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ON EMBEDDINGS OF PROPER AND EQUICONTINUOUS ACTIONS IN ZERO-DIMENSIONAL COMPACTIFICATIONS

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Abstract. We provide a tool for studying properly discontinuous actions of non-compact groups on locally compact, connected and paracompact spaces, by embedding such an action in a suitable zero-dimensional compactification of the underlying space with pleasant properties. Precisely, given such an action (G, X) we construct a zero-dimensional compactification μX of X with the properties: (a) there exists an extension of the action on μX, (b) if $\mu L \subseteq \mu X \setminus X$ is the set of the limit points of the orbits of the initial action in μX, then the restricted action $(G, \mu X \setminus \mu L)$ remains properly discontinuous, is indivisible and equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX, and (c) μX is the maximal among the zero-dimensional compactifications of X with these properties. Proper actions are usually embedded in the endpoint compactification εX of X, in order to obtain topological invariants concerning the cardinality of the space of the ends of X, provided that X has an additional "nice" property of rather local character ("property Z", i.e., every compact subset of X is contained in a compact and connected one). If the considered space has this property, our new compactification coincides with the endpoint one. On the other hand, we give an example of a space not having the "property Z" for which our compactification is different from the endpoint compactification. As an application, we show that the invariant concerning the cardinality of the ends of X holds also for a class of actions strictly containing the properly discontinuous ones and for spaces not necessarily having "property Z".

Introduction

The endpoint compactification of a locally compact space has been proved fruitful for the study of the space in the topological framework, including proper actions. One reason for this is that we have a "clear view" of the embedded space in such a compactification, contrary to the situation when, for example, the Stone-Čech compactification is considered instead. Actually, the endpoint compactification is the quotient space of the Stone-Čech compactification with respect to the equivalence relation whose equivalence classes are the singletons of X and the connected components of $\beta X \setminus X$.

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Our purpose in this paper is to provide an equivariant and analogously useful notion corresponding to the endpoint compactification in order to have a "clear view" of the embedded proper action. By saying a "clear view" of the embedded action we mean that the embedded action has at least the three properties that follow.

Let (G, X) be the initial proper action, (G, Y) the extended action in a zero-dimensional compactification Y of X and let L be the set of the limit points of the orbits of the initial action in Y (i.e., the cluster points of the nets $\{g_i x\}$, for all nets $\{g_i\}$ divergent in G, and $x \in X$). Then the maximal invariant subspace where the extended action can be proper is, obviously, $Y \setminus L \supseteq X$. So, the required properties are: The action $(G, Y \setminus L)$

(a) remains proper,

(b) is equicontinuous with respect to the uniformity induced on $Y \setminus L$ by that of Y, and

(c) is indivisible (i.e., if $\lim g_i y_0 = e \in L$ for some $y_0 \in Y \setminus L$, then $\lim g_i y = e$ for every $y \in Y \setminus L$).

In this direction the main results of the paper at hand are:

(1) If X is a locally compact, connected and paracompact space and G is a non-compact group acting properly discontinuously on X, there always exists a zero-dimensional compactification μX of X which is the maximal (in the ordering of zero-dimensional compactifications of X) that satisfies the properties: (a) the initial action can be extended on μX, and (b) if μL denotes the set of the limit points of the orbits of the initial action in μX, the restricted action $(G, \mu X \setminus \mu L)$ remains proper, is equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX and indivisible as embedded in the action $(G, \mu X)$ (Theorem 6.2).

(2) μL consists of at most two or infinitely many points (Theorem 6.3).

(3) If X has the *"property Z"*, i.e., every compact subset of X is contained in a compact and connected one (for example if X is locally compact, connected and locally connected), then μX coincides with the endpoint compactification εX of X (Corollary in Section 6).

The proof of the results stated above relies on a *new construction*: The action $(G, \mu X)$ is obtained by taking the initial action as an equivariant inverse limit of properly discontinuous G-actions on polyhedra, which are constructed via G-invariant locally finite open coverings of X, generated by locally finite coverings of (always existing) suitable fundamental sets of the initial action (cf. Section 3).

As an application of these results we prove in Theorem 7.1 that the invariant concerning the cardinality of the ends for proper actions of non-compact groups on locally compact and connected spaces with the "property Z" holds also for proper actions on spaces not necessarily satisfying this property:

If either G_0, the connected component of the neutral element of G, is non-compact, or G_0 is compact and G/G_0 contains an infinite discrete subgroup, then X has at most two or infinitely many ends.

Moreover, in Section 2 we give an example of a properly discontinuous action (G, X), where G is a non-compact group and X is a locally compact, connected and paracompact space not satisfying the "property Z" such that μX does not coincide with the endpoint compactification of X: we show that the sets of the limit points of the actions (G, X) and $(G, \varepsilon X \setminus \varepsilon L)$ in εX coincide, but the action

$(G, \varepsilon X \setminus \varepsilon L)$ is neither proper nor equicontinuous with respect to the uniformity induced on $\varepsilon X \setminus \varepsilon L$ by that of εX.

Properties (a) and (b) in (1) above have already been used, especially concerning embeddings in the endpoint compactification, in order to prove that the existence of a proper action (G, X) of a non-compact group on a locally compact, connected and paracompact space with the "property Z" has implications in the structure and the cardinality of the space of ends of X. The following indications trace the known results in this direction.

The first theorem that relates, although indirectly, equicontinuous actions with structural features of spaces, is formulated by Kerékjártó (1934), who proved that, if the abelian group generated by a homeomorphism of the 2-sphere, S^2, acts equicontinuously on S^2 with respect to the metric uniformity of S^2 except for a finite number of points, then the number of these exceptional points is at most two. These points can be viewed as the set of the endpoints of the maximal subspace of S^2 on which the above group acts equicontinuously. This result is considerably generalized by Lam in [7] for equicontinuous actions of non-compact groups on locally compact, connected metric spaces X with respect to uniformities induced, say, by the uniformities of suitable zero-dimensional compactifications of X, i.e., compactifications with zero-dimensional remainder. Roughly speaking it is shown that, if an action (G, X) can be embedded in an action (G, Y), where Y is a zero-dimensional compactification of X, such that

(a) there exists a subset $R \supseteq X$ of Y such that $Y \setminus R$ is exactly the non-empty set of the points where the action (G, Y) is not equicontinuous, and

(b) the restricted action (G, R) is indivisible (i.e., if $\lim g_i y_0 = e \in Y \setminus R$ for some $y_0 \in R$, then $\lim g_i y = e$ for every $y \in R$),

then $Y \setminus R$ consists of at most two or infinitely many points.

On the other hand, similar results are proved by Abels in [2] for proper actions (G, X), where G is a non-compact topological group and X is a locally compact and connected space with the "property Z" (i.e., every compact subset of X is contained in a compact and connected one). The corresponding property in Lam's work requires X to be a semicontinuum, which ensures the indivisibility of the equicontinuous action on R. In [2] is considered the endpoint compactification, εX, of X, the maximal compactification of it with zero-dimensional remainder, instead of an appropriate zero-dimensional compactification Y of X, and it is proved that such a proper action (G, X) has an extension on εX. To be more precise, let εL denote the set of the limit points of the action (G, X) in εX. Then, it is shown that (a) the action $(G, \varepsilon X \setminus \varepsilon L)$ remains proper and (b) it is indivisible. Using this embedding, it is shown that X has at most two or infinitely many ends, a remarkable invariant of the proper action (G, X) of the non-compact group G.

The interconnection of the main results in [7] and [2] is explained in [11], where it is shown that, for spaces with the "property Z", a group acting equicontinuously in Lam's view may be considered as a dense (not necessarily strict) subgroup of a group acting properly as in Abels' view.

1. Preliminaries

1.1. The *Freudenthal* or *endpoint* compactification εX of a locally compact space X may be defined as the quotient space of the Stone-Čech compactification βX of X with respect to the equivalence relation whose equivalence classes are the singletons of X and the connected components of $\beta X \setminus X$. Recall that the zero-dimensional compactifications of X are ordered with respect to the following ordering: Let Y and Z be two zero-dimensional compactifications of X; then $Y \leq Z$ if there exists a surjection from Z onto Y extending the identity map of X. Therefore, the endpoint compactification is the maximal zero-dimensional compactification of X, i.e., for every zero-dimensional compactification Y of X there is a surjection $p : \varepsilon X \to Y$ extending the identity map of X.

The points of $\varepsilon X \setminus X$ are the *ends* of X.

The following theorem, [9], provides an equivalent definition.

Theorem 1.1.1. *If Y is a compactification of X, it is the endpoint compactification of X iff $Y \setminus X$ is totally disconnected and does not disconnect Y locally, i.e., given an open (in Y) neighborhood V of $y \in Y \setminus X$, then there is no decomposition of $V \cap X$ into two open disjoint subsets U_1, U_2 such that $y \in \overline{U_1} \cap \overline{U_2}$.*

The endpoint compactification has the following useful properties.

Proposition 1.1.2. *Let X and Y be two locally compact topological spaces. Then every proper map $f : X \to Y$ may be extended to a unique map $\varepsilon f : \varepsilon X \to \varepsilon Y$ that maps ends of X to ends of Y.*

Proof. By the characteristic property of the Stone-Čech compactification, the map $f : X \to Y$ has a unique extension $\varepsilon f : \varepsilon X \to \varepsilon Y$. The inclusion $\varepsilon f(\varepsilon X \setminus X) \subseteq \varepsilon Y \setminus Y$ follows from the assumption that f is a proper map. □

Proposition 1.1.3. *Let X be a locally compact and connected space and Y be a zero-dimensional compactification of X. Then, whenever a continuous action (G, X) has an extension (G, Y) this extension is continuous.*

Proof. It suffices to show the continuity of the extended action map at the point (e, z), where e is the neutral element of G and $z \in Y$. Let V and U be two open neighborhoods of z in Y with boundaries in X such that $\overline{V} \subseteq U$. Since the boundaries ∂U and ∂V are compact subsets of X, the set $A = \{g \in G \,|\, g\partial V \subseteq U \text{ and } g^{-1}\partial U \subseteq Y \setminus \overline{V}\}$ is an open neighborhood of e in G. We shall show that $g\overline{V} \subseteq U$ for every $g \in A$: The boundary of the set $g\overline{V} \cap (Y \setminus U)$ is contained in $(g\partial V \cap (Y \setminus U)) \cap (g\overline{V} \cap \partial U)$, which is empty by the definition of A. Since Y is connected, this implies that $g\overline{V} \cap (Y \setminus U)$ is either the empty set or coincides with Y. The latter is impossible since, choosing a point $x \in \partial V$, the definition of A implies that $gx \notin Y \setminus U$. Therefore $g\overline{V} \cap (Y \setminus U) = \emptyset$. □

As an immediate consequence of the above two propositions we state the following:

Corollary 1.1.4. *An action (G, X) of a group G on a locally compact and connected space X may be extended to a unique action on the endpoint compactification εX of X.*

1.2. The notion of a proper action is given in [4, III, 4]. Equivalently, an action (G, X) is *proper* if $J(x)$ is the empty set for every $x \in X$, where

$$J(x) = \{y \in X \,|\, \text{there exist nets } \{x_i\} \text{ in } X \text{ and } \{g_i\} \text{ in } G \text{ with } g_i \to \infty, \\ \lim x_i = x \text{ and } \lim g_i x_i = y\}.$$

Here $g_i \to \infty$ means that the net $\{g_i\}$ does not have any limit point in G.

In the special case where G is locally compact, an action (G, X) is proper iff for every $x, y \in X$ there exist neighborhoods U_x and U_y of x and y, respectively, such that the set

$$G(U_x, U_y) = \{g \in G \mid (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G.

The action is called *properly discontinuous* when $G(U_x, U_y)$ is finite.

Remark. Let (G, X) be a proper action of a non-compact group G and $(G, \varepsilon X)$ its extension on the endpoint compactification of X. Then, the set $J(x)$ with respect to the extended action is a non-empty subset of $\varepsilon X \setminus X$ for every $x \in X$. The study of these sets provides useful information. As an example, we note from [2] the following:

Theorem 1.2.1. *Let (G, X) be a proper action of a non-compact group G on a locally compact and connected space X with the "property Z". Then, X has at most two or infinitely many ends. In particular, if G is connected, then X has at most two ends.*

1.3. A characteristic and very useful feature of a proper action is the fundamental set (cf. [6] and [1]).

Definition. Given an action (G, X), a subset F of X is a *fundamental set* for the action if $GF = X$ and for every compact subset K of X the set $\{g \in G \mid (gK) \cap F \neq \emptyset\}$ is relatively compact in G.

The existence of a fundamental set implies that the action (G, X) is proper, but the converse does not hold, in general. The notion of the fundamental set is relative to the well-known notion of a section but is different in general, in the sense that there are cases where a section is a fundamental set, a fundamental set fails to be a section and cases where a section fails to be a fundamental set.

Theorem 1.3.1. *Let (G, X) be a proper action, where X is a locally compact, connected and paracompact space. Then, there exist open fundamental sets F and S for G in X such that $\overline{F} \subseteq S$.*

This follows immediately by [6, Lemma 2, p. 8], because X is σ-compact; hence the orbit space of the action is paracompact.

1.4. Establishing the notation, we recall

Definition. An inverse system $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ consists of a directed set Λ, a family of topological spaces $\{X_\lambda,\ \lambda \in \Lambda\}$, and continuous mappings $p_{\kappa\lambda} : X_\lambda \to X_\kappa$ with the properties that for every $\kappa, \lambda, \mu \in \Lambda$ with $\kappa \leq \lambda$ and $\lambda \leq \mu$ the map $p_{\lambda\lambda} : X_\lambda \to X_\lambda$ is the identity of X_λ, and $p_{\kappa\lambda} \circ p_{\lambda\mu} = p_{\kappa\mu}$. Let $p_\lambda : \prod_\lambda X_\lambda \to X_\lambda$ be the λ-projection. The (possibly empty) space

$$\{x \in \prod_\lambda X_\lambda \mid p_\kappa(x) = p_{\kappa\lambda} \circ p_\lambda(x) \text{ for every } \kappa \leq \lambda\}$$

is called the *inverse limit* of $\{X_\lambda,\ \lambda \in \Lambda\}$ and is denoted by $\varprojlim X_\lambda$.

Proposition 1.4.1 ([5, Pr. 2.3, p. 428])**.** *The sets $\{p_\lambda^{-1}(U) \mid \lambda \in \Lambda,\ U \text{ open in } X_\lambda\}$ form a basis for $\varprojlim X_\lambda$.*

The following notion provides an alternative way to describe locally compact and paracompact spaces using coverings.

Definition. Let X be a paracompact space, $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ be an inverse system and $\{p_\lambda \mid \lambda \in \Lambda\}$ a family of mappings $p_\lambda : X \to X_\lambda$ such that $p_\kappa(x) = p_{\kappa\lambda} \circ p_\lambda(x)$ for every $\kappa \leq \lambda$. We say that the inverse system $(X_\lambda, p_{\kappa\lambda}, \Lambda)$ is a *resolution* of X if the following conditions hold:

(a) For every covering $\mathcal{U}$ of X that admits a subordinated partition of unity, there exist an index $\lambda \in \Lambda$ and a covering $\mathcal{U}_\lambda$ of X_λ that also admits a subordinated partition of unity such that $p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$.

(b) For every $\kappa \in \Lambda$ and every covering $\mathcal{U}_\kappa$ of X_κ, as above, there exists $\lambda \geq \kappa$ such that $p_{\kappa\lambda}(X_\lambda) \subseteq St(p_\kappa(X), \mathcal{U}_\kappa)$, where

$$St(B, \mathcal{U}) = \bigcup \{U_i \mid U_i \cap B \neq \emptyset,\, U_i \in \mathcal{U}\}$$

is the *star* of B with respect to the covering $\mathcal{U}$.

Theorem 1.4.2 ([8, Cor. 4, p. 83])**.** *If the spaces X_λ are normal and X is paracompact, then a resolution of X gives X as an inverse limit.*

2. A counterexample

Following the notation in the introduction, we now give an example showing that, if the space X does not have the "property Z", then the action $(G, \varepsilon X \setminus \varepsilon L)$ is not necessarily proper or equicontinuous.

2.1. The *half-open Alexandroff square* Y is the space $[0,1] \times [0,1] \setminus \{(x,y) \mid x = 0 \,\text{and}\, y \in (0,1] \,\text{or}\, x = 1 \,\text{and}\, y \in [0,1)\}$ endowed with the topology τ defined as follows: A neighborhood basis of a point $(x,x) \in \Delta = \{(x,x) \mid x \in [0,1]\}$ is obtained by the intersection of Y with open (in $Y \subseteq \mathbb{R}^2$) horizontal strips less a finite number of vertical lines; a neighborhood basis for the points $p = (s,t)$ off Δ is obtained by the intersection of $Y \setminus \Delta$ with open vertical segments centered at p (cf. [10, Ex. 101, p. 120]). This space is a compact, connected and not locally connected Hausdorff space. Observe that, if $\{(x_i, y_i)\}$ is a net converging with respect to the Euclidean topology on Y to a point (x,y), then this net converges to (y,y) with respect to τ, unless there is an index i_0 such that $x_i = x$ for all $i \geq i_0$, in which case it converges to (x,y).

2.2. Let X be the subspace $(0,1) \times (0,1)$ of Y. This space is locally compact, connected and paracompact, because the closed horizontal strips are compact subsets of X. It does not have the "property Z", because every closed horizontal strip is not contained in a compact and connected subset of X.

The space Y is the endpoint compactification, εX, of X, and the ends are the points $(x,0)$ for $x \in [0,1)$ and $(x,1)$ for $x \in (0,1]$. In order to prove this, by Theorem 1.1.1, it is sufficient to verify that the set $Y \setminus X$ is totally disconnected and that every point of it does not disconnect Y locally: For the points of the form $(x,0)$ and $(x,1)$ for $x \in (0,1)$ this follows from the fact that a neighborhood basis of every one of these points consists of half-open vertical segments which do not disconnect Y locally. To verify the same for the points $(0,0)$ and $(1,1)$ observe that, if there is a neighborhood V (in Y) for, e.g., $(0,0)$ such that $V \cap X$ is the union of two open sets (in X) having $(0,0)$ as a common point of their closures in Y, then they have common interior points.

2.3. Next we define a properly discontinuous action of the additive group of the integers $\mathbb{Z}$ on X. For convenience, we consider X as $\mathbb{R}^2$ endowed with the topology τ, and we define the action by letting

$$z(x,y) = (x+z, y+z) \text{ for } z \in \mathbb{Z} \text{ and } (x,y) \in \mathbb{R}^2.$$

By Corollary 1.1.4, this action has an extension on $Y = \varepsilon X$. The set εL, of the limit points of this action, consists of the points $(0,0)$ and $(1,1)$. The restricted action $(\mathbb{Z}, \varepsilon X \setminus \varepsilon L)$ is neither proper nor equicontinuous with respect to the uniformity induced on $\varepsilon X \setminus \varepsilon L$ by that of εX. For this, observe that the sequence $\{(x-n,x) \,|\, n \in \mathbb{N}\}$ converges to the point (x,x), while the sequence $\{n(x-n,x) = (x, x+n)\}$ converges to an end e that corresponds to the vertical line $\{(x,y) \,|\, y \in \mathbb{R}\}$. Since $e \in J((x,x))$, the action $(\mathbb{Z}, \varepsilon X \setminus \varepsilon L)$ is not proper. On the other hand, $\lim n(x-n,x) = e$ and $\lim n(x,x) = (1,1)$; therefore this action is not equicontinuous at (x,x).

3. The basic construction

In the sequel we shall proceed to answer the question formulated in the introduction. Our answer will be based on an inverse system of properly discontinuous actions on polyhedra, defined from the initial action on X. This is achieved using appropriate invariant locally finite coverings of the given space, in order to have the initial action as an inverse limit of them. To obtain this, it is reasonable to work with invariant coverings of X extending specific coverings of always existing fundamental sets of the initial action. The construction of this inverse system, which follows, is new and will be given in several steps:

3.1. Let (G,X) be a properly discontinuous action of a non-compact group G on a locally compact, connected and paracompact space X. Recall that a covering $\mathcal{V}$ of X is called a barycentric refinement of a given covering if the covering $\{St(x,\mathcal{V}) \,|\, x \in X\}$ refines it, where $St(x,\mathcal{V})$ has been defined in §1.4. Since X is a locally compact and paracompact space, by [5, Cor. 7.4, p. 242], starting with an open covering of X, we can always find an open locally finite barycentric refinement $\mathcal{V} = \{V_j \,|\, j \in J\}$ of it consisting of relatively compact open sets.

3.2. Theorem 1.3.1 ensures that there exist open fundamental sets F and S such that $\overline{F} \subset S$. With the previous notation, we can choose $\mathcal{V}$ such that (a) if V_j intersects the boundary of the open fundamental set S, then V_j does not intersect the open fundamental set F, and (b) if V_j does not intersect the boundary of S, then either $V_j \subseteq S$ or $V_j \subseteq X \setminus \overline{S}$. The family $\mathcal{U} = \{St(x,\mathcal{V}) \,|\, x \in F\}$ is an open locally finite covering of F (in X); it is also a covering of $\overline{F}$, because if some V_j intersects the boundary of F, then it intersects F, hence is a member of a star of some point of F. From (a) and (b) it is easily seen that each member of $\mathcal{U}$ is a subset of S.

3.3. In the sequel we shall use the following modification of the previous construction, aiming to enrich $\mathcal{U}$ with the property: if $U_i \in \mathcal{U}$ and $gU_i \cap F \neq \emptyset$ for some $g \in G$, then $gU_i \in \mathcal{U}$. To this end, let $\mathcal{W} = \{W_k \,|\, k \in K\}$ be a locally finite refinement of $\mathcal{V}$ with the property that the closures of the stars of it are subsets of corresponding stars of $\mathcal{V}$. Now, we observe that the set $M_i = \{g \in G \,|\, gU_i \cap \overline{F} \neq \emptyset\}$ is non-empty and finite, because the action is properly discontinuous, U_i is relatively compact and $\overline{F} \subseteq S$ (cf. §§1.2 and 1.3).

If $x \in \overline{W}_k \subseteq U_i$ for some $U_i \in \mathcal{U}$, and $gx \in \overline{F}$, then $g \in M_i$ which is finite. So, for $x \in X$ we can find a neighborhood $N_x \subseteq U_i$ of x such that, if $gN_x \cap F \neq \emptyset$, then gN_x is a subset of some U_j. Since $\overline{W}_k$ is compact, we can replace this W_k by a finite number of neighborhoods such as N_x and the corresponding open sets gN_x for $g \in M_i$. In this way, we obtain a refinement of $\mathcal{U}$, which will be denoted again with $\mathcal{U}$ and shall be used in the sequel.

This refinement remains locally finite and, in addition, has the required property, because if $gN_x \cap F \neq \emptyset$, then $gU_i \cap F \neq \emptyset$, from which follows that $g \in M_i$; hence gN_x is a member of our refinement. It is easily seen that this property passes to the family $\{St(x,\mathcal{U}) \,|\, x \in F\}$, because if $gx \in \overline{F}$, then $gSt(x,\mathcal{U}) = St(gx,\mathcal{U})$.

3.4. Next, using the covering $\mathcal{U}$ of $\overline{F}$ defined in §3.3, we consider the invariant covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ of X. We show that it is locally finite: For $x \in X$ there exists $h \in G$ such that $hx \in F$. Since F is open, there exists an open and relatively compact neighborhood $N \subseteq F$ of hx that intersects finitely many members of $\mathcal{U}$. Then, the neighborhood $h^{-1}N$ of x intersects finitely many members of $\mathcal{C}$, because by §3.3, if $gU_i \cap N \neq \emptyset$, then $gU_i \in \mathcal{U}$.

3.5. To each covering $\mathcal{C}$ corresponds a polyhedron $X_{\mathcal{C}}$, namely the nerve of the covering $\mathcal{C}$ with the CW-topology. A subordinated partition of unity $\Phi_{\mathcal{C}} = \{\varphi_U \,|\, U \in \mathcal{C}\}$ determines a *canonical map* $p_{\mathcal{C}} : X \to X_{\mathcal{C}}$ with the property that $p_{\mathcal{C}}$ maps a point $x \in X$ to the point of $X_{\mathcal{C}}$ whose barycentric coordinate corresponding to the vertex U equals $\varphi_U(x)$.

Since $\mathcal{C}$ is invariant, the initial properly discontinuous action (G, X) induces a natural action $(G, X_{\mathcal{C}})$ defined as follows: For $g \in G$ and U a vertex of $\mathcal{C}$ we let $(g, U) \mapsto gU$ and we extend the action map by linearity. This action is properly discontinuous as is easily verified.

3.6. The construction of the desired inverse system of properly discontinuous actions on polyhedra will be based in the proof of the following theorem (cf. [3]; see also [8, Th. 7 and Cor. 5, pp. 84-85]).

Theorem. *Every connected, locally compact and paracompact space is the inverse limit of polyhedra.*

For the convenience of the reader, we outline the proof: Let X be a connected, locally compact paracompact space and $\mathcal{F}$ be the family of all coverings of X admitting a subordinated partition of unity. For every $\mathcal{D} \in \mathcal{F}$ we choose a locally finite partition of unity $\Phi_{\mathcal{D}}$ subordinated to $\mathcal{D}$. Let $X_{\mathcal{D}}$ be the nerve of $\mathcal{D}$ with the CW-topology. Let Λ be the set of all finite subsets $\lambda = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ of $\mathcal{F}$ ordered by inclusion. We denote by X_λ the nerve of the covering

$$\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n = \{V_1 \cap \ldots \cap V_n \,|\, (V_1, \ldots, V_n) \in \mathcal{D}_1 \times \ldots \times \mathcal{D}_n\}.$$

If $\lambda \leq \mu = \{\mathcal{D}_1, \ldots, \mathcal{D}_n, \ldots, \mathcal{D}_l\}$, let $p_{\lambda\mu} : X_\mu \to X_\lambda$ be the simplicial map which maps the vertex $(V_1, \ldots, V_n, \ldots, V_l)$ of the nerve of $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n \wedge \ldots \wedge \mathcal{D}_l$ to the vertex $(V_1, \ldots, V_n)$ of the nerve of $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n$.

As is shown in [3], the family

$$\Phi_{\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n} = \{\varphi_{(V_1, \ldots, V_n)} \,|\, (V_1, \ldots, V_n) \in \mathcal{D}_1 \times \ldots \times \mathcal{D}_n\},$$

where $\varphi_{(V_1, \ldots, V_n)} = \varphi_{V_1} \cdot \ldots \cdot \varphi_{V_n}$, is a partition of unity subordinated to the covering $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n$. Using this, for $\lambda = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$ we define the canonical map $p_\lambda : X \to X_\lambda$ as in §3.5.

In order to obtain a polyhedral resolution of X (cf. §1.4), a slight modification of the above construction is needed:

We replace the previous inverse system $(X_\lambda, p_{\lambda\mu}, \Lambda)$ by a larger system (Y_r, q_{rs}, S) defined as follows: For $\lambda \in \Lambda$ let $\mathcal{V}_\lambda$ be a neighborhood basis of the closure of $p_\lambda(X)$ in X_λ, and let

$$S = \{r = (\lambda, V) \,|\, \lambda \in \Lambda \text{ and } V \in \mathcal{V}_\lambda\}.$$

Let $r \leq s = (\mu, W)$ if $\lambda \leq \mu$ and $p_{\lambda\mu}(W) \subseteq V$. Moreover, letting $Y_r = V$, for $r \leq s$ we define the map $q_{rs} : Y_s \to Y_r$ as the restriction of $p_{\lambda\mu}$ on W.

Taking into account the fact that $\mathcal{F}$ consists of *all* coverings of X admitting subordinated partitions of unity, it is proved that $X = \varprojlim Y_r = \varprojlim X_\lambda$.

3.7. If we replace $\mathcal{F}$ by $\mathcal{P}$, the family of the coverings of X of the form $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ defined in §3.4, and we repeat the previous steps, we obtain an inverse system, denoted (for simplicity) again by $(X_\lambda, p_{\lambda\mu}, \Lambda)$. Since we use star coverings, we note that $St(x, \mathcal{D}_1) \cap St(x, \mathcal{D}_2) = St(x, \mathcal{D}_1 \wedge \mathcal{D}_2)$.

3.8. If we restrict ourselves to the fundamental set $\overline{F} \subseteq X$, the coverings from $\mathcal{P}$ induce a family of coverings on $\overline{F}$ defined by intersections of each one covering with $\overline{F}$. This family is cofinal to the corresponding one defined analogously via $\mathcal{F}$ on $\overline{F}$. Since $\mathcal{P}$ is not cofinal to $\mathcal{F}$, regarded as families of coverings of X, we shall focus on the induced coverings of the fundamental set $\overline{F}$, where we may assume that the members of both families are the same. Note that, by [4, I, Cor., p. 49], $\overline{F} = \varprojlim p_\lambda(\overline{F})$ holds, with respect to both $\mathcal{F}$ and $\mathcal{P}$. Moreover, with respect to $\mathcal{F}$, we have $\varprojlim p_\lambda(\overline{F}) = \overline{F} \subseteq X$, by the theorem in §3.6, while, with respect to $\mathcal{P}$ and the notation from §3.7, $\overline{F} \subseteq \varprojlim X_\lambda$.

4. The initial action as inverse limit of actions on polyhedra

Lemma 4.1. *Let $\mathcal{C}_i \in \mathcal{P}$. For the covering $\mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_n$, there exists a subordinated partition of unity $\Phi_{\mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_n} = \{\varphi_{(V_1,\ldots,V_n)} \,|\, (V_1, \ldots, V_n) \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_n\}$ such that $\varphi_{(V_1,\ldots,V_n)} = \varphi_{(gV_1,\ldots,gV_n)} \circ g$, for every $g \in G$.*

Proof. If the assertion is true for every single covering $\mathcal{C}$, then

$$\varphi_{(gV_1,gV_2,\ldots,gV_n)} \circ g = [(\varphi_{V_1} \circ g^{-1}) \cdot \ldots \cdot (\varphi_{V_n} \circ g^{-1})] \circ g = \varphi_{V_1} \cdot \ldots \cdot \varphi_{V_n} = \varphi_{(V_1,\ldots,V_n)}.$$

So, it suffices to prove the assertion for a covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$ as in §3.4. We follow the usual construction (cf. [5, Th. 4.2, p. 170]): We choose locally finite coverings $\{V_i \,|\, i \in I\}$ and $\{W_i \,|\, i \in I\}$ of the open fundamental set F such that $\overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i$ for every $i \in I$. We can apply Urysohn's Theorem in order to find continuous maps $f_{U_i} : X \to [0, 1]$ which are identically 1 on $\overline{W_i}$ and vanish on $X \setminus V_i$. We set $f_{gU_i} = f_{U_i} \circ g^{-1}$ for every $g \in G$. Since the covering $\{g\overline{W_i} \,|\, i \in I,\, g \in G\}$ is locally finite, it follows that for each $x \in X$ at least one and at most finitely many f_{gU_i} are not zero; therefore $\sum f_{gU_i}$ is a well-defined continuous real-valued map on X and is never zero. So, we can define the required partition of unity by setting

$$\varphi_{gU_i}(y) = \frac{f_{gU_i}(y)}{\sum f_{gU_i}(y)}.$$

Since $x \in U_i$ iff $gx \in gU_i$, we have

$$\varphi_{gU_i}(gx) = \frac{f_{gU_i}(gx)}{\sum f_{gU_i}(gx)} = \frac{f_{U_i} \circ g^{-1}(gx)}{\sum f_{U_i} \circ g^{-1}(gx)} = \frac{f_{U_i}(x)}{\sum f_{U_i}(x)} = \varphi_{U_i}(x).$$

□

Theorem 4.2. *With the notation from §3.7, X is equivariantly homeomorphic to $\varprojlim X_\lambda$.*

Proof. We recall that the actions (G, X_λ) defined in §3.5 induce an action on $\varprojlim X_\lambda$ as follows: Let $g \in G$ and $\mathbf{x} \in \varprojlim X_\lambda$ with coordinates $p_\lambda(x)$. The coordinates of $g\mathbf{x}$ are $gp_\lambda(x)$, i.e., $p_\lambda(gx) = gp_\lambda(x)$. This action is well defined since the maps $p_{\lambda\mu} : X_\mu \to X_\lambda$ have the property $p_{\lambda\mu}(gx_\mu) = gp_{\lambda\mu}(x_\mu)$ for every $g \in G$ and $x_\mu \in X_\mu$, by the definition of the action on each X_λ.

An equivariant homeomorphism $f : X \to \varprojlim X_\lambda$ may be defined in the following way: For $x \in X$ there exists some $g \in G$ such that $gx \in \overline{F}$ (cf. §1.3). We let $f(x)$ be the point with coordinates $p_\lambda(f(x)) = g^{-1}(p_\lambda(gx))$. We will prove that f is well defined: It suffices to prove that, if $x \in X$ and $g \in G$ with $gx \in \overline{F}$, then $g^{-1}(p_\lambda(gx))$ is independent of the choice of g. Indeed, with the notation from §§3.5 and 3.6 let $x \in V_1 \cap \ldots \cap V_n$. Then, by the definition of the actions on the polyhedra and the previous lemma, we have:

$$\varphi_{(g^{-1}gV_1,\ldots,g^{-1}gV_n)}(g^{-1}(p_\lambda(gx))) = \varphi_{(gV_1,\ldots,gV_n)}(p_\lambda(gx)) = \varphi_{(V_1,\ldots,V_n)}(p_\lambda(x)).$$

Using this and the fact that f is the identity map on the open fundamental set F, we can first verify that f is equivariant and then a homeomorphism. □

5. The embedding of the action in $\varprojlim \varepsilon X_\lambda$ and its basic properties

Theorem 5.1. *The space $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of X. Moreover, G acts on $\varprojlim \varepsilon X_\lambda$ and (G, X) is equivariantly embedded in $(G, \varprojlim \varepsilon X_\lambda)$.*

Proof. The simplicial maps $p_{\lambda\mu} : X_\mu \to X_\lambda$ are proper surjections. Hence, by Proposition 1.1.2, they have unique extensions $\varepsilon p_{\lambda\mu} : \varepsilon X_\mu \to \varepsilon X_\lambda$ that map the space of the ends of X_μ onto that of X_λ. Furthermore, $\varepsilon p_{\lambda\lambda} : \varepsilon X_\lambda \to \varepsilon X_\lambda$ is the identity map of εX_λ, and for $\kappa \le \lambda$ and $\lambda \le \mu$ we have $\varepsilon p_{\kappa\lambda} \circ \varepsilon p_{\lambda\mu} = \varepsilon p_{\kappa\mu}$. Hence, they define an inverse limit, $\varprojlim \varepsilon X_\lambda$.

Using the fact that each εX_λ is a zero-dimensional compactification of X_λ and applying Proposition 1.4.1, we see that $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of $\varprojlim X_\lambda$. By Corollary 1.1.4, the action of G on X_λ is extended to an action on εX_λ such that the following equivariant diagram commutes:

$$\begin{array}{ccc}
(G, \varprojlim X_\lambda) & \xrightarrow[\mathrm{id_G \times h}]{} & (G, \varprojlim \varepsilon X_\lambda) \\
{\scriptstyle \mathrm{id_G \times p_\lambda}} \downarrow & & \downarrow {\scriptstyle \mathrm{id_G \times \varepsilon p_\lambda}} \\
(G, X_\lambda) & \xrightarrow[\mathrm{id_G \times i_\lambda}]{} & (G, \varepsilon X_\lambda)
\end{array}$$

where $i_\lambda : X_\lambda \to \varepsilon X_\lambda$ are the inclusion maps, $p_\lambda : \varprojlim X_\lambda \to X_\lambda$ and $\varepsilon p_\lambda : \varprojlim \varepsilon X_\lambda \to \varepsilon X_\lambda$ are projections, id_G is the identity map of G and $h : \varprojlim X_\lambda \to \varprojlim \varepsilon X_\lambda$ is defined by setting $h_\lambda = i_\lambda$.

That (G, X) embeds equivariantly in $(G, \varprojlim \varepsilon X_\lambda)$ is an immediate consequence of Theorem 4.2 and the above diagram. □

Remark. The example in Section 2 shows that $\varprojlim \varepsilon X_\lambda$ does not necessarily coincide with the endpoint compactification εX of X. However,

Proposition 5.2. *If X has finitely many ends, then $\varepsilon X = \varprojlim \varepsilon X_\lambda$.*

Proof. Let e_i for $i = 1, 2, \ldots, n$ be the ends of X and $V_1, V_2, \ldots, V_n$ be open neighborhoods of them in εX, respectively, with disjoint closures and boundaries lying in X. Then, the set $K = \varepsilon X \setminus \bigcup_{i=1}^n V_i$ is a compact subset of X. Let e_1, e_2 be two distinct ends in εX with the same image in $\varprojlim \varepsilon X_\lambda$ via the projection map $p : \varepsilon X \to \varprojlim \varepsilon X_\lambda$. Such a projection exists since, by Theorem 5.1, $\varprojlim \varepsilon X_\lambda$ is a zero-dimensional compactification of X and εX is the maximal one. Therefore, e_1 and e_2 should have the same image under the composition map $\varepsilon p_\lambda \circ p$. With the notation from §3.4, this means that there is a subfamily of $\mathcal{C}$ with infinitely many members gU_i with the property $g_i U_i \cap K \neq \emptyset$. Then, we can find a sequence $\{x_k\}$ with $x_k \in g_k U_k \cap K$. Since K is compact, we may assume that $\lim x_k = x \in K$, from which follows that the covering $\mathcal{C}$ fails to be locally finite at x, a contradiction. □

The following proposition shows that, especially for equicontinuous actions, the sets $J(x)$, defined in §1.2, can be replaced by the *limit sets*

$$L(x) = \{y \in X \,|\, \text{there exists a net } \{g_i\} \text{ in } G \text{ with } g_i \to \infty \text{ and } \lim g_i x = y\},$$

which are simpler to handle. The points of the sets $L(x)$ are the *limit points* of the action.

Proposition 5.3. *Let (G, X) be an equicontinuous action of a locally compact group G on a locally compact space X. Then $J(x) = L(x)$ holds for every $x \in X$. Moreover, if the nets $\{x_i\}$ in X and $\{g_i\}$ in G are such that $\lim x_i = x$, $g_i \to \infty$ and $\lim g_i x_i = y \in J(x)$, then $\lim g_i x = y \in L(x)$.*

Proof. Since (G, X) is equicontinuous, for every entourage U there exists an entourage V such that for $y \in X$,

$$(x, y) \in V \text{ implies } (gx, gy) \in U \text{ for every } g \in G.$$

But $\lim x_i = x$, so we may assume that $(x, x_i) \in V$; therefore $(g_i x, g_i x_i) \in U$ and $(g_i x_i, y) \in U$. So $(g_i x, y) \in U \circ U$; hence $\lim g_i x = y$. □

A kind of converse of the previous proposition is the following:

Proposition 5.4. *Let Y be a zero-dimensional compactification of the locally compact and connected space Z. Let (G, Y) be an action such that Z is an invariant subspace of Y and the action (G, Z) is proper. The restricted action (G, Z) is equicontinuous with respect to the uniformity induced on Z by that of Y iff the following condition is satisfied: If $z \in Z$ is such that there exist a net $\{z_i\}$ in Z with $\lim z_i = z$, and a net $\{g_i\}$ in G with $g_i \to \infty$ and $\lim g_i z_i = e \in Y \setminus Z$, then $\lim g_i z = e$.*

Proof. The necessity may be proved by arguments analogous to those applied in the proof of the previous proposition.

For the sufficiency, note that if the action (G, Z) is not equicontinuous at the point z, then there exists an entourage U such that for every entourage V there exist a point $z_V \in Z$ and some $g_V \in G$ such that

$$(z, z_V) \in V \text{ and } (g_V z, g_V z_V) \notin U.$$

Since the entourages of the uniformity may be directed by setting $V_1 \leq V_2$ if $V_2 \subseteq V_1$, we may assume that $g_V \to \infty$ and $\lim z_V = z$. By the compactness of Y, we may assume that the nets $\{g_V z\}$ and $\{g_V z_V\}$ converge to different points of $Y \setminus Z$, a contradiction to our hypothesis. $\square$

For the formulation of the next basic theorem, we recall that an action (G, X) embedded in an action (G, Y), where Y is a zero-dimensional compactification of X, is *indivisible* if whenever $\lim g_i y_0 = e \in Y \setminus X$ for some $y_0 \in X$, then $\lim g_i y = e$ for every $y \in X$.

Theorem 5.5. *Let $(G, \varprojlim \varepsilon X_\lambda)$ be the action defined in Theorem 5.1 and L be the set of the limit points of the action $(G, X) = (G, \varprojlim X_\lambda)$ in $\varprojlim \varepsilon X_\lambda$. Then, the action $(G, \varprojlim \varepsilon X_\lambda \setminus L)$ is*

(a) *proper,*

(b) *equicontinuous with respect to the uniformity induced on $\varprojlim \varepsilon X_\lambda \setminus L$ by that of $\varprojlim \varepsilon X_\lambda$, and*

(c) *indivisible as embedded in the action $(G, \varprojlim \varepsilon X_\lambda)$.*

For the proof we need the following:

Lemma 5.6. *Let L_λ be the set of the limit points of the action (G, X_λ) in εX_λ. Then $L = \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$.*

Proof. Let $w \in L$, $\lim g_i x = w$ for $x \in X = \varprojlim X_\lambda$, and $g_i \to \infty$. Then $\lim g_i \varepsilon p_\lambda(x) = \lim \varepsilon p_\lambda(g_i x) = \varepsilon p_\lambda(w)$, from which follows that $\varepsilon p_\lambda(w) \in L_\lambda$; therefore $L \subseteq \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$.

For the inverse inclusion, let $v \in \bigcap_\lambda \varepsilon p_\lambda^{-1}(L_\lambda)$, that is, $\varepsilon p_\lambda(v) \in L_\lambda$ for every $\lambda \in \Lambda$. This means that for each $\lambda \in \Lambda$ there exist a net $\{g_i^\lambda\}$ in G with $g_i^\lambda \to \infty$ and $x_\lambda \in X_\lambda$ with $\lim g_i^\lambda x_\lambda = \varepsilon p_\lambda(v)$. Since the polyhedron X_λ is a connected, locally compact and locally connected space, it has the "property Z"; therefore, by [2, 3.4], the action (G, X_λ) is indivisible as a restriction of $(G, \varepsilon X_\lambda)$. So, we may assume that $x_\lambda = \varepsilon p_\lambda(x)$ for a fixed $x \in X$ and every $\lambda \in \Lambda$. By the compactness of $\varprojlim \varepsilon X_\lambda$, we may assume that $\lim g_i^\lambda x = v^\lambda \in L$. So, we have

$$\varepsilon p_\lambda(v) = \lim g_i^\lambda x_\lambda = \lim g_i^\lambda \varepsilon p_\lambda(x) = \lim \varepsilon p_\lambda(g_i^\lambda x) = \varepsilon p_\lambda(v^\lambda).$$

Let $\lim v^\lambda = u \in \varprojlim \varepsilon X_\lambda$. This u is contained in $\varprojlim \varepsilon X_\lambda \setminus X$, because $v^\lambda \in \varprojlim \varepsilon X_\lambda \setminus X$, which, by Theorem 5.1, is a compact set. But, for each $\kappa \in \Lambda$ and every $\lambda \in \Lambda$ with $\kappa \leq \lambda$, by §1.4, we have

$$\varepsilon p_\kappa(u) = \lim \varepsilon p_\kappa(v^\lambda) = \lim \varepsilon p_{\kappa\lambda} \circ \varepsilon p_\lambda(v^\lambda) = \lim \varepsilon p_{\kappa\lambda} \circ \varepsilon p_\lambda(v) = \varepsilon p_\kappa(v),$$

from which it follows that $u = v$. Taking into account that $\lim g_i^\lambda x = v^\lambda$ and applying a diagonal procedure, we may find a net $\{g_j\}$ in G such that $\lim g_j x = v \in \varprojlim \varepsilon X_\lambda \setminus X$. The properness of the action (G, X) implies that this net is divergent, and therefore $v \in L$, as required. $\square$

Proof of Theorem 5.5. (a) Assume that $\{g_i\}$ is a net in G and x, x_i and y are points in $\varprojlim \varepsilon X_\lambda \setminus L$ such that $\lim x_i = x$ and $\lim g_i x_i = y$. By the previous lemma, $\varprojlim \varepsilon X_\lambda \setminus L = \bigcup_\lambda \varepsilon p_\lambda^{-1}(\varepsilon X_\lambda \setminus L_\lambda)$. So, there exist κ and λ such that $x \in \varepsilon p_\kappa^{-1}(\varepsilon X_\kappa \setminus L_\kappa)$ and $y \in \varepsilon p_\lambda^{-1}(\varepsilon X_\lambda \setminus L_\lambda)$.

For an index μ with $\kappa \leq \mu$ and $\lambda \leq \mu$, we may assume that

$$\varepsilon p_{\kappa\mu}^{-1}(\varepsilon X_\kappa \setminus L_\kappa) \cup \varepsilon p_{\lambda\mu}^{-1}(\varepsilon X_\lambda \setminus L_\lambda) \subseteq \varepsilon X_\mu \setminus L_\mu.$$

Indeed, note that if, e.g., $z \in \varepsilon p_{\kappa\mu}^{-1}(\varepsilon X_\kappa \setminus L_\kappa)$ and $z \in L_\mu$, there exist a net $\{h_j\}$ in G with $h_j \to \infty$ and some $x_\mu \in X_\mu$ with $\lim h_j x_\mu = z$; hence $\lim h_j \varepsilon p_{\kappa\mu}(x_\mu) = \varepsilon p_{\kappa\mu}(z) \in L_\kappa$, a contradiction.

By this, we may assume that the points x, x_i and y are contained in the open and invariant set $\varepsilon p_\mu^{-1}(\varepsilon X_\mu \setminus L_\mu)$. Since X_μ is connected, locally compact and locally connected, it has the "property Z"; therefore the action $(G, \varepsilon X_\mu \setminus L_\mu)$ is proper [2, 4.7], and hence $J(\varepsilon p_\mu(x)) = \emptyset$. From this and the fact that $\lim g_i \varepsilon p_\mu(x_i) = \varepsilon p_\mu(y)$, it follows that the net $\{g_i\}$ cannot be divergent. Hence, by §1.2, the action $(G, \varprojlim \varepsilon X_\lambda \setminus L)$ is proper.

(b) We shall use Proposition 5.4 for $Z = \varprojlim \varepsilon X_\lambda \setminus L$ and the notation there. Let $\lim g_i z = e_1$. For every $\lambda \in \Lambda$ we have

$$\lim \varepsilon p_\lambda(z_i) = \varepsilon p_\lambda(z), \ \lim g_i \varepsilon p_\lambda(z_i) = \varepsilon p_\lambda(e), \text{ and } \lim g_i \varepsilon p_\lambda(z) = \varepsilon p_\lambda(e_1).$$

By (a), the action (G, Z) is proper; hence $e, e_1 \in L$. Therefore, by Lemma 5.6, $\varepsilon p_\lambda(e), \varepsilon p_\lambda(e_1) \in L_\lambda$. From this and the indivisibility of the action $(G, \varepsilon X_\lambda \setminus L_\lambda)$ (cf. [2, 3.4]), it follows that $\varepsilon p_\lambda(e) = \varepsilon p_\lambda(e_1)$ for every $\lambda \in \Lambda$, i.e., $e = e_1$, and the assertion follows.

(c) The proof follows by repeating the arguments in the proof of (b). □

Remark. We note that $X \subseteq \varprojlim \varepsilon X_\lambda \setminus L$, because (G, X) is proper and $X = \varprojlim X_\lambda \subseteq \varprojlim \varepsilon X_\lambda$.

6. The maximality of $\varprojlim \varepsilon X_\lambda = \mu X$ and the cardinality of $L = \mu L$

In this paragraph we prove the main results of the paper.

Lemma 6.1. *Let $(X, \mathbf{D})$ be a uniform space, and (G, X) be an equicontinuous action. Then, there exists a finer uniformity $\mathbf{D}^*$ compatible with the topology of X such that G acts on X by pseudoisometries with respect to the pseudometrics generating $\mathbf{D}^*$.*

Proof. Let $\{d_i,\, i \in I\}$ be a saturated family of bounded pseudometrics on X which generates $\mathbf{D}$ [4, II, Th. 1, p. 142]. We obtain a pseudometric d_i^* on X such that every $h \in G$ acts on X as a d_i^*-pseudoisometry, by letting $d_i^*(x,y) = \sup_{g \in G} d_i(gx, gy)$. Let $\mathbf{D}^*$ denote the uniformity generated by the family $\{d_i^* \mid i \in I\}$. The topologies τ, τ^* induced on X by $\mathbf{D}$ and $\mathbf{D}^*$, respectively, coincide: Since $d_i^*(x,y) \geq d_i(x,y)$, we have $\mathbf{D} \subseteq \mathbf{D}^*$ and $\tau \subseteq \tau^*$. Conversely, if $U_x^* = \bigcap_{k=1}^n S_k(x, \epsilon)$ is a neighborhood of x in τ^*, where $S_k(x,\epsilon)$ denotes a $d_{i_k}^*$-ball of radius ϵ, centered at x, then the equicontinuity of G implies the existence of a neighborhood U_x of x in τ, such that $U_x \subseteq U_x^*$. □

Theorem 6.2. *The compactification $\varprojlim \varepsilon X_\lambda = \mu X$ is maximal among the zero-dimensional compactifications of X satisfying simultaneously the following properties:*

(a) *The initial action (G, X) is extended to an action $(G, \mu X)$.*

(b) *The action $(G, \mu X \setminus \mu L)$, where μL is the set of the limit points of the orbits of the initial action (G, X) in μX, is proper, equicontinuous with respect to the uniformity induced on $\mu X \setminus \mu L$ by that of μX, and indivisible.*

Proof. By Theorem 5.5, the zero-dimensional compactification μX of X satisfies the properties (a) and (b). So, it remains to prove the maximality of μX: Suppose that Y is a zero-dimensional compactification of X also satisfying these properties, such that $q : Y \to \varprojlim \varepsilon X_\lambda$ is a surjection extending the identity map of X (cf. §1.1). We have to show that q is bijective.

Claim 1. The restriction of q on the set L_Y of the limit points of the action (G, X) in Y is a bijection.

Let L_Y be the set of the limit points of the action (G, X) in Y, and c_1, $c_2 \in L_Y$ be two distinct points such that $q(c_1) = q(c_2)$. Then, there are open neighborhoods V_1 and V_2 of c_1 and c_2, respectively, in Y with disjoint closures.

Due to the indivisibility of the action $(G, Y \setminus L_Y)$, we may assume that $\lim g_j x = c_1$ and $\lim h_j x = c_2$ with $g_j x \in V_1$ and $h_j x \in V_2$. If there exists a covering $\mathcal{C} = \{gU_i \,|\, U_i \in \mathcal{U},\, g \in G\}$, as in §3.7, such that the members of it containing $g_j x$, respectively $h_j x$, are pairwise disjoint, then it is easily seen that $\lim \varepsilon p_\lambda(q(g_j x)) = \lim \varepsilon p_\lambda(g_j x) \neq \lim \varepsilon p_\lambda(q(h_j x))$, a contradiction to the assumption that $q(c_1) = q(c_2)$. Therefore, there exist cofinal families $\{A_j\}$ and $\{B_j\}$ of members of $\mathcal{C}$ such that $g_j x \in A_j$, $h_j x \in B_j$ and $A_j \cap B_j \neq \emptyset$.

Since we consider star coverings, using refinements if necessary, we may assume that $A_j \cup B_j = f_j U_j$ is a member of our covering intersecting both V_1 and V_2. Then $g_j x \in f_j U_j$; hence $x \in g_j^{-1} f_j U_j \in \mathcal{C}$. Since $\mathcal{C}$ is a locally finite covering, passing if necessary to a subnet, we may assume that $x \in g_j^{-1} f_j U_j = gU_r$ for suitable g and r. It follows that $h_j x = g_j g x_j$, where $x_j \in U_r$. Since U_r is relatively compact in X, we may assume that $\lim x_j = y \in X$. Thus, $c_2 \in J(y)$ with respect to the action (G, Y), because $\lim h_j x = c_2$. From this and the assumption that the action $(G, Y \setminus L_Y)$ is equicontinuous, taking into account Proposition 5.4, we conclude that $\lim g_j g y = c_2$. This contradicts the fact that $gy \in X$, $\lim g_j x = c_1$ and the action $(G, Y \setminus L_Y)$ is indivisible.

Claim 2. The restriction of q on the set $Y \setminus L_Y$ is also a bijection.

Since q is, by definition, the identity map on X, we have to show that it is bijective on $Y \setminus (L_Y \cup X)$. The action $(G, Y \setminus L_Y)$ is equicontinuous; hence, by Lemma 6.1, we may assume that G acts by pseudoisometries. So, we are allowed to assume that the invariant covering $\mathcal{C}$ consists of open sets leading to invariant entourages.

Let $b_1, b_2 \in Y \setminus (L_Y \cup X)$ be two distinct points such that $q(b_1) = q(b_2)$, and $V = \{(x, y) \in (Y \setminus L_Y) \times (Y \setminus L_Y) \,|\, d_k(x, y) < \epsilon,\, k = 1, 2, \ldots, n\}$ be an entourage such that $(b_1, b_2) \notin V$. Moreover, we may assume that $\mathcal{C}$ consists of open sets leading to invariant entourages of the form

$$W = \{(x, y) \in (Y \setminus L_Y) \times (Y \setminus L_Y) \,|\, d_k(x, y) < \epsilon/2,\, k = 1, 2, \ldots, n, n+1, \ldots, m\}.$$

As in the proof of Claim 1 and the notation there, since $b_1, b_2 \notin X$, we can find families $\{A_j\}$ and $\{B_j\}$ of members of $\mathcal{C}$ and $x_j \in A_j$, $y_j \in B_j$ such that $\lim x_j = b_1$, $\lim y_j = b_2$ and $A_j \cup B_j$ is a member of our covering. From this and the specific choice of the entourages V and W, it follows that $(b_1, b_2) \in V$, a contradiction. □

Corollary. *If X has the "property Z", then $\mu X = \varepsilon X$.*

Proof. We have to show that if (G, X) is a properly discontinuous action, then the properties (a) and (b) of the previous theorem are satisfied for εX, the maximal zero-dimensional compactification of X, instead of μX. This follows from Corollary 1.1.4, the already mentioned results of [2] in the introduction (cf. [2, 4.7 and 3.7]), and from Proposition 5.4 for $Y = \varepsilon X$ and $Z = \varepsilon X \setminus \varepsilon L$. □

Example. In our counterexample the set μL consists of the two endpoints of the diagonal and the zero-dimensional compactification μX may obtained as a quotient space of the half-open Alexandroff square by identifying on the one hand the points $\{(x,1)\,|\,x\in(0,1]\}$, and on the other hand the points $\{(x,0)\,|\,x\in[0,1)\}$.

Theorem 6.3. *The space μL of the limit points of the action (G,X) in μX either consists of at most two points or it is a perfect compact set. In the case where the group G is abelian, μL has at most two points.*

Proof. Let μL have infinitely many points. We have to show that for every point e of it and every neighborhood $V=\varepsilon p_\lambda^{-1}(U)$ of e (cf. Proposition 1.4.1), we can find a point $e_1\in\mu L$ with $e_1\in\overline{V}$ and $e_1\neq e$. By [2, 4.2], the action $(G,\varepsilon X_\lambda\setminus L_\lambda)$ is proper. In the case under consideration, L_λ is a perfect compact set, by [2, 4.11, Satz D, 4]. Since, by Lemma 5.6, $\varepsilon p_\lambda(e)\in L_\lambda$, we can find $e_\lambda\in L_\lambda$ with $e_\lambda\in U$ and $e_\lambda\neq\varepsilon p_\lambda(e)$. By the indivisibility of the action (G,X_λ) (cf. [2, 3.4]), there is a net $\{g_i\}$ in G with $g_i\to\infty$ and $\lim g_i\varepsilon p_\lambda(x)=e_\lambda$ for every $x\in X$. Since $\lim\varepsilon p_\lambda(g_ix)=e_\lambda$ and μX is compact, fixing an $x\in X$, we may assume that $g_ix\in V$ and $\lim g_ix=e_1\in\mu L$. Therefore $e_1\in\overline{V}$. Since $\varepsilon p_\lambda(e_1)=\lim\varepsilon p_\lambda(g_ix)=e_\lambda\neq\varepsilon p_\lambda(e)$, we have $e_1\neq e$.

Now, assume that μL has finitely many points. Since, by Lemma 5.6, $L=\bigcap_\lambda\varepsilon p_\lambda^{-1}(L_\lambda)$, the set μL is the inverse limit of the inverse system $(L_\lambda,\varepsilon p_{\lambda\mu},\Lambda)$. By Theorem 1.2.1, every L_λ consists of at most two points. From this and the fact that every simplicial map $p_{\lambda\mu}:X_\mu\to X_\lambda$ is defined by deleting the last coordinates (cf. §3.6), we conclude that μX also has at most two points.

If the acting group is abelian, then the action (G,X_λ) fulfills the assumptions of Theorem 1.17 of [7]; therefore every L_λ consists of one or two points. From this and using the same arguments as before, we see that μL consists of at most two points. □

7. An application

In this section we apply our main results to show that the already known necessary condition for the existence of a proper action of a non-compact group on a locally compact and connected space with the "property Z" (cf. Theorem 1.2.1) remains also necessary in a broad class of actions, containing the properly discontinuous ones, on spaces that do not have the "property Z".

Theorem 7.1. *Let X be a locally compact, connected and paracompact space, and G be a non-compact group acting properly on X such that either G_0, the connected component of the neutral element of G, is non-compact, or G_0 is compact and G/G_0 contains an infinite discrete subgroup. Then X has*

(a) *at most two or infinitely many ends, and*
(b) *at most two ends, if G_0 is not compact.*

Proof. We begin with the proof of (b) and we shall restrict ourselves in the proof of (a) only in the case where G_0 is compact.

(b) If G_0 is non-compact, we consider the restricted action (G_0,X). By Iwasawa's Decomposition Theorem, G_0 contains a closed subgroup isomorphic to $\mathbb{R}$; therefore it contains a closed subgroup isomorphic to $\mathbb{Z}$, the additive group of the integers. The restricted action $(\mathbb{R},X)$ is proper; therefore the action $(\mathbb{Z},X)$ is properly discontinuous.

Since the space of the ends of X is totally disconnected, every end is a fixed point for the action $(G_0, \varepsilon X)$, therefore for the restricted action $(\mathbb{Z}, \varepsilon X)$ too. Since the projection $p : \varepsilon X \to \mu X$ is equivariant, every point of $\mu X \setminus X$ is a fixed point for the action $(\mathbb{Z}, \mu X)$, where μX is the zero-dimensional compactification of X that corresponds to the action $(\mathbb{Z}, X)$ by Theorem 6.2. From this and Theorem 6.3, there exist at most two limit points for the action $(\mathbb{Z}, \mu X \setminus \mu L)$. The set $\mu X \setminus X$ cannot have any other point except these limit points, because by Theorem 5.5, the action $(\mathbb{Z}, \mu X \setminus \mu L)$ is properly discontinuous, therefore has compact isotropy groups.

We claim that in this case $\varepsilon X = \mu X$ holds, which implies (b). To this end, we have to prove that p is injective. In order to be able to repeat the arguments in the proof of Theorem 6.2, Claim 2, replacing Y by εX, we need the following:

Claim. The action $(\mathbb{R}, X)$ is equicontinuous with respect to the uniformity induced on X by that of εX.

We shall use Proposition 5.4. Let $x \in X$ and $\lim x_i = x$ for $x_i \in X$. To arrive at a contradiction, assume that there exists a net $\{t_i\}$ in $\mathbb{R}$ with $t_i \to +\infty$ and $\lim t_i x = e_1 \in \varepsilon X \setminus X$, while $\lim t_i x_i = e_2 \in \varepsilon X \setminus X$, where $e_1 \neq e_2$. Let U and V_1 be disjoint neighborhoods in εX of x and e_1, respectively, with boundaries in X. Then, there exists t_0 such that $tx \in V_1$ for every $t \geq t_0$, because otherwise, by the connectedness of the orbits, we can find a net $\{r_i x\}$ in the boundary of V_1 with $\lim r_i x = y \in X$ and $r_i \to +\infty$; this is not possible, because the action $(\mathbb{R}, X)$ is proper; hence $L(x) \subseteq J(x) = \emptyset$ for every $x \in X$ (cf. §1.2 and Section 5). So, we can find a neighborhood V_2 in εX of e_2 with boundary in X, disjoint from U such that $tx \notin \overline{V_2}$ for every $t \geq 0$. Since $\lim x_i = x \in U$ and $\lim t_i x_i = e_2 \in V_2$ there exists a net $\{s_i x_i\}$ in the boundary of V_2 such that $\lim s_i x_i = z \in X$. As before, the net $\{s_i\}$ cannot be divergent; therefore we may assume that $\lim s_i = s \geq 0$. Hence $z = sx \in \overline{V_2}$, a contradiction.

(a) We have to consider only the case where G_0 is compact and G/G_0 contains an infinite discrete subgroup. Since X is connected and σ-compact, the orbit space $X \backslash G_0$ of the action (G_0, X) is connected and σ-compact, therefore paracompact. The group G/G_0 acts on $X \backslash G_0$, by letting

$$(gG_0, G_0(x)) \mapsto G_0(gx), \text{ for every } g \in G \text{ and } x \in X.$$

This action is well defined, since G_0 is a normal subgroup of G. Moreover, it is proper: Since the initial action is proper, G is locally compact. Therefore, by §1.2, there exist compact neighborhoods U_x, U_y in X of x and y, respectively, such that the set

$$G(U_x, U_y) = \{g \in G \mid (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G. Then $W_1 = \{G_0(z) \mid z \in U_x\}$ and $W_2 = \{G_0(z) \mid z \in U_y\}$ are compact neighborhoods of the points $G_0(x)$, $G_0(y)$ in $X \backslash G_0$, respectively. The set

$$(G/G_0)(W_1, W_2) = \{gG_0 \in G/G_0 \mid (gG_0 W_1) \cap W_2 \neq \emptyset\}$$

is relatively compact in G/G_0. Indeed, let $\{g_i G_0\}$ be a net in $(G/G_0)(W_1, W_2)$. Then, there exist h_i, $q_i \in G_0$ and $x_i \in U_x$, $y_i \in U_y$ such that $g_i h_i x_i = q_i y_i$, i.e., $q_i^{-1} g_i h_i \in G(U_x, U_y)$. Therefore $g_i \in G_0 \cdot G(U_x, U_y) \cdot G_0$, which is a relatively compact subset of G. This means that $\{g_i G_0\} \to \infty$ is not possible. Hence

$(G/G_0)(W_1, W_2)$ is relatively compact. Therefore, every non-compact discrete subgroup F of G/G_0 acts properly discontinuously on $X\backslash G_0$, which is a locally compact, connected and paracompact space.

So, we can apply our results for the action $(F, X\backslash G_0)$. The surjective map $q : X \to X\backslash G_0$ with $q(x) = G_0(x)$ is proper, because G_0 is compact. Therefore, by Proposition 1.1.2, it has a unique extension $\varepsilon q : \varepsilon X \to \varepsilon(X\backslash G_0)$ that maps the ends of X onto those of $X\backslash G_0$. So, this map relates the ends of X with those of $X\backslash G_0$.

Claim. The restriction of the map εq on the set of the ends of X is a bijection.

Since G_0 is connected, as before, the ends of X are fixed points for the action $(G_0, \varepsilon X)$. The map εq is equivariant; therefore the ends of $X\backslash G_0$ are also fixed points with respect to the action $(G/G_0, \varepsilon(X\backslash G_0))$. Since every end of X is a G_0-orbit, the assertion follows.

If X has infinitely many ends there is nothing to prove. If X has finitely many ends, let $\mu(X\backslash G_0)$ be the zero-dimensional compactification of $X\backslash G_0$ that corresponds to the action $(F, X\backslash G_0)$ by Theorem 6.2. According to Proposition 5.2, we have that $\mu(X\backslash G_0) = \varepsilon(X\backslash G_0)$, and by Theorem 6.3, the set L^* of the limit points of the action $(F, X\backslash G_0)$ consists of at most two points. There are no other ends except those of L^*, because by Theorem 6.2(b), the non-compact group F acts properly on $\varepsilon(X\backslash G_0) \setminus L^*$ which has finitely many points. This and the previous claim prove the theorem. $\square$

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