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Splitting a simple homotopy equivalence along a submanifold with filtration

A. Bak and Yu. V. Muranov

Abstract. A simple homotopy equivalence $f\colon M^n \to X^n$ of manifolds splits along a submanifold $Y \subset X$ if it is homotopic to a map that is a simple homotopy equivalence on the transversal preimage of the submanifold and on the complement of this preimage. The problem of splitting along a submanifold with filtration is a natural generalization of this problem. In this paper we define groups LSF_* of obstructions to splitting along a submanifold with filtration and describe their properties. We apply the results obtained to the problem of the realization of surgery and splitting obstructions by maps of closed manifolds and consider several examples.

Bibliography: 36 titles.

§ 1. Introduction

The classical surgery theory [1]–[5] has a natural generalization to the case of stratified manifolds [6]–[11]. A pair of closed manifolds is the simplest case of a stratified manifold [5]. The surgery theory of a manifold pair and the problem of splitting a (simple) homotopy equivalence along a submanifold (see [2], [4], [5], [12]–[15]) provide effective methods of the solution of many problems in geometric topology (see [13]–[22]). The splitting problem is closely related to the classification of manifolds of a fixed homotopy type, (see [16], [20]–[22]), the computation of the Wall groups and natural maps between them (see [23] and [24]), problems of the classification of involutions and group actions on manifolds (see [2], [12]–[14]), and the realization of elements of the Wall group by normal maps of closed manifolds (see [10], [14]–[16], [18] and [19]).

The problem of splitting a simple homotopy equivalence along a submanifold with filtration is a natural generalization of the problem of splitting along a submanifold. In this paper we define groups LSF_* of obstructions to splitting along a submanifold with filtration and investigate their properties. We consider topological manifolds and the surgery obstruction groups $L_* = L^s_*$ [4], [5]. The orientation homomorphism of the fundamental group of a manifold coincides with the homomorphism given by the first Stiefel-Whitney class.

Let π be a group equipped with an orientation homomorphism $w\colon \pi \to \{\pm 1\}$. We denote by $L_*(\pi) = L^s_*(\pi, w)$ the Wall group of obstructions to surgery up to

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a simple homotopy equivalence and by $C_*(\pi) \subset L_*(\pi)$ the subgroup generated by the elements that can be realized by normal maps of closed manifolds. Let $C^p_*(\pi) \subset L^p_*(\pi)$ be the image of the subgroup $C_*(\pi)$ in the projective group L^p_* under the natural map $L^s_* \to L^p_*$.

Let X^n be a closed topological manifold of dimension $n \geqslant 5$ with fundamental group $\pi_1(X) = \pi$ and $Y^{n-q} \subset X^n$ a submanifold of codimension q. A simple homotopy equivalence

$$f\colon M^n \to X^n \tag{1.1}$$

of n-dimensional manifolds splits along the submanifold Y if the homotopy class of the map f contains a map g transversal to Y with $N = g^{-1}(Y)$ for which the restrictions

$$g\big|_N\colon N \to Y \quad \text{and} g\big|_{M\setminus N}\colon M \setminus N \to X \setminus Y$$

are simple homotopy equivalences.

A manifold pair $Y^{n-1} \subset X^n$ is called a *Browder-Livesay pair* if Y is a one-sided submanifold and the inclusion maps induce isomorphisms $\pi_1(Y) \cong \pi_1(X)$ and $\pi_1(\partial U) \to \pi_1(X \setminus Y)$, where ∂U is the boundary of a tubular neighbourhood of Y in X. In this case the map $\pi_1(X \setminus Y) \to \pi_1(X)$ is an inclusion of index 2 and the Browder-Livesay splitting obstruction group $LN_{n-1}\bigl(\pi_1(X\setminus Y) \to \pi_1(X)\bigr)$ is defined (see [2], [5], [12]–[15]). Let $\mathbb{R}P^N$ be a real projective space of high dimension $N > n$. For an arbitrary subgroup $\rho \to \pi$ of index 2 consider a map

$$\phi\colon X^n \to \mathbb{R}P^N,$$

inducing the epimorphism of fundamental groups with kernel ρ. Let $Y^{n-1} \subset X^n$ be the transversal preimage of $\mathbb{R}P^{N-1}$. Using surgery inside the manifold X (see [2] and [15]) we can assume that the pair $Y \subset X$ is a Browder-Livesay pair and the group of obstructions to splitting along Y is equal to $LN_{n-1}(\rho \to \pi)$. For the inclusion $\rho \to \pi$ of index 2 we have the rooted transfer map $\partial\colon L_n(\pi) \to LN_{n-2}(\rho \to \pi)$ (see [2], [5], [14], [17], [23]–[25]), which is called the *Browder-Livesay invariant*. The map ∂ gives the first obstruction to the realization of elements of the group $L_n(\pi)$ by normal maps of closed manifolds [14].

In [15] Hambleton constructed the second Browder-Livesay invariant and proved that it is sufficient to use these invariants for detecting the group $C^p_*(\pi)$ in the case of an arbitrary finite 2-group π. From the geometric point of view, the second Browder-Livesay invariant arises in the case of the consecutive consideration of two splitting problems for the Browder-Livesay pairs: first, for the pair $Y \subset X$ and, afterwards, for the pair $Z \subset Y$. These manifolds fit in the triple

$$Z \subset Y \subset X \tag{1.2}$$

giving a filtration $\mathscr{X}$ of the manifold X. We shall call the triple (1.2) a *Browder-Livesay triple* if the pairs (X, Y) and (Y, Z) are Browder-Livesay pairs. The filtration (1.2) is a stratified manifold in the sense of Browder-Quinn [6]–[8]; hence the Browder-Quinn stratified surgery obstruction groups

$$L^{BQ}_{n-2}(\mathscr{X}) = LT_{n-2}(X, Y, Z).$$

are defined.

In the general case of a manifold with filtration we present the definition of the Browder-Quinn groups below, at the end of § 2.

Let (X, Y, Z) be a triple of manifolds. The groups $LSP_{n-2}(X, Y, Z)$ of obstructions to splitting a simple homotopy equivalence $f\colon M^n \to X^n$ along the pair of submanifolds $(Z \subset Y)$ are defined in [11]. The obstruction $\sigma(f) \in LSF_{n-2}(X, Y, Z)$ is trivial if and only if the homotopy class of the map f contains a map g split along every pair of manifolds in the triple (X, Y, Z). The Browder-Quinn groups and the LSP_*-groups fit in the exact sequence [11]

$$\cdots \to LT_{n-q}(X, Y, Z) \longrightarrow L_n(\pi_1(X)) \xrightarrow{\partial_p} LSP_{n-q+1}(X, Y, Z) \to \cdots . \tag{1.3}$$

For the Browder-Livesay triple (1.2) the map

$$L_n(\pi_1(X)) \xrightarrow{\partial_p} LSP_{n-2}(X, Y, Z), \qquad n - 2 \geqslant 5, \tag{1.4}$$

from the exact sequence (1.3) gives an invariant forbidding the realization of elements of the group $L_n(\pi_1(X))$ by normal maps of closed manifolds (see [11] and [26]). This invariant is equivalent to the pair of Browder-Livesay invariants constructed in [15] (see [11] and [26]).

Iterated Browder-Livesay invariants forbidding realization of elements of the Wall groups by normal maps of closed manifolds were defined by Kharshiladze (see [18], [23], [27], and [28]). Kharshiladze's construction uses a filtration of a manifold X by a system of submanifolds in which every pair of adjacent manifolds is a Browder-Livesay pair for the fixed inclusion $\rho \to \pi = \pi_1(X)$ of index 2.

Consider a filtration $\mathscr{X}$:

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{1.5}$$

of a closed topological manifold X by closed locally flat submanifolds.

In this paper we define groups $LSF_*(\mathscr{X})$ of obstructions to splitting a simple homotopy equivalence f in (1.1) along the subfiltration $\mathscr{Y}$:

$$X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1. \tag{1.6}$$

For $\dim X_k = n_k \geqslant 5$, the obstruction $\sigma_k(f) \in LSF_{n_k}(\mathscr{X})$ is trivial if and only if the homotopy class of the map f contains a map g that is split for all pairs of submanifolds appearing in the filtration $\mathscr{X}$. The introduced groups are a natural generalization of the groups LS_* and LSP_*.

In this paper we describe relations between the LSF_*-groups, the classical surgery obstruction groups, and the Browder-Quinn groups. The main relations are given by commutative diagrams and braids of exact sequences. We obtain also relations between the groups LSF_* and various structure sets for the filtration (1.5). We use the algebraic surgery theory of Ranicki (see [4], [7]–[11] and [28]–[31]) yielding realization of obstruction groups, structure sets and natural maps on the spectrum level. The groups $LSF_*(\mathscr{X})$ are also realized by the spectra $\mathbb{L}SF(\mathscr{X})$.

We call a filtration $\mathscr{X}$ a *Browder-Livesay filtration* if every pair of manifolds (X_i, X_{i+1}) from (1.5) is a Browder-Livesay pair with respect to the inclusion $\pi_1(X_i \setminus X_{i+1}) \to \pi_1(X_i)$ of index 2. In this case the spectra $\mathbb{L}SF(\mathscr{X})$ are closely related to the surgery spectral sequence of Hambleton-Kharshiladze [32]. For the case of a Browder-Livesay filtration we obtain a map into the group $LSF_*(\mathscr{X})$ that is equivalent to the system of iterated Browder-Livesay invariants given by the filtration $\mathscr{X}$. After that we apply the groups LSF_* to the investigation of the realization of splitting obstructions by simple homotopy equivalences of closed manifolds (see [13], [28] and [33]). We give several explicit results for the groups $LN_*(\rho \to \pi)$.

§ 2. Algebraic surgery theory and manifolds with filtration

In this paper we use the algebraic surgery theory of topological manifolds and manifolds with filtration (see [2], [4], [7]–[9], [17], [29]–[31] and [34]). In the present section we recall necessary preliminary facts about the realization of groups and maps on the spectrum level. In the homotopy category of spectra the concepts of push-out and pull-back squares are equivalent (see [10], [31] and [35]); hence we shall call them push-out squares. All obstruction groups under consideration are equipped with the decoration 's'; for example, $LS_*(F) = LS^s_*(F)$ (see [2], [4] and [5]).

Let X be a connected topological space with fundamental group $\pi = \pi_1(X)$ and orientation homomorphism $w\colon \pi \to \{\pm 1\}$. Let

$$\mathbb{L}(\pi) = \mathbb{L}(\pi, w) = \{\mathbb{L}_{-k}(\pi) : k \in \mathbb{Z}\}$$

be the Ω-spectrum constructed in [4] (see [2], [5] and [12]). The spectrum $\mathbb{L}(\pi)$ is 4-periodic and its homotopy groups are isomorphic to Wall groups:

$$\mathbb{L}_m(\pi) \simeq \mathbb{L}_{m+4}(\pi), \qquad L_n(\pi) = \pi_n(\mathbb{L}(\pi)).$$

We denote by $\mathbf{L}_\bullet$ the 1-connected cover of the spectrum $\mathbb{L}(1)$. Then $\mathbf{L}_{\bullet 0} \simeq G/\mathrm{TOP}$ and there is a cofibration of spectra (see [4])

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi) \to \mathbb{S}(X). \tag{2.1}$$

Let

$$\mathscr{S}_m(X) = \pi_m(\mathbb{S}(X)).$$

The homotopy long exact sequence of the cofibration (2.1) yields the algebraic surgery exact sequence [4]

$$\cdots \to L_{m+1}(\pi) \to \mathscr{S}_{m+1}(X) \to H_m(X; \mathbf{L}_\bullet) \to L_m(\pi) \to \cdots, \tag{2.2}$$

where $H_m(X; \mathbf{L}_\bullet) = \pi_m(X_+ \wedge \mathbf{L}_\bullet)$.

For a closed n-dimensional topological manifold X the left-hand part ($m \geqslant n$) of the exact sequence (2.2) coincides with the surgery exact sequence (see [2], [4], [5] and [10]). Under this identification $H_n(X; \mathbf{L}_\bullet) = [X, G/\mathrm{TOP}] = \mathscr{T}^{\mathrm{TOP}}(X)$ is the set of normal invariants of the manifold X and $\mathscr{S}_{n+1}(X) = \mathscr{S}^{\mathrm{TOP}}(X)$ is the set of classes of s-cobordance of simple homotopy equivalences $f\colon M^n \to X$.

An orientation-preserving group homomorphism $g\colon \pi \to \pi'$ induces a cofibration of spectra

$$\mathbb{L}(\pi) \to \mathbb{L}(\pi') \to \mathbb{L}(g). \tag{2.3}$$

The homotopy long exact sequence of the cofibration (2.3) is the relative exact sequence of L-groups with $\pi_n(\mathbb{L}(g)) = L_n(g)$ (see [2], [4], [29] and [31]).

The transfer map and its relative groups are also realized on the spectrum level [29]. The disc bundle

$$(D^q, S^{q-1}) \to (E, \partial E) \xrightarrow{p} X$$

over a closed manifold X induces the homotopy commutative diagram of spectra (see [2], [5], [10], [29] and [31])

$$\begin{array}{ccc} \mathbb{L}(\pi_1(X)) & \xrightarrow{p^\sharp} & \Omega^q\mathbb{L}\big(\pi_1(\partial E) \to \pi_1(E)\big) \\ & {\scriptstyle p_1^\sharp}\searrow & \downarrow{\scriptstyle \delta^\sharp} \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(\partial E)) \end{array} \tag{2.4}$$

in which $p^\sharp$ and $p_1^\sharp$ are the transfer maps and $\delta^\sharp$ is the connecting map in the cofibration exact sequence of $\mathbb{L}$-spectra that is induced by the natural map $\pi_1(\partial E) \to \pi_1(E)$ of fundamental groups.

Recall (see [4]) that a closed manifold pair (X^n, Y^{n-q}, ξ) of codimension q is given by a locally flat submanifold Y^{n-q} of the manifold X^n together with a normal block bundle

$$\xi = \xi_{Y\subset X}\colon Y \to \widetilde{\mathrm{BTOP}}(q), \qquad X = E(\xi) \cup_{S(\xi)} Z, \quad Z = \overline{X \setminus E(\xi)}.$$

In addition, the pair $(E(\xi), S(\xi))$ fits in the associated (D^q, S^{q-1}) fibration (see [4], § 7.2)

$$(D^q, S^{q-1}) \to (E(\xi), S(\xi)) \to Y. \tag{2.5}$$

We assume in what follows that all pairs of manifolds of filtrations (1.2) and (1.5) are manifold pairs in the sense of this definition.

Let

$$(f, b)\colon M^n \to X^n \tag{2.6}$$

be a topological normal map (t-triangulation) (see [4], [5; § 7], [8] and [10]), where b is a map of topological normal bundles covering the map f. The map (2.6) defines a t-triangulation

$$\big((f, b), (g, c)\big)\colon (M, N) \to (X, Y) \tag{2.7}$$

of the manifold pair (X, Y, ξ) (see [5]) if the following conditions are satisfied:

a) the map f is transversal to Y with $N = f^{-1}(Y)$;

b) the pair (M, N) is a manifold pair with normal block bundle

$$\nu\colon N \xrightarrow{f|_N} Y \xrightarrow{\xi} \widetilde{\mathrm{BTOP}}(q), \qquad M = E(\nu) \cup_{S(\nu)} P, \quad P = \overline{M \setminus E(\nu)};$$

c) the restriction

$$(g, c) = (f, b)\big|_N\colon N \to Y$$

is a t-triangulation of the manifold Y;

d) the restriction

$$(h,d) = (f,b)\big|_P \colon (P, S(\nu)) \to (Z, S(\xi))$$

is a t-triangulation of the pair $(Z, S(\xi))$ for which the restriction

$$(h,d)\big|_{S(\nu)} \colon S(\nu) \to S(\xi)$$

coincides with the induced map

$$(g,c)^!\big|_{S(\nu)} \colon S(\nu) \to S(\xi)$$

and $(f,b) = (g,c)^! \cup (h,d)$.

By [5], Proposition 7.2.3, the set of concordance classes $\mathscr{T}^{\mathrm{TOP}}(X,Y,\xi)$ of t-triangulations of a manifold pair (X,Y,ξ) coincides with the set of normal invariants $\mathscr{T}^{\mathrm{TOP}}(X)$ of the manifold X.

By definition [5] a t-triangulation (2.7) of a manifold pair (X,Y,ξ) is an s-triangulation if the above-defined maps

$$f \colon M \to X, \quad g \colon N \to Y, \qquad (P, S(\nu)) \to (Z, S(\xi)) \tag{2.8}$$

are simple homotopy equivalences (s-triangulations). We denote by $\mathscr{S}^{\mathrm{TOP}}(X,Y,\xi)$ the set of concordance classes of s-triangulations of a manifold pair (X,Y,ξ) (see [5]). By definition a simple homotopy equivalence $f \colon M \to X$ splits along the submanifold Y if it is homotopic to an s-triangulation of the pair (X,Y). The natural forgetful maps

$$\mathscr{S}^{\mathrm{TOP}}(X,Y,\xi) \to \mathscr{S}^{\mathrm{TOP}}(X), \qquad \mathscr{S}^{\mathrm{TOP}}(X,Y,\xi) \to \mathscr{S}^{\mathrm{TOP}}(Y) \tag{2.9}$$

are well defined (see [5], § 7.2).

For a manifold pair (X,Y,ξ) we denote by $\partial U = S(\xi)$ the boundary of a tubular neighbourhood $U = E(\xi)$ of the submanifold Y in X and by $X \setminus Y$ the closure $\overline{X \setminus U}$. Let

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \to & \pi_1(X) \end{pmatrix} \tag{2.10}$$

be the square of fundamental groups with orientations. The maps in (2.10) are induced by maps of the corresponding spaces. The groups $LS_*(F)$ of obstructions to splitting of the simple homotopy equivalence (1.1) along the submanifold Y are defined in [2], § 11. These groups depend on $n-q \pmod 4$ and, functorially, on the square F. We denote by Θ the map

$$\mathscr{S}^{\mathrm{TOP}}(X) \to LS_{n-q}(F), \tag{2.11}$$

assigning an obstruction to splitting.

For the normal map (2.6) the groups $LP_{n-q}(F)$ of obstructions to surgery on the manifold pair (X, Y, ξ) are defined (see [2] and [5]). The groups $LP_{n-q}(F)$ also depend only on $n-q \pmod 4$ and, functorially, on the square F. We denote by σ the map

$$\mathscr{T}^{\mathrm{TOP}}(X) \to LP_{n-q}(F) \tag{2.12}$$

assigning the corresponding obstruction.

The groups $LS_*(F)$ and $LP_*(F)$ are realized by the spectra $\mathbb{L}S(F)$ and $\mathbb{L}P(F)$. There exists a commutative diagram of spectra (see [10] and [31])

$$\begin{array}{ccccc} \Omega\mathbb{L}(\pi_1(Y)) & \xrightarrow{\Omega p^\sharp} & \Omega^{q+1}\mathbb{L}\big(\pi_1(X\setminus Y)\to\pi_1(X)\big) & \longrightarrow & \mathbb{L}S(F) \\ \| & & \downarrow{\scriptstyle \Omega\delta^\sharp} & & \downarrow \\ \Omega\mathbb{L}(\pi_1(Y)) & \xrightarrow{\Omega p_1^\sharp} & \Omega^{q}\mathbb{L}(\pi_1(X\setminus Y)) & \longrightarrow & \mathbb{L}P(F) \end{array} \tag{2.13}$$

in which the left-hand square follows from the diagram (2.4). The right-hand square of the diagram (2.13) is a push-out. The homotopy long exact sequences of the maps in this square give us the commutative braid of exact sequences (see [2], [3] and [31])

$$\begin{array}{ccccccccc} \to L_n(\pi_1(X\setminus Y)) & & \longrightarrow & & L_n(\pi_1(X)) & & \longrightarrow & & LS_{n-q-1}(F) \to \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & LP_{n-q}(F) & & & & L_n^{rel} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ \to LS_{n-q}(F) & & \longrightarrow & & L_{n-q}(\pi_1(Y)) & & \longrightarrow & & L_{n-1}(\pi_1(X\setminus Y)) \to \end{array} \tag{2.14}$$

where $L_n^{rel} = L_n\big(\pi_1(X\setminus Y)\to\pi_1(X)\big)$. Note that the maps

$$LP_{n-q}(F)\to L_n(\pi_1(X)) \quad \text{and} \quad LP_{n-q}(F)\to L_{n-q}(\pi_1(Y))$$

in (2.14) are the natural forgetful maps [2].

The map (2.12) is realized by the map of spectra (see [2], [4], [5], § 7.2, [7], [10] and [31])

$$X_+\wedge \mathbf{L}_\bullet \to \Sigma^q\mathbb{L}P(F),$$

fitting in a cofibration

$$X_+\wedge \mathbf{L}_\bullet \to \Sigma^q\mathbb{L}P(F) \to \mathbb{S}(X,Y,\xi). \tag{2.15}$$

Let $\mathscr{S}_i(X,Y,\xi) = \pi_i(\mathbb{S}(X,Y,\xi))$; then by § 7.2 of [5] there exists an isomorphism

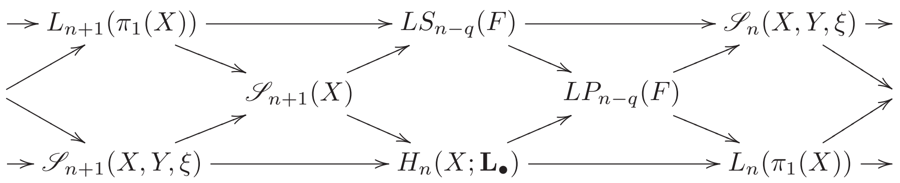
$$\mathscr{S}_{n+1}(X,Y,\xi) \cong S^{\mathrm{TOP}}(X,Y,\xi).$$

The maps from (2.9), (2.11) and (2.12) are also realized on the spectrum level by maps fitting in a homotopy commutative diagram of spectra (see [5], § 7.2, [10] and [31]):

$$\begin{array}{ccccc} \Omega\mathbb{S}(X,Y,\xi) & \longrightarrow & \Omega\mathbb{S}(X) & \longrightarrow & \Sigma^q\mathbb{L}S(F) \\ \| & & \downarrow & & \downarrow \\ \Omega\mathbb{S}(X,Y,\xi) & \longrightarrow & X_+\wedge\mathbf{L}_\bullet & \longrightarrow & \Sigma^q LP(F), \end{array} \tag{2.16}$$

in which the right-hand square is a push-out. The homotopy cofibres of the vertical map in the right-hand square in (2.16) are equal to $\mathbb{L}(\pi_1(X))$.

The homotopy long exact sequences of maps in the right-hand square in (2.16) give us the commutative braid of exact sequences (see [5], Proposition 7.2.6)



(2.17)

We now present necessary facts on the surgery on manifolds with filtration in a form suitable for the development of splitting theory (see [6]–[9], [11], [26] and [28]).

Let (X, Y, Z) be a triple of topological manifolds (1.2) with $n = \dim X$, $n-q = \dim Y$, $n-q-q' = \dim Z \geqslant 5$. Let ξ be the normal bundle of Y in X, η the normal bundle of Z in Y and ν the normal bundle of Z in X.

Similarly to (2.5) we obtain the associated fibrations

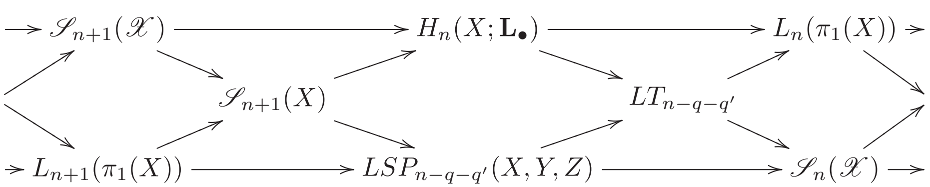
$$
\begin{aligned}
(D^q, S^{q-1}) &\to (E(\xi), S(\xi)) \to Y,\\
(D^{q'}, S^{q'-1}) &\to (E(\eta), S(\eta)) \to Z,\\
(D^{q+q'}, S^{q+q'-1}) &\to (E(\nu), S(\nu)) \to Z.
\end{aligned}
$$

We assume that the space $E(\nu)$ of the normal bundle ν is identified with the space $E(\xi|_{E(\eta)})$ of the restriction of the bundle ξ to the space $E(\eta)$ of the normal bundle η so that

$$
E(\nu) = E(\xi|_{E(\eta)}), \qquad S(\nu) = S(\xi|_{E(\eta)}) \cup E(\xi|_{S(\eta)}) \tag{2.18}
$$

(see [6]–[8], [11], [26] and [28]).

Note that for the filtration $\mathscr{X}$ given by (1.2) there exists a commutative braid of exact sequences (see [11])



(2.19)

which is realized on the spectrum level. The diagram (2.19) is a natural generalization of the diagram (2.17).

Consider the filtration (1.5). In what follows we suppose that conditions similar to (2.18) hold for any triple of manifolds from this filtration. Then the filtration (1.5) is a stratified manifold in the sense of Browder-Quinn [6], [7]. Let n_j ($n_0 = n$) be the dimension of the manifold X_j, $q_j = n_{j-1} - n_j$ ($1 \leqslant j \leqslant k$) the codimension of the submanifold X_j in X_{j-1}, and s_j the codimension of X_j in $X_0 = X$. We assume that $n_k \geqslant 5$. Let F_i (for $0 \leqslant i \leqslant k-1$) be the square of fundamental groups in the splitting problem for the pair of manifolds (X_i, X_{i+1}).

We denote by $\overline{\mathscr{X}}$ the filtration

$$\big(X_{k-1}\setminus X_k, \partial(X_{k-1}\setminus X_k)\big) \subset \cdots \subset \big(X\setminus X_k, \partial(X\setminus X_k)\big) \tag{2.20}$$

of the manifold $X\setminus X_k$ with boundary $\partial(X\setminus X_k)$. Every pair of manifolds in (2.20) is a pair of manifolds with boundaries in the sense of Ranicki's definition (see [5]). The boundaries of manifolds in the filtration (2.20) give us the filtration $\partial\overline{\mathscr{X}}$:

$$\partial(X_{k-1}\setminus X_k) \subset \partial(X_{k-2}\setminus X_k) \subset \cdots \subset \partial(X\setminus X_k) \tag{2.21}$$

of the closed manifold $\partial(X\setminus X_k)$. For $0\leqslant j\leqslant k$ we denote by $\mathscr{X}_j$ the subfiltration

$$X_j \subset X_{j-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X \tag{2.22}$$

of the filtration $\mathscr{X}$. As above, the filtrations $\overline{\mathscr{X}}_j$ and $\partial\overline{\mathscr{X}}_j$ are defined.

Recall the inductive definition of the spectrum $\mathbb{L}^{BQ}(\mathscr{X})$ for the Browder-Quinn groups $L_*^{BQ}(\mathscr{X})$ (see [6]–[8] and [28]). Note that in our definition the subscript $*$ for the obstruction group $L_*^{BQ}(\mathscr{X})$ is equal to the dimension n_k of the smallest manifold in the filtration $\mathscr{X}$ and

$$\pi_n(\mathbb{L}^{BQ}(\mathscr{X})) = L_n^{BQ}(\mathscr{X}). \tag{2.23}$$

Let

$$\mathbb{L}^{BQ}(\mathscr{X}_0) = \mathbb{L}(\pi_1(X_0)). \tag{2.24}$$

We defined the spectrum $\mathbb{L}^{BQ}(\mathscr{X}_1)$ as the homotopy fibre of the composition

$$\mathbb{L}(\pi_1(X_1)) \xrightarrow{p_1^\sharp} \Omega^{q_1-1}\mathbb{L}\big(\pi_1(\partial(X_0\setminus X_1))\big) \longrightarrow \Omega^{q_1-1}\mathbb{L}(\pi_1(X_0\setminus X_1)),$$

where the first map is the transfer and the second map is induced by the inclusion $\partial(X_0\setminus X_1)\subset(X_0\setminus X_1)$. We point out (see (2.13)) that

$$\mathbb{L}^{BQ}(\mathscr{X}_1) \simeq \mathbb{L}P(F_0). \tag{2.25}$$

Assume that the spectra $\mathbb{L}^{BQ}(\mathscr{X}_{j-1})$ are already defined for $0\leqslant j\leqslant k$. We define the spectrum $\mathbb{L}^{BQ}(\mathscr{X}_j)$ as the homotopy fibre of the composition

$$\mathbb{L}(\pi_1(X_j)) \to \Omega^{q_j-1}\mathbb{L}^{BQ}(\partial\overline{\mathscr{X}}_j) \to \Omega^{q_j-1}\mathbb{L}^{BQ}(\overline{\mathscr{X}}_j), \tag{2.26}$$

where the first map is the transfer (see [7], § 6) and the second map is induced by the natural inclusion of filtrations $\partial\overline{\mathscr{X}}_j\to\overline{\mathscr{X}}_j$.

Directly from the definition we obtain the cofibrations

$$\mathbb{L}^{BQ}(\mathscr{X}_j) \to \mathbb{L}(\pi_1(X_j)) \to \Omega^{q_j-1}\mathbb{L}^{BQ}(\overline{\mathscr{X}}_j), \qquad 1\leqslant j\leqslant k, \tag{2.27}$$

and the homotopy long exact sequences

$$\cdots \to L_n^{BQ}(\mathscr{X}_j) \to L_n(\pi_1(X_j)) \to L_{n+q_j-1}^{BQ}(\overline{\mathscr{X}}_j) \to \cdots. \tag{2.28}$$

For $0 \leqslant j \leqslant k$ there exist cofibrations (see [7], §§ 6, 7) and [8])

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^{n_0-n_j}\mathbb{L}^{BQ}(\mathscr{X}_j) \to \mathbb{S}(\mathscr{X}_j). \tag{2.29}$$

For $j = 0$ the cofibration (2.29) coincides with (2.1), and for $j = 1$ the cofibration (2.29) coincides with (2.15) for the pair (X, X_1).

Let $\mathscr{S}_i(\mathscr{X}_j) = \pi_i(\mathbb{S}(\mathscr{X}_j))$. Then there exist isomorphisms (see [7], [8])

$$\mathscr{S}_{n+1}(\mathscr{X}_j) = \mathscr{S}^{\mathrm{TOP}}(\mathscr{X}_j), \qquad 0 \leqslant j \leqslant k,$$

where the set $\mathscr{S}^{\mathrm{TOP}}(\mathscr{X}_j)$ consists of the concordance classes of s-triangulations of the filtration $\mathscr{X}_j$ (see [7], [8]).

For $0 \leqslant i \leqslant j \leqslant k$ we denote by $\mathscr{X}_j^i$ the subfiltration

$$X_j \subset X_{j-1} \subset \cdots \subset X_i \tag{2.30}$$

of the filtration $\mathscr{X}$. We shall write the filtration $\mathscr{X}_k^i$ as $\mathscr{X}^i$, and $\mathscr{X}_k^1 = \mathscr{Y}$ (see (1.6)).

Let

$$G_i = \pi_1(X_i), \quad 0 \leqslant i \leqslant k, \qquad G_0 = G, \qquad \rho_i = \pi_1(X_i \setminus X_{i+1}), \quad 0 \leqslant i \leqslant k-1.$$

By [8] there are the following push-out squares of spectra:

$$\begin{array}{ccc} \mathbb{L}^{BQ}(\mathscr{X}_j^i) & \longrightarrow & \Omega^{q_j}\mathbb{L}^{BQ}(\mathscr{X}_{j-1}^i) \\ \downarrow & & \downarrow \\ \mathbb{L}^{BQ}(\mathscr{X}_j^{i+1}) & \longrightarrow & \Omega^{q_j}\mathbb{L}^{BQ}(\mathscr{X}_{j-1}^{i+1}) \end{array} \qquad 0 \leqslant i < i+1 < j \leqslant k, \tag{2.31}$$

and

$$\begin{array}{ccc} \mathbb{L}^{BQ}(\mathscr{X}_{i+1}^i) & \longrightarrow & \Omega^{q_{i+1}}\mathbb{L}^{BQ}(\mathscr{X}_i^i) \\ \downarrow & & \downarrow \\ \mathbb{L}^{BQ}(\mathscr{X}_{i+1}^{i+1}) & \longrightarrow & \Omega^{q_{i+1}}\mathbb{L}(\rho_i \to G_i) \end{array} \qquad 0 < i+1 \leqslant k. \tag{2.32}$$

Note that the square (2.32) coincides with the square

$$\begin{array}{ccc} \mathbb{L}P(F_i) & \longrightarrow & \Omega^{q_{i+1}}\mathbb{L}(G_i) \\ \downarrow & & \downarrow \\ \mathbb{L}(G_{i+1}) & \longrightarrow & \Omega^{q_{i+1}}\mathbb{L}(\rho_i \to G_i) \end{array} \tag{2.33}$$

which realizes the central square in the diagram (2.14) for the pair (X_i, X_{i+1}).

The horizontal maps in the diagram (2.31) correspond to the map of forgetting the submanifolds X_j, and the vertical maps correspond to the map of forgetting the submanifold X_i. In the diagram (2.31) the homotopy fibre of the vertical maps is equal to $\Omega^{s_j}\mathbb{L}(\rho_i)$ and the homotopy fibre of the horizontal maps is equal to $\mathbb{L}S(F_{j-1})$.

§ 3. Splitting along a submanifold with filtration

In this section we consider a filtration $\mathscr{X}$ (see (1.5)). We define the spectra $\mathbb{L}SF(\mathscr{X}_i)$, $1 \leqslant i \leqslant k$, and the groups

$$\pi_n(\mathbb{L}SF(\mathscr{X}_i)) = LSF_n(\mathscr{X}_i), \tag{3.1}$$

where $\mathscr{X}_i$ is the subfiltration defined in (2.22). The group $LSF_{n_k}(\mathscr{X})$, where n_k is the dimension of the smallest manifold X_k of the filtration $\mathscr{X}$, is the group of obstructions to splitting a simple homotopy equivalence (1.1) along the subfiltration $\mathscr{Y} = \mathscr{X}^1$ from (1.6). We describe relations between the groups LSF_*, the Browder-Quinn groups, and various structure sets for the filtration $\mathscr{X}$. We preserve the notation and the agreements of the previous section.

The subfiltration $\mathscr{X}^i_{i+1}$ of the filtration $\mathscr{X}$ is given by the manifold pair $X_{i+1} \subset X_i$ $(0 \leqslant i < i+1 \leqslant k)$. By definition, let

$$\mathbb{L}SF(\mathscr{X}^i_{i+1}) = \mathbb{L}S(F_i). \tag{3.2}$$

Consider the composition

$$\mathbb{L}^{BQ}(\mathscr{X}^1_2) \to \Omega^{q_2}\mathbb{L}(\mathscr{X}^1_1) \to \Omega^{q_1+q_2}\mathbb{L}\bigl(\pi_1(X \setminus X_1) \to \pi_1(X)\bigr), \tag{3.3}$$

in which the first map is the forgetting map from the diagram (2.32) (for $i = 1$) and the second is the q_2-looped map of the transfer $p^\sharp$ for the pair (X_0, X_1) (see the diagrams (2.4) and (2.13)). Denote by

$$\mathbb{L}SF(\mathscr{X}_2) = \mathbb{L}SF(\mathscr{X}^0_2) = \mathbb{L}SP(X_0, X_1, X_2)$$

the homotopy fibre of the composition (3.3) (see [11]). The diagram (2.31) and this notation give rise to the following homotopy commutative diagram of spectra [11]):

$$\begin{array}{ccccc} \mathbb{L}^{BQ}(\mathscr{X}^1_2) & \longrightarrow & \Omega^{s_2}\mathbb{L}\bigl(\pi_1(X \setminus X_1) \to \pi_1(X)\bigr) & \longrightarrow & \Omega^{-1}\mathbb{L}SF(\mathscr{X}_2) \\ \| & & \downarrow & & \downarrow \\ \mathbb{L}^{BQ}(\mathscr{X}^1_2) & \longrightarrow & \Omega^{s_2-1}\mathbb{L}(\pi_1(X \setminus X_1)) & \longrightarrow & \Omega^{-1}\mathbb{L}^{BQ}(\mathscr{X}_2) \end{array} \tag{3.4}$$

where $s_2 = q_1 + q_2$. The rows in the diagram (3.4) are cofibrations and the right-hand square is a push-out.

The upper row of the diagram (2.31) yields a sequence of maps of spectra:

$$\mathbb{L}^{BQ}(\mathscr{X}^1_k) \to \Omega^{q_k}\mathbb{L}^{BQ}(\mathscr{X}^1_{k-1}) \to \cdots \to \Omega^{s_k-q_1}\mathbb{L}(\mathscr{X}^1_1) = \Omega^{s_k-q_1}\mathbb{L}(G_1). \tag{3.5}$$

For $1 \leqslant j \leqslant k$ consider the composition

$$\mathbb{L}^{BQ}(\mathscr{X}^1_j) \to \Omega^{s_j-q_1}\mathbb{L}(G_1) \to \Omega^{s_j}\mathbb{L}\bigl(\pi_1(X \setminus X_1) \to \pi_1(X)\bigr) \tag{3.6}$$

in which the first map follows from (3.5) by means of the corresponding delooping and the second map is similar to the map in (3.3). We define the spectrum $\mathbb{L}SF(\mathscr{X}_j)$ as the homotopy fibre of the composition (3.6). Then we obtain a cofibration

$$\mathbb{L}^{BQ}(\mathscr{X}^1_j) \to \Omega^{s_j}\mathbb{L}\bigl(\pi_1(X \setminus X_1) \to \pi_1(X)\bigr) \to \Omega^{-1}\mathbb{L}SF(\mathscr{X}_j). \tag{3.7}$$

Now, (3.1) defines the groups $LSF_n(\mathscr{X}_j)$, $1 \leqslant j \leqslant k$, of the filtration $\mathscr{X}_j$. It follows from the definition that the groups LSF_n depend on $n \pmod 4$.

Proposition 1. *The following commutative diagram of spectra holds*:

$$\begin{array}{ccccc} \mathbb{L}^{BQ}(\mathscr{X}_j^1) & \longrightarrow & \Omega^{s_j}\mathbb{L}\big(\pi_1(X\setminus X_1)\to\pi_1(X)\big) & \longrightarrow & \Omega^{-1}\mathbb{L}SF(\mathscr{X}_j) \\ \| & & \downarrow & & \downarrow \\ \mathbb{L}^{BQ}(\mathscr{X}_j^1) & \longrightarrow & \Omega^{s_j-1}\mathbb{L}(\pi_1(X\setminus X_1)) & \longrightarrow & \Omega^{-1}\mathbb{L}^{BQ}(\mathscr{X}_j) \end{array} \tag{3.8}$$

in which the horizontal rows are cofibrations and the right-hand square is a push-out. The homotopy fibre of the vertical maps of the right-hand square is equal to $\Omega^{s_j}\mathbb{L}(\pi_1(X))$.

Proof. This follows from the diagrams (2.31), (3.5) and the definition of the spectrum $\mathbb{L}SF(\mathscr{X}_j)$.

Corollary 1. *The groups* $LSF_* = LSF_*(\mathscr{X})$ *fit in the commutative braid of exact sequences*

$$\begin{array}{ccccccccccc} \to & L_n(\rho_0) & & \longrightarrow & & L_n(G_0) & & \longrightarrow & & LSF_{m-1} & \to \\ \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & L_m^{BQ}(\mathscr{X}) & & & & L_n(\rho_0\to G_0) & & & \\ \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \to & LSF_m & & \longrightarrow & & L_m^{BQ}(\mathscr{Y}) & & \longrightarrow & & L_{n-1}(\rho_0) & \to \end{array} \tag{3.9}$$

where $\rho_0 = \pi_1(X\setminus X_1)$, $G_0 = \pi_1(X)$ *and* $m = n - s_k$.

Proof. The homotopy long exact sequences of the maps in the right-hand square in (3.8) give us the diagram (3.9).

The diagram (3.9) is a natural generalization of the diagram (2.14).

Corollary 2. *Let* X_1 *be a submanifold of* X *of codimension* $q_1 \geqslant 3$. *Then*

$$LSF_n(\mathscr{X}) \cong L_n^{BQ}(\mathscr{Y})$$

for all n.

Proof. In the case under consideration $L_n(\rho_0 \to G_0) = 0$ since the map $\rho_0 \to G_0$ is an isomorphism. The result now follows from the diagram (3.9).

Theorem 1. *For* $2 \leqslant j \leqslant k$ *there exists a push-out square of spectra*

$$\begin{array}{ccc} \mathbb{L}SF(\mathscr{X}_j) & \longrightarrow & \mathbb{L}^{BQ}(\mathscr{X}_j^1) \\ \downarrow & & \downarrow \\ \Omega^{q_j}\mathbb{L}SF(\mathscr{X}_{j-1}) & \longrightarrow & \Omega^{q_j}\mathbb{L}^{BQ}(\mathscr{X}_{j-1}^1) \end{array} \tag{3.10}$$

in which the homotopy fibre of the vertical maps is equal to $\mathbb{L}S(F_{j-1})$ *and the homotopy fibre of the horizontal maps is equal to*

$$\Omega^{s_j+1}\mathbb{L}\big(\pi_1(X\setminus X_1)\to\pi_1(X)\big).$$

The square (3.10) *gives rise to the braid of exact sequences*

$$
\begin{array}{ccccccccc}
\to LS_n(F_{j-1}) & \longrightarrow & & & L_n^{BQ}(\mathscr{X}_j^1) & \longrightarrow & & & L_m^{rel} \longrightarrow \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & LSF_n(\mathscr{X}_j) & & & & L_l^{BQ}(\mathscr{X}_{j-1}^1) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\longrightarrow L_{m+1}^{rel} & \longrightarrow & & & LSF_l(\mathscr{X}_{j-1}) & \longrightarrow & & & LS_{n-1}(F_{j-1}) \to
\end{array}
\tag{3.11}
$$

where $m = n + s_j$, $l = n + q_j$ *and* $L_*^{rel} = L_*\big(\pi_1(X \setminus X_1) \to \pi_1(X)\big)$.

Proof. The definition of the spectrum $\mathbb{L}SF$ and the upper map for $i = 1$ in the diagram (2.31) yield the homotopy commutative diagram of spectra

$$
\begin{CD}
\mathbb{L}SF(\mathscr{X}_j) @>>> \mathbb{L}^{BQ}(\mathscr{X}_j^1) @>>> \Omega^{s_j}\mathbb{L}\big(\pi_1(X\setminus X_1) \to \pi_1(X)\big) \\
@VVV @VVV @| \\
\Omega^{q_j}\mathbb{L}SF(\mathscr{X}_{j-1}) @>>> \Omega^{q_j}\mathbb{L}^{BQ}(\mathscr{X}_{j-1}^1) @>>> \Omega^{s_j}\mathbb{L}\big(\pi_1(X\setminus X_1) \to \pi_1(X)\big)
\end{CD}
\tag{3.12}
$$

in which the left-hand vertical map is induced by the other two vertical maps (see [35]). The homotopy fibre of the map $\mathbb{L}^{BQ}(\mathscr{X}_j^1) \to \Omega^{q_j}\mathbb{L}^{BQ}(\mathscr{X}_{j-1}^1)$ is equal to $\mathbb{L}S(F_{j-1})$ as follows from the diagram (2.31). The left-hand square is a push-out since the cofibres of the horizontal maps are naturally homotopy equivalent.

Theorem 2. *There exists a push-out square of spectra*

$$
\begin{CD}
\Omega\mathbb{S}(X) @>>> X_+ \wedge \mathbf{L}_\bullet \\
@VVV @VVV \\
\Omega^{-s_k}\mathbb{L}SF(\mathscr{X}) @>>> \Omega^{-s_k}\mathbb{L}^{BQ}(\mathscr{X})
\end{CD}
\tag{3.13}
$$

in which the homotopy cofibre of the horizontal maps is equal to $\mathbb{L}(\pi_1(X))$ *and the homotopy cofibre of the vertical maps is equal to* $\mathbb{S}(\mathscr{X})$.

The square (3.13) *gives rise to the braid of exact sequences*

$$
\begin{array}{ccccccccc}
\to \mathscr{S}_{n+1}(\mathscr{X}) & \longrightarrow & & & H_n(X, \mathbf{L}_\bullet) & \longrightarrow & & & L_n(\pi_1(X)) \to \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \mathscr{S}_{n+1}(X) & & & & L_{n_k}^{BQ}(\mathscr{X}) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\to L_{n+1}(\pi_1(X)) & \longrightarrow & & & LSF_{n_k}(\mathscr{X}) & \longrightarrow & & & \mathscr{S}_n(\mathscr{X}) \longrightarrow
\end{array}
\tag{3.14}
$$

Proof. The required result follows from the homotopy commutative diagram

$$
\begin{CD}
\Omega\mathbb{S}(X) @>>> X_+ \wedge \mathbf{L}_\bullet @>>> \mathbb{L}(\pi_1(X)) \\
@VVV @VVV @VVV \\
\Omega^{-s_k}\mathbb{L}SF(\mathscr{X}) @>>> \Omega^{-s_k}\mathbb{L}^{BQ}(\mathscr{X}) @>>> \mathbb{L}(\pi_1(X))
\end{CD}
\tag{3.15}
$$

in which the maps in the right-hand square are defined in (2.29) and (3.5), the upper row follows from the cofibration (2.1), the bottom row is defined in Proposition 1, and the left-hand vertical map is induced by the other two vertical maps.

The commutative diagram (3.14) generalizes the diagrams from [5], Proposition 7.2.6, iv, and [11], Proposition 3.2. The map

$$\Theta_k\colon S_{n+1}(X) \longrightarrow LSF_{n_k}(\mathscr{X}) \tag{3.16}$$

in the diagram (3.14) fits in the exact sequence

$$\cdots \to \mathscr{S}_{n+1}(\mathscr{X}) \longrightarrow \mathscr{S}_{n+1}(X) \longrightarrow LSF_{n_k}(\mathscr{X}) \to \cdots$$

and corresponds algebraically to the map assigning the obstruction to splitting a simple homotopy equivalence $f\colon M \to X$ along the subfiltration $\mathscr{Y}$.

Theorem 3. *For $2 \leqslant j \leqslant k$ there exists a push-out square of spectra*

$$\begin{array}{ccc} \mathbb{L}SF(\mathscr{X}_j) & \longrightarrow & \Omega^{s_j}\mathbb{S}(\mathscr{X}_j) \\ \downarrow & & \downarrow \\ \Omega^{q_j}\mathbb{L}SF(\mathscr{X}_{j-1}) & \longrightarrow & \Omega^{s_j}\mathbb{S}(\mathscr{X}_{j-1}) \end{array} \tag{3.17}$$

in which the homotopy cofibre of the horizontal maps is equal to $\Omega^{-1}\mathbb{L}S(F_{j-1})$ and the homotopy cofibre of the vertical maps is equal to $\Omega^{s_j}\mathbb{S}(\mathscr{X})$.

The square (3.17) gives rise to the braid of exact sequences

$$\begin{array}{ccccccccc} \to & \mathscr{S}_{m+1}(X) & & \longrightarrow & & LSF_{m-s_{j-1}}(\mathscr{X}_{j-1}) & \longrightarrow & LS_{m-s_j-1}(F_{j-1}) & \to \\ \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \searrow \\ & & & LSF_{m-s_j}(\mathscr{X}_j) & & & & \mathscr{S}_m(\mathscr{X}_{j-1}) & \\ \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \nearrow \\ \to & LS_{m-s_j}(F_{j-1}) & & \longrightarrow & & \mathscr{S}_m(\mathscr{X}_j) & \longrightarrow & \mathscr{S}_m(X) & \to \end{array} \tag{3.18}$$

Proof. Consider a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Omega\mathbb{S}(X) & \longrightarrow & \Omega^{-s_j}\mathbb{L}SF(\mathscr{X}_j) & \longrightarrow & \mathbb{S}(\mathscr{X}_j) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega\mathbb{S}(X) & \longrightarrow & \Omega^{-s_{j-1}}\mathbb{L}SF(\mathscr{X}_{j-1}) & \longrightarrow & \mathbb{S}(\mathscr{X}_{j-1}) \end{array} \tag{3.19}$$

in which the maps of the right-hand square follow from (3.15) and (3.17). The description of the cofibres of the horizontal maps follows from Theorem 2. By Theorem 1 the cofibre of the middle vertical map in (3.19) is equal to $\Omega^{-1-s_j}\mathbb{L}S(F_{j-1})$.

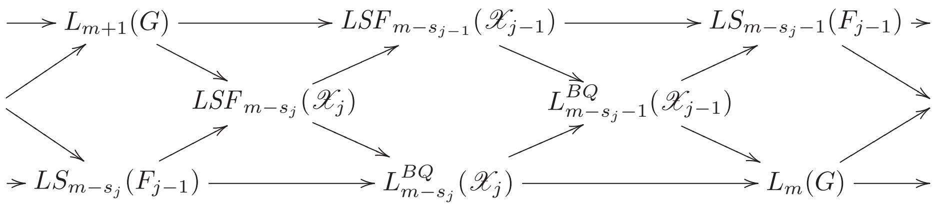
Theorem 4. *There exists a homotopy commutative diagram of spectra*

$$\begin{array}{ccccccccc} \Omega\mathbb{S}(X) & \longrightarrow & \Sigma^{s_k}\mathbb{L}SF(\mathscr{X}) & \longrightarrow & \cdots & \longrightarrow & \Sigma^{s_2}\mathbb{L}SF(\mathscr{X}_2) & \longrightarrow & \Sigma^{s_1}\mathbb{L}S(F_0) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^{s_k}\mathbb{L}^{BQ}(\mathscr{X}) & \longrightarrow & \cdots & \longrightarrow & \Sigma^{s_2}\mathbb{L}^{BQ}(\mathscr{X}_2) & \longrightarrow & \Sigma^{s_1}\mathbb{L}P(F_0) \end{array} \tag{3.20}$$

in which all the squares are push-outs and the cofibres of all the vertical maps are naturally homotopy equivalent to the spectrum $\mathbb{L}(\pi_1(X))$.

Proof. The diagram (3.20) follows from the diagrams (2.29), (2.31), (3.15) and (3.17).

Corollary 3. *For* $2 \leqslant j \leqslant k$ *there are braids of exact sequences*

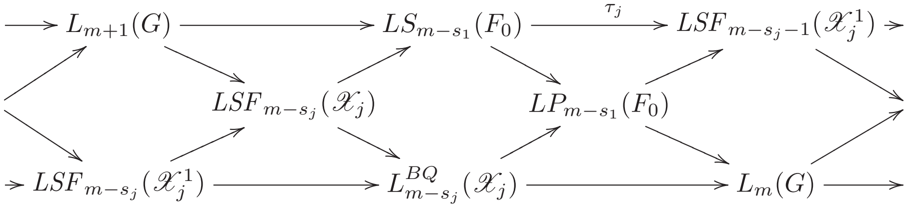


(3.21)

which are realized on the spectrum level.

Proof. This follows from the diagram (3.20).

Corollary 4. *For* $2 \leqslant j \leqslant k$ *there are braids of exact sequences*



(3.22)

which are realized on the spectrum level.

Proof. The central square of the diagram (3.22) is realized by the square of spectra from the diagram (3.20).

Corollary 5. *Let*

$$\Theta\colon \mathscr{S}^{\mathrm{TOP}}(X) \to LS_{n-q_1}(F_0)$$

be the map (2.11) *for a pair* (X_0, X_1) *and let* $x \in LS_{n-q_1}(F_0)$. *If* $\tau_j(x) \neq 0$ *for some* j, $2 \leqslant j \leqslant k$, *then the element* x *cannot be realized as an obstruction to splitting a simple homotopy equivalence into the closed manifold* X.

Proof. It follows from [5], Proposition 7.2.7 that the image of the map Θ coincides with the image of the map

$$\Theta_1\colon \mathscr{S}_{n+1}(X) \to LS_{n-q_1}(F_0),$$

which is realized on the spectrum level by the map

$$\Omega\mathbb{S}(X) \longrightarrow \Sigma^{q_1}\mathbb{L}S(F_0). \tag{3.23}$$

The map (3.23) is the composition of the maps in the upper row of the diagram (3.20). The result now follows from the exact sequence with the map τ_j in the diagram (3.22).

§ 4. The Browder-Livesay filtration

We apply the groups LSF_* to the problem of the realization of splitting obstructions by simple homotopy equivalences of closed manifolds (see [13], [28] and [33]) and describe their relation to the iterated Browder-Livesay invariants. Afterwards, we give several examples.

Assume that a manifold pair (X^n, Y^{n-1}) is a Browder-Livesay pair. In this case the square (2.10) has the following form:

$$F = \begin{pmatrix} \rho & \xrightarrow{\cong} & \rho \\ \downarrow & & \downarrow \\ G^- & \xrightarrow{\cong} & G^+ \end{pmatrix}, \tag{4.1}$$

where $\rho = \pi_1(\partial U) \cong \pi_1(X \setminus Y)$, $G^- = \pi_1(Y)$ and $G^+ = \pi_1(X)$ are groups with orientation homomorphisms. The vertical maps in (4.1) are inclusions of index 2. The orientation homomorphisms on the groups $G^\pm$ coincide on the images of the vertical maps and differ outside these images. In this case the groups $LS_*(F)$ are called the Browder-Livesay groups and are denoted by $LN_*(\rho \to G^+)$ (see [12]–[15], [17], [18], [23]–[25] and [34]). Throughout this section we suppose that the filtration $\mathscr{X}$ in (1.5) is a Browder-Livesay filtration. In this case the dimension of the submanifold X_i is $n-i$, where $n = \dim X$, the codimension s_i of the submanifold X_i in X is i, and

$$F_i = \begin{pmatrix} \rho_i & \xrightarrow{\cong} & \rho_i \\ \downarrow & & \downarrow \\ G_{i+1} & \xrightarrow{\cong} & G_i \end{pmatrix}, \tag{4.2}$$

where all the groups $G_i = \pi_1(X_i)$ are isomorphic to the group $G_0 = \pi_1(X_0)$ if we forget the orientation homomorphism.

We now state more algebraically several results of surgery theory for the Browder-Livesay filtration (1.5).

Consider an infinite diagram $\mathscr{G}$ of groups with orientations (which is commutative as a diagram of groups):

$$\begin{array}{ccccccccc} \cdots & & & \rho_2 & & \rho_1 & & \rho_0 & \\ & \searrow & \swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow \\ & & G_3 & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 \end{array} \tag{4.3}$$

in which the maps $\rho_i \to G_i$ and $\rho_i \to G_{i+1}$ are inclusions of index 2 of oriented groups. Every horizontal map is an isomorphism preserving the orientation on the image of the corresponding group ρ_i and reversing the orientation outside this image.

The commutative triangles of groups in (4.3) define the sequence of squares $\mathscr{F}$

$$F_0, F_1, \dots, F_i, \dots, \qquad i \geqslant 0. \tag{4.4}$$

For $0 \leqslant i \leqslant j$ we denote by $\mathscr{F}_j^i$ the finite subset

$$F_i, F_{i+1}, \dots, F_{j-1}, F_j \tag{4.5}$$

of the sequence (4.4). Let $\mathscr{F}_j = \mathscr{F}_j^0$. For any square of groups (4.1) the Browder-Livesay groups $LN_*(\rho \to G)$ are defined in [25]. These algebraically defined groups and the diagram (2.14) are realized on the spectrum level (see [10], [17], [29], [31] and [34]). We can transfer the inductive definition of the spectra $\mathbb{L}SF(\mathscr{X}_j)$ to the case of the sequence $\mathscr{F}$ using the push-out squares (2.31) and (2.32). Thus, we obtain the spectra $\mathbb{L}^{BQ}(\mathscr{F}_j^i)$ and $\mathbb{L}SF(\mathscr{F}_j^i)$.

Note that the Browder-Livesay filtration (1.5) yields the finite set $\mathscr{F}_{k-1} = \mathscr{F}_{k-1}^0$. In this case for $0 \leqslant i \leqslant j < k$ we have

$$\mathbb{L}^{BQ}(\mathscr{F}_j^i) = \mathbb{L}^{BQ}(\mathscr{X}_{j+1}^i), \qquad \mathbb{L}SF(\mathscr{F}_j^i) = \mathbb{L}SF(\mathscr{X}_{j+1}^i).$$

For the sequence $\mathscr{F}$ the diagram (3.20) yields the homotopy commutative diagram of spectra

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Sigma^{k+1}\mathbb{L}SF(\mathscr{F}_k) & \longrightarrow & \cdots & \longrightarrow & \Sigma\mathbb{L}SF(\mathscr{F}_0) & \longrightarrow & * \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \Sigma^{k+1}\mathbb{L}^{BQ}(\mathscr{F}_k) & \longrightarrow & \cdots & \longrightarrow & \Sigma\mathbb{L}^{BQ}(\mathscr{F}_0) & \longrightarrow & \mathbb{L}(G_0) \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathbb{L}(G_0) & = & \cdots & = & \mathbb{L}(G_0) & = & \mathbb{L}(G_0)
\end{array} \tag{4.6}$$

columns in which are cofibrations. We point out that in (4.6) we have $\mathbb{L}SF(\mathscr{F}_0) = \mathbb{L}S(F_0)$ and $\mathbb{L}^{BQ}(\mathscr{F}_0) = \mathbb{L}P(F_0)$ and

The diagram (4.6) yields the filtration of spectra

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Sigma^{k+1}\mathbb{L}^{BQ}(\mathscr{F}_k) & \longrightarrow & \cdots & \longrightarrow & \Sigma\mathbb{L}^{BQ}(\mathscr{F}_0) & \longrightarrow & \mathbb{L}(G_0) \\
 & & \| & & & & \| & & \| \\
\cdots & \longrightarrow & \mathbb{X}_{k+1} & \longrightarrow & \cdots & \longrightarrow & \mathbb{X}_1 & \longrightarrow & \mathbb{X}_0
\end{array}$$

which defined the surgery spectral sequence (see [8] and [32])

$$E_r^{p,q} = E_r^{p,q}(\mathscr{F}), \qquad p \geqslant 0, \quad q \in \mathbb{Z},$$

such that

$$E_1^{p,q} = \pi_{q-p}(\mathbb{X}_p, \mathbb{X}_{p+1}) \cong LN_{q-2p-2}(\rho_p \to G_p). \tag{4.7}$$

The first differential

$$d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}$$

of the surgery spectral sequence coincides with the composition

$$LN_{q-2p-2}(\rho_p \to G_p) \to L_{q-2p-2}(G_{p+1}) \to LN_{q-2p}(\rho_{p+1} \to G_{p+1}). \tag{4.8}$$

The first map of the composition (4.8) fits in the diagram (2.14) for the square F_p, and the second map fits in the diagram (2.14) for the square F_{p+1}.

The upper row of the diagram (4.6) yields the filtration of spectra

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Sigma^{k+1}\mathbb{L}SF(\mathscr{F}_k) & \longrightarrow \cdots \longrightarrow & \Sigma\mathbb{L}SF(\mathscr{F}_0) & \longrightarrow & * \\ & & \| & & \| & & \| \\ \cdots & \longrightarrow & \mathbb{Y}_{k+1} & \longrightarrow \cdots \longrightarrow & \mathbb{Y}_1 & \longrightarrow & \mathbb{Y}_0 \end{array}$$

The homotopy spectral sequence $\mathscr{E}^r_{p,q}(\mathscr{F})$ of this filtration is similar to the surgery spectral sequence (see [8], [32] and [36]). In this case we obtain

$$\mathscr{E}_1^{p,q} = \pi_{q-p}(\mathbb{Y}_p, \mathbb{Y}_{p+1})$$

with the first differential

$$\partial_1^{p,q} \colon \mathscr{E}_1^{p,q} \longrightarrow \mathscr{E}_1^{p+1,q}.$$

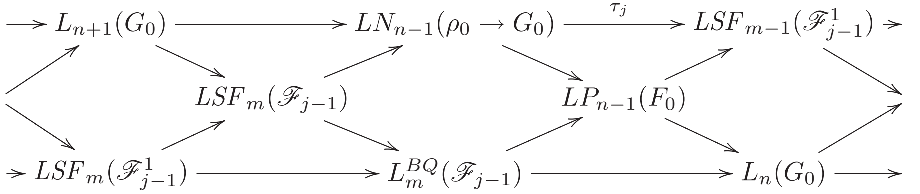
Proposition 2. *For $p \geqslant 0$ and $q \in \mathbb{Z}$ the vertical maps in the diagram (4.6) induce an isomorphism*

$$\mathscr{E}_r^{p,q} \cong E_r^{p,q}$$

of spectral sequences. In particular, the differential $\partial_1^{p,q}$ coincides with the composition (4.8).

Proof. All the top squares in (4.6) are push-outs; hence the fibres of all the corresponding horizontal maps coincide (see [32] and [36]).

Proposition 3. *For $j \geqslant 2$ there is a braid of exact sequences*



(4.9)

where $m = n - j$. The diagram (4.9) is realized on the spectrum level.

Proof. This follows from the diagram (4.6).

Theorem 5. *Let $\mathscr{G}$ be a diagram of groups (4.3) and $x \in LN_{n-1}(\rho_0 \to G_0)$ an element of the Browder-Livesay group. If $\tau_j(x) \neq 0$ for some $j \geqslant 2$, then the element x cannot be realized as an obstruction to splitting a simple homotopy equivalence $f \colon M^n \to X^n$ of closed manifolds with $\pi_1(X) = G_0$.*

Proof. Let $f \colon M^n \to X_0^n$ be a simple homotopy equivalence, and (X_0, X_1) a Browder-Livesay pair of closed manifolds such that the splitting obstruction $\Theta(f)$ is equal to $x \in LN_{n-1}(\rho_0 \to G_0)$, where $G_0 = \pi_1(X_0)$, $\rho_0 = \pi_1(X_0 \setminus X_1)$. Using the geometric definition of 4-periodicity (see [2], Theorem 11.6.1) we can assume that $n \geqslant j + 5$. Let $\mathbb{R}P^N$ be a real projective space of high dimension. Consider a map

$$\phi \colon X_1 \to \mathbb{R}P^N,$$

which induces an epimorphism of fundamental groups $G_1 = \pi_1(X_1) \to \mathbb{Z}/2$ with kernel ρ_1. Without loss of generality we can assume (see the introduction) that the map ϕ is transversal to $\mathbb{R}P^{N-1}$ with $\phi^{-1}(\mathbb{R}P^{N-1}) = X_2$ and that the pair $X_2 \subset X_1$ is a Browder-Livesay pair with the square F_1 in the splitting problem. Iterating this construction we obtain a Browder-Livesay filtration $\mathscr{X}$

$$X_j \subset X_{j-1} \subset \cdots \subset X_1 \subset X_0, \tag{4.10}$$

in which F_i is the square in the splitting problem for the pair (X_i, X_{i+1}). Similarly to the proof of Corollary 5 we obtain that the element x lies in the image of the map

$$LSF_{n-j}(\mathscr{F}_{j-1}) \to LN_{n-1}(\rho_0 \to G_0)$$

from the diagram (4.9). Hence $\tau_j(x) = 0$. We obtain a contradiction, which proves the theorem.

For $j \geqslant 1$ we have the homotopy long exact sequences

$$\cdots \longrightarrow L^{BQ}_{n-j}(\mathscr{F}_{j-1}) \to L_n(G_0) \xrightarrow{\beta_j} LSF_{n-j-1}(\mathscr{F}_{j-1}) \longrightarrow \cdots \tag{4.11}$$

of the vertical cofibrations from the diagram (4.6).

Theorem 6. *Let $x \in L_n(G_0)$. If $\beta_j \neq 0$ for some $j \geqslant 1$, then the element x cannot be realized by a normal map of closed n-dimensional manifolds.*

Proof. If $\beta_j(x) \neq 0$, then it follows from the exact sequence (4.11) that x does not lie in the image of the map $L^{BQ}_{n-j}(\mathscr{F}_{j-1}) \to L_n(G_0)$. Then, by [8] and [28], the element x cannot be realized by a normal map of closed n-dimensional manifolds.

Every diagram $\mathscr{G}$ yields the set of iterated Browder-Livesay invariants (see [8], [11], [18], [27] and [28]). Note that the map β_j, $j \geqslant 1$, is equivalent to the full set of iterated Browder-Livesay invariants of order up to j (see [8], [11], [18], [27] and [28]).

Proposition 4. *There is a homotopy commutative diagram of spectra*

$$\begin{array}{ccccccc}
\cdots \longrightarrow \Sigma^{k+1}\mathbb{L}SF(\mathscr{F}_k) & \longrightarrow \cdots \longrightarrow & \Sigma^{3}\mathbb{L}SF(\mathscr{F}_2) & \longrightarrow & \Sigma^{2}\mathbb{L}SF(\mathscr{F}_1) \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow \Sigma^{1}\mathbb{L}N(\rho_0 \to G_0) & = \cdots = & \Sigma^{1}\mathbb{L}N(\rho_0 \to G_0) & = & \Sigma^{1}\mathbb{L}N(\rho_0 \to G_0) \\
\downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow \Sigma^{k+2}\mathbb{L}SF(\mathscr{F}^1_k) & \longrightarrow \cdots \longrightarrow & \Sigma^{4}\mathbb{L}SF(\mathscr{F}^1_2) & \longrightarrow & \Sigma^{3}\mathbb{L}S(F_1)
\end{array} \tag{4.12}$$

columns in which are cofibrations.

Proof. This follows from the diagram (4.6) and Proposition 3.

Corollary 6. *Let $x \in LN_{n-1}(\rho_0 \to G_0)$. If $\tau_j(x) \neq 0$ for some $j \geqslant 2$, then $\tau_i(x) \neq 0$ for all $i \geqslant j$.*

Proof. For $j \geqslant 2$ the map

$$\Sigma^1 \mathbb{L}N(\rho_0 \to G_0) \to \Sigma^{j+1} \mathbb{L}SF(\mathscr{F}^1_{j-1})$$

in the diagram (4.12) induces the map τ_j. The required result now follows from (4.12).

The forbidding invariant τ_2 is similar to the first Browder-Livesay invariant.

Theorem 7. *The invariant τ_2 fits in the braids of exact sequences*

$$\begin{array}{ccccccccc}
\longrightarrow & L_{n+1}(G_0) & & \longrightarrow & LN_{n-1}(\rho_0 \to G_0) & \xrightarrow{\tau_2} & LN_{n-3}(\rho_1 \to G_1) & \to & \\
\nearrow & & \searrow & \nearrow & & \searrow \quad \nearrow & & & \searrow \\
& & & LSF_{n-2}(\mathscr{F}_1) & & LP_{n-1}(F_0) & & & \\
\searrow & & \nearrow & \searrow & & \nearrow \quad \searrow & & & \nearrow \\
\to & LN_{n-2}(\rho_1 \to G_1) & & \longrightarrow & L^{BQ}_{n-2}(\mathscr{F}_1) & \longrightarrow & L_n(G_0) & \longrightarrow &
\end{array} \tag{4.13}$$

and

$$\begin{array}{ccccccccc}
\to & L_{n+1}(\rho_0 \to G_0) & & \longrightarrow & LN_{n-1}(\rho_0 \to G_0) & \xrightarrow{\tau_2} & LN_{n-3}(\rho_1 \to G_1) & \to & \\
\nearrow & & \searrow & \nearrow & & \searrow \quad \nearrow & & & \searrow \\
& & & LSF_{n-2}(\mathscr{F}_1) & & L_{n-1}(G_1) & & & \\
\searrow & & \nearrow & \searrow & & \nearrow \quad \searrow & & & \nearrow \\
\to & LN_{n-2}(\rho_1 \to G_1) & & \longrightarrow & LP_{n-2}(F_1) & \longrightarrow & L_n(\rho_0 \to G_0) & \longrightarrow &
\end{array} \tag{4.14}$$

which are realized on the spectrum level. The map τ_2 coincides with the first differential

$$d_1^{0,n+1} \colon E_1^{0,n+1} \to E_1^{1,n+1}$$

of the surgery spectral sequence for the diagram $\mathscr{G}$.

Proof. The diagram (4.9) with $j = 2$ yields the diagram (4.13). The diagram (3.11) with $j = 2$ yields the diagram (4.14). The push-out square (3.10) yields the homotopy commutative diagram

$$\begin{array}{ccccc}
\mathbb{L}SF(\mathscr{F}_1) & \longrightarrow & \mathbb{L}P(F_1) & & \\
\downarrow & & \downarrow & & \\
\Omega^1 \mathbb{L}N(\rho_0 \to G_0) & \longrightarrow & \Omega^1 \mathbb{L}(G_1) & \longrightarrow & \Omega^2 \mathbb{L}(\rho_0 \to G_0) \\
& \overset{\delta}{\searrow} & \downarrow & & \\
& & \Omega^{-1} \mathbb{L}S(F_1) & = & \Omega^{-1} \mathbb{L}N(\rho_1 \to G_1)
\end{array} \tag{4.15}$$

the central row and the central column in which are cofibrations. The map δ in (4.15) realizes the map τ_2 on the spectrum level. It now follows from (4.8) that the map of homotopy groups

$$\begin{array}{ccc}\pi_{n-2}\big(\Omega^1\mathbb{L}N(\rho_0\to G_0)\big) & \xrightarrow{\delta_*} & \pi_{n-2}\big(\Omega^{-1}\mathbb{L}N(\rho_1\to G_1)\big)\\ \| & & \|\\ LN_{n-1}(\rho_0\to G_0) & & LN_{n-3}(\rho_1\to G_1)\end{array}$$

induced by δ coincides with $d_1^{0,n+1}$.

We now consider several examples of the application of the results obtained.

Let $i\colon \rho\to G^-$ be an inclusion of index 2 of Abelian groups, where the orientation homomorphism is trivial on the group ρ and non-trivial on the group G^-.

Theorem 8. *No element of infinite order of the group $LN_{2k}(\rho\to G^-)$ can be realized as the splitting obstruction of a simple homotopy equivalence of closed manifolds.*

Proof. The inclusion i yields the diagram of groups $\mathscr{G}$, in which all inclusions of index 2 coincide with i and the orientation on the groups G_j changes on each successive group outside the image of the map i. In particular, the orientation homomorphisms of all the groups G_{2k} coincide with the orientation homomorphism of the group $G_0=G^-$. By Theorem 7 the map τ_2 coincides in dimension $2k$ with the first differential of the surgery spectral sequence, which is yielded by the composition

$$LN_{2k}(\rho\to G_0)\to L_{2k}(\pi)\to LN_{2k+2}(\rho\to G_1)\overset{\cong}{\to} LN_{2k}(\rho\to G_0) \tag{4.16}$$

and is multiplication by 2 (see [15] and [32]). The required result now follows from Theorem 5.

Corollary 7. *Only the trivial element of the group*

$$LN_{2k}(\mathbb{Z}/2^n\to\mathbb{Z}/2^{n+1^-})=\mathbb{Z}^{2^{n-1}}$$

can be realized as the splitting obstruction of a simple homotopy equivalence of closed manifolds.

Corollary 8. *Only the trivial elements of the groups*

$$LN_0(1\to\mathbb{Z}/2^-)=\mathbb{Z},\qquad LN_0(\mathbb{Z}/2\to\mathbb{Z}\oplus\mathbb{Z}/2^-)=\mathbb{Z}\oplus\mathbb{Z},$$
$$LN_0(\mathbb{Z}/2\oplus\mathbb{Z}/2\to\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2^-)=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$$

can be realized as the splitting obstructions of simple homotopy equivalences of closed manifolds.

Corollary 9. *Let $\mathbb{R}P^{4k}$ be a real projective space. Then for any manifold X^{4m+1} with $\pi_1(X)=0$ and $4m+4k+1>5$ a simple homotopy equivalence*

$$f\colon M\to X\times\mathbb{R}P^4$$

of $(4m+4k+1)$-dimensional manifolds splits along the submanifold $X\times\mathbb{R}P^3$.

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Bibliography

[1] S. P. Novikov, "Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. I, II", *Izv. Akad. Nauk SSSR Ser. Mat.* **34**:2 (1970), 253–288; **34**:3 (1970), 475–500; English transl. in *Math. USSR-Izv.* **4**:2 (1970), 257–292; **4**:3 (1970), 479–505.

[2] C. T. C. Wall, *Surgery on compact manifolds*, London Math. Soc. Monogr. Ser., vol. 1, Academic Press, London–New York 1970.

[3] W. Browder, *Surgery on simply-connected manifolds*, Ergeb. Math. Grenzgeb., vol. 65, Springer-Verlag, Berlin–Heidelberg–New York 1972.

[4] A. Ranicki, "The total surgery obstruction", *Algebraic Topology, Aarhus* 1978 (Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer-Verlag, Berlin–Heidelberg 1979, pp. 275–316.

[5] A. Ranicki, *Exact sequences in the algebraic theory of surgery*, Math. Notes, vol. 26, Princeton Univ. Press, Princeton, NJ; Univ. of Tokyo Press, Tokyo 1981.

[6] W. Browder and F. Quinn, "A surgery theory for G-manifolds and stratified sets", *Manifolds – Tokyo* 1973 (Tokyo, 1973), Proc. Internat. Conf. Manifolds, Univ. Tokyo Press, Tokyo 1975, pp. 27–36.

[7] Sh. Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, IL 1994.

[8] Yu. V. Muranov, D. Repovs and R. Jimenez, "A spectral sequence in surgery theory and manifolds with filtrations", Tr. Mosk. Mat. Obs., vol. 67, URSS, Moscow 2006, pp. 294–325; English transl. in *Trans. Moscow Math. Soc.*, 2006, 261–288.

[9] Yu. V. Muranov, D. Repovs and F. Spaggiari, "Surgery on triples of manifolds", *Mat. Sb.* **194**:8 (2003), 139–160; English transl. in *Sb. Math.* **194**:8 (2003), 1251–1271.

[10] A. Bak and Yu. V. Muranov, "Normal invariants of manifold pairs and assembly maps", *Mat. Sb.* **197**:6 (2006), 3–24; English transl. in *Sb. Math.* **197**:6 (2006), 791–811.

[11] R. Jimenez, Yu. V. Muranov and D. Repovš, "Splitting along a submanifold pair", *K-Theory* (to appear).

[12] W. Browder and G. R. Livesay, "Fixed point free involutions on homotopy spheres", *Bull. Amer. Math. Soc.* **73** (1967), 242–245.

[13] S. López de Medrano, *Involutions on manifolds*, Ergeb. Math. Grenzgeb., vol. 59, Springer-Verlag, Berlin–Heidelberg–New York 1971.

[14] S. E. Cappell and J. L. Shaneson, "Pseudo-free actions. I", *Algebraic topology, Aarhus* 1978 (Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer-Verlag, Berlin–Heidelberg 1979, pp. 395–447.

[15] I. Hambleton, "Projective surgery obstructions on closed manifolds", *Algebraic K-theory* (Oberwolfach, 1980), Part II, Lecture Notes in Math., vol. 967, Springer-Verlag, Berlin–New York 1982, pp. 101–131.

[16] A. F. Kharshiladze, "Smooth and piecewise-linear structures on products of projective spaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **47**:2 (1983), 366–383; English transl. in *Math. USSR-Izv.* **22**:2 (1984), 339–355.

[17] Yu. V. Muranov, "The splitting problem", *Tr. Mat. Inst. Steklova*, vol. 212, Nauka, Moscow 1996, pp. 123–146; English transl. in *Proc. Steklov Inst. Math.* **212** (1996), 115–137.

[18] A. F. Kharshiladze, "Surgery on manifolds with finite fundamental groups", *Uspekhi Mat. Nauk* **42**:4 (1987), 55–85; English transl. in *Russian Math. Surveys* **42**:4 (1987), 65–103.

[19] I. Hambleton, R. J. Milgram, L. Taylor and B. Williams, "Surgery with finite fundamental group", *Proc. London Math. Soc.* (3) **56**:2 (1988), 349–379.

[20] H. K. Mukerjee, "Classification of homotopy Dold manifolds", *New York J. Math.* **9** (2003), 271–293.

[21] H. K. Mukerjee, "Classification of homotopy real Milnor manifolds", *Topology Appl.* **139**:1–3 (2004), 151–184.

[22] A. Cavicchioli, Yu. V. Muranov and F. Spaggiari, "Mixed structures on a manifold with boundary", *Glasg. Math. J.* **48**:1 (2006), 125–143.

[23] Yu. V. Muranov and A. F. Kharshiladze, "Browder–Livesay groups of Abelian 2-groups", *Mat. Sb.* **181**:8 (1990), 1061–1098; English transl. in *Math. USSR-Sb.* **70**:2 (1991), 499–540.

[24] I. Hambleton, L. Taylor and B. Williams, "An introduction to maps between surgery obstruction groups", *Algebraic topology, Aarhus* 1982 (Aarhus, 1982), Lecture Notes in Math., vol. 1051, Springer-Verlag, Berlin–Heidelberg–New York 1984, pp. 49–127.

[25] A. Ranicki, "The L-theory of twisted quadratic extensions", *Canad. J. Math.* **39**:2 (1987), 345–364.

[26] Yu. V. Muranov and R. Himenez (Jimenez), "Transfer maps for triples of manifolds", *Mat. Zametki* **79**:3 (2006), 420–433; English transl. in *Math. Notes* **79**:3–4 (2006), 387–398.

[27] A. F. Kharshiladze, "Iterated Browder–Livesay invariants and the oozing problem", *Mat. Zametki* **41**:4 (1987), 557–563; English transl. in *Math. Notes* **41**:4 (1987), 312–315.

[28] A. Cavicchioli, Yu. V. Muranov and F. Spaggiari, *On the elements of the second type in surgery groups*, Preprint no. 111, Max-Planck-Institut für Mathematik, Bonn 2006.

[29] I. Hambleton, A. Ranicki and L. Taylor, "Round L-theory", *J. Pure Appl. Algebra* **47**:2 (1987), 131–154.

[30] A. A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Math., vol. 102, Cambridge Univ. Press, Cambridge 1992.

[31] A. Bak and Yu. V. Muranov, "Splitting along submanifolds and L-spectra", *Sovrem. Mat. Prilozh.* **1** (2003), 3–18; English transl. in *J. Math. Sci. (N.Y.)* **123**:4 (2004), 4169–4184.

[32] I. Khèmblton (Hambleton) and A. F. Kharshiladze, "A spectral sequence in surgery", *Mat. Sb.* **183**:9 (1992), 3–14; English transl. in *Russian Acad. Sci. Sb. Math.* **77**:1 (1994), 1–9.

[33] P. M. Akhmetiev, A. Cavicchioli and D. Repovš, "On realization of splitting obstructions in Browder–Livesay groups for closed manifold pairs", *Proc. Edinb. Math. Soc.* (2) **43**:1 (2000), 15–25.

[34] Yu. V. Muranov, "Splitting obstruction groups and quadratic extensions of anti-structures", *Izv. Ross. Akad. Nauk Ser. Mat.* **59**:6 (1995), 107–132; English transl. in *Izv. Math.* **59**:6 (1995), 1207–1232.

[35] R. M. Switzer, *Algebraic topology–homotopy and homology*, Grundlehren Math. Wiss., vol. 212, Springer-Verlag, Berlin–Heidelberg–New York 1975.

[36] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin–Heidelberg–New York 1972.

A. Bak
Bielefeld University, Germany
E-mail: bak@mathematik.uni-bielefeld.de

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Yu. V. Muranov
Universidad Tecnológica de la Mixteca, México
E-mail: ymuranov@mixteco.utm.mx