isometric isomorphism).

For $A = \sum U_{\mathbf{n}} f_{\mathbf{n}} \in \ell^1(\Sigma, C_0(X))$ we call $f_{\mathbf{n}} \equiv E_{\mathbf{n}}(A)$ the $\mathbf{n}$*th Fourier coefficient* of A. We have shown above that the maps $E_{\mathbf{n}}\colon \ell^1(\Sigma, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $\mathscr{A}$, hence they extend to contractions $E_{\mathbf{n}}\colon \mathscr{A} \to C_0(X)$.

Moreover,

$$U_{\mathbf{m}} E_{\mathbf{m}}(A) = \frac{1}{(2\pi)^d} \int_{([-\pi,\pi])^d} \theta_{\mathbf{t}}(A) \exp(-i\mathbf{m}.\mathbf{t})\, d\mathbf{t},$$

where $\mathbf{m}.\mathbf{t} = m_1 t_1 + \ldots + m_d t_d$ and the automorphism $\theta_{\mathbf{t}}$ is defined first on the dense subalgebra $\ell^1(\Sigma, C_0(X))$ by

$$\theta_{\mathbf{t}}\left(\sum U_{\mathbf{n}} f_{\mathbf{n}}\right) = \sum U_{\mathbf{n}}(\exp(i\mathbf{t}.\mathbf{n})\, f_{\mathbf{n}})$$

and then extended to $\mathscr{A}$ by continuity.

Thus, by injectivity of the Fourier transform on $C(([-\pi,\pi])^d)$, if a continuous linear form η on $\mathscr{A}$ satisfies $\eta(E_{\mathbf{m}}(A)) = 0$ for all $\mathbf{m}$ then (the function $\mathbf{t} \to \eta(\theta_{\mathbf{t}}(A))$ vanishes and hence) $\eta(A) = 0$. The Hahn–Banach Theorem yields the following remark.

Remark. Any $A \in \mathscr{A}$ belongs to the closed linear span of the set $\{U_{\mathbf{m}} E_{\mathbf{m}}(A) \colon \mathbf{m} \in \Sigma\}$ of its "associated monomials".

In particular, $\mathscr{A}$ is the closure of the subalgebra $\mathscr{A}_0$ of trigonometric polynomials, i.e., finite sums of monomials.

As $\theta_{\mathbf{t}}$ is an automorphism of $\mathscr{A}$, we conclude that if $\mathscr{J} \subseteq \mathscr{A}$ is a closed automorphism invariant ideal (in particular, the Jacobson radical) then for all $B \in \mathscr{J}$ and $\mathbf{m} \in \Sigma$ we obtain $U_{\mathbf{m}} E_{\mathbf{m}}(B) \in \mathscr{J}$. Thus, an element $\sum U_{\mathbf{n}} f_{\mathbf{n}}$ is in $\mathscr{J}$ if and only if each monomial $U_{\mathbf{n}} f_{\mathbf{n}}$ is in $\mathscr{J}$; this was first observed (for $d = 1$) in [16, Proposition 2.1]. It now follows from the remark that any such ideal is the closure of the trigonometric polynomials it contains.

0.2 *Radicals in Banach algebras.* Recall that an ideal $\mathscr{J}$ of an algebra $\mathscr{A}$ is said to be *primitive* if it is the kernel of an (algebraically) irreducible representation. The intersection of all primitive ideals of $\mathscr{A}$ is the *Jacobson radical* of $\mathscr{A}$, denoted Rad $\mathscr{A}$.

An ideal $\mathscr{J}$ is *prime* if it cannot factor as the product of two distinct ideals, i.e., if $\mathscr{J}_1, \mathscr{J}_2$ are ideals of $\mathscr{A}$ such that $\mathscr{J}_1 \mathscr{J}_2 \subseteq \mathscr{J}$ then either $\mathscr{J}_1 \subseteq \mathscr{J}$ or $\mathscr{J}_2 \subseteq \mathscr{J}$. The intersection of all prime ideals is the *prime radical* of $\mathscr{A}$, denoted PRad $\mathscr{A}$. An algebra $\mathscr{A}$ is *semisimple* if Rad $\mathscr{A} = \{0\}$ and *semiprime* if PRad $\mathscr{A} = \{0\}$, or equivalently, if there are no (nonzero) nilpotent ideals.

As a primitive ideal is prime, $\mathrm{PRad}\,\mathscr{A} \subseteq \mathrm{Rad}\,\mathscr{A}$. Thus a semisimple algebra is semiprime. If $\mathscr{A}$ is a Banach algebra, then the Jacobson radical is closed; indeed every primitive ideal is the kernel of some *continuous* representation of $\mathscr{A}$ on a Banach space. In fact an element $A \in \mathscr{A}$ is in $\mathrm{Rad}\,\mathscr{A}$ if and only if the spectral radius of AB vanishes for all $B \in \mathscr{A}$.

The prime radical need not be closed; it is closed if and only if it is a nilpotent ideal (see [8] or [17, Theorem 4.4.11]). Thus for a general Banach algebra, $\mathrm{PRad}\,\mathscr{A} \subseteq \overline{\mathrm{PRad}\,\mathscr{A}} \subseteq \mathrm{Rad}\,\mathscr{A}$.

1. RECURRENCE AND MONOMIALS

Our main results will be proved for metrisable dynamical systems; hence we make the blanket assumption that X *will be a locally compact metrisable space*. As in the one-dimensional case, we say that a point $x \in X$ is *recurrent* for the dynamical system (X, Φ) if there exists a sequence $(\mathbf{n}_k)$ tending to infinity so that $\phi_{\mathbf{n}_k}(x) \to x$. We will need the following variant:

Definition 3. Let $J \subseteq \{1, 2, ..., d\}$. Say $x \in X$ is *J-recurrent* if there exists a sequence $(\mathbf{n}_k)$ which is strictly increasing in the directions of J (that is, the jth entry of $\mathbf{n}_{k+1}$ is greater than the jth entry of $\mathbf{n}_k$ for every $j \in J$ and $k \in \mathbb{N}$) such that $\lim_k \phi_{\mathbf{n}_k}(x) = x$. Denote the set of all J-recurrent points by X_{Jr}.

We say that a point $x \in X$ is *strongly recurrent* if it is $\{1, 2, ..., d\}$-recurrent. Finally, Σ_J denotes $\{\mathbf{n} \in \mathbb{Z}^d_+ : \mathbf{n}_j > 0 \text{ for all } j \in J\}$.

In the multidimensional case, the Jacobson radical cannot be characterised in terms of either the recurrent points (in the traditional sense) or the strongly recurrent points. To justify this, we give the following example.

Example 4. Let $X = X_0 \cup X_1 \cup X_2$ where $X_i = \mathbb{R} \times \{i\}$. Consider the dynamical system $(X, (\phi_1, \phi_2))$, where ϕ_1 acts as translation by 1 on X_1 and as the identity on $X_0 \cup X_2$ while ϕ_2 acts as translation by 1 on X_2 and as the identity on $X_0 \cup X_1$. It is easy to see that the set of $\{1\}$-recurrent points is $X_0 \cup X_2$, the set of $\{2\}$-recurrent points is $X_0 \cup X_1$ and the set of strongly recurrent points is X_0.

Choose small neighbourhoods $V_1 \subseteq X_1$ and $V_2 \subseteq X_2$ of $(0, 1)$ and $(0, 2)$ respectively such that $\phi_1(V_1) \cap V_1 = \varnothing$ and $\phi_2(V_2) \cap V_2 = \varnothing$. Let $f \in C_0(X)$ be any function supported on $V_1 \cup V_2$ such that $f(0, 1) = f(0, 2) = 1$.

Then one can verify (as in the proof of Lemma 8 in the next section) that $U_1 U_2 f$ is in the prime radical. On the other hand, neither $U_1 f$ nor $U_2 f$ belong to the Jacobson radical (they are not even quasinilpotent).

Here, the associated semicrossed product has nonzero Jacobson radical, although every point is recurrent. Also, the monomial $U_1 f$ is not in the Jacobson radical, although f vanishes on the strongly recurrent points. The next lemma shows that for such a monomial to be in the Jacobson radical, f must vanish on the $\{1\}$-recurrent points.

The main result of this section is the following lemma, which is crucial to our analysis.

LEMMA 5. *Let $U_{\mathbf{q}} f \in Rad(C_0(X) \times_\phi \mathbb{Z}_+^d)$. If J contains the support of* $\mathbf{q}$, *then f vanishes on each J-recurrent point of (X, Φ).*

In order to prove this lemma, we need a basic property of recurrent points, adapted to our circumstances.

DEFINITION 6. Given a sequence $\bar{\mathbf{n}} = (\mathbf{n}_k) \subseteq \mathbb{Z}_+^d$ we define recursively the *family of indices associated to* $\bar{\mathbf{n}}$, denoted $\mathscr{S}(\bar{\mathbf{n}}) = (S_0, S_1, S_2, \ldots)$ as follows: $S_0 = \{\mathbf{0}\}$, $S_1 = \{\mathbf{n}_1\}$ and generally

$$S_{k+1} = \left\{ \mathbf{n}_{k+1} + \mathbf{m}_k + \mathbf{j} : \mathbf{j} \in \bigcup_{i=0}^{k} S_i \right\}$$

where $\mathbf{m}_0 = \mathbf{0}$ and $\mathbf{m}_k = \mathbf{n}_k + 2\mathbf{m}_{k-1}$.

The sets in $\mathscr{S}(\bar{\mathbf{n}})$ will be needed in the proof of Lemma 5: they are the indices of ϕ occurring in the simplification of the inductive sequence of products given by $P_1 = U_{\mathbf{n}_1} g$ and $P_k = P_{k-1}(U_{\mathbf{n}_k}(g/2^{k-1}))\, P_{k-1}$. We should also point out that $\bigcup_i S_i$ is an IP-set (see [6, Section 8.4]) and the next lemma is a variant on [6, Theorem 2.17].

Recall Σ_J denotes $\{(n_1, n_2, \ldots, n_d) : n_j \neq 0 \text{ for all } j \in J\}$. Let Δ_J be the subset of Σ_J with entries in the directions of J^c identically zero.

LEMMA 7. *Let x be in X, J be a subset of $\{1, 2, \ldots, d\}$. Suppose that $\lim_k \phi_{\mathbf{p}_k}(x) = x$, where $(\mathbf{p}_k)$ is a sequence whose restriction to J is strictly increasing while its restriction to J^c is constant.*

For each open neighbourhood V of x and each $k \in \mathbb{N}$, there is $\mathbf{n}_k \in \Delta_J$ and $x_k \in V$ with

$$\phi_{\mathbf{s}}(x_k) \in V \quad \textit{for all} \quad \mathbf{s} \in \bigcup_{i=0}^{k} S_i,$$

where $\mathscr{S}(\bar{\mathbf{n}}) = (S_0, \ldots)$ is the family of indices associated to the sequence $(\mathbf{n}_k)$.

Proof. We inductively find indices $\mathbf{n}_1, \mathbf{n}_2, \ldots,$ as above, open sets $V \supseteq V_1 \supseteq V_2 \supseteq \cdots$ and points $x_1, x_2, \ldots$ with $x_i \in V_i$ and $x_i = \phi_{\mathbf{k}_i}(x)$ for some index $\mathbf{k}_i$, so that

$$\phi_{\mathbf{s}}(V_i) \subseteq V \quad \text{for all} \qquad \mathbf{s} \in S_i.$$

This will prove the lemma, for if $k \in \mathbb{N}$ and $\mathbf{s} \in S_i$ for some $i \leqslant k$ then, since $x_k \in V_k \subseteq V_i$ it will follow that $\phi_{\mathbf{s}}(x_k) \in \phi_{\mathbf{s}}(V_k) \subseteq \phi_{\mathbf{s}}(V_i) \subseteq V$.

Since $\lim_k \phi_{\mathbf{p}_k}(x) = x \in V$, there is $\mathbf{p}_{i_1}$ with $x_1 = \phi_{\mathbf{p}_{i_1}}(x) \in V$. Let $\mathbf{k}_1 = \mathbf{p}_{i_1}$. Using $\lim_k \phi_{\mathbf{p}_k}(x) = x \in V$ and the form of the $\mathbf{p}_k$, it follows that there is $\mathbf{n}_1 \in \varDelta_J$ so that $\phi_{\mathbf{n}_1 + \mathbf{k}_1}(x) \in V$. Now

$$\phi_{\mathbf{n}_1}(x_1) = \phi_{\mathbf{n}_1}(\phi_{\mathbf{k}_1}(x)) = \phi_{\mathbf{n}_1 + \mathbf{k}_1}(x) \in V,$$

and so there is $V_1 \subseteq V$, an open neighbourhood of x_1, so that $\phi_{\mathbf{n}_1}(V_1) \subseteq V$. Since $S_1 = \{\mathbf{n}_1\}$, this establishes the base step.

For the inductive step, assume we have chosen indices $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_q$, open subsets of V, $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_q$ and points $x_1, x_2, \ldots, x_q$, with $x_i \in V_i$ and $x_i = \phi_{\mathbf{k}_i}(x)$, so that, for $i = 1, \ldots, q$, we have

$$\phi_{\mathbf{s}}(V_i) \subseteq V \quad \text{for all} \qquad \mathbf{s} \in S_i. \tag{1}$$

Since $\lim_k \phi_{\mathbf{p}_k}(x_q) = \phi_{\mathbf{k}_q}(\lim_k \phi_{\mathbf{p}_k}(x)) = x_q \in V_q$, there is $\mathbf{k}_{q+1} = \mathbf{p}_{i_q}$ so that $x_{q+1} = \phi_{\mathbf{k}_{q+1}}(x_q) \in V_q$. Notice that $\mathbf{m}_q$ (as in Definition 6) is in $\varDelta_J$. It follows that there exists $\mathbf{n}_{q+1} \in \varDelta_J$ such that $\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q + \mathbf{k}_{q+1}}(x_q) \in V_q$ and so $\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(x_{q+1}) \in V_q$. Hence there exists an open neighbourhood V_{q+1} of x_{q+1}, contained in V_q, so that

$$\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(V_{q+1}) \subseteq V_q. \tag{2}$$

It remains only to show that $\phi_{\mathbf{s}}(V_{q+1}) \subseteq V$ for all $\mathbf{s} \in S_{q+1}$. An element $\mathbf{s}$ in S_{q+1} is of the form $\mathbf{s} = \mathbf{n}_{q+1} + \mathbf{m}_q + \mathbf{j}$ for some $\mathbf{j} \in \bigcup_{i=0}^{q} S_i$. Assuming $\mathbf{j} \in S_i$ for some i, we have

$$\begin{aligned} \phi_{\mathbf{s}}(V_{q+1}) &= \phi_{\mathbf{j}}(\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(V_{q+1})) \subseteq \phi_{\mathbf{j}}(V_q) && \text{by (2)} \\ &\subseteq \phi_{\mathbf{j}}(V_i) \subseteq V && \text{by (1)} \end{aligned}$$

completing the induction. ∎

Proof of Lemma 5. Assume that $f(x) \neq 0$ for some J-recurrent point x. We will find $B \in \mathscr{A}$ such that $BU_{\mathbf{q}} f$ has nonzero spectral radius. We may scale f so that there exists a relatively compact open neighbourhood V of x such that $|f(y)| \geqslant 1$ for all $y \in V$. Since $U_{\mathbf{q}} \, |f|^2 = (U_{\mathbf{q}} f) \, f^* \in \operatorname{Rad} \mathscr{A}$ when $U_{\mathbf{q}} f \in \operatorname{Rad} \mathscr{A}$, we may also assume that $f \geqslant 0$.

Since x is J-recurrent, there exists a sequence $(\mathbf{p}_k)$ which is strictly increasing in the directions of J such that $\lim_k \phi_{\mathbf{p}_k}(x) = x$. Deleting some initial segment, we may assume that $\phi_{\mathbf{p}_k}(x) \in V$ for all $k \in \mathbb{N}$.

If $(\mathbf{p}_k)$ has all entries going to infinity, then we may apply Lemma 7 with $J = \{1, 2, ..., d\}$, to find a strictly increasing sequence $(\mathbf{n}_k)$ such that $\mathbf{n}_k > \mathbf{q}$ for all k and points $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s}$ in $\bigcup_{i=0}^k S_i$.

If not, enlarging J and passing to a subsequence if necessary, we may assume that the restriction of $(\mathbf{p}_k)$ to J^c takes only finitely many values. Passing to another subsequence, we may further assume that this restriction is constant. Applying Lemma 7, we may find a strictly increasing sequence $(\mathbf{n}_k)$ in $\mathbb{Z}_+^d$ with $\mathbf{n}_k \in \Delta_J$ and points $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s}$ in $\bigcup_{i=0}^k S_i$. We may suppose that $\mathbf{n}_k - \mathbf{q} \in \Sigma_J$ for all k. Thus $U_{\mathbf{n}_k - \mathbf{q}}$ is an admissible term in the formal power series of an element of $C_0(X) \times_\phi \mathbb{Z}_+^d$.

Fix a nonnegative function $h \in C_0(X)$ such that $h(\phi_{\mathbf{q}}(y)) = 1$ for all $y \in V$ and consider

$$B = \sum_{k=1}^{\infty} U_{\mathbf{n}_k - \mathbf{q}} \frac{h}{2^{k-1}}.$$

This is an element of $\mathscr{A}$ since the series converges absolutely. To complete the proof, it suffices to show that the spectral radius of $A \equiv BU_{\mathbf{q}} f$ is strictly positive. Note that

$$A = \sum U_{\mathbf{n}_k} \frac{g}{2^{k-1}},$$

where g is $f.(h \circ \phi_{\mathbf{q}})$, a nonnegative function satisfying $g(y) \geqslant 1$ for all $y \in V$. Thus each Fourier coefficient $E_{\mathbf{n}}(A^m)$ of A^m is a finite sum of non-negative functions, and hence its norm dominates the (supremum) norm of each summand. Since $\|A^{2^k-1}\| \geqslant \|E_{\mathbf{n}}(A^{2^k-1})\|$, it suffices to find $\varepsilon > 0$ such that for each k there exists $\mathbf{n}$ such that the norm of *some* summand of $E_{\mathbf{n}}(A^{2^k-1})$ exceeds ε^{2^k-1}.

If we let $P_1 = U_{\mathbf{n}_1} g$, then trivially P_1 is a term in A. In the next product, $A^3 = A(\sum U_{\mathbf{n}_k} \frac{g}{2^{k-1}})\, A = \sum A(U_{\mathbf{n}_k} \frac{g}{2^{k-1}})\, A$, we have the term

$$P_2 = U_{\mathbf{n}_1} g \left(U_{\mathbf{n}_2} \frac{g}{2} \right) U_{\mathbf{n}_1} g.$$

Generally, one term in the expansion of $A^{2^k-1} = A^{2^{k-1}-1} A A^{2^{k-1}-1}$ is

$$P_k = P_{k-1} \left(U_{\mathbf{n}_k} \frac{g}{2^{k-1}} \right) P_{k-1}.$$

CLAIM. *If $\lambda_1 = 1$ and $\lambda_{k+1} = \lambda_k^2/2^k$, then $P_k = U_{\mathbf{m}_k}\lambda_k \prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}}$ where $\mathbf{m}_k$ is as in the definition of $\mathscr{S}(\bar{\mathbf{n}})$ and the product is over all $\mathbf{s}$ in $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$.*

Proof of Claim. For $k = 1$, the claim holds trivially as $(S_0 \cup S_1) \setminus \{\mathbf{m}_1\} = \{0\}$. Assuming the claim is true for some k, we have

$$\begin{aligned} P_{k+1} &= P_k \left(U_{\mathbf{n}_{k+1}} \frac{g}{2^k} \right) P_k \\ &= U_{\mathbf{m}_k} \lambda_k \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right) \left(U_{\mathbf{n}_{k+1}} \frac{g}{2^k} \right) U_{\mathbf{m}_k} \lambda_k \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{t}} \right) \end{aligned}$$

(where $\mathbf{s}, \mathbf{t}$ range over $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$)

$$\begin{aligned} &= U_{\mathbf{m}_k} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right) U_{\mathbf{n}_{k+1}+\mathbf{m}_k} (g \circ \phi_{\mathbf{m}_k}) \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{s}} \right) \\ &= U_{2\mathbf{m}_k+\mathbf{n}_{k+1}} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}+\mathbf{n}_{k+1}+\mathbf{m}_k} \right) (g \circ \phi_{\mathbf{m}_k}) \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{t}} \right) \\ &= U_{2\mathbf{m}_k+\mathbf{n}_{k+1}} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}'} g \circ \phi_{\mathbf{s}'} \right) \left(\prod_{\mathbf{t}'} g \circ \phi_{\mathbf{t}'} \right), \end{aligned}$$

where $\mathbf{s}'$ ranges over $\{\mathbf{n}_{k+1} + \mathbf{m}_k + \mathbf{s}\}$, for $\mathbf{s} \in (\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$, and $\mathbf{t}'$ ranges over $(\bigcup_{i=0}^k S_i)$. Therefore

$$P_{k+1} = U_{\mathbf{m}_{k+1}} \lambda_{k+1} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right)$$

for $\mathbf{s}$ in $(\bigcup_{i=0}^{k+1} S_i) \setminus \{\mathbf{m}_{k+1}\}$, proving the claim.

Recall that for each $k \in \mathbb{N}$ there exists $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s} \in \bigcup_{i=0}^k S_i$. Since $g|_V \geqslant 1$, we have $\prod_{\mathbf{s}} g(\phi_{\mathbf{s}}(x_k)) \geqslant 1$ where $\mathbf{s}$ ranges over $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$ and hence $\|\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}}\| \geqslant 1$. From the claim, it follows that $\|P_k\| \geqslant \lambda_k$ and so, by the earlier remarks,

$$\|A^{2^k-1}\| \geqslant \|E_{\mathbf{m}_k}(A^{2^k-1})\| \geqslant \|P_k\| \geqslant \lambda_k.$$

Thus the proof will be complete if we show that $\lambda_k \geqslant (\frac{1}{2})^{2^k-1}$ or equivalently $\log_2 \lambda_k^{-1} \leqslant 2^k - 1$ for all k. Setting $\mu_k = \log_2 \lambda_k^{-1}$, the recurrence relation for λ_k becomes $\mu_{k+1} = 2\mu_k + k$ and $\mu_1 = 0$, which has solution $\mu_k = 2^k - k - 1$. ∎

2. WANDERING SETS AND SEMISIMPLICITY

We characterise semisimplicity of analytic crossed products and show this is equivalent to being semiprime. Part of this characterisation is of course a special case of our main result, Theorem 18, but we will need the preliminary results in any case.

A *wandering open set* is an open set $V \subset X$ so that $\phi_{\mathbf{n}}^{-1}(V) \cap V = \varnothing$ whenever $\mathbf{n} \in \mathbb{Z}_+^d$ is nonzero. A *wandering point* is a point with a wandering neighbourhood.

We will need the following variant: let $J \subseteq \{1, ..., d\}$. An open set $V \subseteq X$ is said to be *wandering in the directions of* J, or J-*wandering*, if $\phi_{\mathbf{n}}^{-1}(V) \cap V = \varnothing$ whenever $\mathbf{n}$ is in Σ_J. It is easily seen that, if X_{Jw} denotes the set of all J-wandering points (those with a J-wandering neighbourhood), then X_{Jw} is open and its complement is invariant and contains the set X_{Jr} of J-recurrent points.

Note, however, that it is possible for a recurrent point (in the usual sense) to have a neighbourhood that is J-wandering (for some J). For example, if $X = \mathbb{R}^2$ and $\phi_1(x, y) = (x+1, y)$ while $\phi_2(x, y) = (x, 3y)$, then the origin is recurrent for the dynamical system $(X, (\phi_1, \phi_2))$, but it also has a $\{1\}$-wandering neighbourhood.

The idea of the following Lemma comes from [16, Theorem 4.2].

LEMMA 8. *Suppose $V \subseteq X$ is an open set which is J-wandering and $g \in C_0(X)$ is a nonzero function with support contained in V. If $\mathbf{e}_J$ denotes the characteristic function of J, then $B = U_{\mathbf{e}_J} g$ generates a nonzero ideal $\mathscr{A}B\mathscr{A}$ whose square is* 0.

Proof. Let $C \in \mathscr{A}$ be arbitrary and $h = E_{\mathbf{k}}(C)$. Then

$$BU_{\mathbf{k}}E_{\mathbf{k}}(C)\,B = U_{\mathbf{e}_J}\,gU_{\mathbf{k}}hU_{\mathbf{e}_J}\,g = U_{\mathbf{k}+2\mathbf{e}_J}(\alpha_{\mathbf{k}+\mathbf{e}_J}(g)\,\alpha_{\mathbf{e}_J}(h)\,g),$$

which is zero since g is supported on V and $\alpha_{\mathbf{k}+\mathbf{e}_J}(g)$ is supported on the disjoint set $\phi_{\mathbf{k}+\mathbf{e}_J}^{-1}(V)$. This shows that all Fourier coefficients of BCB will vanish, and hence $BCB = 0$. It follows that all products $(C_1BC_2)(C_3BC_4)$ vanish and hence $(\mathscr{A}B\mathscr{A})^2 = 0$. On the other hand, choosing functions $h_1 \in C_0(X)$ equal to 1 on $\phi_{\mathbf{e}_J}^{-1}(V)$ and h_2 equal to 1 on V, we find $E_{\mathbf{e}_J}(h_1Bh_2) = \alpha_{\mathbf{e}_J}(h_1)\,gh_2 = g \neq 0$, so the ideal $\mathscr{A}B\mathscr{A}$ is nonzero. ∎

The following proposition is known for the usual notions of recurrence and wandering in the case $d = 1$; see [6, Theorem 1.27].

PROPOSITION 9. *Suppose X is a locally compact metrisable space. If (X, Φ) has no nonempty J-wandering open sets, then the J-recurrent points are dense.*

Proof. Let $V \subseteq X$ be a relatively compact open set. We wish to find a J-recurrent point in V.

Since V is not J-wandering, there exists $\mathbf{n}_1 \in \Sigma_J$ such that $\phi_{\mathbf{n}_1}^{-1}(V) \cap V \neq \emptyset$. Hence there is a nonempty, relatively compact, open set V_1 with $\operatorname{diam}(V_1) < 1$ such that $\overline{V_1} \subseteq \phi_{\mathbf{n}_1}^{-1}(V) \cap V$.

Since V_1 contains no J-wandering subsets, a similar argument shows that there exists $\mathbf{n}_2$ such that $\phi_{\mathbf{n}_2}^{-1}(V_1) \cap V_1 \neq \emptyset$ and the jth entry of $\mathbf{n}_2$ is greater than that of $\mathbf{n}_1$ for every $j \in J$.

Inductively one obtains a sequence of open sets V_k and $\mathbf{n}_k$ strictly increasing in the directions of J with $\overline{V_k} \subseteq \phi_{\mathbf{n}_k}^{-1}(V_{k-1}) \cap V_{k-1}$ and $\operatorname{diam}(V_k) < 1/k$ all contained in the compact metrisable space $\overline{V_0}$. It follows from Cantor's theorem that the intersection $\bigcap_{n \geqslant 1} \overline{V_n}$ is a singleton, say x. Since $x \in \overline{V_k} \subseteq \phi_{\mathbf{n}_k}^{-1}(V_{k-1})$ we have $\phi_{\mathbf{n}_k}(x) \in V_{k-1}$ for all k and so $\phi_{\mathbf{n}_k}(x) \to x$; hence $x \in X_{Jr}$. ∎

THEOREM 10. *If X is a metrisable, locally compact space, then the following are equivalent:*

1. *the strongly recurrent points are dense in X,*
2. *$C_0(X) \times_\phi \mathbb{Z}_+^d$ is semisimple, and*
3. *$C_0(X) \times_\phi \mathbb{Z}_+^d$ is semiprime.*

Proof. If the strongly recurrent points are dense in X, then by Lemma 5 there are no nonzero monomials in the Jacobson radical of $C_0(X) \times_\phi \Sigma$. But we have already observed that an element A is in the Jacobson radical if and only if each monomial $U_{\mathbf{n}} E_{\mathbf{n}}(A)$ is. Thus $C_0(X) \times_\phi \Sigma$ is semisimple and hence semiprime.

Suppose that $C_0(X) \times_\phi \Sigma$ is semiprime. Then Lemma 8 shows that there are no nonempty J-wandering open sets for $J = \{1, 2, ..., d\}$. Thus, by Proposition 9, the strongly recurrent points are dense. ∎

3. CENTRES AND THE JACOBSON RADICAL

In order to describe the Jacobson radical of an analytic crossed product, we need to characterise the closure of the J-recurrent points, for a dynamical system (X, Φ) with X a locally compact metrisable space.

LEMMA 11. (i) *If $Y \subseteq X$ is a closed invariant set, the set Y_{Jr} of J-recurrent points for the dynamical system (Y, Φ) equals $X_{Jr} \cap Y$.*

(ii) *The set $\overline{X_{Jr}}$ is the largest closed invariant set $Y \subseteq X$ such that (Y, Φ) has no J-wandering points.*

Proof. (i) To see that $Y_{Jr} \subseteq X_{Jr}$, note that if $y \in Y_{Jr}$ then for every neighbourhood V of y (in X) the set $V \cap Y$ is a neighbourhood of y in the relative topology of Y, so there exists $\mathbf{n} \in \Sigma_J$ such that $\phi_{\mathbf{n}}(y) \in V \cap Y$. Thus $\phi_{\mathbf{n}}(y) \in V$ showing that $y \in X_{Jr}$. On the other hand if $y \in Y \cap X_{Jr}$ then for each relative neighbourhood $V \cap Y$ of y, since V is a neighbourhood of y in X there exists $\mathbf{n} \in \Sigma_J$ such that $\phi_{\mathbf{n}}(y) \in V$. Since $y \in Y$ and Y is invariant, $\phi_{\mathbf{n}}(y) \in V \cap Y$ establishing (i).

(ii) Given a closed invariant set $Y \subseteq X$, if $(Y, \Phi|_Y)$ has no J-wandering points, then Y_{Jr} is dense in Y by Proposition 9, and hence $Y \subseteq \overline{X_{Jr}}$. On the other hand, $(\overline{X_{Jr}}, \Phi)$ clearly has no J-wandering open sets. ▮

The set $\overline{X_{Jr}}$ is found by successively "peeling off" the J-wandering parts of the dynamical system. This construction and Lemma 13 generalise the well known concept of the centre of a dynamical system (X, ϕ) [7, 7.19].

If $V \subseteq X$ is the union of the J-wandering open subsets of X, then let $X_{J,1}$ be the closed invariant set $X \backslash V$. Consider the dynamical system $(X_{J,1}, \Phi_{J,1})$, where $\Phi_{J,1} \equiv \Phi|_{X_{J,1}}$. Let $X_{J,2}$ be the complement of the union of all J-wandering open sets of $(X_{J,1}, \Phi_{J,1})$. Again we have a closed invariant set, and we may form the dynamical subsystem $(X_{J,2}, \Phi_{J,2})$ where $\Phi_{J,2} \equiv \Phi|_{X_{J,2}}$. By transfinite recursion, we obtain a decreasing family $(X_{J,\gamma}, \Phi_{J,\gamma})$ of dynamical systems: indeed, if $(X_{J,\gamma}, \Phi_{J,\gamma})$ has been defined, we let $X_{J,\gamma+1} \subseteq X_{J,\gamma}$ be the set of points in $(X_{J,\gamma}, \Phi_{J,\gamma})$ having no J-wandering neighbourhood and we define $\Phi_{J,\gamma+1} = \Phi|_{X_{J,\gamma+1}}$; if β is a limit ordinal and the systems $(X_{J,\gamma}, \Phi_{J,\gamma})$ have been defined for all $\gamma < \beta$, then we set $X_{J,\beta} = \bigcap_{\gamma<\beta} X_{J,\gamma}$ and $\Phi_{J,\beta} = \Phi|_{X_{J,\beta}}$. (We write $X_{J,0} = X$ and $\Phi_{J,0} = \Phi$.) This process must stop, for the cardinality of the family $\{X_{J,\gamma}\}$ cannot exceed that of the power set of X.

Definition 12. By the above argument, there exists a least ordinal γ such that $X_{J,\gamma+1} = X_{J,\gamma}$. The set $X_{J,\gamma}$ is called *the strong J-centre* of the dynamical system, and γ is called *the depth* of the strong J-centre.

Lemma 13. *If X is metrisable, then the strong J-centre of the dynamical system is the closure of the J-recurrent points.*

Proof. As a J-recurrent point cannot be J-wandering, $X_{Jr} \subseteq X_{J,1}$. If $X_{Jr} \subseteq X_{J,\gamma}$ for some γ, then by Lemma 11 the set $(X_{J,\gamma})_{Jr}$ of J-recurrent points of the subsystem $(X_{J,\gamma}, \Phi_{J,\gamma})$ equals $X_{Jr} \cap X_{J,\gamma}$, so $(X_{J,\gamma})_{Jr} = X_{Jr}$; but $(X_{J,\gamma})_{Jr} \subseteq X_{J,\gamma+1}$, and so $X_{Jr} \subseteq X_{J,\gamma+1}$. Finally, if γ is a limit ordinal and we assume that $X_{Jr} \subseteq X_{J,\delta}$ for all $\delta < \gamma$ then $X_{Jr} \subseteq \bigcap_{\delta<\gamma} X_{J,\delta} = X_{J,\gamma}$. This shows that $X_{Jr} \subseteq \bigcap_\gamma X_{J,\gamma}$ and so $\overline{X_{Jr}} \subseteq \bigcap_\gamma X_{J,\gamma}$ since the sets $X_{J,\gamma}$ are closed.

But on the other hand, if γ_0 is the depth of the strong J-centre we have $\bigcap_\gamma X_{J,\gamma} = X_{J,\gamma_0}$, a closed invariant set. Since $X_{J,\gamma_0+1} = X_{J,\gamma_0}$, the dynamical system $(X_{J,\gamma_0}, \Phi_{J,\gamma_0})$ can have no J-wandering points. Thus it follows from Lemma 11 that $X_{J,\gamma_0} \subseteq \overline{X_{Jr}}$ and hence equality holds. ∎

Remark. If X is a locally compact (not necessarily metrisable) space and $\{\phi_{\mathbf{n}}: \mathbf{n} \in \mathbb{Z}^d\}$ is an action of an *equicontinuous* group of homeomorphisms (with respect to a uniformity compatible with the topology of X) then $X_{Jr} = X \setminus X_{Jw}$ (see [14, Proposition 4.15]).

LEMMA 14. *For any ordinal δ, any $f \in C_c(X^c_{J,\delta+1})$ (i.e., f has compact support disjoint from $X_{J,\delta+1}$) can be written as a* finite *sum $f = \sum f_k$ where each f_k has compact support contained in a set V_k such that $V_k \cap X_{J,\delta}$ is J-wandering set for $(X_{J,\delta}, \Phi_{J,\delta})$.*

Proof. If K is the support of f then $K \cap X_{J,\delta} \subseteq X_{J,\delta} \setminus X_{J,\delta+1}$; in other words the compact set $K \cap X_{J,\delta}$ consists of J-wandering points for $(X_{J,\delta}, \Phi_{J,\delta})$. This means that each $x \in K \cap X_{J,\delta}$ has an open neighbourhood V_x so that the (relatively open) set $V_x \cap X_{J,\delta}$ is J-wandering for $(X_{J,\delta}, \Phi_{J,\delta})$. Each $y \in K \setminus X_{J,\delta}$ has an open neighbourhood V_y such that $V_y \cap X_{J,\delta}$ is empty (and so J-wandering).

The family $\{V_x: x \in K\}$ is an open cover for K. Thus, there is a partition of unity for f, i.e., a finite subcover, $\{V_k: 1 \leqslant k \leqslant m\}$, and functions f_k, $1 \leqslant k \leqslant m$, with $\operatorname{supp}(f_k)$ a compact subset of V_k, so that $f = f_1 + \cdots + f_m$. ∎

DEFINITION 15. We denote by $\mathcal{R}_{J,\gamma}$ the closed ideal generated by all monomials of the form $U_{\mathbf{n}} f$ where $\mathbf{n}$ is in Σ_J and $f \in C_0(X)$ vanishes on the set $X_{J,\gamma}$ and by $\mathcal{S}_{J,\gamma}$ the set of all elements of the form Bf where $B \in \mathcal{R}_{J,\gamma}$ and f has compact support disjoint from $X_{J,\gamma}$.

Note that a monomial $U_{\mathbf{n}} f \in \mathcal{R}_{J,\gamma}$ may be written in the form $CU_{\mathbf{e}_J} f$ with $C \in \mathcal{A}$, since $\mathbf{n} \in \Sigma_J$.

Also observe that $\mathcal{S}_{J,\gamma}$ is dense in $\mathcal{R}_{J,\gamma}$. Indeed if $U_{\mathbf{n}} f \in \mathcal{R}_{J,\gamma}$, then f can be approximated by some $g \in C_c(X^c_{J,\gamma})$; now $U_{\mathbf{n}} g$ is in $\mathcal{S}_{J,\gamma}$ and approximates $U_{\mathbf{n}} f$.

PROPOSITION 16. *For each ordinal γ and each $J \subseteq \{1, 2, ..., d\}$, the set $\mathcal{S}_{J,\gamma}$ is contained in Rad $\mathcal{A}$. Hence $\mathcal{R}_{J,\gamma}$ is contained in Rad $\mathcal{A}$.*

If PRad $\mathcal{A}$ is closed, then $\mathcal{R}_{J,\gamma}$ is contained in PRad $\mathcal{A}$.

Proof. Since $\mathcal{S}_{J,\gamma}$ is dense in $\mathcal{R}_{J,\gamma}$, it suffices to prove that any $A = Bf \in \mathcal{S}_{J,\gamma}$ is contained in Rad $\mathcal{A}$.

Suppose $\gamma = 1$. By Lemma 14 we may write A as a finite sum $A = \sum_k Bf_k$ where each f_k is supported on a compact set that is J-wandering. Since $A_k \equiv Bf_k = DU_{e_J} f_k$ for some $D \in \mathscr{A}$ as observed above, by Lemma 8 we have $(\mathscr{A}A_k\mathscr{A})^2 = 0$ and so $A_k \in \operatorname{PRad} \mathscr{A}$. Thus $A \in \operatorname{PRad} \mathscr{A} \subseteq \operatorname{Rad} \mathscr{A}$.

Suppose the result has been proved for all ordinals less than some γ.

Let γ be a limit ordinal. If supp $f = K \subseteq X^c_{J,\gamma}$, we have $K \subseteq X^c_{J,\gamma} = \bigcup_{\delta<\gamma} X^c_{J,\delta}$; hence K can be covered by finitely many of the $X^c_{J,\delta}$, hence (since they are decreasing) by one of them. Thus f has compact support contained in some $X^c_{J,\delta}$ $(\delta < \gamma)$ and so $Bf \in \mathscr{S}_{J,\delta}$. Therefore $A = Bf \in \operatorname{Rad} \mathscr{A}$ by the induction hypothesis.

Now suppose that γ is a successor, $\gamma = \delta + 1$. By Lemma 14, we may write $f = \sum f_k$ where the support of f_k is compact and contained in an open set V_k such that $V_k \cap X_{J,\delta}$ is J-wandering for $(X_{J,\delta}, \Phi_{J,\delta})$, i.e.,

$$\phi_{\mathbf{n}}^{-1}(V_k \cap X_{J,\delta}) \cap (V_k \cap X_{J,\delta}) = \varnothing$$

when $\mathbf{n} \in \Sigma_J$. This can easily be seen to imply $\phi_{\mathbf{n}}^{-1}(V_k) \cap V_k \subseteq X^c_{J,\delta}$.

Let $C \in \mathscr{A}$ be arbitrary. Writing $A_k = DU_{e_J} f_k$ as above, it follows as in the proof of Lemma 8 that for each k all Fourier coefficients of A_kCA_k are supported in $V_k \cap \phi_{\mathbf{n}}^{-1}(V_k)$ (for some $\mathbf{n} \in \Sigma_J$) which is contained in $X^c_{J,\delta}$ by the previous paragraph.

Thus $A_kCA_k \in \mathscr{R}_{J,\delta}$. By the induction hypothesis, A_kCA_k must be contained in $\operatorname{Rad} \mathscr{A}$. Thus $(A_kC)^2$ is quasinilpotent, hence so is A_kC (by the spectral mapping theorem). Since $C \in \mathscr{A}$ is arbitrary, it follows that $A_k \in \operatorname{Rad} \mathscr{A}$ for each k, so that $A \in \operatorname{Rad} \mathscr{A}$.

Finally, we suppose that $\operatorname{PRad} \mathscr{A}$ is closed. Then the argument above can be repeated exactly up to the previous paragraph, changing $\operatorname{Rad} \mathscr{A}$ to $\operatorname{PRad} \mathscr{A}$. The previous paragraph can be replaced by the following argument.

Thus $A_kCA_k \in \mathscr{R}_{J,\delta}$. By the induction hypothesis, A_kCA_k must be contained in $\operatorname{PRad} \mathscr{A}$. Thus all products $(C_1A_kC_2)(C_3A_kC_4)$ are in $\operatorname{PRad} \mathscr{A}$ and so the (possibly non-closed) ideal $\mathscr{J}_k$ generated by A_k satisfies $\mathscr{J}_k\mathscr{J}_k \subseteq \operatorname{PRad} \mathscr{A}$. For every prime ideal $\mathscr{P}$, we have $\mathscr{J}_k\mathscr{J}_k \subseteq \mathscr{P}$ and so $\mathscr{J}_k \subseteq \mathscr{P}$. Hence $\mathscr{J}_k \subseteq \operatorname{PRad} \mathscr{A}$, and therefore $A_k \in \operatorname{PRad} \mathscr{A}$ for each k, so that $A \in \operatorname{PRad} \mathscr{A}$. ∎

One cannot conclude that $\mathscr{R}_{J,\gamma} \subseteq \operatorname{PRad} \mathscr{A}$ in general, even for finite γ, as the following example shows. Thus the prime radical is not always closed. Note that Hudson has given examples of TAF algebras in which the prime radical is not closed [11, Example 4.9].

EXAMPLE 17. We use a continuous dynamical system $(X, \{\phi_t\}_{t\in\mathbb{R}})$ based on [3, Example 3.3.4, p. 20] and look at the discrete system given by the maps $\{\phi_t\}$ for $t \in \mathbb{Z}_+$. The space X is the closed unit disc in $\mathbb{R}^2$. For the continuous system, the trajectories consist of: (i) three fixed points, namely the origin O and the points $A(1,0)$ and $B(-1,0)$ on the unit circle, (ii) the two semicircles on the unit circle joining A and B and (iii) spiraling trajectories emanating at the origin and converging to the boundary.

Let $\phi = \phi_1$. The recurrent points for the (discrete) dynamical system (X, ϕ) are $X_r = \{A, B, O\}$ and the set of wandering points is the open unit disc except the origin. Hence $X_2 = X_r$ and so the depth of the dynamical system is 2.

Now choose small disjoint open neighbourhoods V_A, V_B, V_O around the fixed points and let $f \in C(X)$ be a nonnegative function which is 1 outside these open sets and vanishes only at A, B and O. Then the element $Uf \in \mathscr{A}$ is clearly not nilpotent, so $Uf \notin \mathrm{PRad}\,\mathscr{A}$. However $Uf \in \mathrm{Rad}\,\mathscr{A}$ by the next theorem.

THEOREM 18. *Let (X, Φ) be a dynamical system with X metrisable. The Jacobson radical, $Rad(C_0(X) \times_\phi \mathbb{Z}_+^d)$, is the closed ideal generated by all monomials $U_{\mathbf{n}} f$ $(\mathbf{n} \neq \mathbf{0})$ where f vanishes on the set X_{Jr} of J-recurrent points corresponding to the support J of $\mathbf{n}$.*

Moreover, $PRad\,\mathscr{A} = Rad\,\mathscr{A}$ if and only if $PRad\,\mathscr{A}$ is closed.

Proof. Let $U_{\mathbf{n}} f$ be a monomial contained in $\mathrm{Rad}\,\mathscr{A}$ and let J be the support of $\mathbf{n}$. Then Lemma 5 shows that f must vanish on X_{Jr}.

On the other hand, let $U_{\mathbf{n}} f$ be as in the statement of the Theorem, so that f vanishes on X_{Jr} (where $J = \operatorname{supp} \mathbf{n}$). We will show that $U_{\mathbf{n}} f$ is in $\mathrm{Rad}\,\mathscr{A}$. It is enough to suppose that the support K of f is compact. Since K is contained in $(\overline{X_{Jr}})^c = \bigcup_\gamma X_{J,\gamma}^c$, it is contained in finitely many, hence one, $X_{J,\gamma}^c$. It follows by Proposition 16 that $U_{\mathbf{n}} f \in \mathrm{Rad}\,\mathscr{A}$.

In the final statement of the theorem, one direction is obvious. For the other, suppose $\mathrm{PRad}\,\mathscr{A}$ is closed. Then by the final statement of Proposition 16, we have $\mathscr{R}_{J,\gamma} \subseteq \mathrm{PRad}\,\mathscr{A}$. ∎

This theorem leaves open the possibility that the closure of the prime radical is always equal to the Jacobson radical.

REFERENCES

1. W. B. Arveson, Analyticity in operator algebras, *Amer. J. Math.* **89** (1967), 578–642.
2. W. B. Arveson and K. B. Josephson, Operator algebras and measure preserving transformations II, *J. Funct. Anal.* **4** (1969), 100–134.
3. N. P. Bhatia and G. P. Szegő, "Stability Theory of Dynamical Systems," Springer-Verlag, Berlin, 1970.
4. A. P. Donsig, Semisimple triangular AF algebras, *J. Funct. Anal.* **111** (1993), 323–349.

5. E. G. Effros and F. Hahn, "Locally Compact Transformation Groups and C^*-algebras," Mem. Amer. Math. Soc., 75, Amer. Math. Soc., Providence, RI, 1967.
6. H. Furstenberg, "Recurrence in Ergodic Theory and Combinatorial Number Theory," M. B. Porter Lectures, Princeton Univ. Press, Princeton, NJ, 1981.
7. W. H. Gottschalk and G. A. Hedlund, "Topological Dynamics," Amer. Math. Soc. Coll. Publ., Vol. 36, Amer. Math. Soc., Providence, RI, 1955.
8. S. Grabiner, The nilpotency of Banach nil algebras, *Proc. Amer. Math. Soc.* **21** (1969), 510.
9. D. W. Hadwin and T. B. Hoover, Operator algebras and the conjugacy of transformations, *J. Funct. Anal.* **77** (1988), 112–122.
10. D. W. Hadwin and T. B. Hoover, Operator algebras and the conjugacy of transformations II, *Proc. Amer. Math. Soc.* **106** (1989), 365–369.
11. T. D. Hudson, Radicals and prime ideals in limit subalgebras of AF algebras, *Quart. J. Math. Oxford Ser. (2)* **48** (1997), 213–233.
12. M. P. Lamoureux, Nest representations and dynamical systems, *J. Funct. Anal.* **114** (1993), 467–492.
13. M. P. Lamoureux, Ideals in some continuous nonselfadjoint crossed product algebras, *J. Funct. Anal.* **142** (1996), 211–248.
14. A. Manoussos, "Contribution to the Study of D-Stable Actions," Ph.D. thesis, University of Athens, Greece, 1993. [In Greek]
15. L. Mastrangelo, P. S. Muhly, and B. Solel, Locating the radical of a triangular operator algebra, *Math. Proc. Cambridge Philos. Soc.* **115** (1994), 27–38.
16. P. S. Muhly, Radicals, crossed products, and flows, *Ann. Polon. Math.* **43** (1983), 35–42.
17. T. W. Palmer, "Banach Algebras and the General Theory of *-Algebras, Vol. I," Encyclopedia of mathematics and its applications, No. 149, Cambridge Univ. Press, Cambridge, 1994.
18. J. R. Peters, Semi-crossed products of C^*-algebras, *J. Funct. Anal.* **59** (1984), 498–534.
19. J. R. Peters, The ideal structure of certain nonselfadjoint operator algebras, *Trans. Amer. Math. Soc.* **305** (1988), 333–352.
20. S. C. Power, Classification of analytic crossed products, *Bull. London Math. Soc.* **24** (1992), 368–372.