HOW THE CONCEPT OF A NET ARRIVES FROM RIEMANN INTEGRATION IN CALCULUS AND WHY NETS DESCRIBE "CONVERGENCE"

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Abstract. In this talk we explain how the concept of a net in general topology arrives naturally from the theory of Riemann integration in Calculus and why nets describe "convergence". This example is historically important; it is what first led E. H. Moore and H. L. Smith to the concept of a net.

1. How the notion of nets arrives naturally from the theory of Riemann integration in Calculus

Riemann integration theory is one of the basics in Calculus. And in the theory of real functions we work with sequences of real numbers. So, how do nets arise naturally in this setting?

Let us describe what a Riemann integrable function $f : [a, b] \to \mathbb{R}$ is. We start with the notion of a tagged partition of a closed interval $[a, b]$. A (tagged) partition is a finite sequence of numbers of the form $a = x_0 < x_1 < \ldots < x_n = b$. The set of all partitions of $[a, b]$ will be denoted by $\mathcal{P}$. The lower Riemann sum of f on $\Delta \in \mathcal{P}$ is defined by

$$L_\Delta := \sum_{i=1}^{n} \Big((x_i - x_{i-1}) \cdot \inf\{f(x) \,:\, x \in [x_{i-1}, x_i]\} \Big)$$

and, similarly, the upper Riemann sum of f on Δ is

$$U_\Delta := \sum_{i=1}^{n} \Big((x_i - x_{i-1}) \cdot \sup\{f(x) \,:\, x \in [x_{i-1}, x_i]\} \Big).$$

We say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if the limits of the lower and upper Riemann sums of a function exist and are equal as the partitions get finer. Intuitively, a partition gets finer if we add more points. To be more precise, a partition Δ_2 is finer (or bigger) than a partition Δ_1 if $\Delta_1 \subseteq \Delta_2$. When Δ_2 is finer that Δ_1 we write $\Delta_1 \leq \Delta_2$. It is important to notice that given two partitions $\Delta_1, \Delta_2 \in \mathcal{P}$ it is not always true that $\Delta_1 \leq \Delta_2$ or $\Delta_2 \leq \Delta_1$. For example take as $[a, b]$ the unit interval $[0, 1]$, take as $\Delta_1 = \{0, 1/2, 1\}$

and $\Delta_2 = \{0, 1/3, 1\}$. What is always true is that given $\Delta_1, \Delta_2 \in \mathcal{P}$ there always exists $\Delta_3 \in \mathcal{P}$ such that $\Delta_1 \subseteq \Delta_3$ and $\Delta_1 \subseteq \Delta_3$. As Δ_3 we may take a partition consisting of the union of the points of Δ_1 and Δ_2. For instance, in the previous example we may take as $\Delta_3 = \{0, 1/3, 1/2, 1\}$. Hence, formally, a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if there exists a real number c such that for every $\varepsilon > 0$ there exists a partition $\Delta_0 \in \mathcal{P}$ such that

$$|L_\Delta - c| < \varepsilon \quad \text{and} \quad |U_\Delta - c| < \varepsilon \qquad \text{for every } \Delta \geq \Delta_0.$$

However, there is an unfortunate problem with this definition: it is very difficult to work with since we must know beforehand what is the value of c, i.e. the value of the Riemann integral. In order to check more easily if a function is Riemann integrable, without knowing beforehand the Riemann integral, Riemann gave a nice criterion that carries his name.

Proposition 1.1 (Riemann's criterion). *A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if for every $\varepsilon > 0$ there is a partition $\Delta_0 \in \mathcal{P}$ such that $|U_\Delta - L_\Delta| < \varepsilon$ for every $\Delta \geq \Delta_0$.*

Sooooo? Where are the nets? Let us recall the definition of a net. We need firstly to define what a directed set is.

Definition 1.2. A non empty set $\mathcal{P}$ equipped with a relation $\leq$ is called a directed set with direction $\leq$ if the following hold.

(1) $x \leq x$ for every $x \in \mathcal{P}$.
(2) If $x \leq y$ and $y \leq z$ then $x \leq z$, where $x, y, z \in \mathcal{P}$ and
(3) for every $x, y \in \mathcal{P}$ there exists $z \in \mathcal{P}$ such that $x \leq z$ and $y \leq z$.

A typical example for a directed set is the set $\mathcal{P}$ of all finite partitions of a closed interval $[a, b]$.

Now, do you remember what a sequence of real numbers is? An easy way to reply is to say that a sequence of real numbers is a function $p : \mathbb{N} \to \mathbb{R}$. As usual we denote a sequence by $(x_n)_{n\in\mathbb{N}}$. So now we are ready to say what a net is.

Definition 1.3. Let X be a non empty set. A net in X is a map $p : \mathcal{P} \to X$, where $\mathcal{P}$ is a directed set with a direction $\leq$. A net will be denoted by $(x_i)_{i\in\mathcal{P}}$ and a point of $\mathcal{P}$ is called an index.

An example? The upper and the lower Riemann sums of a function $f : [a, b] \to \mathbb{R}$!

And what about convergence of nets? Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if for every open neighborhood U of x we can find $n_0 \in \mathbb{N}$

such that $x_n \in U$ for every $n \geq n_0$. We can the same say for nets! A net $(x_i)_{i\in\mathcal{P}}$ converges to x if for every open neighborhood U of x we can find $i_0 \in \mathcal{P}$ such that $x_i \in U$ for every $i \geq i_0$! So now, we can revisit Riemann integration and say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if there exists a real number c such that the nets $L_\Delta \to c$ and $U_\Delta \to c$. The Riemann's criterion takes the following form: A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if the net $U_\Delta - L_\Delta \to 0$.

2. Why nets describe "convergence"?

As we saw, "a sequence or a net comes close to a point x" means that for any choice of an open neighborhood of x our sequence or net is "finally" inside this neighborhood. So open neighborhoods of points are closely related with nets and their convergence. The reason that nets describe convergence is a "special" net which is related to the family of all open neighborhoods of a point: Let x be a point in our space and $\mathcal{U}$ be the family of all open neighborhoods of x. We can define a natural direction on $\mathcal{U}$ in such a way that "$\geq$" means to be "closer" to x: Let U_1, U_2 be open neighborhoods of x. We say that $U_1 \leq U_2$ if $U_2 \subseteq U_1$. It is easy to check that this really defines a direction on $\mathcal{U}$. Now if we pick a point x_U from each $U \in \mathcal{U}$ we have a net $x_U \to x$! Indeed, just notice that for every open neighborhood V of x there is an index $U_0 := V \in \mathcal{U}$ such that for every $U \geq U_0$ it holds $x_U \in U \subseteq U_0 = V$!!

Are the nets "good" enough to describe convergence? They will be "good" if the "good" maps of the theory of convergence respect them! And what are the "good" maps of the theory of convergence? Before we give an answer we must say what are the "good" maps (the "morphisms") in any theory. Good maps are those that respect the underlying structures! For example, in the theory of linear spaces we have two operations, an addition of vectors and a multiplication of a vector by a scalar. So a "good" map must respect them, that is $f(x + y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$. Such a map is called a linear map. In the theory of convergence we have $x_i \to x$ for a net (or a sequence) $(x_i)_{i\in\mathcal{P}}$. So the "good" maps must respect convergence, that is whenever $(x_i)_{i\in\mathcal{P}}$ is a net with $x_i \to x$ for a point x then $f(x_i) \to f(x)$. Such a map is called a continuous map. But we have seen that a continuous map at a point x is defined in another way, namely a map f is continuous at x if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f(U) \subseteq V$. Next theorem says that the two definitions are equivalent.

Theorem 2.1. *A map $f : X \to Y$ is continuous at a point $x \in X$ if and only if whenever $(x_i)_{i\in\mathcal{P}}$ is a net in X with $x_i \to x$ then $f(x_i) \to f(x)$.*

Proof. Assume that f is continuous at a point x and $(x_i)_{i\in\mathcal{P}}$ is a net in X with $x_i \to x$. We will show that $f(x_i) \to f(x)$. Let V be an open neighborhood of $f(x)$ in Y. Since f is continuous there is an open neighborhood U of x in X such that $f(U) \subseteq V$. The net $(x_i)_{i\in\mathcal{P}}$ converges to x, hence there is an index $i_0 \in \mathcal{P}$ such that $x_i \in U$ for every $i \geq i_0$. Therefore, $f(x_i) \in f(U) \subseteq V$ for every $i \geq i_0$. So, $f(x_i) \to f(x)$.

For the converse implication we argue by contradiction. Assume that whenever $(x_i)_{i\in\mathcal{P}}$ is a net in X with $x_i \to x$ then $f(x_i) \to f(x)$ but f is not continuous at x. Then, there is an open neighborhood V of $f(x)$ such that if U is any open neighborhood of x then $f(U) \nsubseteq V$. So, for each open neighborhood U of x there exists a point $x_U \in U$ such that $f(x_U) \notin V$. As we saw before, the points (x_U) form a "natural" net which converges to x. But $f(x_U) \notin V$ for every U, hence $f(x_U) \nrightarrow f(x)$ which is a contradiction to our assumption. □

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