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LOCAL-GLOBAL PRINCIPLE FOR TRANSVECTION GROUPS

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ABSTRACT. In this article we extend the validity of Suslin's Local-Global Principle for the elementary transvection subgroup of the general linear group $\mathrm{GL}_n(R)$, the symplectic group $\mathrm{Sp}_{2n}(R)$, and the orthogonal group $\mathrm{O}_{2n}(R)$, where $n > 2$, to a Local-Global Principle for the elementary transvection subgroup of the automorphism group $\mathrm{Aut}(P)$ of either a projective module P of global rank > 0 and constant local rank > 2, or of a nonsingular symplectic or orthogonal module P of global hyperbolic rank > 0 and constant local hyperbolic rank > 2. In Suslin's results, the local and global ranks are the same, because he is concerned only with free modules. Our assumption that the global (hyperbolic) rank > 0 is used to define the elementary transvection subgroups. We show further that the elementary transvection subgroup $\mathrm{ET}(P)$ is normal in $\mathrm{Aut}(P)$, that $\mathrm{ET}(P) = \mathrm{T}(P)$, where the latter denotes the full transvection subgroup of $\mathrm{Aut}(P)$, and that the unstable K_1-group $\mathrm{K}_1(\mathrm{Aut}(P)) = \mathrm{Aut}(P)/\mathrm{ET}(P) = \mathrm{Aut}(P)/\mathrm{T}(P)$ is nilpotent by abelian, provided R has finite stable dimension. The last result extends previous ones of Bak and Hazrat for $\mathrm{GL}_n(R)$, $\mathrm{Sp}_{2n}(R)$, and $\mathrm{O}_{2n}(R)$.

An important application to the results in the current paper can be found in a preprint of Basu and Rao in which the last two named authors studied the decrease in the injective stabilization of classical modules over a nonsingular affine algebra over perfect C_1-fields. We refer the reader to that article for more details.

1. INTRODUCTION

In 1956, J-P. Serre asked if a finitely generated projective module over a polynomial ring over a field is free. This is known as Serre's problem on projective modules. It was affirmatively proved by D. Quillen and A. Suslin independently in 1976. Now it is known as the Quillen-Suslin Theorem. Quillen established the following Local-Global Principle in his proof of Serre's problem in [12].

Quillen's local-global principle. *A finitely presented module over a polynomial ring $R[X]$ over a commutative ring R is extended if and only if it is locally extended over the localization of $R[X]$ at every maximal ideal of R.*

We shall be concerned with the matrix-theoretic version of this theorem. It was established by Suslin in his second proof of Serre's problem in [13].

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Suslin's local-global principle. *Let R be a commutative ring with identity and let $\alpha(X) \in \mathrm{GL}_n(R[X])$ with $\alpha(0) = \mathrm{I}_n$. If $\alpha_{\mathfrak{m}}(X) \in \mathrm{E}_n(R_{\mathfrak{m}}[X])$, for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$, then $\alpha(X) \in \mathrm{E}_n(R[X])$.*

Shortly after his proof of Serre's problem, Suslin-Kopeiko in [14] established an analogue of the Local-Global Principle for the elementary subgroup of the orthogonal group. Around the same time, V.I. Kopeiko proved the analogous result for the elementary subgroup of the symplectic group. In this note we establish an analogous Local-Global Principle for the elementary transvection subgroup of the automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3. All previous work on this topic assumed that the global rank is at least 3. By definition the global rank or simply rank of a finitely generated projective R-module (resp. symplectic or orthogonal R-module) is the largest integer r such that $\overset{r}{\oplus} R$ (resp. $\overset{r}{\perp}\, \mathbb{H}(R)$) is a direct summand (resp. orthogonal summand) of the module. $\mathbb{H}(R)$ denotes the hyperbolic plane.

Using this principle one can generalize well known facts regarding the group $\mathrm{GL}_n(R)$ ($\mathrm{Sp}_{2n}(R)$ or $\mathrm{O}_{2n}(R)$) of automorphisms of the free module $\overset{n}{\oplus} R$ of rank n (free hyperbolic module $\overset{n}{\perp}\, \mathbb{H}(R)$ of rank n) to the automorphism group of finitely generated projective (symplectic or orthogonal) modules of global rank at least 1 and satisfying the local condition mentioned above. Specifically, we shall show that the elementary transvection subgroup is normal and the full automorphism group modulo its elementary transvection subgroup is nilpotent-by-abelian whenever the stable dimension is finite. These generalize results by Suslin and Kopeiko in [9], [13], [14], Taddei in [15], the first author in [1], Vavilov and Hazrat in [8], and others. We treat the above three groups uniformly.

Our main results are as follows:

Let Q denote a projective, symplectic or orthogonal module of global rank ≥ 1 and satisfying the local conditions stated above. Let

$$\begin{aligned} \mathrm{G}(Q) &= \text{the automorphism group of } Q, \\ \mathrm{T}(Q) &= \text{the subgroup generated by transvections, and} \\ \mathrm{ET}(Q) &= \text{the subgroup generated by elementary transvections.} \end{aligned}$$

Theorem 1. *Let R be a commutative ring with identity and let $\alpha(X) \in \mathrm{G}(Q[X])$, with $\alpha(0) = \mathrm{I}_n$. If $\alpha_{\mathfrak{m}}(X) \in \mathrm{ET}(Q_{\mathfrak{m}}[X])$, for every maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$, then $\alpha(X) \in \mathrm{ET}(Q[X])$.*

Theorem 2. $\mathrm{T}(Q) = \mathrm{ET}(Q)$. *Hence* $\mathrm{ET}(Q)$ *is a normal subgroup of* $\mathrm{G}(Q)$.

By applying the Local-Global Principle (Theorem 1) we prove

Theorem 3. *The factor group $\frac{\mathrm{G}(Q)}{\mathrm{ET}(Q)}$ is nilpotent-by-abelian when the stable dimension (i.e. Bass-Serre dimension) is finite.*

To prove the result we use the ideas of the first author in [1], where he has shown that the group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ is nilpotent-by-abelian for $n \geq 3$, but we avoid the functorial construction of the descending central series.

2. Preliminaries

Definition 2.1. Let R be an associative ring with identity. The following condition was introduced by H. Bass:

(R_m) for every $(a_1, \ldots, a_{m+1}) \in \mathrm{Um}_{m+1}(R)$, there are $\{x_i\}_{(1\leq i\leq m)} \in R$ such that $(a_1 + a_{m+1}x_1)R + \cdots + (a_m + a_{m+1}x_m)R = R$.

The condition $(R_m) \Rightarrow (R_{m+1})$ for every $m > 0$. Moreover, for any $n \geq m+1$ the condition (R_m) implies (R_n) with $x_i = 0$ for $i \geq m+1$.

By **stable range** for an associative ring R we mean the least n such that (R_n) holds. The integer $n-1$ is called the **stable dimension** of R and is denoted by $\mathrm{sdim}(R)$.

Lemma 2.2 (*cf.* [2])**.** *If R is a commutative Noetherian ring with identity of Krull dimension d, then* $\mathrm{sdim}(R) \leq d$.

Definition 2.3. A row vector $(a_1, \ldots, a_n) \in R^n$ is said to be **unimodular** in R if $\sum_{i=1}^n Ra_i = R$. The set of unimodular vectors of length n in R is denoted by $\mathrm{Um}_n(R)$. For an ideal I, $\mathrm{Um}_n(R, I)$ will denote the set of those unimodular vectors which are $(1, 0, \ldots, 0)$ modulo I.

Definition 2.4. Let M be a finitely generated left module over a ring R. An element m in M is said to be **unimodular** in M if $Rm \cong R$ and Rm is a direct summand of M, *i.e.* if there exists a finitely generated R-submodule M' such that $M \cong Rm \oplus M'$.

Definition 2.5. For an element $m \in M$, one can attach an ideal, called the **order ideal** of m in M, *viz.* $\mathrm{O}_M(m) = \{f(m) | f \in M^* = \mathrm{Hom}(M, R)\}$. Clearly, m is unimodular if and only if $Rm = R$ and $\mathrm{O}_M(m) = R$.

Definition 2.6. Following H. Bass ([2], pg. 167) we define a transvection of a finitely generated left R-module as follows: Let M be a finitely generated left R-module. Let $q \in M$ and $\varphi \in M^*$ with $\varphi(q) = 0$. An automorphism of M of the form $1 + \varphi_q$ (defined by $\varphi_q(p) = \varphi(p)q$, for $p \in M$), will be called a **transvection** of M if either $q \in \mathrm{Um}(M)$ or $\varphi \in \mathrm{Um}(M^*)$. We denote by $\mathrm{Trans}(M)$ the subgroup of $\mathrm{Aut}(M)$ generated by transvections of M.

Definition 2.7. Let M be a finitely generated left R-module. The automorphisms of the form $(p, a) \mapsto (p+ax, a)$ and $(p, a) \mapsto (p, a+\psi(p))$, where $x \in M$ and $\psi \in M^*$, are called **elementary transvections** of $M \oplus R$. (It is easily verified that these automorphisms are transvections.) The subgroup of $\mathrm{Trans}(M \oplus R)$ generated by the elementary transvections is denoted by $\mathrm{ETrans}(M \oplus R)$.

Definition 2.8. Let R be an associative ring with identity. To define other classical modules, we need an involutive antihomomorphism (**involution**, in short) $* : R \to R$ (*i.e.*, $(x-y)^* = x^* - y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$, for any $x, y \in R$). We assume that $1^* = 1$. For any left R-module M the involution induces a left module structure to the right R-module $M^* = \mathrm{Hom}(M, R)$ given by $(xf)v = (fv)x^*$, where $v \in M$, $x \in R$ and $f \in M^*$. Any right R-module can be viewed as a left R-module via the convention $ma = a^*m$ for $m \in M$ and $a \in R$. Hence if M is a left R-module, then $\mathrm{O}_M(m)$ has a left R-module structure with scalar multiplication given by $\lambda f(m) = f(\lambda m)$.

Blanket assumption. Let A be an R-algebra, where R is a commutative ring with identity, such that A is finite as a left R-module. Let A possesses an involution $* : r \mapsto \bar{r}$, for $r \in A$. For a matrix $M = (m_{ij})$ over A we define $\overline{M} = (\overline{m}_{ij})^t$. Let $\psi_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\psi_n = \psi_{n-1} \perp \psi_1$ for $n > 1$; and $\widetilde{\psi}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\widetilde{\psi}_n = \widetilde{\psi}_{n-1} \perp \widetilde{\psi}_1$, for

$n > 1$. For a column vector $v \in A^n$ we write $\widetilde{v} = \bar{v}^t \psi_n$ in the symplectic case and $\widetilde{v} = \bar{v}^t \widetilde{\psi_n}$ in the orthogonal case. We define a form $\langle\, , \rangle$ as follows:

$$\langle v, w\rangle = \begin{cases} v^t \cdot w & \text{in the linear case,} \\ \widetilde{v} \cdot w & \text{otherwise.} \end{cases}$$

(Viewing M as a right A-module we can assume the linearity.)

Since R is commutative, we can assume that the involution "$*$" defined on A is trivial over R. We shall always assume that 2 is invertible in the ring R while dealing with the symplectic and the orthogonal cases.

Definition 2.9. A **symplectic** (**orthogonal**) A-module is a pair $(P, \langle\, , \rangle)$, where P is a projective left A-module of even rank and $\langle\, , \rangle : P \times P \to A$ is a nonsingular (*i.e.* $P \cong P^*$ by $x \mapsto \langle x, \cdot\rangle$) alternating (symmetric) bilinear form.

Definition 2.10. Let $(P_1, \langle\, , \rangle_1)$ and $(P_2, \langle\, , \rangle_2)$ be two symplectic (orthogonal) left A-modules. Their **orthogonal sum** is the pair $(P, \langle\, , \rangle)$, where $P = P_1 \oplus P_2$ and the inner product is defined by $\langle (p_1, p_2), (q_1, q_2)\rangle = \langle p_1, q_1\rangle_1 + \langle p_2, q_2\rangle_2$. Since this form is also non-singular we shall henceforth denote $(P, \langle\, , \rangle)$ by $P_1 \perp P_2$ and call it the orthogonal sum of $(P_1, \langle\, , \rangle_1)$ and $(P_2, \langle\, , \rangle_2)$ (if $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$ are clear from the context).

Definition 2.11. For a projective left A-module P of rank n, we define $\mathbb{H}(P)$ of global rank rank n supported by $P \oplus P^*$, with form $\langle (p, f), (p', f')\rangle = f(p') - f'(p)$ for the symplectic modules and $f(p') + f'(p)$ for the orthogonal modules. There is a unique nonsingular alternating (symmetric) bilinear form $\langle\, , \rangle$ on the A-module $\mathbb{H}(A) = A \oplus A^*$ (up to scalar multiplication by A^*); namely, $\langle (a_1, b_1), (a_2, b_2)\rangle = a_1 b_2 - a_2 b_1$ in the symplectic case and $a_1 b_2 + a_2 b_1$ in the orthogonal case.

Remark 2.12. A bilinear form $\langle\, , \rangle$ induces a homomorphism $\Psi : P \to P^* = \mathrm{Hom}(P, A)$, defined by $\Psi(p)(q) = \langle p, q\rangle$. The converse is also true since 2 is invertible in A. If $\langle\, , \rangle$ is symmetric, then one has $\Psi = \Psi^*$, and if $\langle\, , \rangle$ is alternating, then one has $\Psi + \Psi^* = 0$, under the canonical isomorphism $P \cong P^{**}$.

Definition 2.13. An **isometry** of a symplectic (orthogonal) module $(P, \langle\, , \rangle)$ is an automorphism of P which fixes the bilinear form. The group of isometries of $(P, \langle\, , \rangle)$ is denoted by $\mathrm{Sp}(P)$ for the symplectic modules and by $\mathrm{O}(P)$ for the orthogonal modules.

Definition 2.14. Following Bass [3] we define a symplectic transvection as follows: Let $\Psi : P \to P^*$ be an induced isomorphism. Let $\alpha : A \to P$ be an A-linear map defined by $\alpha(1) = u$. Then $\alpha^*\Psi \in P^*$ is defined by $\alpha^*\Psi(p) = \langle u, p\rangle$. Let $v \in P$ be such that $\alpha^*\Psi(v) = \langle u, v\rangle = 0$. An automorphism $\sigma_{(u,v)}$ of $(P, \langle\, , \rangle)$ of the form

$$\sigma_{(u,v)}(p) = p + \langle u, p\rangle v + \langle v, p\rangle u + \langle u, p\rangle u,$$

for $u, v \in P$ with $\langle u, v\rangle = 0$, will be called a **symplectic transvection** of $(P, \langle\, , \rangle)$ if either $v \in \mathrm{Um}(P)$ or $\alpha^*\Psi \in \mathrm{Um}(P^*)$. (Viewing P as a right A-module we can assume the linearity.)

Since $\langle \sigma_{(u,v)}(p_1), \sigma_{(u,v)}(p_2)\rangle = \langle p_1, p_2\rangle$, $\sigma_{(u,v)} \in \mathrm{Sp}(P, \langle\, , \rangle)$. Note that $\sigma_{(u,v)}^{-1}(p) = p - \langle u, p\rangle v - \langle v, p\rangle u - \langle u, p\rangle u$.

The subgroup of $\mathrm{Sp}(P, \langle\, , \rangle)$ generated by the symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P)$.

Definition 2.15. The (symplectic) transvections of $(P \perp A^2)$ of the form $(p, b, a) \mapsto (p + aq, b - \langle p, q\rangle + a, a)$ and $(p, b, a) \mapsto (p + bq, b, a + \langle p, q\rangle - b)$, where $a, b \in A$ and $p, q \in P$, are called **elementary symplectic transvections**. The subgroup of $\mathrm{Trans}_{\mathrm{Sp}}(P \perp A^2)$ generated by the elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(P \perp A^2)$.

In a similar manner we can define a transvection $\tau_{(u,v)}$ for an orthogonal module $(P, \langle , \rangle)$. For this we need to assume that $u, v \in P$ are isotropic, *i.e.* $\langle u, u\rangle = \langle v, v\rangle = 0$.

Definition 2.16. An automorphism $\tau_{(u,v)}$ of $(P, \langle\, ,\rangle)$ of the form

$$\tau_{(u,v)}(p) = p - \langle u, p\rangle v + \langle v, p\rangle u$$

for $u, v \in P$ with $\langle u, v\rangle = 0$ will be called an **isotropic (orthogonal) transvection** of $(P, \langle\, ,\rangle)$ if either $v \in \mathrm{Um}(P)$ or $\alpha^*\Psi \in \mathrm{Um}(P^*)$ (see [7], pg. 214).

One checks that $\tau_{(u,v)} \in \mathrm{O}(P, \langle\, ,\rangle)$ and $\tau^{-1}_{(u,v)}(p) = p + \langle u, p\rangle v - \langle v, p\rangle u$.

The subgroup of $\mathrm{O}(P, \langle\, ,\rangle)$ generated by the isotropic orthogonal transvections is denoted by $\mathrm{Trans}_{\mathrm{O}}(P)$.

Definition 2.17. The isotropic orthogonal transvections of $(P \perp A^2)$ of the form $(p, b, a) \mapsto (p - aq, b + \langle p, q\rangle, a)$ and $(p, b, a) \mapsto (p - bq, b, a - \langle p, q\rangle)$, where $a, b \in A$ and $p, q \in P$, are called **elementary orthogonal transvections**. The subgroup of $\mathrm{Trans}_{\mathrm{O}}(P \perp A^2)$ generated by elementary orthogonal transvections is denoted by $\mathrm{ETrans}_{\mathrm{O}}(P \perp A^2)$.

Notation 2.18. In the sequel, P will denote either a finitely generated projective left A-module of rank n, a symplectic left A-module or an orthogonal left A-module of even rank $n = 2r$ with a fixed form $\langle , \rangle$. Q will denote $P \oplus A$ in the linear case, and $P \perp A^2$, otherwise. To denote $(P \oplus A)[X]$ in the linear case and $(P \perp A^2)[X]$, otherwise, we will use the notation $Q[X]$. We assume that the rank of the projective module is $n \geq 2$ when dealing with the linear case, and $n \geq 6$ when considering the symplectic and the orthogonal cases. For a finitely generated projective A-module M we use the notation $\mathrm{G}(M)$ to denote $\mathrm{Aut}(M)$, $\mathrm{Sp}(M, \langle\, ,\rangle)$ and $\mathrm{O}(M, \langle\, ,\rangle)$ respectively; $\mathrm{S}(M)$ to denote $\mathrm{SL}(M)$, $\mathrm{Sp}(M, \langle\, ,\rangle)$ and $\mathrm{SO}(M, \langle\, ,\rangle)$ respectively; $\mathrm{T}(M)$ to denote $\mathrm{Trans}(M)$, $\mathrm{Trans}_{\mathrm{Sp}}(M)$ and $\mathrm{Trans}_{\mathrm{O}}(M)$ respectively; and $\mathrm{ET}(M)$ to denote $\mathrm{ETrans}(M)$, $\mathrm{ETrans}_{\mathrm{Sp}}(M)$ and $\mathrm{ETrans}_{\mathrm{O}}(M)$ respectively.

The reader should be able to easily verify that if R is a reduced ring and P is a free R-module, i.e. if $P = R^r$ (in the symplectic and the orthogonal cases we assume that P is free with the standard bilinear form), then $\mathrm{ETrans}(P) \supset \mathrm{E}_r(R)$, $\mathrm{ETransSp}(P) \supset \mathrm{ESp}(R)$ and $\mathrm{ETrans}_O(P) \supset \mathrm{EO}_r(R)$, for $r \geq 3$, in the linear case, and for $r \geq 6$, in the symplectic (and orthogonal) case.

Equality in all these cases will follow from Lemma 2.20 *below.*

We shall assume

(H1) For every maximal ideal $\mathfrak{m}$ of A, the symplectic (orthogonal) module $Q_{\mathfrak{m}}$ is isomorphic to $A_{\mathfrak{m}}^{2n+2}$ with the standard bilinear form $\mathbb{H}(A_{\mathfrak{m}}^{n+1})$.

(H2) For every nonnilpotent $s \in A$, if the projective module Q_s is a free A_s-module, then the symplectic (orthogonal) module Q_s is isomorphic to A_s^{2n+2} with the standard bilinear form $\mathbb{H}(A_s^{n+1})$.

Remark 2.19. Note that $\mathrm{T}(P)$ is a normal subgroup of $\mathrm{G}(P)$. Indeed, for $\alpha \in \mathrm{G}(P)$ in the linear case we have $\alpha(1+\varphi_q)\alpha^{-1} = 1+(\varphi\alpha^{-1})_{\alpha(q)}$. In the symplectic case we can write $\sigma_{(u,v)} = 1+\sigma'_{(u,v)}$ and similarly, $\alpha(1+\sigma'_{(u,v)})\alpha^{-1} = 1+(\sigma'\alpha^{-1})_{(\sigma'(u),\sigma'(v))}$. A similar argument will also hold for the orthogonal case.

Lemma 2.20. *If the projective module P of finite rank n is free (in the symplectic and the orthogonal cases we assume that the projective module is free with the standard bilinear form), then* $\mathrm{Trans}(P) = \mathrm{E}_n(R)$*,* $\mathrm{Trans}_{\mathrm{Sp}}(P) = \mathrm{ESp}_n(R)$ *and* $\mathrm{Trans}_{\mathrm{O}}(P) = \mathrm{EO}_n(R)$ *for $n \geq 3$ in the linear case and for $n \geq 6$ otherwise.*

Proof. In the linear case, for $p \in P$ and $\varphi \in P^*$, if $P = R^n$, then $\varphi_p : R^n \to R \to R^n$. Hence $1 + \varphi_p = I_n + v.w^t$ for some row vector v and column vector w in R^n. Since $\varphi(p) = 0$, it follows that $\langle v, w\rangle = 0$. Since either v or w is unimodular, it follows that $1 + \varphi_p = I_n + v.w^t \in \mathrm{E}_n(R)$. Similarly, in the nonlinear cases we have $\sigma_{(u,v)}(p) = I_n + v.\widetilde{w} + w.\widetilde{v}$ and $\tau_{(u,v)}(p) = I_n + v.\widetilde{w} - w.\widetilde{v}$, where either v or w is unimodular and $\langle v, w\rangle = 0$. (Here $\sigma_{(u,v)}$ and $\tau_{(u,v)}$ are as in the definition of symplectic and orthogonal transvections.) Historically, these are known to be elementary matrices; for details see [13] for the linear case, [9] for the symplectic case, and [14] for the orthogonal case. □

Remark 2.21. Lemma 2.20 holds for $n = 4$ in the symplectic and the orthogonal cases. This will follow from Remark 2.22.

Remark 2.22. $\mathrm{ESp}_4(A)$ is a normal subgroup of $\mathrm{Sp}_4(A)$ by ([9], Corollary 1.11). Also $\mathrm{ESp}_4(A[X])$ satisfies the Dilation Principle and the Local-Global Principle by ([9], Theorem 3.6). Since we were intent on a uniform proof, these cases have not been covered by us.

Notation 2.23. When $P = A^n$ (n is even in the nonlinear cases), we also use the notation $\mathrm{G}(n, A)$, $\mathrm{S}(n, A)$ and $\mathrm{E}(n, A)$ for $\mathrm{G}(P)$, $\mathrm{S}(P)$ and $\mathrm{T}(P)$, respectively. We denote the usual standard elementary generators of $\mathrm{E}(n, A)$ by $ge_{ij}(x)$, $x \in A$; e_i will denote the column vector $(0, \ldots, 1, \ldots, 0)^t$ (1 at the i-th position).

Remark 2.24. Note that if $\alpha \in \mathrm{End}(Q)$, then α can be considered as a matrix of the form $\left(\begin{smallmatrix} \mathrm{End}(P) & \mathrm{Hom}(P,A) \\ \mathrm{Hom}(A,P) & \mathrm{End}(A) \end{smallmatrix}\right)$ in the linear case. In the nonlinear cases one has a similar matrix for α of the form $\left(\begin{smallmatrix} \mathrm{End}(P) & \mathrm{Hom}(P,A\oplus A) \\ \mathrm{Hom}(A\oplus A,P) & \mathrm{End}(A\oplus A) \end{smallmatrix}\right)$.

3. Local-Global Principle for the transvection groups

In this section we deduce an analogue of Quillen's Local-Global Principle for the linear, symplectic and isotropic orthogonal transvection groups.

Proposition 3.1 (Dilation Principle)**.** *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let P and Q be as in 2.18. Assume that* (**H2**) *holds. Let s be a nonnilpotent element in R such that P_s is free, and let $\sigma(X) \in \mathrm{G}(Q[X])$ with $\sigma(0) = \mathrm{Id}$. Suppose*

$$\sigma_s(X) \in \begin{cases} \mathrm{E}(n+1, A_s[X]) & \text{in the linear case,} \\ \mathrm{E}(2n+2, A_s[X]) & \text{otherwise.} \end{cases}$$

Then there exists $\hat{\sigma}(X) \in \mathrm{ET}(Q[X])$ and $l > 0$ such that $\hat{\sigma}(X)$ localizes to $\sigma(bX)$ for some $b \in (s^l)$ and $\hat{\sigma}(0) = \mathrm{Id}$.

First we state the following useful lemmas.

Lemma 3.2. *Let R be a ring and M be a finitely presented left (right) R-module and let N be any R-module. Then we have a natural isomorphism:*

$$\gamma : \mathrm{Hom}_R(M, N)[X] \to \mathrm{Hom}_{R[X]}(M[X], N[X]).$$

Lemma 3.3. *Let S be a multiplicative closed subset of a ring R. Let M be a finitely presented left (right) R-module and N be any R-module. Then we have a natural isomorphism:*

$$\eta : S^{-1}(\mathrm{Hom}_R(M, N)) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Lemma 3.4 (see [5])**.** *If $\varepsilon = \varepsilon_1\varepsilon_2\cdots\varepsilon_r$, where each ε_j is a standard elementary generator, then for any $r > 0$, and for any $(p, q) \in \mathbb{N}\times\mathbb{N}$,*

$$\varepsilon ge_{pq}(X^{2^r m}Y)\varepsilon^{-1} = \prod_{t=1}^{k} ge_{p_t q_t}(X^m h_t(X, Y)),$$

for some $h_t(X, Y) \in R[X, Y]$, $(p_t, q_t) \in \mathbb{N}\times\mathbb{N}$, and for some $k > 0$.

Proof of Proposition 3.1. Since elementary transvections can always be lifted, we can and hence assume that R is reduced. We show that there exists $l > 0$ such that $\sigma(bX) \in \mathrm{ET}(Q[X])$ for all $b \in (s^k)R$, for all $k \geq l$.

As $\sigma(0) = \mathrm{Id}$, we can write $\sigma_s(X) = \prod_k \gamma_k ge_{i_k j_k}(X\lambda_k(X))\gamma_k^{-1}$, where $\lambda_k(X) \in A_s[X]$. Hence for $d > 0$, $\sigma_s(XT^{2d}) = \prod_k \gamma_k ge_{i_k j_k}(XT^{2d}\lambda_k(XT^{2d}))\gamma_k^{-1}$, for some γ_k in $\mathrm{E}(n+1, R_s)$ in the linear case, and in $\mathrm{E}(2n+n, R_s)$ in the nonlinear cases. Let $\gamma_k = \varepsilon_1\varepsilon_2\ldots\varepsilon_r$, and $d = 2^{r-1}$. Now apply Lemma 3.4 for $m = 1$, $X = T$, and $Y = 1$. Then use the fact that for $i \neq 1 \neq j$,

$$ge_{ij}(T^2\mu(X)) = [ge_{i1}(T\mu(X)), ge_{1j}(T)]$$

in the linear case, and

$$ge_{ij}(T^2\mu(X)) = \begin{cases} [ge_{i1}(T\mu(X)), ge_{1j}(T)] & \text{if } i \neq \sigma(j) \\ ge_{1j}(T\mu(X))[ge_{1l}(-T\mu(X)), ge_{i1}(-T)] & \text{if } i = \sigma(j) \\ [ge_{\sigma(i)1}(T\mu(X)), se_{1\sigma(j)}(T)] & \text{if } i \neq \sigma(j) \\ ge_{1\sigma j}(T\mu(X))[ge_{1\sigma(l)}(-T\mu(X)), ge_{\sigma(i)1}(-T)] & \text{if } i = \sigma(j) \end{cases}$$

in the nonlinear cases for some $l \leq n$, when $i+1$ is even and when $\sigma(i)+1$ is even respectively. Then for $d \gg 0$ we get $\sigma_s(XT^{2d}) = \prod_t ge_{p_t q_t}(T\mu_t(X))$, for some $\mu_t(X) \in A_s[X]$ with $p_t = 1$ or $q_t = 1$.

Since P_s is a free A_s-module,

$$P_s[X, T] \cong A_s^n[X, T] \cong P_s[X, T]^* \text{ in the linear case,}$$
$$P_s[X, T] \cong A_s^{2n}[X, T] \cong P_s[X, T]^* \text{ in the nonlinear cases.}$$

Thus using the isomorphism, polynomials in $P_s[X, T]$ can be regarded as linear forms which act as follows: For $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in A_s^k[X, T]$ ($k = n$ in the linear case and $k = 2n$ in the symplectic case),

$$\langle x, y\rangle = \begin{cases} xy^t & \text{in the linear case,} \\ x\psi_n y^t & \text{in the symplectic case,} \\ x\widetilde{\psi_n} y^t & \text{in the orthogonal case} \end{cases}$$

(where ψ_n denotes the alternating matrix corresponding to the standard symplectic form $\sum_{i=1}^{2n} e_{2i-1,i} - \sum_{i=1}^{2n} e_{2i,2i-1}$ and $\tilde{\psi}_n$ denotes the symmetric matrix corresponding to the standard hyperbolic form $\sum_{i=1}^{2n} e_{2i-1,i} + \sum_{i=1}^{2n} e_{2i,2i-1}$).

First we consider the case when $p_t = 1$. Let $p_1^*, \ldots, p_k^*$ be the standard basis of P_s and let $s^m p_i^* \in P$ for some $m > 0$ and $i = 1, \ldots, k$. Let e_i^* be the standard basis of A_s. For $q_t = i$, consider the element $T\mu_t(X)e_i^* \in A_s^k[X,T]$ as an element in $P_s[X,T]^*$. Using Lemma 3.2, we may say that $T\mu_t(X)e_i^*$ is actually a polynomial in T. By Lemma 3.3, there exists $k_1 > 0$ such that k_1 is the maximum power of s occurring in the denominator of $\mu_t(X)e_i^*$. Choose $l_1 \geq \max(k_1, m)$.

Next suppose $q_t = 1$. For $p_t = j$, $T\mu_t(X)e_j^* \in P_s[X,T]$. From Lemma 3.3 it follows that we can choose $k_2 > 0$ such that k_2 is the maximum power of s occurring in $\mu_t(X)e_j^*$. Again, using Lemma 3.2, we can regard $T\mu_t(X)e_j^*$ as a polynomial in T. Choose $l_2 \geq \max(k_2, m)$ and $l \geq \max(l_1, l_2)$. Now applying the homomorphism $T \mapsto s^l T$, it follows that $\sigma(bXT^{2d})$ is defined over $Q[X]$. Putting $T = 1$, by the usual Dilation Principle there exists $l > 0$ such that $\hat{\sigma}(X) \in \mathrm{ET}(Q[X])$ localizes to $\sigma(bX)$ for some $b \in (s^l)$ and $\hat{\sigma}(0) =$ Id. □

Lemma 3.5. *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let $\alpha \in \mathrm{S}(n,R)$ and let I be an ideal contained in the nil radical $\mathrm{Nil}(R)$ of R. Let 'bar' denote the reduction modulo I. If $\overline{\alpha} \in \mathrm{E}(n, \overline{A})$, then $\alpha \in \mathrm{E}(n, A)$.*

Proof. This is easy to verify. (The proof of Lemma 4.6 later is similar; its argument can be used to prove this lemma.) □

Theorem 3.6 (Local-Global Principle)**.** *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let P and Q be as in* 2.18*. Assume that* (**H1**) *holds. Suppose $\sigma(X) \in \mathrm{G}(Q[X])$ with $\sigma(0) = \mathrm{Id}$. If*

$$\sigma_{\mathfrak{p}}(X) \in \begin{cases} \mathrm{E}(n+1, A_{\mathfrak{p}}[X]) & \textit{in the linear case,} \\ \mathrm{E}(2n+2, A_{\mathfrak{p}}[X]) & \textit{otherwise} \end{cases}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\sigma(X) \in \mathrm{ET}(Q[X])$.

Proof. This follows from a similar argument as in the proof of (4) ⇒ (3) in Theorem 3.1 of [5]. □

Corollary 3.7. *Assume that* (**H2**) *holds. Let $\tau(X) \in \mathrm{G}(Q[X])$, with $\tau(0) = \mathrm{Id}$. If $\tau_s(X) \in \mathrm{ET}(Q_s[X])$, and $\tau_t(X) \in \mathrm{ET}(Q_t[X])$, for some $s, t \in R$ such that $Rs + Rt = R$, then $\tau(X) \in \mathrm{ET}(Q[X])$.*

Lemma 3.8. *Let R be a commutative ring with identity. If $\alpha = (a_{ij})$ is an $r \times r$ matrix over R with all entries nilpotent, then α is nilpotent.*

Proof. Let $a_{ij}^l = 0$, for all $i, j \in \{1, \ldots, r\}$. Now α^2 has entries which are homogeneous polynomials of degree 2 in the a_{ij}'s. Consequently, α^4 has entries which are homogeneous polynomials of degree 4 in the a_{ij}'s, and so on. Therefore, $\alpha^{2^m} = 0$, if $2^m > lr^2$, by the Pigeon Hole Principle. □

Corollary 3.9. *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let 'bar' denote the reduction modulo* $\mathrm{Nil}(R)$. *Assume that* $(\mathbf{H1})$ *holds. Then for* $\tau \in \mathrm{G}(Q)$, $\overline{\tau} \in \mathrm{ET}(\overline{Q})$ *if and only if* $\tau \in \mathrm{ET}(Q)$.

Proof. First we show that for $\sigma(X) \in \mathrm{G}(Q[X])$, $\overline{\sigma(X)} \in \mathrm{ET}(\overline{Q[X]}) \Leftrightarrow \sigma(X) \in \mathrm{ET}(Q[X])$. Suppose that $\overline{\sigma(X)} \in \mathrm{ET}(\overline{Q[X]})$. Then $\overline{\sigma_{\mathfrak{p}}(X)} \in \mathrm{ET}(\overline{A_{\mathfrak{p}}}[X]) = \mathrm{E}(n+1, \overline{A_{\mathfrak{p}}}[X])$ and $\mathrm{E}(2n+2, \overline{A_{\mathfrak{p}}}[X])$ in the linear and the nonlinear cases respectively, for all $\mathfrak{p} \in \mathrm{Spec}(R)$. But then from Lemma 3.5 it follows that

$$\sigma_{\mathfrak{p}}(X) \in \begin{cases} \mathrm{E}(n+1, A_{\mathfrak{p}}[X]) \text{ in the linear case, and} \\ \mathrm{E}(2n+2, A_{\mathfrak{p}}[X]) \text{ otherwise.} \end{cases}$$

Hence by Theorem 3.6, $\sigma(X) \in \mathrm{ET}(Q[X])$.

Now modifying τ by some $\varepsilon \in \mathrm{ET}(Q)$ we assume that $\tau = \mathrm{Id} + \gamma$, where $\gamma \equiv 0$ modulo $\mathrm{Nil}(R)$. As the nilpotent entries are in R, which is a commutative ring, by Lemma 3.8, γ is nilpotent. Define $\theta(X) = \mathrm{Id} + X\gamma$. As $\overline{\theta(X)} = \mathrm{Id} \in \mathrm{ET}(\overline{Q[X]})$, from the above it follows that $\theta(X) \in \mathrm{ET}(Q[X])$. Whence $\tau = \theta(1) \in \mathrm{ET}(Q)$, as required. $\square$

Theorem 3.10. *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let Q be as in* 2.18. *Assume that* $(\mathbf{H1})$ *holds. Then* $\mathrm{T}(Q) = \mathrm{ET}(Q)$.

Proof. Using Corollary 3.9 we assume that A is reduced. By definition $\mathrm{ET}(Q) \subset \mathrm{T}(Q)$. To prove the converse assume $\tau \in \mathrm{T}(Q)$. Then there exists $\sigma(X) \in \mathrm{T}(Q[X])$ such that $\sigma(0) = \mathrm{Id}$ and $\sigma(1) = \tau$. Now, for every $\mathfrak{p} \in \mathrm{Spec}(R)$,

$$\sigma_{\mathfrak{p}} \in \begin{cases} \mathrm{E}(n+1, A_{\mathfrak{p}}[X]), \text{ in the linear case, and} \\ \mathrm{E}(2n+2, A_{\mathfrak{p}}[X]), \text{ otherwise.} \end{cases}$$

Therefore, by Theorem 3.6 it follows that $\sigma(X) \in \mathrm{ET}(Q[X])$. Whence $\tau = \sigma(1) \in \mathrm{ET}(Q)$, as required. $\square$

The next lemma was proved in the linear case in ([11], Proposition 4.1). The authors do not assume the existence of a unimodular element in Q though and only get a unipotent lift. In ([6], Lemma 2.1) it is mentioned that if Q has a unimodular element, then the lift is a transvection.

Corollary 3.11. *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let Q be as in* 2.18. *If I in an ideal in R, then the map* $\mathrm{T}(Q) \to \mathrm{T}(Q/IQ)$ *is surjective.*

Proof. By Proposition 3.10, $\mathrm{T}(Q/IQ) = \mathrm{ET}(Q/IQ)$. Since an elementary transvection can always be lifted to an elementary transvection, the result follows. $\square$

As a consequence of Theorem 3.1 and Theorem 3.10, following L.N. Vaserstein's proof of Serre's conjecture (see [10], Chapter III, §2) we deduce the following Local-Global Principle for the action of the elementary subgroups of an extended projective, symplectic and orthogonal module.

Theorem 3.12. *Let A be an associative R-algebra such that A is finite as a left R-module and let R be a commutative ring with identity. Let Q be as in* 2.18. *Assume that* $(\mathbf{H1})$ *and* $(\mathbf{H2})$ *hold. Let* $v(X) = (p(X), a(X))$ *be a unimodular* *element in* $Q[X]$ *with* $v(0) = (0, 1)$. *Suppose that for every prime ideal* $\mathfrak{p} \in \mathrm{Spec}(R)$ *there exists*

$$\tau_{\mathfrak{p}}(X) \in \mathrm{T}(Q_{\mathfrak{p}}[X]) = \begin{cases} \mathrm{E}(n+1, A_{\mathfrak{p}}[X]) & \textit{in the linear case, and} \\ \mathrm{E}(2n+2, A_{\mathfrak{p}}[X]) & \textit{in nonlinear cases} \end{cases}$$

with $\tau_{\mathfrak{p}}(0) = I_n$ *such that* $v(X)\tau(X) = v(0)$. *Then there exists* $\tau(X) \in \mathrm{ET}(Q[X])$ *with* $\tau(0) = \mathrm{Id}$ *such that* $v(X)\tau(X) = v(0)$.

4. The unstable K_1-groups $\frac{\mathrm{S}(Q)}{\mathrm{ET}(Q)}$ are nilpotent

In this section, earlier results on unstable K_1-groups of classical groups of A. Bak, R. Hazrat, and H. Vavilov have been uniformly generalized to classical modules.

We prove Theorem 3 mentioned in the introduction. Before that we give a brief historical sketch about our result. Throughout this section we assume R is a commutative ring with identity.

In [1], A. Bak defines a functorial filtration $\mathrm{GL}_n(R) = \mathrm{S}^{-1}\mathrm{L}_n(R) \supset \mathrm{S}^0\mathrm{L}_n(R) \supset \cdots \supset \mathrm{S}^i\mathrm{L}_n(R) \supset \cdots \supset \mathrm{E}_n(R)$ of the general linear group $\mathrm{GL}_n(R)$, where R is an associative ring with identity and $n \geq 3$, which is a descending central series. His construction has its own merits, which we do not study here other than the fact that the quotient $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ is nilpotent-by-abelian. A. Bak uses a localization-completion method; we show that the localization part suffices to get the desired result.

In [8], R. Hazrat and N. Vavilov have shown: Let Φ be a reduced irreducible root system of rank ≥ 2 and R be a commutative ring such that its Bass-Serre dimension $\delta(R)$ is finite. Then for any Chevalley group $\mathrm{G}(\Phi, R)$ of type Φ over R the quotient $\mathrm{G}(\Phi, R)/\mathrm{E}(\Phi, R)$ is nilpotent-by-abelian. In particular, $\mathrm{K}_1(\Phi, R)$ is nilpotent of class at most $\delta(R)+1$. They use the localization-completion method of A. Bak in [1], who showed that $\mathrm{K}_1(n, R)$ is nilpotent-by-abelian. Their main result is to construct a descending central series in the Chevalley group, indexed by the Bass-Serre dimension of the factor-rings of the ground ring. Our approach shows that for classical groups the localization part suffices.

The precise statement of our theorem is the following:

Theorem 4.1. *Assume the notation in* 2.18. *We assume that* (**H1**) *and* (**H2**) *hold and that* R *is Noetherian. Let* $d = \dim(R)$ *and* $t = \text{local rank of } Q$.

The quotient group $Q/\mathrm{T}(Q)$ *is nilpotent of class at most* $\max(1, d+3-t)$ *in the linear case and* $\max(1, d+3-t/2)$ *otherwise.*

(We assume that the global rank of Q is at least 1 and that the local rank of Q is at least 3 in all the above cases.)

Corollary 4.2. *Let* $d = \dim(R)$ *and* $t = \text{local rank of } Q$. *The quotient group* $Q/\mathrm{ET}(Q)$ *is nilpotent of class at most* $\max(1, d+3-t)$ *in the linear case and* $\max(1, d+3-t/2)$ *otherwise.*

Recall

Definition 4.3. Let H be a group. Define $Z^0 = H$, $Z^1 = [H, H]$ and $Z^i = [H, Z^{i-1}]$. Then H is said to be nilpotent if $Z^r = \{e\}$ for some $r > 0$, where e denotes the identity element of H.

Remark 4.4. The standard generators of $\mathrm{E}(n,R)$ satisfy the following relation:

$$[ge_{ik}(x), ge_{kj}(y)] = ge_{ij}(zxy)$$

for $x, y \in R$ and some $z \in \mathbb{N}$ (fixed for the group), and $1 \leq i \neq j \neq k \leq n$, $j \neq \sigma i, \sigma j$, where σ is the permutation given by $2l \mapsto 2l-1$ and $2l-1 \mapsto 2l$.

Notation 4.5. Let $\mathrm{S}(n, s^l R)$ be the subgroup of $\mathrm{S}(n,R)$ consisting of matrices which are the identity modulo $s^l R$ and let $\mathrm{S}(Q, s^l R)$ be the subgroup of $\mathrm{S}(Q)$ consisting of the automorphisms with determinant 1 which are the identity modulo $s^l R$.

Let $J(R)$ denote the Jacobson radical of R.

Lemma 4.6. *Let $\beta \in \mathrm{S}(n,R)$, with $\beta \equiv I_n$ modulo I, where I is contained in the Jacobson radical $J(R)$ of R. Then there exists $\varepsilon \in \mathrm{E}(n,R)$ such that $\beta\varepsilon =$ the diagonal matrix $[d_1, d_2, \ldots, d_n]$, where each d_i is a unit in R and each d_i satisfies $d_{\sigma i} d_i^* = 1$, and ε is a product of elementary generators with each congruent to the identity modulo I.*

Proof. The diagonal elements are units. Using this, one can establish the result easily in the linear case. In the symplectic and the orthogonal cases, by multiplying from the right side by suitable standard elementary generators each of which is congruent to the identity modulo I, we can make all the $(1,j)$-th entries zero for $j = 3, \ldots, n$. Since char $R \neq 2$, in the orthogonal case the $(1,2)$-th entry will then be automatically zero. In the symplectic case, again by multiplying from the right side by suitable elementary generators, we can make the $(1,2)$-th entry zero. Similarly, multiplying by the left we can make the first column of β to be $(d_1, 0, \ldots, 0)^t$, where d_1 is a unit in R and $d_1 \equiv 1$ modulo I. Repeating the above process we can reduce the size of β. Note that after modifying the first row and the first column in the symplectic and the orthogonal cases the second row and column will automatically become $(0, d_2, 0, \ldots, 0)^t$ for some unit d_2 in R, with $d_2 \equiv 1$ modulo I. Repeating the process we can modify β to the required form. $\square$

Blanket assumption. Henceforth we shall assume that the matrices have size at least 3×3 when dealing with the linear case and at least 6×6 when dealing with the symplectic and the orthogonal cases.

Lemma 4.7. *Let R be a commutative ring and let s be a nonzero divisor in R. Let D denote the diagonal matrix $[d_1, \ldots, d_n]$, where d_i should be units and satisfy $d_{\sigma i} d_i^* = 1$, and $d_i \equiv 1$ modulo (s^l) for $l \geq 2$. Then*

$$\left[ge_{ij}\left(\frac{a}{s}X\right), D\right] \subset \mathrm{E}(n, R[X]) \cap \mathrm{S}(n, (s^{l-1})R).$$

Proof. Let $d = d_i d_j^{-1}$. Then $\left[ge_{ij}\left(\frac{a}{s}X\right), D\right] = ge_{ij}\left(\frac{a}{s}X\right) ge_{ij}\left(-\frac{a}{s}dX\right)$. Since $d_i, d_j \equiv 1$ modulo (s^l) for $l \geq 2$, we can write $d = 1 + s^m \lambda$ for some $m > 2$ and $\lambda \in R$. Hence

$$\begin{aligned} ge_{ij}\left(\frac{a}{s}X\right) ge_{ij}\left(-\frac{a}{s}dX\right) &= ge_{ij}\left(\frac{a}{s}X\right) ge_{ij}\left(-\frac{a}{s}X\right) ge_{ij}\left(-\frac{a}{s}s^m \lambda X\right) \\ &= ge_{ij}\left(-\frac{a}{s}s^m \lambda X\right) \in \mathrm{E}(n, R[X]) \cap \mathrm{S}(n, (s^{m-1})R). \end{aligned}$$ $\square$

Lemma 4.8. *Let R be a ring, $s \in R$ a nonzero divisor in R and $a \in R$. Then for $l \geq 2$,*

$$\left[ge_{ij}\left(\frac{a}{s}X\right), \mathrm{S}(n, s^l R)\right] \subset \mathrm{E}(n, R[X]).$$

More generally, $\big[\varepsilon(X), \mathrm{S}(n, s^l R[X])\big] \subset \mathrm{E}(n, R[X])$ *for* $l \gg 0$ *and* $\varepsilon(X) \in \mathrm{E}(n, R_s[X])$.

Proof. First fix (i,j) for $i \neq j$. Let $\alpha(X) = [e_{ij}\left(\frac{a}{s}X\right), \beta]$ for some $\beta \in \mathrm{S}(n, s^l R)$. As $l \geq 2$, it follows that $\alpha(X) \in \mathrm{S}(n, R[X])$. Since $\mathrm{E}(n, R[X])$ is a normal subgroup of $\mathrm{S}(n, R[X])$, we get $\alpha_s(X) \in \mathrm{E}(n, R_s[X])$. Let $B = 1 + sR$. We show that $\alpha_B(X) \in \mathrm{E}(n, R_B[X])$. Since $s \in J(R_B)$, it follows from Lemma 4.6 that we can decompose $\beta_B = \varepsilon_1 \cdots \varepsilon_t D$, where $\varepsilon_i = ge_{p_i q_i}(s^l \lambda_i) \in \mathrm{E}(n, R_B)$, $\lambda_i \in R_B$ and D = the diagonal matrix $[d_1, \ldots, d_n]$ with d_i a unit in R and $d_i \equiv 1$ modulo (s^l) for $l \geq 2$, $i = 1, \ldots, n$. If $t = 1$, then using the commutator law and Lemma 4.7 it follows that $\alpha_B(X) \in \mathrm{E}(n, R_B[X])$. Suppose $t > 1$. Then

$$\begin{aligned}\alpha_B(X) &= \left[ge_{ij}\left(\frac{a}{s}X\right), \varepsilon_1 \cdots \varepsilon_t D\right] \\ &= \left[ge_{ij}\left(\frac{a}{s}X\right), \varepsilon_1\right] \varepsilon_1 \left[ge_{ij}\left(\frac{a}{s}X\right), \varepsilon_2 \cdots \varepsilon_t D\right] \varepsilon_1^{-1}\end{aligned}$$

and by induction each term is in $\mathrm{E}(n, R_B[X])$; hence $\alpha_B(X) \in \mathrm{E}(n, R_B[X])$. Since $\alpha(0) = I_n$, by the Local-Global Principle for the classical groups it follows that $\alpha(X) \in \mathrm{E}(n, R[X])$. $\square$

Corollary 4.9. *Let R be a ring, let $s \in R$ be a nonzero divisor in R and $a \in R$. Then for $l \geq 2$,*

$$\left[ge_{ij}\left(\frac{a}{s}\right), \mathrm{S}(n, s^l R)\right] \subset \mathrm{E}(n, R).$$

More generally, $\big[\varepsilon, \mathrm{S}(n, s^l R)\big] \subset \mathrm{E}(n, R)$ *for* $l \gg 0$ *and* $\varepsilon \in \mathrm{E}(n, R_s)$.

Lemma 4.10. *Fix the notation as in* 2.18. *Let s be a nonzero divisor in R such that P_s is free. Assume that* (**H2**) *holds. Suppose $\tau \in \mathrm{T}(Q_s)$. Then for $l \gg 0$, $[\tau, \mathrm{S}(Q, s^l R)] \subset \mathrm{T}(Q)$.*

Proof. Let $\eta \in \mathrm{S}(Q, s^l R)$ and $\widetilde{\tau}(X) \in \mathrm{T}(Q_s[X])$ be an isotopy between the identity map and τ; *i.e.* $\widetilde{\tau}(0) = \mathrm{Id}$ and $\widetilde{\tau}(1) = \tau$. Let $\alpha(X) = [\widetilde{\tau}(X), \eta]$. Now, since $\eta \equiv \mathrm{Id}$ modulo (s^l), $\eta = \mathrm{Id} + s^l \psi$ for some $\psi \in \mathrm{End}(Q)$. Therefore, ψ can be considered as a matrix as in Remark 2.24. Hence $\widetilde{\tau}(X)\, \eta\, \widetilde{\tau}(X)^{-1} = \mathrm{Id} + s^l \widetilde{\tau}(X) \psi \widetilde{\tau}(X)^{-1} \in \mathrm{S}(Q[X])$ for $l \gg 0$. As 'T' is a normal subgroup of 'S', it follows that $\alpha_s(X) \in \mathrm{T}(Q_s[X])$. Let $B = 1 + sR$. We show that $\alpha_B(X) \in \mathrm{T}(Q_B[X])$. Note that $s \in \mathrm{Jac}(R_B)$. Hence for all $\mathfrak{m} \in \mathrm{Max}(R_B)$,

$$(\eta_B)_{\mathfrak{m}} \in \mathrm{S}((Q_B)_{\mathfrak{m}}, s^l (R_B)_{\mathfrak{m}}) = \begin{cases} \mathrm{E}(n+1, (R_B)_{\mathfrak{m}}) & \text{in the linear case,} \\ \mathrm{E}(2n+2, (R_B)_{\mathfrak{m}}) & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 4.7, $(\eta_B)_{\mathfrak{m}}$ can be expressed as a product of elementary matrices over $(R_B)_{\mathfrak{m}}$ with each being the identity modulo (s^l), and a diagonal matrix $D = [d_1, \ldots, d_t]$, where $t = r + 1$ in the linear case and $r + 2$ otherwise, and the d_i are units in R for $i = 1, \ldots, t$. Let $(\eta_B)_{\mathfrak{m}} = \prod_{i=1}^k \varepsilon_i D$, where ε_i is in $\mathrm{E}_{n+1}((R_B)_{\mathfrak{m}})$ in the linear case and in $\mathrm{E}_{2n+2}((R_B)_{\mathfrak{m}})$ otherwise, and $\varepsilon_i = \mathrm{Id}$ mod (s^l). So, $(\alpha_B)_{\mathfrak{m}}(X) = [\widetilde{\tau}(X), \varepsilon_1 \cdots \varepsilon_k D]$. Hence by Lemma 4.7 and Lemma 4.8, we get

$$(\alpha_B)_{\mathfrak{m}}(X) \in \begin{cases} \mathrm{E}(n+1, (R_B)_{\mathfrak{m}}[X]) & \text{in the linear case,} \\ \mathrm{E}(2n+2, (R_B)_{\mathfrak{m}}[X]) & \text{otherwise.} \end{cases}$$

Hence by the L-G Principle for the tranvection groups we get $\alpha_B(X) \in \mathrm{T}(Q_B[X])$. Therefore, it follows from Corollary 3.7 that $\alpha(X) \in \mathrm{T}(Q[X])$. Hence, $[\tau, \eta] \in \mathrm{T}(Q)$. $\square$

Proof of Theorem 4.1. Using Corollary 3.9 we may and do assume that R is a reduced ring. Note that if $t \geq d+3$, then the group $\mathrm{S}(Q)/\mathrm{T}(Q) = \mathrm{K}_1(Q)$, which is abelian and hence nilpotent. So we consider the case $t \leq d+3$. Let us first fix a t. We prove the theorem by induction on $d = \dim R$. Let $H = \mathrm{S}(Q)/\mathrm{T}(Q)$. Let $m = d+3-t$ and $\alpha = [\beta, \gamma]$ for some $\beta \in H$ and $\gamma \in Z^{m-1}$. Clearly, the result is true for $d = 0$. Let $\widetilde{\beta}$ be the preimage of β under the map $\mathrm{S}(Q) \to \mathrm{S}(Q)/\mathrm{T}(Q)$. Choose a nonnilpotent element s in R such that P_s is free and $\widetilde{\beta}_s \in \mathrm{E}(n, A_s)$. We define $\overline{H} = \mathrm{S}(\overline{Q})/\mathrm{T}(\overline{Q})$, where the bar denotes reduction modulo s^l for some $l \gg 0$. By the induction hypothesis, $\overline{\gamma} = \{1\}$ in $\overline{\mathrm{G}}(Q)$. Since $\mathrm{T}(Q)$ is a normal subgroup of $\mathrm{S}(Q)$ for $n \geq 3$ in the linear case and for $n \geq 4$ otherwise, by modifying γ we may assume that $\widetilde{\gamma} \in \mathrm{S}(Q, s^l A)$, where $\widetilde{\gamma}$ is the preimage of γ in $\mathrm{S}(Q)$. Now by Lemma 4.10 it follows that $[\widetilde{\beta}, \widetilde{\gamma}] \in \mathrm{T}(Q)$. Hence $\alpha = \{1\}$ in H. $\square$

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