A BRIEF ANALYSIS OF MY PUBLICATIONS RESEARCH STATEMENT

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List of Publications in chronological order

[1] G. Costakis and A. Manoussos, *J-class operators and hypercyclicity*, J. Operator Theory. **67** (2012), 101-119.

Abstract. The purpose of the present work is to treat a new notion related to linear dynamics, which can be viewed as a "localization" of the notion of hypercyclicity. In particular, let T be a bounded linear operator acting on a Banach space X and let x be a non-zero vector in X such that for every open neighborhood $U \subset X$ of x and every non-empty open set $V \subset X$ there exists a positive integer n such that $T^n U \cap V \neq \emptyset$. In this case T will be called a J-class operator. We investigate the class of operators satisfying the above property and provide various examples. It is worthwhile to mention that many results from the theory of hypercyclic operators have their analogues in this setting. For example we establish results related to the Bourdon-Feldman theorem and we characterize the J-class weighted shifts. We would also like to stress that even some non-separable Banach spaces which do not support topologically transitive operators, as for example $l^\infty(\mathbb{N})$, do admit J-class operators.

[2] H. Abels and A. Manoussos *Topological generators of abelian Lie groups and hypercyclic finitely generated abelian semigroups of matrices*, Adv. Math. **229** (2012), 1862-1872.

Abstract. In this paper we bring together results about the density of subsemigroups of abelian Lie groups, the minimal number of topological generators of abelian Lie groups and a result about actions of algebraic groups. We find the minimal number of generators of a finitely generated abelian semigroup or group of matrices with a dense or a somewhere dense orbit by computing the minimal number of generators of a dense subsemigroup (or subgroup) of the connected component of the identity of its Zariski closure.

[3] H. Abels, A. Manoussos and G. Noskov, *Proper actions and proper invariant metrics*, J. London Math. Soc. (2) **83** (2011), 619-636.

Abstract. We show that if a locally compact group G acts properly on a locally compact σ-compact space X, then there is a family of G-invariant proper (Heine-Borel) continuous finite-valued pseudometrics which induces the topology of X. If X is, furthermore, metrizable, then G acts properly on X if and only if there exists a G-invariant proper compatible metric on X.

[4] A. Manoussos, *On the action of the group of isometries on a locally compact metric space*, Münster J. Math. **3** (2010), 209-212.

ABSTRACT. In this short note we give an answer to the following question. Let X be a locally compact metric space with group of isometries G. Let $\{g_i\}$ be a net in G for which $g_i x$ converges to y, for some $x, y \in X$. What can we say about the convergence of $\{g_i\}$? We show that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_y$, where C_x and C_y denote the pseudo-components of x and y respectively. Recall that pseudo-components are the equivalence classes C_x of the following equivalence relation: $x \sim y$ if and only if x and y, as also y and x, can be connected by a finite sequence of intersecting open balls with compact closure. Applying this we give short proofs of the van Dantzig–van der Waerden theorem (1928) and Gao–Kechris theorem (2003).

[5] A. Manoussos, *The group of isometries of a locally compact metric space with one end*, Topology Appl. **157** (2010), 2876-2879.

ABSTRACT. In this note we study the dynamics of the natural evaluation action of the group of isometries G of a locally compact metric space (X, d) with one end. Using the notion of pseudo-components introduced by S. Gao and A.S. Kechris we show that X has only finitely many pseudo-components exactly one of which is not compact and G acts properly on this pseudo-component. The complement of the non-compact component is a compact subset of X and G may fail to act properly on it.

[6] G. Costakis, D. Hadjiloucas and A. Manoussos, *On the minimal number of matrices which form a locally hypercyclic, non-hypercyclic tuple*, J. Math. Anal. Appl. **365** (2010), 229-237.

ABSTRACT. In this paper we extend the notion of a locally hypercyclic operator (J-class) to that of a locally hypercyclic tuple of operators. We then show that the class of hypercyclic tuples of operators forms a proper subclass to that of locally hypercyclic tuples of operators. What is rather remarkable is that in every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, a pair of commuting matrices exists which forms a locally hypercyclic, non-hypercyclic tuple. This comes in direct contrast to the case of hypercyclic tuples where the minimal number of matrices required for hypercyclicity is related to the dimension of the vector space. In this direction we prove that the minimal number of diagonal matrices required to form a hypercyclic tuple on $\mathbb{R}^n$ is $n+1$, thus complementing a recent result due to Feldman.

[7] G. Costakis, D. Hadjiloucas and A. Manoussos, *Dynamics of tuples of matrices*, Proc. Amer. Math. Soc. **137** (2009), 1025-1034.

ABSTRACT. In this article we answer a question raised by N. Feldman in 2008 concerning the dynamics of tuples of operators on $\mathbb{R}^n$. In particular, we prove that for every positive integer $n \geq 2$ there exist n-tuples $(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices over $\mathbb{R}$ such that $(A_1, A_2, \ldots, A_n)$ is hypercyclic. We also establish related results for tuples of 2×2 matrices over $\mathbb{R}$ or $\mathbb{C}$ being in Jordan form.

[8] G. Costakis and A. Manoussos, *J-class weighted shifts on the space of bounded sequences of complex numbers*, Integral Equations Operator Theory **62** (2008), 149-158.

Abstract. We provide a characterization of J-class and J^{mix}-class unilateral weighted shifts on $l^{\infty}(\mathbb{N})$ in terms of their weight sequences. In contrast to the previously mentioned result we show that a bilateral weighted shift on $l^{\infty}(\mathbb{Z})$ cannot be a J-class operator.

[9] A. Manoussos and P. Strantzalos, *On embeddings of proper and equicontinuous actions in zero-dimensional compactifications*, Trans. Amer. Math. Soc. **359** (2007), 5593-5609.

Abstract. We provide a tool for studying properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces, by embedding such an action in a suitable zero-dimensional compactification of the underlying space with pleasant properties. Precisely, given such an action (G, X) we construct a zero-dimensional compactification μX of X with the properties: (a) there exists a continuous extension of the action on μX, (b) if $\mu L \subset \mu X \setminus X$ is the set of the limit points of the orbits of the initial action in μX, then the restricted action $(G, \mu X \setminus \mu L)$ remains properly discontinuous, is indivisible and equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX and (c) μX is the maximal among the zerodimensional compactifications of X with these properties. Proper actions are usually embedded in the endpoint compactification εX of X, in order to obtain topological invariants concerning the cardinality of the space of the ends of X, provided that X has an additional "nice" property of rather local character ("property Z", i.e. every compact subset of X is contained in a compact and connected one). If the considered space has this property, our new compactification coincides with the endpoint one. On the other hand, we give an example of a space not having the "property Z" for which our compactification is different from the endpoint compactification. As an application, we show that the invariant concerning the cardinality of the ends of X holds also for a class of actions strictly containing the properly discontinuous ones and for spaces not necessarily having "property Z".

[10] A. Manoussos and P. Strantzalos, *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), 7-12.

Abstract. In the present paper we study conditions under which the group of isometries on a locally compact metric space is locally compact, or acts properly.

[11] A.P. Donsig, A. Katavolos and A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. **187** (2001), 129-145.

Abstract. We characterise the Jacobson radical of an analytic crossed product $C_0(X)\times_{\phi}\mathbb{Z}_{+}$, answering a question first raised by Arveson and Josephson in 1969. In fact, we characterise the Jacobson radical of analytic crossed products $C_0(X)\times_{\phi}\mathbb{Z}_{+}^{d}$. This consists of all elements whose "Fourier coefficients" vanish on the recurrent points of the dynamical system (and the first one is zero). The multidimensional version requires a variation of the notion of recurrence, taking into account the various degrees of freedom.

[12] K. Athanassopoulos and A. Manoussos, *Minimal flows on multipunctured surfaces*, Bull. London Math. Soc. **27** (1995), 595-598.

ABSTRACT. In this paper we make a full study of minimal flows on open 2-manifolds of the type $M \setminus F$, where M is a compact 2-manifold and F is a non-empty, closed and totally disconnected subset of M (i.e. F is the set of ends of M).

[13] H. Abels and A. Manoussos, *A group of isometries with non-closed orbits*, arXiv:0910.4717, submitted.

ABSTRACT. In this note we give an example of a one-dimensional manifold with two connected components and a complete metric whose group of isometries has an orbit which is not closed. This answers a question of S. Gao and A.S. Kechris.

[14] H. Abels and A. Manoussos, *Linear semigroups with coarsely dense orbits*, arXiv:1108.2221, submitted.

ABSTRACT. Let S be a finitely generated abelian semigroup of invertible linear operators on a finite dimensional vector space V. We show that every coarsely dense orbit of S is actually dense in V. More generally, if the orbit contains a coarsely dense subset of some open cone C in V then the closure of the orbit contains the closure of C. In the complex case the orbit is then actually dense in V. For the real case we give precise information about the possible cases for the closure of the orbit.

[15] A. Manoussos, *A Birkhoff's transitivity type theorem for non-separable complete metric spaces with applications to Linear Dynamics*, arXiv:1105.2429, submitted.

ABSTRACT. In this note we prove a Birkhoff's transitivity type theorem for non-separable complete metric spaces and we give some applications for dynamics of bounded linear operators on F-spaces (i.e. complete and metrizable vector spaces). Among them we show that any positive power and any unimodular multiple of a topologically transitive linear operator is topologically transitive, generalizing similar results of Ansari and León-Müller for hypercyclic operators.

[16] A. Manoussos and P. Strantzalos, *Properness, Cauchy-indivisibility and the Weil completion of a group of isometries*, arXiv:1105.0557, submitted.

ABSTRACT. Investigating the impact of local compactness and connectedness in the theory of proper actions on locally compact and connected spaces, we introduce a new class of isometric actions on separable metric spaces called "Cauchy-indivisible" actions. The new class coincides with that of proper actions on locally compact metric spaces, without assuming connectivity, and, as examples show, may be different in general. In order to provide some basic theory for this new class of actions, we embed a "Cauchy-indivisible" action in a proper action of a semigroup in the completion of the underlying space. We show that, in case this semigroup is a group, there are remarkable connections between "Cauchy-indivisibility" and properness, while the original group has a "Weil completion" and vice versa. Further connections in this direction

establish a relation between “Borel sections” for “Cauchy-indivisible” actions and “fundamental sets” for proper actions. Some open questions are added.

Other publications

(1) *Limit sets and asymptotic methods in Operator Theory, Topological Transformation Groups and Dynamical Systems*, Habilitation Thesis, Dept. of Mathematics, University of Bielefeld, Germany, 2010.
(2) *Contribution to the study of D-stable Actions*, Ph.D. Thesis, Dept. of Mathematics, University of Athens, 1993 (in Greek).

Research Interests

The main characteristic of my research work is that I bring together methods and ideas from the theories of Topological Transformation Groups, Dynamical Systems, Actions of Algebraic and Lie Groups, Dynamics of Linear Operators and Operator Theory in order to study the dynamic behavior of various objects in these theories.

A brief analysis of my publications

For our convenience we divided the publications into three categories, namely papers on Linear Dynamics on finite and infinite dimensional vector spaces, Topological Transformation Groups and Dynamical Systems. They share common ideas, problems and even common methods when possible. A general characteristic is the use of asymptotic methods and various concepts of limit sets coming from the stability theory of Dynamical Systems which we describe in brief in the following.

Let X be a Hausdorff locally compact space or a (complex or real) Hilbert or Banach space and G be a locally compact group acting on X or G be the semigroup of non-negative integers generated by a continuous map or a bounded linear operator on X. For $x \in X$ the *limit set* $L(x)$ is defined by

$$L(x) = \{y \in X \,|\, \text{there exists a divergent net } \{g_i\}_{i\in I} \subset G \text{ such that } g_i x \to y\}$$

and the *extended (prolongational) limit set* $J(x)$ is defined by

$$J(x) = \{y \in X \,|\, \text{there exist a divergent net} \{g_i\}_{i\in I} \subset G \text{ and a net } \{x_i\}_{i\in I} \subset X \text{ converging to } x \text{ such that } g_i x_i \to y\}.$$

So we can say that limit sets describe the limit behavior of an orbit and generalized limit sets describe the asymptotic behavior of the orbits of nearby points to $x \in X$. In the cases we study the limit and the generalized limit sets are closed and invariant sets. Limit and extended limit sets have their roots in the Qualitative Theory of Dynamical Systems when they are used mainly to describe the Lyapunov and the asymptotic stability of an equilibrium point or, more generally, of a compact minimal set. They, also, "encode" information which allows us to connect the global structure of the underlying space with local properties, like we do with proper and properly discontinuous actions for example where the limit and the extended sets are empty.

1. Linear Dynamics on finite and infinite dimensional vector spaces

Before, we present our results, and for our convenience, let us recall some concepts. A topologically transitive operator is a bounded linear operator T on a Banach space X such that $J(x) = X$ for every $x \in X$ or, in other words, for every pair of non-empty open sets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \emptyset$. A bounded linear operator on a separable Banach space is hypercyclic if it has the property that $L(x) = X$ for some non-zero vector $x \in X$ (i.e. the orbit of x is dense in X). Actually the existence of one (non-zero) vector $x \in X$ such that $L(x) = X$ is enough to ensure that the set of vectors with this property is a dense G_δ subset of X. Obviously in the case of a hypercyclic operator $T : X \to X$, the space X must be separable and T is a topologically transitive operator. For separable spaces the converse is also true: Birkhoff's Transitivity Theorem says that a topologically transitive operator on a separable Banach space is hypercyclic. Some examples of hypercyclic operators are the following (a) the translation operator $T_\alpha : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $T_\alpha(f) = f(z+\alpha)$, where $z \in \mathbb{C}$, α is a non-zero complex number and $H(\mathbb{C})$ is the space of holomorphic functions on $\mathbb{C}$ (G. D. Birkhoff 1929), (b) the differentiation operator on $H(\mathbb{C})$ (G. R. MacLane 1952) and (c) for every scalar λ of modulus greater than 1 the operator λB on $l^p(\mathbb{N})$ for each $1 < p < +\infty$ where B is the backward shift on $l^p(\mathbb{N})$ (S. Rolewicz 1969). Actually the hypercyclic operators in the previous examples have also the additional property that the set of periodic points is dense and they are chaotic (in the sense of R. L. Devaney).

In the main result in [1] we used information of local nature (the generalized set of a cyclic vector has non-empty interior) and we got, as a result, the global behavior of an operator (that it is topologically transitive). Precisely, we showed that if x is a cyclic vector for an operator $T : X \to X$ and the set $J(x)$ has non-empty interior then $J(y) = X$ for every $y \in X$, hence T is topologically transitive (hypercyclic), without x being necessarily a hypercyclic vector (i.e. a vector with dense orbit). An important implication of this theorem is that it gives the Bourdon-Feldman Theorem as a corollary. Bourdon-Feldman's Theorem [1, reference 11] says that somewhere dense orbits are everywhere dense and plays an important role in the theory of hypercyclic operators. This result gave us also the idea to "localize" the notion of a topological transitive operator by introducing and studying a new class of operators called locally topologically transitive or J-class operators. This class of operators is characterized by the property that there exists a non-zero vector $x \in X$ with $J(x) = X$. The reason we excluded the zero vector is to avoid certain trivialities, as for example the multiples of the identity operator acting on a finite or infinite dimensional space. Hypercyclic and J-class operators can occur only in infinite dimensional spaces. As it turns out this new notion of operators although different from the notion of hypercyclic operators shares some similarities with the behavior of hypercyclic operators. Note that no compact, positive or normal operators can be J-class. We would like to stress that some non-separable Banach spaces, like the space $l^\infty(\mathbb{N})$ of bounded sequences, supports J-class operators (in [1] we showed that the operator λB for every scalar λ of modulus greater than 1 is J-class, where B is the backward shift on $l^\infty(\mathbb{N})$), while it is known [1, reference 3] that the space $l^\infty(\mathbb{N})$ does not support topologically transitive operators. The arguments we used in this work are quite similar to those we used to study isometric actions plus the additional structure of linearity.

The scope of [15] was to provide a "tool" for studying topologically transitive operators on non-separable F-spaces (i.e. complete and metrizable vector spaces) by using technics and already known results from the theory of hypercyclic operators. This "tool" is the following theorem in which we showed that each vector in the underlying space X is contained in "plenty" closed invariant hypercyclic subspaces: "Let $T : X \to X$ be a topological transitive operator on an F-space X and let y be a vector of X. Then there exists a G_δ dense subset D of X such that for each $z \in D$ there exists a T-invariant (separable) closed subspace Y_z of X with $y, z \in Y_z$ and such that the restriction $T : Y_z \to Y_z$ is hypercyclic." The previous theorem arrives from a Birkhoff's transitivity type theorem for topologically transitive continuous selfmaps on (not necessarily separable) complete metric spaces: "Let $T : X \to X$ be a continuous map on a complete metric space X without isolated points and let $x \in X$. If T is topologically transitive there exists a G_δ dense subset D of X with the following properties: (i) Every point $z \in D$ is recurrent, that is there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $T^{k_n}z \to z$ and (ii) The point x belongs to the orbit closure $\overline{O(z,T)}$ of z for every $z \in D$; especially x belongs to the limit set of z for every $z \in D$, i.e.

there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $T^{k_n}z \to x$." Recall that Birkhoff's transitivity theorem states that topological transitivity is equivalent to hypercyclicity, i.e. there exists a point in $x \in X$ which has a dense orbit. The applications we gave in [15] indicate the possibility to use the previous mentioned results to work on non-separable F-spaces by applying technics and already known results from the theory of hypercyclic operators. As an example we showed that any positive power and any unimodular multiple of a topologically transitive linear operator is topologically transitive, generalizing similar results of Ansari and León-Müller for hypercyclic operators.

In [8] we provided a characterization of J-class unilateral weighted shifts on $l^\infty(\mathbb{N})$ in terms of their weight sequences and we described the set of the J-vectors (i.e. vectors $x \in l^\infty(\mathbb{N})$ such that $J(x) = l^\infty(\mathbb{N})$). In contrast to the previously mentioned result we showed that a bilateral weighted shift on $l^\infty(\mathbb{Z})$ cannot be a J-class operator.

As we mentioned above, hypercyclic and J-class operators can occur only in infinite dimensional spaces. This is in contrast with the case of hypercyclic and J-class commuting tuples of matrices. In [6] we extended the notion of a J-class operator to that of a J-class tuple of operators. We then showed that the class of hypercyclic tuples of operators forms a proper subclass to that of J-class tuples of operators. What is rather remarkable is that in every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, a pair of commuting matrices exists which forms a J-class non-hypercyclic tuple. This comes in direct contrast to the case of hypercyclic tuples where the minimal number of matrices required for hypercyclicity is related to the dimension of the vector space. Finally in [6], as also in [7], we gave some complementing results concerning hypercyclic and J-class commuting pairs of matrices in diagonal or in upper triangular form.

In [2] we brought together results about the density of subsemigroups of abelian Lie groups, the minimal number of topological generators of abelian Lie groups and a result about actions of algebraic groups. We determine the minimal number of generators of a finitely generated group or semigroup of commuting matrices with real or complex entries with a dense or a somewhere dense orbit for several classes of matrices by computing the minimal number of generators of a dense subsemigroup (or subgroup) of the connected component of the identity of its Zariski closure. We would like to point out that for this number there is no difference between dense and somewhere dense orbits and also no difference between the group and the semigroup cases.

This follows from the following theorem which is the basic result of the present paper: “Let V be a finite dimensional real vector space and let S be a subsemigroup of $GL(V)$. For a point $x \in V$ we say that x has a somewhere dense orbit if the closure $\overline{S(x)}$ of its orbit $S(x)$ contains a non-empty open subset of V. Let S be a finitely generated commutative subsemigroup of $GL(V)$ and let $x \in V$ be a point which has a somewhere dense orbit. Let G be the Zariski closure of S and let G^0 be its connected component of the identity with respect to the Euclidean topology. Then the orbit $G(x)$ of x is an open subset of V, the natural map $G \to G(x)$, $g \mapsto gx$, is a diffeomorphism and the closure of S is a subgroup of G and contains G^0.”

Let S be a finitely generated abelian semigroup of invertible linear operators on a finite dimensional vector space V. Recall that a subset Y of a metric space (X, d) is called coarsely dense if there is a positive constant D such that the union of balls with radius D and center at points of Y covers X. In [14] we showed that every coarsely dense orbit of S is actually dense in V. More generally, if the orbit contains a coarsely dense subset of some open cone C in V then the closure of the orbit contains the closure of C. In the complex case the orbit is then actually dense in V. For the real case we give precise information about the possible cases for the closure of the orbit. Actually we showed a more general result, namely that if an orbit $\mathcal{O}$ of S has a subset which is coarsely dense in some open cone C in V then the closure of $\mathcal{O}$ is a cone which contains C. We also gave detailed information about the structure of the orbits of $\overline{S}$, as follows. There are only finitely many maximal $\overline{S}$-invariant vector subspaces of V. Let U be the complement of their union. Then U has a finite number of connected components which are open cones in V. For every $v \in U$ the orbit $\overline{S} \cdot v$ is a union of connected components of U, in particular an open cone.

2. Topological Transformation Groups

One of the most important notions related to the limit and to the extended limit sets in the theory of Topological Transformation Groups is the notion of a proper action. Proper actions are characterized by the property $J(x) = L(x) = \emptyset$ for every $x \in X$. In case G is a locally compact group we have the usual definition: an action is proper if for every $x, y \in X$ there exist neighborhoods U and V of x and y, respectively, such that the set $\{g \in G \,|\, gU \cap V \neq \emptyset\}$ has compact closure in G. The next interesting class of actions related to the limit and generalized limit sets is the class where $J(x) = L(x)$ holds for every

$x \in X$ but $J(x)$ may not be empty. This class contains the isometric actions.

In [3] we characterized proper actions in terms of the geometry of the underlying space. Namely, we showed that a locally compact group G acts properly on a locally compact σ-compact metrizable space X if and only if there exists a G-invariant proper (Heine-Borel) compatible metric on X. A few words concerning terminology. A σ-compact space is a topological space that can be written as a countable union of compact sets. For locally compact metrizable spaces this is equivalent to separability. By a proper (or Heine-Borel) metric we mean a metric such that all balls of bounded radius have compact closures. In other words the previous result says that we can consider the group G, modulo the kernel of the action, as a closed subgroup of the group of isometries of a locally compact σ-compact metrizable space. Removing the assumption about metrizability for X we generalized the previous result as follows. If a locally compact group G acts properly on a locally compact σ-compact space X then there is a family of G-invariant proper continuous finite-valued pseudometrics which induces the topology of X. We showed also a converse result: let X be a topological space and let $\mathcal{D}$ be a family of proper continuous finite-valued pseudometrics on X, which induces the topology of X. Let G be the group of all bijective maps $X \to X$, leaving every $d \in \mathcal{D}$ invariant. If we endow G with the compact-open topology then G is a locally compact topological group and acts properly on X.

Another important class of transformation groups is the class in which $J(x) = L(x)$ holds for every $x \in X$. As we mentioned above this class contains the isometric actions. One of the first problems studied in this direction was the problem of the local compactness of the group of isometries and the way it acts on the underlying space. A classic result is the theorem of D. van Dantzig and B. L. van der Waerden which says that the group G of isometries of a connected, locally compact metric space X is locally compact (with respect to the compact-open topology) and acts properly on X (via the natural action $(g, x) \mapsto g(x)$ $g \in G$, $x \in X$). Proper and isometric actions are closely related as we showed in [3] but in general isometric actions are not proper. In [4] and [10] we studied the dynamic behavior of the action of the group of isometries of a locally compact metric space. Since such an action is not necessarily proper the idea is to look for "thick" (i.e. closed-open) invariant subsets of the underlying space where the action behaves like a proper one. To be more precise, in [10] we generalized the results of D. van Dantzig and B. L. van der Waerden for the case of a locally compact metric space which has quasi-compact (i.e. compact but not necessarily Hausdorff) space of connected components (or quasi-components). In particular it is shown that the group of isometries of X is locally compact but may fail to act properly on X even for the case that X has only two connected components. In [4] we gave an answer to the following question: Let X be a locally compact metric space with group of isometries G and let $\{g_i\}$ be a net in G for which $g_i x$ converges to y, for some $x, y \in X$. What can we say about the convergence of $\{g_i\}$? In [4] we showed that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_y$, where C_x and C_y denote the pseudo-components of x and y respectively. Recall that pseudo-components are the equivalence classes C_x of the following equivalence relation: $x \sim y$ if and only if x and y, as also y and x, can be connected by a finite sequence of intersecting open balls with compact closure. Applying the theorem above we gave short proofs of the van Dantzig-van der Waerden theorem (1928), the main result of [10] and a theorem of S. Gao and A.S. Kechris (2003).

In [13, reference 3] S. Gao and A.S. Kechris asked the following question. Let (X, d) be a locally compact complete metric space with finitely many pseudo-components or connected components. Does its group of isometries have closed orbits? This is the case if X is connected since then the group of isometries acts properly by the result of van Dantzig and van der Waerden we mentioned above and hence all of its orbits are closed. The above question arose in the following context. Suppose a locally compact group with a countable base acts on a locally compact space with a countable base. Then the action has locally closed orbits (i.e. orbits which are open in their closures) if and only if there exists a Borel section for the action (see [13, reference 4], [13, reference 2]) or, in other terminology, the corresponding orbit equivalence relation is smooth. For isometric actions it is easy to see that an orbit is locally closed if and only if it is closed. In [13] we gave a negative answer to the question of Gao and Kechris. Our space is a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. In the course of the construction we gave an example of a 2-dimensional manifold with two connected components one compact and one non-compact and a complete metric whose group G of isometries also has non-closed dense orbits on the compact component. The difference is that G contains a subgroup of index 2 which is isomorphic to $\mathbb{R}$.

In [16], investigating the impact of local compactness and connectedness in the theory of proper actions on locally compact and connected spaces, we introduced a new class of isometric actions on separable metric spaces called "Cauchy-indivisible" actions. In particular, let (G, X) be a continuous action of a topological group G on a metric space X. The action is said to be Cauchy-indivisible if the following holds: If $\{g_i\}$ is a net in G such that $g_i \to \infty$ in G and $\{g_i x\}$ is a Cauchy net in X for some $x \in X$ then $\{g_i x\}$ is a Cauchy net for every $x \in X$. The new class coincides with that of proper actions on locally compact metric spaces, without assuming connectivity, and, as examples show, may be different in general. In order to provide some basic theory for this new class of actions, we embed a "Cauchy-indivisible" action in a proper action of a semigroup in the completion of the underlying space. We show that, in case this semigroup is a group, there are remarkable connections between "Cauchy-indivisibility" and properness, while the original group has a "Weil completion" and vice versa. Finally in the following theorem we established a relation between "Borel sections" for "Cauchy-indivisible" actions and "fundamental sets" for proper actions: "Let G be a group which acts properly on a locally compact space X, and suppose that the orbit space $G\backslash X$ is paracompact. Let S be a section for the action (G, X). Then the following hold: (i) For every open neighborhood U of S we can construct a closed fundamental set F_c and an open fundamental set F_o such that $F_c \subset F_o \subset U$ and (ii) If, in addition, (X, d) is a separable metric space, in which case the action (G, X) is Cauchy-indivisible, then there exists a Borel section S_B, which is also a fundamental set, such that $S_B \subset F_c \subset F_o \subset U$.

There is a remarkable invariant concerning the cardinality of the ends of a locally compact and connected space with the "property Z" which admits a proper action of a non-compact group. "Property Z" is a certain technical connectedness assumption: a space X has "property Z" if every compact subset of X is contained in a compact and connected one, for instance every locally compact connected and locally connected space has "property Z". When we say ends we mean the remainder of X in the end-point (Freudenthal) compactification εX of X. As it is proved in [9, reference 2] X has at most two or infinitely many ends. Combining this result with the van Dantzig-van der Waerden theorem we get the following remarkable implication. For locally compact locally connected and connected metric space (e.g. a finite dimensional manifold) with finitely many but more than two ends the group of isometries is compact. In [9] we provided a tool for studying properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces, by embedding such an action in a suitable zero-dimensional compactification (i.e. a compactification such that X has compact totally disconnected remainder) of the underlying space with pleasant properties. Precisely, given such an action we constructed a zero-dimensional compactification μX of X which is the maximal (in the ordering of zero-dimensional compactifications of X) with respect to the following properties: (a) the action has a continuous extension on μX, (b) if μL denotes the set of the limit points of the orbits of the initial action in μX, the restricted action of G on $\mu X \setminus \mu L$ remains properly discontinuous, is equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX (so all the information concerning the invariants is contained in the set μL) and (c) the action is indivisible, i.e. if $\lim g_i x_0 = e \in \mu L$ for some $x_0 \in \mu X \setminus \mu L$ and a net $\{g_i\}$ in G, then $\lim g_i y = e$ for every $x \in \mu X \setminus \mu L$ (so actually there is a correspondence between divergent nets in G and limit points in μL). As we showed by an example there is a locally compact, connected and paracompact space not having the "property Z" for which our compactification is different from the end point compactification. So, if X doesn't have the "property Z" εX may fail to have the above mentioned properties. The construction of the compactification μX stated above relies on a new construction: The action of G on μX is obtained by taking the initial action as an equivariant inverse limit of properly discontinuous G-actions on polyhedra, which are constructed via G-invariant locally finite open coverings of X, generated by locally finite coverings of (always existing) suitable fundamental sets of the initial action. As an application of the previously mentioned construction we have that μL consists of at most two or infinitely many points. Another result is that if X has the "property Z" then μX coincides with the end point compactification εX of X. Finally, we gave an application concerning the cardinality of the ends of X. To be more precise, let X be a locally compact, connected and paracompact space, and G be a non-compact group acting properly on X such that either G_1, the connected component of the neutral element of G, is non-compact, or G_1 is compact and G/G_1 contains an infinite discrete subgroup. Then X has at most two or infinitely many ends, and has at most two ends, if G_1 is not compact.

In [5] we studied the action of the group of isometries G of a locally compact metric space X with one end. Using technics we developed in [4] , we showed that X has only finitely many pseudo-components exactly one of which is not compact and G acts properly on this pseudo-component. The complement of the non-compact component is a compact subset of X and G may fail to act properly on it.

3. Dynamical Systems

In [11] **we answered a long standing question (open for more than 30 years)** asked by W.B. Arveson and K.B. Josephson in 1969 concerning the problem of the description of the radical of the analytic crossed product of a classical dynamical system in terms of the dynamic behavior of the system. The analytic crossed product of a classical dynamical system is a non self adjoint algebra of operators that characterizes the dynamical system. Two dynamical systems are topologically conjugate if and only if the corresponding analytic crossed products are isomorphic as algebras. There is a rich interplay between operator algebras and dynamical systems, going back to the founding work of F.J. Murray and J. von Neumann in the 1930's. Crossed product constructions continue to provide fundamental examples of von Neumann algebras and C^*-algebras as also remarkable results in the theory of dynamical systems. In 1967 W.B. Arveson introduced a non-selfadjoint crossed product construction, called the analytic crossed product or the semi-crossed product, which has the remarkable property of capturing all of the information about the dynamical system. By this we mean that two analytic crossed product algebras are isomorphic as complex algebras if and only if the underlying dynamical systems are topologically conjugate, i.e. there is a homeomorphism between the spaces that intertwines the two actions. The construction of an analytic crossed product starts with a dynamical system, i.e. a locally compact Hausdorff space X and a continuous, proper surjection $\phi : X \to X$. Consider the algebra generated by $C_0(X)$ (i.e. the space of continuous functions of X that vanish at infinity) and a symbol U, where U satisfies the relation $fU = U(f \circ \phi)$, $f \in C_0(X)$. The elements F of this algebra can be viewed as noncommutative polynomials in U of the form $F = \sum_{n=0}^{N} U^n f_n$, $f_n \in C_0(X)$, $N \in \mathbb{N}$. Let us call this algebra $\mathcal{A}_0$. We formed the Banach Algebra $l_1(\mathcal{A}_0)$ by providing a norm to elements F as above by setting $\|F\|_1 = \sum_{n=0}^{N} \|f_n\|_{C_0(X)}$ and then completing $\mathcal{A}_0$ in this norm. On the other hand, we can define the class of covariant representations of $\mathcal{A}_0$ and complete $\mathcal{A}_0$ in the resulting norm. Either approach yields the same analytic crossed product $C_0(X) \times_\phi \mathbb{Z}_+$. By a covariant representation of $\mathcal{A}_0$ we mean a homomorphism π of $\mathcal{A}_0$ into the bounded operators of a Hilbert space, which is a $*$-representation when restricted to $C_0(X)$, viewed as a subalgebra of $\mathcal{A}_0$, and such that $\pi(U)$ is an isometry. Let us denote an element of the analytic crossed product by $\sum_{n=0}^{+\infty} U^n f_n$, $f_n \in C_0(X)$ and let us call the sequence $\{f_n\}$ the corresponding Fourier coefficients. A long standing question asked by W.B. Arveson and K.B. Josephson in 1969 was to characterize the Jacobson radical of the analytic crossed product in terms of the dynamic behavior of the system. Recall that the Jacobson radical of an algebra is the intersection of all primitive ideals, i.e. the intersection of kernels of all irreducible representations of the algebra. If the Jacobson radical is zero then the algebra is called semisimple. In [11] we solved this problem. We showed that the Jacobson radical consists of all elements of the form $\sum_{n=1}^{+\infty} U^n f_n$ such that each Fourier coefficient f_n vanishes on the set of recurrent points of the dynamical system (a point $x \in X$ is called recurrent if $x \in L(x)$). We generalized also this result for the case of a multivariable dynamical system, that is a locally compact Hausdorff space with a d-tuple of commuting proper surjections. In this case we need a modification of the notion of a recurrent point (as also a modification of the notion of the Birkhoff center of the dynamical system we used in the case of one variable). Namely, let $I \subset \{1, 2, \dots, d\}$. A point $x \in X$ is called I-recurrent if there is a sequence $\{\mathbf{n_k}\} \subset \mathbb{N}^d$ such that the i-th entry of $\mathbf{n_{k+1}}$ is greater than the i-th coordinate of $\mathbf{n_k}$ for every $i \in I$ such that $\phi_{\mathbf{n_k}} x \to x$. In this case the Jacobson radical is the closed ideal generated by all monomials of the form $U_{\mathbf{n}} f$, $\mathbf{n} \neq \mathbf{0}$ where f vanishes on the set of recurrent point corresponding to the support of $\mathbf{n}$. Some interesting corollaries of the previous results are the following: (a) The analytic crossed product is semisimple if and only if it is semiprime and (b) The prime radical of the analytic crossed product coincides with the Jacobson radical if and only if it is closed. Recall that the prime radical is the intersections of all prime ideals and the algebra is called semiprime if the prime radical is zero or, equivalently, if there are no (non-zero) nilpotent ideals.

In [12] we made a full study of minimal flows on multipunctured surfaces, i.e. open 2-manifolds of the type $M \setminus F$, where M is a compact 2-manifold and F is a non-empty, closed and totally disconnected subset of M (i.e. F is the set of ends of M). We reduced the study to the case where F is finite by proving that if $M \setminus F$ carries a minimal flow and F has infinite cardinality, there exists a finite set $K \subset F$, a smooth vector field ξ on $M \setminus K$ with minimal flow, and a smooth function $f : M \setminus K \to [0, 1]$ with $f^{-1}(0) = F \setminus K$, such that the flow is topologically equivalent to the extension of the flow of $f \cdot \xi$ to M. After proving that every minimal flow on a multipunctured torus $T^2 \setminus F$, is topologically equivalent to the restriction on $T^2 \setminus F$ of the flow of the product of an irrational vector field by a smooth function $f : T^2 \to [0, 1]$ such that $f^{-1}(0) = F$, we showed, by gluing together minimal flows on multipunctured tori, that all possible cases of behavior at infinity can occur if the Euler characteristic of M is negative.

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