A PROOF OF TYCHONOFF'S THEOREM

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1. Nets and Compactness

Definition 1.1. We say that a net $\{x_\lambda\}$ has $x \in X$ as a *cluster point* if and only if for each neighborhood U of x and for each $\lambda_0 \in \Lambda$ there exist some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. In this case we say that $\{x_\lambda\}$ is *cofinally* (or *frequently*) in each neighborhood of x.

Theorem 1.2. *A net $\{x_\lambda\}$ has $y \in X$ as a cluster point if and only if it has a subnet which converges to y.*

Proof. Let y be a cluster point of $\{x_\lambda\}$. Define

$$M := \{(\lambda, U) : \lambda \in \Lambda, U \text{ a neighborhood of } y \text{ such that } x_\lambda \in U\},$$

and order M as follows: $(\lambda_1, U_1) \leq (\lambda_2, U_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $U_2 \subseteq U_1$. This is easily verified to be a direction on M. Define $\varphi : M \to \Lambda$ by $\varphi(\lambda, U) = \lambda$. Then φ is increasing and cofinal in Λ, so φ defines a subnet of $\{x_\lambda\}$. Let U_0 be any neighborhood of y and find $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} \in U_0$. Then $(\lambda_0, U_0) \in M$, and moreover, $(\lambda, U) \geq (\lambda_0, U_0)$ implies $U \subseteq U_0$, so that $x_\lambda \in U \subseteq U_0$. It follows that the subnet defined by φ converges to y.

Suppose $\varphi : M \to \Lambda$ defines a subnet of $\{x_\lambda\}$ which converges to y. Then for each neighborhood U of y, there is some u_U in M such that $u \geq u_U$ implies $x_{\varphi(u)} \in U$. Suppose a neighborhood U of y and a point $\lambda_0 \in \Lambda$ are given. Since $\varphi(M)$ is cofinal in Λ, there is some $u_0 \in M$ such that $\varphi(u_0) \geq \lambda_0$. But there is also some $u_U \in M$ such that $u \geq u_U$ implies $x_{\varphi(u)} \in U$. Pick $u^* \geq u_0$ and $u^* \geq u_U$. Then $\varphi(u^*) = \lambda^* \geq \lambda_0$, since $\varphi(u^*) \geq \varphi(u_0)$, and $x_{\lambda^*} = x_{\varphi(u^*)} \in U$, since $u^* \geq u_U$. Thus for any neighborhood U of y and any $\lambda_0 \in \Lambda$, there is some $\lambda^* \geq \lambda$ with $x_{\lambda^*} \in U$. It follows that y is a cluster point of $\{x_\lambda\}$. □

Theorem 1.3. *A topological space X is compact if and only if every net on X has a convergent subnet on X.*

Proof. Assume that X is compact, and suppose that we have a net $\{x_\lambda\}$ that does not have any convergent subnet. Hence, using the previous theorem, the net $\{x_\lambda\}$ does not have cluster points. This means that for each $x \in X$ we can find a neighborhood U_x of x and an index λ_x such that $x_\lambda \notin U_x$ for every $\lambda \geq \lambda_x$. Since X is compact then there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U_{x_i}$. Take any $\lambda \geq \lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_n}$. Then $x_\lambda \notin X$ which is a contradiction.

Assume that every net on X has a convergent subnet on X. We will show that X is compact. To this end take a family $\mathcal{F} = \{F_i : i \in I\}$ of closed subsets of X with the finite intersection property, that is $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n} \neq \emptyset$ for every $\{i_1, i_2, \ldots, i_n\} \subseteq I$. We will show that $\bigcap_{i \in I} F_i \neq \emptyset$. Define a net as follows: Let

$$\Lambda = \{\ \{i_1, i_2, \ldots, i_n\} : i_1, i_2, \ldots, i_n \in I \text{ and } n \in \mathbb{N}\},$$

and order Λ as follows: $\lambda_1 = \{i_1, i_2, \ldots, i_k\} \leq \lambda_2 = \{j_1, j_2, \ldots, j_n\}$ if and only if $\{i_1, i_2, \ldots, i_k\} \subseteq \{j_1, j_2, \ldots, j_n\}$. This is easily verified to be a direction on Λ. Since the family $\mathcal{F}$ has the finite intersection property then for every $\lambda = \{i_1, i_2, \ldots, i_n\} \in \Lambda$ we can find $x_\lambda \in F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n}$. Using our hypothesis, the net $\{x_\lambda\}$ has a convergent subnet, let say $\{x_{\lambda_m}\}$. That is, there exists $x \in X$ such that $x_{\lambda_m} \to x$. We will show that $x \in F_i$ for all $i \in I$. Fix some F_i. Hence, there exists m_0 such that $\lambda_{m_0} \geq \{i\}$. Thus, for every $\lambda_m = \{i_1, i_2, \ldots, i_n, i\} \geq \lambda_{m_0} \geq \{i\}$ we have that $x_{\lambda_m} \in F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n} \cap F_i \subseteq F_i$. Since $x_{\lambda_m} \to x$ and F_i is closed then $x \in F_i$. This finishes the proof of the theorem. □

2. Ultranets and Tychonoff's Theorem

Definition 2.1. A net $\{x_\lambda\}$ in a set X is an *ultranet* (*universal net*) if and only if for each subset E of X, $\{x_\lambda\}$ is either residually in E or residually in $X \setminus E$.

Remark 2.2. It follows from this definition that an ultranet must converge to each of its cluster points since if an ultranet is frequently in a set E then it is residually in E. A trivial example of an ultranet is the following: For any directed set Λ, the map $P : \Lambda \to X$, defined by $P(\lambda) = x$ for a fixed point $x \in X$ and for all $\lambda \in \Lambda$, gives an ultranet on X, called the *trivial ultranet*.

Theorem 2.3. *Every net $\{x_\lambda\}$ has a subnet which is an ultranet.*

Proof. The proof follows by *Zorn's Lemma* but this is beyond the scope of these short notes. □

Theorem 2.4. *Let X, Y be two non-empty sets. If $\{x_\lambda\}$ is an ultranet in X and $f : X \to Y$ is a map, then $\{f(x_\lambda)\}$ is an ultranet.*

Proof. If $B \subseteq Y$, then $f^{-1}(B) = X \setminus f^{-1}(Y \setminus B)$, so $\{x_\lambda\}$ is eventually in either $f^{-1}(B)$ or $f^{-1}(Y \setminus B)$, from which it follows that $\{f(x_\lambda)\}$ is eventually in B or $Y \setminus B$. Thus, $\{f(x_\lambda)\}$ is an ultranet. □

Theorem 2.5 (**Tychonoff**). *A non-empty product* $\prod_{i \in I} X_i$ *is compact if and only if each factor* X_i *is compact.*

Proof. If the product space is non-empty, then the projection maps $pr_i : \prod_{i \in I} X_i \to X_i$ are all continuous surjections, so each factor X_i is compact.

For the converse implication assume that X_i is compact for all $i \in I$. Let $\{x_\lambda\}$ be a net in $\prod_{i \in I} X_i$. By Theorem 2.3, $\{x_\lambda\}$ has a subnet $\{x_{\lambda_m}\}$ which is an ultranet. Then, by Theorem 2.4, for each fixed $i \in I$, the net $\{pr_i(x_{\lambda_m})\}$ is an ultranet in X_i, hence has a convergent subnet in X_i (see Theorem 1.3). So, by Remark 2.2, the net $\{pr_i(x_{\lambda_m})\}$ converges in X_i from which it follows that $\{x_{\lambda_m}\}$ converges in $\prod_{i \in I} X_i$. Thus, by Theorem 1.3, the product $\prod_{i \in I} X_i$ is compact. □

References

[1] Willard S., *General Topology.* Reprint of the 1970 original [Addisson-Wesley, Reading, MA]. Dover Publications, Inc., Mineola, NY, 1995.

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