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Nonabelian K-Theory: The Nilpotent Class of K_1 and General Stability

Dedicated to Alexander Grothendieck on his sixtieth birthday

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Abstract. A functorial filtration $\mathrm{GL}_n = \mathrm{S}^{-1}\mathrm{L}_n \supseteq \mathrm{S}^0\mathrm{L}_n \supseteq \cdots \supseteq \mathrm{S}^i\mathrm{L}_n \supseteq \cdots \supseteq E_n$ of the general linear group GL_n, $n \geqslant 3$, is defined and it is shown for any algebra A, which is a direct limit of module finite algebras, that $\mathrm{S}^{-1}\mathrm{L}_n(A)/\mathrm{S}^0\mathrm{L}_n(A)$ is abelian, that $\mathrm{S}^0\mathrm{L}_n(A) \supseteq \mathrm{S}^1\mathrm{L}_n(A) \supseteq \cdots$ is a descending central series, and that $\mathrm{S}^i\mathrm{L}_n(A) = E_n(A)$ whenever $i \geqslant$ the Bass–Serre dimension of A. In particular, the K-functors $K_1\mathrm{S}^i\mathrm{L}_n := \mathrm{S}^i\mathrm{L}_n/E_n$ are nilpotent for all $i \geqslant 0$ over algebras of finite Bass–Serre dimension. Furthermore, without dimension assumptions, the canonical homomorphism $\mathrm{S}^i\mathrm{L}_n(A)/\mathrm{S}^{i+1}\mathrm{L}_n(A) \to \mathrm{S}^i\mathrm{L}_{n+1}(A)/\mathrm{S}^{i+1}\mathrm{L}_{n+1}(A)$ is injective whenever $n \geqslant i+3$, so that one has stability results without stability conditions, and if A is commutative then $\mathrm{S}^0\mathrm{L}_n(A)$ agrees with the special linear group $\mathrm{SL}_n(A)$, so that the functor $\mathrm{S}^0\mathrm{L}_n$ generalizes the functor SL_n to noncommutative rings. Applying the above to subgroups H of $\mathrm{GL}_n(A)$, which are normalized by $E_n(A)$, one obtains that each is contained in a sandwich $\mathrm{GL}'_n(A, \mathfrak{q}) \supseteq H \supseteq E_n(A, \mathfrak{q})$ for a unique two-sided ideal $\mathfrak{q}$ of A and there is a descending $\mathrm{S}^0\mathrm{L}_n(A)$-central series $\mathrm{GL}'_n(A, \mathfrak{q}) \supseteq \mathrm{S}^0\mathrm{L}_n(A, \mathfrak{q}) \supseteq \mathrm{S}^1\mathrm{L}_n(A, \mathfrak{q}) \supseteq \cdots \supseteq \mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q}) \supseteq \cdots \supseteq E_n(A, \mathfrak{q})$ such that $\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q}) = E_n(A, \mathfrak{q})$ whenever $i \geqslant$ Bass–Serre dimension of A.

Key words. Nonabelian K_1, noncommutative homotopy, general linear group, superspecial linear groups, descending central series, stability, relative normal subgroups, nilpotent sandwich classifications, quasi-finite algebras.

1. Introduction

Let R denote a commutative ring with identity, let A denote an associative R-algebra, and let $\mathrm{GL}_n(A)$, $n \geqslant 3$, denote the general linear group of rank n. Beginning with the pioneering paper 'K-Theory and Stable Algebra' [2] of H. Bass, published a quarter century ago, the problem of locating and analyzing near normal subgroups of $\mathrm{GL}_n(A)$ for finite R-algebras A has received considerable attention. The combined results of H. Bass [2], [3, §5], J. S. Wilson [17], and L. Vaserstein [16] show that a group $H \subseteq \mathrm{GL}_n(A)$ is normalized by the elementary subgroup $E_n(A)$ of $\mathrm{GL}_n(A)$ if and only if for some (unique) two-sided ideal $\mathfrak{q}$ of A, H fits into a sandwich $E_n(A, \mathfrak{q}) \subseteq H \subseteq \mathrm{GL}'_n(A, \mathfrak{q})$, where $E_n(A, \mathfrak{q})$ denotes the relative elementary subgroup of level $\mathfrak{q}$ and $\mathrm{GL}'_n(A, \mathfrak{q})$ the kernel of the canonical homomorphism $\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{q})/\mathrm{center}(\mathrm{GL}_n(A/\mathfrak{q}))$. The unique ideal $\mathfrak{q}$ is called the *level* of H. The properties and structure of subgroups of level $\mathfrak{q}$ depends obviously on those of $E_n(A, \mathfrak{q})$ and the coset space $\mathrm{GL}'_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$. Bass [2], [3, §5] showed that this coset space is a group whenever n is in the 'stable range' of A and that for such n, $\mathrm{GL}_n(A)$ acts trivially via conjugation on the subquotient $\mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ where $\mathrm{GL}_n(A, \mathfrak{q}) = \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{q}))$. Thus, for n in the stable range of A, every H of level $\mathfrak{q}$ has a filtration $H \supseteq H \cap \mathrm{GL}_n(A, \mathfrak{q}) \supseteq E_n(A, \mathfrak{q})$ such that the quotients $H/H \cap \mathrm{GL}_n(A, \mathfrak{q})$ and $H \cap \mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ are abelian and the action of $\mathrm{GL}_n(A)$ on them, via conjugation, is trivial. Moreover, stability results of Bass [2], [3, V] and later Vaserstein [15] show that the quotient $\mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ is canonically isomorphic to the stable K-group $K_1(A, \mathfrak{q}) := \mathrm{GL}_\infty(A, \mathfrak{q})/E_\infty(A, \mathfrak{q})$ providing n is in the stable range of A. Thus, one can use stable K-theory to study the consecutive quotients $H/H \cap \mathrm{Gl}_n(A, \mathfrak{q})$ and $H \cap \mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ in the sandwich theorem above.

The current article will show that the coset space $\mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ is a group providing $n \geqslant 3$ (without stable range assumptions) and A is quasi-finite (a direct limit of algebras which are module finite over their centers) and will investigate the group $\mathrm{GL}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ by constructing a functorial filtration

$$\mathrm{GL}_n(A, \mathfrak{q}) = \mathrm{S}^{-1}\mathrm{L}_n(A, \mathfrak{q}) \supseteq \mathrm{S}^0\mathrm{L}_n(A, \mathfrak{q}) \supseteq \cdots \supseteq \mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q}) \supseteq \cdots \supseteq E_n(A, \mathfrak{q})$$

of $\mathrm{GL}_n(A, \mathfrak{q})$ such that

(i) $\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q})$ $(i \geqslant -1)$ is normal in $\mathrm{GL}_n(A)$,
(ii) the action of $\mathrm{S}^0\mathrm{L}_n(A)$ on $\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q})/\mathrm{S}^{i+1}\mathrm{L}_n(A, \mathfrak{q})$ $(i \geq -1)$ via conjugation is trivial.
(iii) the canonical homomorphisms

$$\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q})/\mathrm{S}^{i+1}\mathrm{L}_n(A, \mathfrak{q}) \to \mathrm{S}^i\mathrm{L}_{n+1}(A, \mathfrak{q})/\mathrm{S}^{i+1}\mathrm{L}_{n+1}(A, \mathfrak{q})$$

are injective for all $n \geqslant i+3$,
(iv) $\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q}) = E_n(A, \mathfrak{q})$ whenever $i \geqslant$ the Bass–Serre dimension of A,
(v) the nilpotent class of $\mathrm{S}^0\mathrm{L}_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$ relative to $\mathrm{S}^0\mathrm{L}_n(A)$ is $\leqslant$ infimum $(\delta(A), [\delta(A) + 2 - n])$ whenever the Bass–Serre dimension $\delta(A)$ of A is finite, and
(vi) if A is commutative then $\mathrm{S}^0\mathrm{L}_n(A, \mathfrak{q}) = \mathrm{SL}_n(A, \mathfrak{q})$, where $\mathrm{SL}_n(A, \mathfrak{q})$ denotes the $\mathfrak{q}$-congruence subgroup of the special linear group $\mathrm{SL}_n(A)$.

The result (iii) uncouples injective stability from stability conditions on A and allows one to use stable K_1 to compute the quotients $\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q})/\mathrm{S}^{i+1}\mathrm{L}_n(A, \mathfrak{q})$ of consecutive layers of the filtration above when $-1 \leqslant i \leqslant n+2$; namely, defining the stable K-groups $K_1^i(A, \mathfrak{q}) := \mathrm{S}^i\mathrm{L}_\infty(A, \mathfrak{q})/E_\infty(A, \mathfrak{q})$, one obtains from (iii) a canonical injection

$$\mathrm{S}^i\mathrm{L}_n(A, \mathfrak{q})/\mathrm{S}^{i+1}\mathrm{L}_n(A, \mathfrak{q}) \to K_1^i(A, \mathfrak{q})/K_1^{i+1}(A, \mathfrak{q}) \qquad \text{for } -1 \leqslant i \leqslant n+2.$$

Results (ii) and (iv) show that the filtration

$$\mathrm{GL}_n(A, \mathfrak{q}) = \mathrm{S}^{-1}\mathrm{L}_n(A, \mathfrak{q}) \supseteq \mathrm{S}^0\mathrm{L}_n(A, \mathfrak{q}) \supseteq \cdots \supseteq \mathrm{S}^{\delta(A)}\mathrm{L}_n(A, \mathfrak{q}) = E_n(A, \mathfrak{q})$$

is a descending $\mathrm{S}^0\mathrm{L}_n(A)$-central series of finite length. Results (ii) and (v) put an upper bound on the $S^0L_n(A)$-nilpotent class of the (nonstable) K-groups

$$K_1 S^iL_n(A, \mathfrak{q}) := S^iL_n(A, \mathfrak{q})/E_n(A, \mathfrak{q}) \quad (i \geqslant -1),$$

which can be much smaller than that given by the functorial filtration above. The result (vi) shows that the functor S^0L_n is a natural extension of the functor SL_n to noncommutative rings.

Applying the results above to subgroups H of $GL_n(A)$, which are normalized by $E_n(A)$, one obtains for each H of level $\mathfrak{q}$ a canonical filtration

$$H = H \cap GL_n'(A, \mathfrak{q}) \supseteq H \cap S^{-1}L_n(A, \mathfrak{q}) \supseteq \cdots \supseteq H \cap S^iL_n(A, \mathfrak{q}) \supseteq \cdots \supseteq E_n(A\mathfrak{q})$$

such that any quotient formed by consecutive layers of the filtration is abelian and such that any subgroup of $S^0L_n(A)$, which normalizes H, acts trivially via conjugation on these quotients. This is the nilpotent sandwich classification theorem of Section 6. Moreover, for $-1 \leqslant i \leqslant n+2$, there is a canonical injection $H \cap S^iL_n(A, \mathfrak{q})/H \cap S^{i+1}L_n(A, \mathfrak{q}) \to K_1^i(A, \mathfrak{q})/K_1^{i+1}(A, \mathfrak{q})$. All of the above concerns the nilpotent structure of the lattice of subgroups of $GL_n(A)$ normalized by the elementary subgroup $E_n(A)$. Results on the nilpotent structure of lattices of subgroups of $GL_n(A)$ normalized by congruence or relative elementary subgroups of $GL_n(A)$ are found also in Section 6.

In a sequel to the current paper, the results above will be extended to classical and Kac–Moody groups by introducing higher 'nonabelian' K-functors and using Mayer–Vietoris sequences for these functors to replace some of the computations in this article. In another article, it will be shown that similar results hold for nonabelian analytic K-groups.

The remainder of the paper is organized as follows. In Section 2, basic notation is fixed and standard commutator formulas are recalled. In Section 3, quasi-finite algebras are introduced and $S^0L_n(A)$ is defined. In Section 4, the principal tools of the paper are forged. In Section 5, the filtration

$$S^{-1}L_n(A) \supseteq S^0L_n(A) \supseteq \cdots \supseteq S^dL_n(A) \supseteq \cdots$$

is defined and its basic properties are established. In Section 6, the relative filtration

$$S^{-1}L_n(A, \mathfrak{q}) \supseteq S^0L_n(A, \mathfrak{q}) \supseteq \cdots \supseteq S^dL_n(A, \mathfrak{q}) \supseteq \cdots$$

is defined and its properties are deduced from the corresponding ones of the absolute filtration. Results on the absolute and relative nilpotent structure of GL_n are proved, including the nilpotent sandwich classification theorem. One important consequence of the results in Sections 4–6 are upper bounds on the nilpotent class of $S^0L_n(A, \mathfrak{q})/E_n(A, \mathfrak{q})$. In Section 7, $S^0L_n(A)/E_n(A)$ is tied to noncommutative homotopy theory and the latter is used to establish lower bounds on the nilpotent class of this group. The results show that the group $SL_n(R)/E_n(R)$ can be nonabelian for commutative rings R of finite Bass–Serre dimension $\delta(R)$ and allow one to construct nonnormal subgroups, as one expects, of $GL_n(R)$ which are normalized by $E_n(R)$. The latter settles positively a question posed by Bass.

2. Notation

Let G denote a group. If σ, $\rho \in G$, let ${}^{\sigma}\rho = \sigma\rho\sigma^{-1}$ denote the *σ-conjugate* of ρ and let $[\sigma, \rho] = \sigma\rho\sigma^{-1}\rho^{-1}$ denote the *commutator* of σ and ρ. The following formulas will be used frequently.

(2.1) (a) $[\sigma, \rho\tau] = [\sigma, \rho]({}^{\rho}[\sigma, \tau])$,

(b) $[\sigma\rho, \tau] = ({}^{\sigma}[\rho, \tau])[\sigma, \tau]$.

Let H and K denote subgroups of G. Let $[H, K]$ denote the subgroup of G generated by all commutators $[\sigma, \rho]$ such that $\sigma \in H$ and $\rho \in K$. $[H, K]$ is called the *mixed commutator group* generated by H and K. Define inductively $D_H^0(K) = K$ and for $i > 0$, $D_H^i(K) = [H, D_H^{i-1}(K)]$. If $G = H$, we shall write $D^i(K)$ in place of $D_G^i(K)$. $D^0(G) \supset D^1(G) \supset D^2(G) \supset \cdots$ is the *descending central series* of G. G is *nilpotent* if $D^i(G) = 1$ for some i. The smallest i such that $D^i(G) = 1$ is the *nilpotent class* of G. If U and V are subsets of G, let ${}^{U}V$ denote the set $\{{}^{\sigma}\rho \mid \sigma \in U, \rho \in V\}$.

Let A denote an associative ring with identity 1. Let n denote a natural number and let $G(A) = \mathrm{GL}_n(A)$ denote the *general linear group* of rank n over A. By definition it is the group of all $n \times n$ invertible matrices over A, i.e. the units in the ring $M_n(A)$ of all $n \times n$ matrices over A. Let $\mathfrak{q}$ denote a subgroup of A, closed under multiplication. ($\mathfrak{q}$ is not necessarily an ideal of A.) Let $i \neq j$ be natural numbers such that $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. A *$\mathfrak{q}$-elementary matrix* $\varepsilon_{ij}(q)$ is an $n \times n$ matrix whose diagonal coefficients are 1, whose (i,j)th coefficient is q where $q \in \mathfrak{q}$, and whose other nondiagonal coefficients are zero. (If $n = 1$, then 1 is the only elementary matrix.) Let $G(\mathfrak{q})$ denote the subgroup of $G(A)$ of all invertible matrices whose diagonal coefficients are of the form $1 + q$ where $q \in \mathfrak{q}$ and whose nondiagonal coefficients lie in $\mathfrak{q}$. $G(\mathfrak{q})$ is not necessarily normal in $G(A)$. Let $E(\mathfrak{q})$ denote the subgroup of $G(\mathfrak{q})$ generated by all $\mathfrak{q}$-elementary matrices. If $\mathfrak{q}$ is a two-sided ideal in A, then $G(\mathfrak{q})$ is a normal subgroup of $G(A)$, because $G(\mathfrak{q}) = \mathrm{Ker}(G(A) \to G(A/\mathfrak{q}))$. For such $\mathfrak{q}$'s, let $E(A, \mathfrak{q})$ denote the normal subgroup of $E(A)$ generated by $E(\mathfrak{q})$. Clearly $E(\mathfrak{q}) \subset E(A, \mathfrak{q}) \subset G(\mathfrak{q})$. Let L denote a non-negative integer. For $\mathfrak{q}$ arbitrary, let $E^{L}(\mathfrak{q})$ denote the subset of $E(\mathfrak{q})$ consisting of all products of L or fewer $\mathfrak{q}$-elementary matrices. Thus, $E^1(\mathfrak{q})$ is the set of all $\mathfrak{q}$-elementary matrices and, by convention, $E^0(\mathfrak{q}) = 1$. The letter L stands intuitively for word length. The following formulas for elementary matrices will be used frequently.

(2.2) (a) $\varepsilon_{ij}(a)\varepsilon_{ij}(b) = \varepsilon_{ij}(a + b)$.

(b) $[\varepsilon_{ij}(a), \varepsilon_{kl}(b)] = 1$ *providing* $i \neq l$, $j \neq k$.

(c) $[\varepsilon_{ij}(a), \varepsilon_{jk}(b)] = \varepsilon_{ik}(a)$ *providing* $i \neq k$.

From (2.2)(c), it follows that $E(A)$ is perfect whenever $n \geqslant 3$, i.e. $E(A) = [E(A), E(A)]$.

3. Quasi-Finite Algebras and the Special Linear Group $\mathrm{SL}_{n \geqslant 2}$

The goal of this section is to introduce the notions of quasi-finite algebra and of special linear group over such an algebra. It will be shown that the latter notion generalizes the usual one over commutative rings.

Let R denote a commutative associative ring with identity 1. An R-algebra is an associative ring A with identity and a fixed ring homomorphism $R \to \mathrm{center}(A)$. A homomorphism $A_1 \to A_2$ of R_i-algebras ($i = 1, 2$) is a pair of ring homomorphisms $R_1 \to R_2$ and $A_1 \to A_2$ which commute with the canonical homomorphisms $R_i \to A_i$ ($i = 1, 2$).

Suppose A is an R-algebra and I is an index set. By a *direct system* of subalgebras A_i/R_i ($i \in I$) of A, we shall mean a set of subrings R_i of R and a set of subrings A_i of A such that each A_i is naturally on R_i-algebra and such that given $i, j \in I$, there is a $k \in I$ such that $R_i \subseteq R_k$, $R_j \subseteq R_k$, $A_i \subseteq A_k$, and $A_j \subseteq A_k$.

An R-algebra A is called *module finite* or simply *finite* over R if A is finitely generated as an R-module.

DEFINITION-PROPOSITION (3.1). *An R-algebra A is called quasi-finite over R if it satisfies one of the following equivalent conditions*:

(i) *There is a direct system of finite R-subalgebras A_i of A such that $\varinjlim_i A_i = A$.*

(ii) *There is a direct system of subalgebras A_i/R_i of A such that each A_i is finite over R_i and such that $\varinjlim_i R_i = R$ and $\varinjlim_i A_i = A$.*

(iii) *There is a direct system of subalgebras A_i/R_i of A such that each A_i is finite over R_i and each R_i is finitely generated as a $\mathbb{Z}$-algebra and such that $\varinjlim_i R_i = R$ and $\varinjlim_i A_i = A$.*

Proof. The proof that conditions (i), (ii), and (iii) are equivalent is straightforward and is left to the reader.

If A is a ring and $\mathfrak{q}$ a two-sided ideal of A then the *smash product* $A \ltimes \mathfrak{q}$ is the ring whose elements are all pairs (a, q) such that $a \in A$ and $q \in \mathfrak{q}$ and whose addition and multiplication are given, respectively, by

$$(a, q) + (a', q') = (a + a', q + q')$$

and

$$(a, q)(a', q') = (aa', aq' + qa' + qq').$$

The next corollary is surprising, since it is not valid for finite R-algebras, and justifies replacing the category of finite algebras by the bigger category of quasi-finite algebras. Most of the results in Section 6 depend on the corollary including, for example, the results (i)–(vi) in the introduction and the nilpotent sandwich classification theorem.

COROLLARY (3.2). *If A is quasi-finite over R and $\mathfrak{q}$ is a two-sided ideal of A, then $A \ltimes \mathfrak{q}$ is quasi-finite over R.*

Proof. The proof is straightforward, providing one uses the formulation (3.1) (iii) of a quasi-finite algebra.

Let *f-algebra* and *q-f-algebra* denote respectively the categories of finite algebras and of quasi-finite algebras. It is routine to check that *q-f-algebra* is closed under direct limits, whereas *f-algebra* is not.

LEMMA (3.3). *If* $\mathfrak{c}$ *is a category with direct limits and* F: *f-algebra* $\to \mathfrak{c}$ *is a functor then* F *has a unique-up-to-isomorphism extension to a functor* $\tilde{F}$: *q-f-algebras* $\to \mathfrak{c}$ *such that* $\tilde{F}$ *commutes with direct limits. Moreover,* $\tilde{F}$ *is universal among all extensions of* F *to q-f-algebras, i.e. if* $\tilde{F}'$ *is another extension, not necessarily commuting with direct limits, then there is a unique natural transformation* $\tilde{F} \to \tilde{F}'$ *of functors such that if* $A = \varinjlim_i A_i$ $(A_i \in$ *f-algebra*$)$ *then the diagram below commutes for each* i

$$\begin{tikzcd} \tilde{F}(A) \arrow[rr] & & \tilde{F}'(A) \\ & F(A_i) \arrow[ul] \arrow[ur] & \end{tikzcd} .$$

Proof. The proof is routine. One begins by noting that if $\{A_i \mid i \in I\}$ and $\{A'_j \mid j \in J\}$ are direct systems of finite subalgebras of A such that $\varinjlim_i A_i = A = \varinjlim_j A'_j$, then each system is cofinal in the other, i.e. given $i \in I$, $\exists j \in J$ such that $A_i \subset A'_j$, and conversely. Thus, $\varinjlim_i F(A_i)$ and $\varinjlim_j F(A'_j)$ are canonically isomorphic. Define $\tilde{F}(A) = \varinjlim_k F(A''_k)$ where A''_k runs over all finite subalgebras of A. The remaining details of the proof are straightforward and are left to the reader.

DEFINITION (3.4). Suppose $n \geqslant 2$. If A is a finite algebra, define the special linear group $\mathrm{SL}_n(A) = \{\sigma \mid \sigma \in G(A)\ (= \mathrm{GL}_n(A))$, image of σ under $G(f)\colon G(A) \to G(B)$ lies in $E(B)\ (= E_n(B))$, $f\colon A \to B$ a homomorphism of finite algebras, B semilocal [3, p. 86]$\}$. Extend the definition of SL_n to all quasi-finite algebras, via Lemma (3.3).

Let $SG(A) = \mathrm{SL}_n(A)$. It is clear that SG defines a functor *q-f-algebras* $\to$ *groups*. It will be shown next that the B's in Definition (3.4) can be restricted to a certain set of semilocal rings and then it will be shown that for A commutative, the definition of SL_n above agrees with the usual one.

Suppose A is a finite R-algebra and B' is a semilocal ring which is finite over its center. Suppose B is another semilocal ring which is finite over its center. If an algebra homomorphism $f'\colon A \to B'$ can be factored as a composite of algebra homomorphisms $A \xrightarrow{f} B \to B'$ then clearly the condition that $Gf(\sigma) \in E(B)$ implies the condition that $Gf'(\sigma) \in E(B')$. It will be shown that an arbitrary f' can be factored as above where B belongs to a certain set of semilocal rings which are finite over their centers. Let $R' = \mathrm{center}(B')$. Let $\mathfrak{q}$ denote the Jacobson radical (B') [3, II §2]. Let $R'' = \mathrm{center}(B'/\mathfrak{q})$. Since $B'/\mathfrak{q}$ is semisimple, R'' is a finite product of fields and is, therefore, semilocal. Let $\mathfrak{p}''_1, \ldots, \mathfrak{p}''_k$ denote the maximal ideals of R'' and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_k$ denote, respectively, their inverse images in R and R'. Let $R_{\mathfrak{p}_1, \ldots, \mathfrak{p}_k}$ denote the localization of R at the multiplicative set

$R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$ and similarly $R'_{\mathfrak{p}'_1,\ldots,\mathfrak{p}'_k}$. Since an element of B' is invertible if and only if its image in $B'/\mathfrak{q}$ is invertible, it follows that the elements of $R' - \mathfrak{p}'_1 \cup \cdots \cup \mathfrak{p}'_k$ are invertible in B'. Thus, there is a canonical isomorphism

$$R'_{\mathfrak{p}'_1,\ldots,\mathfrak{p}'_k} \otimes_{R'} B' \xrightarrow{\cong} B', \qquad x \otimes b' \mapsto xb',$$

and $f'\colon A \to B'$ factors canonically as the composite

$$A \to R_{\mathfrak{p}_1,\ldots,\mathfrak{p}_k} \otimes_R A \to R'_{\mathfrak{p}'_1,\ldots,\mathfrak{p}'_k} \otimes B' \to B'.$$

On the other hand, if $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ are maximal ideals of R containing, respectively, $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, then the homomorphism $A \to R_{\mathfrak{p}_1,\ldots,\mathfrak{p}_k} \otimes_R A$ factors canonically as the composite

$$A \to R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A \to R_{\mathfrak{p}_1,\ldots,\mathfrak{p}_k} \otimes_R A.$$

$R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k}$ is semilocal and thus, $R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A$ is semilocal, because it is finite over $R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k}$ (cf. [3, III(2.5)]). Thus, the following lemma has been shown.

LEMMA (3.5). *Suppose A is a finite R-algebra. Then $SG(A) = \{\sigma \mid \sigma \in G(A)$, value of σ in $G(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)$ lies in $E(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)$, $\mathfrak{m}_1,\ldots,\mathfrak{m}_k$ any finite set of maximal ideals of $R\}$.*

LEMMA (3.6). *$SG(A)$ is a normal subgroup of $G(A)$ containing* $[G(A), G(A)]$.

Proof. Whenever B is semilocal, $E(B)$ is a normal subgroup of $G(B)$ containing $[G(B), G(B)]$, by [3, IV (9.1)]. The lemma follows.

LEMMA (3.7). *If A is commutative then* $SG(A) = \mathrm{Ker}(\det\colon G(A) \to \mathrm{GL}_1(A))$ *where* det *denotes the usual determinant map.*

Proof. Let $D(A) = \mathrm{Ker}(\det\colon G(A) \to \mathrm{GL}_1(A))$. It suffices by Lemma (3.5) to show that if A is finite over R then

$$D(A) = \bigcap_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_k)} \mathrm{Ker}(G(A) \to G(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)/E(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A))$$

where $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ runs through all finite sets of maximal ideals of R. But, since the canonical homomorphism

$$A \rightarrowtail \prod_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_k)} R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A$$

is injective, it follows that

$$\begin{aligned} D(A) &= \mathrm{Ker}(G(A) \to \prod_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_k)} G(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)/D(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)) \\ &= \bigcap_{(\mathfrak{m}_1,\ldots,\mathfrak{m}_k)} \mathrm{Ker}(G(A) \to G(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)/D(R_{\mathfrak{m}_1\ldots,\mathfrak{m}_k} \otimes_R A)) \\ &= (\text{by } [3, \mathrm{IV}(9.2)]) \bigcap_{(\mathfrak{m}_1,\ldots\mathfrak{m}_k)} \mathrm{Ker}(G(A) \\ &\qquad \to G(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)/E(R_{\mathfrak{m}_1,\ldots,\mathfrak{m}_k} \otimes_R A)). \end{aligned}$$

4. The First Results on $\mathrm{GL}_{n\geqslant 3}$

Let A denote an associative R-algebra with identity 1. Let n denote a fixed natural number $\geqslant 3$ and let $G=\mathrm{GL}_n$, $E=E_n$, and $SG=\mathrm{SL}_n$, as in Sections 2 and 3. If $z\in\mathbb{Z}$, let $[z]$ denote the smallest nonnegative integer $\geqslant z$.

The main result of the section is the following:

THEOREM (4.1). *Suppose that the space of maximal ideals of R, with the usual topology is a finite union of irreducible Noetherian subspaces of dimension $\leq d$ (finite) [3, III §3]. Suppose that A is quasi-finite over R. Then $D^{1+[d+2-n]}SG(A)=E(A)$.*

COROLLARY (4.2). *Assume the hypotheses of (4.1). Then $E(A)$ is the largest perfect subgroup of $SG(A)$ and the quotient $SG(A)/E(A)$ is nilpotent of class $\leqslant 1+[d+2-n]$.*

Proof. The corollary follows directly from (2.2) and (4.1)

COROLLARY (4.3). *Assume the hypotheses of (4.1). Then $E(A)$ is the largest perfect subgroup of $G(A)$ and the quotient $G(A)/E(A)$ is solvable of degree $\leqslant 2+[d+2-n]$.*

Proof. By (3.4), $SG(A)\supset[G(A),G(A)]$. By (2.2) and (4.1),

$$D^{1+[d+2-n]}_{[G(A),G(A)]}([G(A),G(A)])=E(A).$$

The corollary follows.

COROLLARY (4.4). *A natural transformation $G\to G$ or $SG\to SG$ over q-f-algebras induces a natural transformation $E\to E$ over q-f-algebras.*

Proof. I shall prove only the case SG. The proof for G is similar and is left to the reader.

Let $\sigma\colon SG\to SG$ denote the natural transformation. It suffices to show that $\sigma_A(E(A))\subset E(A)$ for any q-f-algebra A. Suppose $A=\varinjlim_i A_i$. Define

$$D^{\infty}SG(A')=\bigcap_{k=0}^{\infty}D^kSG(A').$$

Clearly,

$$\sigma_{A_i}(D^kSG(A_i))\subset D^kSG(A_i).$$

Thus,

$$\sigma_{A_i}(D^{\infty}SG(A_i))\subset D^{\infty}SG(A_i).$$

If one knew that $D^{\infty}SG(A_i)=E(A_i)$, it would follow that $\sigma_{A_i}(E(A_i))\subset E(A_i)$ and thus, that $\sigma_A(E(A))\subset E(A)$, because E commutes with direct limits. It is not difficult to show that there is a direct system of finite R_i-subalgebras A_i of A such that $A=\varinjlim_i A_i$ and such that each R_i is finitely generated as an algebra over its prime ring. If R_i is generated by g_i elements over its prime ring, then its space of prime ideals is Noetherian of dimension $\leqslant g_i + 1$. (This follows from the fact that R_i is Noetherian of Krull dimension $\leqslant g_i + 1$, cf. [3, III §3].) It follows that the space of maximal ideals of R_i is Noetherian of dimension $\leqslant g_i + 1$. Thus, by (2.2) and (4.1), $D^{\infty}SG(A_i) = E(A_i)$.

COROLLARY (4.5). *A natural transformation $G \to G$ over q-f-algebras induces a natural transformation $SG \to SG$ over q-f-algebras.*

Proof. By (4.4), the natural transformation $G \to G$ induces a natural transformation $E \to E$. A straightforward functorial argument will show that the pair $G \to G$, $E \to E$ of natural transformations induces a natural transformation $SG \to SG$. Details are left to the reader.

The proof of Theorem (4.1) will be based on several lemmas. An important aspect of Lemmas (4.6), (4.7), and (4.11) is the control between certain coefficients. This control is based on the principle that if σ and ρ are two matrices such that σ is t-adically small and ρ s-adically small, then the commutator $[\sigma, \rho]$ is st-adically small.

If $s \in R$, let $\langle s \rangle$ denote the multiplicative set $1, s, s^2, \ldots$ defined by s. If $t \in R$, let $(t/s)A$ denote the subgroup of $\langle s \rangle^{-1}A$ consisting all quotients ta/s, where $a \in A$. The letters K, L, M, k, l, and m denote nonnegative integers.

LEMMA (4.6). *Let $(t/s)A$ be as above and let $s^m tA$ denote the subgroup of $(t/s)A$ consisting of all $s^m ta$, where $a \in A$. If K, L and m are given, there are k and M, e.g.*

$$k = (m+1)4^K + 4^{K-1} + \cdots + 4 \quad \textit{and} \quad M = 14^K L,$$

such that ${}^{E^K((t/s)A)}E^L(s^k tA) \subset E^M(s^m tA)$.

Proof. The case $K = 0$ or $L = 0$ is trivial. Suppose $K > 0$ and $L > 0$. Since ${}^{E^K((t/s)A)}E^L(s^k tA)$ is the set of all products of L or fewer elements of ${}^{E^K((t/s)A)}E^1(s^k tA)$, it follows that the assertion of the lemma is true for a pair (K, L) whenever it is true for the pair $(K, 1)$. Thus, it suffices by induction to show that

$${}^{E^1((t/s)A)}E^1(s^{(m+1)4}tA) \subset E^{14}(s^m tA)$$

and that for $K > 1$,

$$\begin{aligned} &{}^{E^K((t/s)A)}E^1(s^{((m+1)4^{K-1}+4^{K-2}+\cdots+1)4}tA) \\ &\quad \subset {}^{E^{K-1}((t/s)A)}E^{14}(s^{(m+1)4^{K-1}+4^{K-2}+\cdots+4}tA). \end{aligned}$$

To prove the latter inclusion, it suffices to show that

$${}^{E^1((t/s)A)}E^1(s^{(m'+1)4}tA) \subset E(s^{m'}tA)$$

where

$$m' = (m+1)4^{K-1} + 4^{K-2} + \cdots + 4.$$

But this is just a special case of the former inclusion. The former inclusion is proved as follows. The formulas in (2.2) will be used repeatedly in the proof.

Let

$$\rho = {}^{\varepsilon_{ij}(ta/s)}\varepsilon_{i'j'}(s^k tb)\ (k = (m+1)4).$$

If $i \neq j'$ or $j \neq i'$, then $\rho = \varepsilon_{i'j'}(s^k tb)$. If $i \neq j'$, but $j = i'$, then

$$\rho = \varepsilon_{ij'}(s^{k-1}t^2ab)\varepsilon_{i'j'}(s^k tb).$$

If $i = j'$, but $j \neq i'$, then

$$\rho = \varepsilon_{i'j}(-s^{k-1}t^2ba)\varepsilon_{i'j'}(s^k tb).$$

If $i = j'$ and $j = i'$, choose $h \neq i, j$. Then

$$\begin{aligned}\rho &= {}^{\varepsilon_{ij}(ta/s)}[\varepsilon_{jh}(s^{k/2}), \varepsilon_{hi}(s^{k/2}tb)] \\ &= \varepsilon_{ih}(s^{((k/2)-1)}ta)\varepsilon_{jh}(s^{k/2})\varepsilon_{hj}(-s^{((k/2)-1)}t^2ba) \times \\ &\quad \times \varepsilon_{hi}(s^{k/2tb)}\varepsilon_{ih}(-s^{((k/2)-1)}ta)\varepsilon_{jh}(-s^{k/2})\varepsilon_{hj}(s^{((k/2)-1)}t^2ba)\varepsilon_{hi}(-s^{k/2}tb).\end{aligned}$$

But,

$$\begin{aligned}&\varepsilon_{jh}(s^{k/2})\varepsilon_{hj}(-s^{((k/2)-1)}t^2ba)\varepsilon_{jh}(-s^{k/2}) \\ &= (\text{setting } k' = (k/2-1))^{\varepsilon_{jh}(s^{k/2})}[\varepsilon_{hi}(-s^{k'/2}t), \varepsilon_{ij}(s^{k'/2}tba)] \\ &= [\varepsilon_{ji}(-s^{(k+k'/2)}t)\varepsilon_{hi}(-s^{k'/2}t), \varepsilon_{ih}(-s^{(k+k')/2}tba)\varepsilon_{ij}(s^{k'/2}tba)]\end{aligned}$$

and

$$\begin{aligned}&\varepsilon_{jh}(s^{k/2})\varepsilon_{hi}(s^{k/2}tb)\varepsilon_{ih}(-s^{((k/2)-1)}ta)\varepsilon_{jh}(-s^{k/2}) \\ &= \varepsilon_{ji}(s^k tb)\varepsilon_{hi}(s^{k/2}tb)\varepsilon_{jh}(-s^{((k/2)-1)}ta).\end{aligned}$$

Thus, $\rho \in E^{14}(s^m tA)$.

If U and V are subsets of some group, let $]U, V[$ denote the *set* of all commutators $[u, v]$ such that $u \in U$, $v \in V$. Let p and q denote nonnegative integers.

LEMMA (4.7). *Let $s, t \in R$. Let $(t^l/s)A$, $(s^k/t)A$, and $s^p t^q A$ denote the subgroups of $\langle st\rangle^{-1}A$ consisting, respectively, of all quotients $(t^l/s)a$, $(s^k/t)a$, and $(s^p t^q/1)a$, where $a \in A$. If K, L, p and q are given, there are k, l, and M,* e.g.

$$k = (p+1)4^{K+1} + 4^K + \cdots + 4, \qquad l = (q+1)4^{L+1} + 4^L + \cdots + 4,$$

and $M = 14^{K+L+2}KL$, such that

$$\left]E^K\!\left(\frac{t^l}{s}A\right), E^L\!\left(\frac{s^k}{t}A\right)\right[\subset E^M(s^p t^q A).$$

Proof. The case $K = 0$ or $L = 0$ is trivial. Suppose $K > 0$ and $L > 0$. If U is a subset of a group and N is a nonnegative integer, let $\mathrm{Prod}^N(U)$ denote the set of all products of N or fewer elements of U. Let $S = (s^k/t)A$ and $T = (t^l/s)A$. From the commutator formulas (2.1), it follows that

$$]E^K(T), E^L(S)[\subset ({}^{E^{K-1}(T)}]E^1(T), E^L(S)[) \cdots ({}^{E^0(T)}]E^1(T), E^L(S)[)$$

and

$$]E^1(T), E^L(S)[\subset ({}^{E^0(S)}]E^1(T), E^1(S)[) \cdots ({}^{E^{L-1}(S)}]E^1(T), E^1(S)[).$$

Thus

$$]E^K(T), E^L(S)[\subset \mathrm{Prod}^{KL}({}^{E^{K-1}(T)E^{L-1}(S)}]E^1(T), E^1(S)[).$$

I shall show that

$$]E^1(T), E^1(S)[\subset E^{14^4}(s^{(p+1)4^{K-1}+4^{K-2}+\cdots+4}t^{(q+1)4^{L-1}+4^{L-2}+\cdots+4}A).$$

The assertion of the lemma will then follow from (4.6).

Let

$$\rho = \left[\varepsilon_{ij}\left(\frac{t^l}{s}a\right), \varepsilon_{i'j'}\left(\frac{s^k}{t}b\right)\right]$$

and suppose that k and l are as in the statement of the lemma. The formulas of (2.2) will be used repeatedly to evaluate ρ. If $j \neq i'$ or $i \neq j'$, then $\rho = 1$. If $i \neq j'$, but $j = i$ then $\rho = \varepsilon_{ij'}, (s^{k-1}t^{l-1}ab)$. If $j \neq i'$, but $i = j'$, then $\rho = \varepsilon_{i'j}(-s^{k-1}t^{l-1}ab)$. The remaining case to consider is $i = j'$ and $j = i'$. Choose $h \neq i, j$ and let $\varepsilon = \varepsilon_{ij}((t^l/s)a)$. Then

$$\begin{aligned}
\rho &= \left[\varepsilon, \left[\varepsilon_{jh}\left(\frac{s^{k/2}}{t}b\right), \varepsilon_{hi}(s^{k/2})\right]\right] \\
&= (\text{by } (2.1))\left[\varepsilon, \varepsilon_{jh}\left(\frac{s^{k/2}}{t}b\right)\right]({}^{\varepsilon_{jh}((s^{k/2}/t)b)}[\varepsilon, \varepsilon_{hi}(s^{k/2})]) \times \\
&\quad \times ({}^{\varepsilon_{jh}((s^{k/2}/t)b)\varepsilon_{hi}(s^{k/2})}[\varepsilon, \varepsilon_{jh}((-s^{k/2}/t)b)]) \times \\
&\quad \times ({}^{\varepsilon_{jh}((s^{k/2}/t)b)\varepsilon_{hi}(s^{k/2})\varepsilon_{jh}((-s^{k/2}/t)b)}[\varepsilon, \varepsilon_{hi}(-s^{k/2})]) \\
&= \varepsilon_{ih}(t^{l-1}s^{k/2-1}ab) \times \\
&\quad \times ({}^{\varepsilon_{jh}((s^{k/2}/t)b)}\varepsilon_{hj}(-t^l s^{((k/2)-1)}a)) \cdot ({}^{\varepsilon_{jh}((s^{k/2}/t)b)\varepsilon_{hi}(s^{k/2})}\varepsilon_{ih}(-t^{l-1}s^{((k/2)-1)}ab)) \times \\
&\quad \times ({}^{\varepsilon_{jh}((s^{k/2}/t)b)\varepsilon_{hi}(s^{k/2})\varepsilon_{jk}((-s^{k/2}/t)b)}\varepsilon_{hj}(t^l s^{(k/2-1)}a)) \in
\end{aligned}$$

(letting $p' = (p+1)4^{K-1} + 4^{K-2} + \cdots + 4$, $q' = (q+1)4^{L-1} + 4^{L-2} + \cdots + 4$, and applying (4.6) to each of the four factors above)

$$E(s^{p'}t^{q'}A)E^{14}(s^{p'}t^{q'}A)E^{14^2}(s^{p'}t^{q'}A)E^{14^3}(s^{p'}t^{q'}A) \subset E^{14^4}(s^{p'}t^{q'}A).$$

LEMMA (4.8). *Let* $\mathfrak{q}$ *be a two-sided ideal of* A. *Then* $E(A, \mathfrak{q})$ *is generated as a group by the elements* ${}^{\varepsilon_{ji}(a)}\varepsilon_{ij}(q)$ *such that* $i \neq j$, $a \in A$, *and* $q \in \mathfrak{q}$.

Proof. If i, j, k are distinct natural numbers and $a, b \in A$ and $q \in \mathfrak{q}$, then one can check by straightforward multiplication that

$$(4.8') \quad {}^{\varepsilon_{ij}(a)\varepsilon_{ji}(b)}\varepsilon_{ij}(q)$$
$$= \varepsilon_{kj}(-q(1+ba))\varepsilon_{ki}(qb)\varepsilon_{ik}(-abqb)\varepsilon_{ij}(abq) \times$$
$$\times ({}^{\varepsilon_{jk}(b)}\varepsilon_{kj}(q))\varepsilon_{ij}(q)\varepsilon_{ik}((ab-1)qb)\varepsilon_{jk}(bqb)({}^{\varepsilon_{ij}(a)}\varepsilon_{ji}(-bqb)) \times$$
$$\times ({}^{\varepsilon_{ki}(1)}\varepsilon_{ik}(qb))\varepsilon_{kj}(qba)\varepsilon_{ij}(qba).$$

By definition, $E(A, \mathfrak{q})$ is generated by the elements ${}^{\varepsilon}\varepsilon_{ij}(q)$ such that $i \neq j$, $\varepsilon \in E(A)$, and $q \in \mathfrak{q}$. If ε is the identity matrix, let $l(\varepsilon) = 0$ and otherwise, let $l(\varepsilon)$ denote the least number of elementary matrices required to write ε as a product of elementary matrices. The proof is by induction on $l(\varepsilon)$. If $l(\varepsilon) = 0$, there is nothing to prove.

Suppose $l(\varepsilon) = 1$. Then $\varepsilon = \varepsilon_{kl}(a)$ for some elementary matrix $\varepsilon_{kl}(a)$. If $(k, l) = (j, i)$, there is nothing to prove. If $(k, l) \neq (j, i)$ then by (2.2), ${}^{\varepsilon_{kl}(a)}\varepsilon_{ij}(q) =$ either $\varepsilon_{ij}(q)$ or $\varepsilon_{i'j'}(q')\varepsilon_{ij}(q)$ for an elementary matrix $\varepsilon_{i'j'}(q')$ *such that* $q' \in \mathfrak{q}$.

Suppose $l(\varepsilon) \geqslant 2$. Write $\varepsilon = \varepsilon'\varepsilon_{mn}(b)\varepsilon_{kl}(a)$ where $l(\varepsilon') = l(\varepsilon) - 2$. If $(k, l) \neq (j, i)$, then applying the paragraph above, one can finish by induction on $l(\varepsilon)$. Suppose $(k, l) = (j, i)$. If $(m, n) = (i, j)$ then applying (4.8′), one can finish by induction on $l(\varepsilon)$. Suppose $(m, n) \neq (i, j)$. If $m \neq i$ and $n \neq j$, then by (2.2) $\varepsilon_{mn}(b)\varepsilon_{ji}(a) = \varepsilon_{ji}(a)\varepsilon_{mn}(b)$. It is not possible that $(m, n) = (j, i)$, because then it would follow that $\varepsilon = \varepsilon'\varepsilon_{ji}(b + a)$ and thus, that $l(\varepsilon) \leqslant l(\varepsilon') + 1$. Since $(m, n) \neq (j, i)$, it follows from (2.2) that

$${}^{\varepsilon_{mn}(b)}\varepsilon_{ij}(q) = \text{either } \varepsilon_{ij}(q) \text{ or } \varepsilon_{i'j'}(q')\varepsilon_{ij}(q)$$

for an elementary matrix $\varepsilon_{i'j'}(q')$ such that $q' \in \mathfrak{q}$ and one is done again by induction on $l(\varepsilon)$. There remain now two cases to check; namely, $(m, n) = (m, j)$ with $m \neq i$ and $(m, n) = (i, n)$ with $n \neq j$. In the first case,

$${}^{\varepsilon_{mj}(b)\varepsilon_{ji}(a)}\varepsilon_{ij}(q)$$
$$= (\text{by } (2.2))\, {}^{\varepsilon_{mi}(ba)\varepsilon_{ji}(a)\varepsilon_{mj}(b)}\varepsilon_{ij}(q)$$
$$= (\text{by } (2.2))\, {}^{\varepsilon_{mi}(ba)\varepsilon_{ji}(a)}\varepsilon_{ij}(q)$$
$$= (\text{by } (2.2))\, {}^{\varepsilon_{ji}(a)\varepsilon_{mi}(ba)}\varepsilon_{ij}(q)$$
$$= (\text{by } (2.2))\, {}^{\varepsilon_{ji}(a)}(\varepsilon_{mj}(baq)\varepsilon_{ij}(q)).$$

Thus, one can finish by induction on $l(\varepsilon)$. The second case is checked similarly.

COROLLARY (4.9). (a) (G. Habdank) *If* $\mathfrak{q}$ *is a two-sided ideal of* A *then* $E(A, \mathfrak{q}^2) \subset E(\mathfrak{q})$.

(b) *Suppose* A *is semilocal. If* $m \geqslant 2$ *then for any* h $(1 \leqslant h \leqslant n)$, $G(s^m A) \subset \Delta_h(s^m A)E(s^{[m/2]}A)$, *where* $\Delta_h(s^m A)$ *denotes all diagonal matrices whose* hth *coefficient lies in* $1 + s^m A$ *and whose other coefficients are* 1.

Proof. (a) By (4.8), $E(A, \mathfrak{q}^2)$ is generated as a group by all matrices ${}^{\varepsilon_{ji}(a)}\varepsilon_{ij}(q_1 q_2)$ such that $a \in A$ and $q_1, q_2 \in \mathfrak{q}$. If $k \neq i, j$ then by (2.2),

$$\begin{aligned}
{}^{\varepsilon_{ji}(a)}\varepsilon_{ij}(q_1 q_2) &= {}^{\varepsilon_{ji}(a)}[\varepsilon_{ik}(q_1),\ \varepsilon_{kj}(q_2)] \\
&= [{}^{\varepsilon_{ji}(a)}\varepsilon_{ik}(q_1),\ \ {}^{\varepsilon_{ji}(a)}\varepsilon_{kj}(q_2)] \\
&= [\varepsilon_{jk}(aq_1)\varepsilon_{ik}(q_1),\ \varepsilon_{ki}(-q_2 a)\varepsilon_{kj}(q_2)].
\end{aligned}$$

(b) By conjugating both sides of the inclusion in (b) by a suitable permutation matrix, one can reduce to the case $h = 1$. By [3, V(3.3)(1)], $G(s^m A) = \Delta_h(s^m A)E(A, s^m A)$ and, by part (a), $E(A, s^m A) \subset E(s^{[m/2]} A)$.

LEMMA (4.10). *Suppose R is Noetherian and A is finite over R. Then given $s \in R$, there is a nonnegative integer k such that the homomorphism $G(s^k A) \to G(\langle s \rangle^{-1} A)$ induced by the canonical homomorphism $A \to \langle s \rangle^{-1} A$ is injective.*

Proof. Clearly $G(s^k A) \to G(\langle s \rangle^{-1} A)$ is injective whenever $s^k A \to \langle s \rangle^{-1} A$ is injective. For $i \geqslant 0$, let $\mathfrak{a}_i = \{a \mid a \in A, s^i a = 0\}$. Clearly, $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$. Since R is Noetherian and A is finite over R, there is a k such that $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \mathfrak{a}_{k+2} = \cdots$. Suppose $a \in A$ such that $s^k a$ vanishes in $\langle s \rangle^{-1} A$. Then $s^j s^k a$ vanishes in A for some j. Thus, $a \in \mathfrak{a}_{j+k} = \mathfrak{a}_k$. Thus, $s^k a = 0$. Thus, $s^k A \to \langle s \rangle^{-1} A$ is injective.

LEMMA (4.11). *Suppose A is quasi-finite over R. Let $s^m A$ denote the subgroup of $(1/s)A$ consisting of all $s^m a$ where $a \in A$. Let "G"$(s^k A)$ denote the image of $G(s^k A)$ $(\subset G(A))$ in $G(\langle s \rangle^{-1} A)$. Given K and m, there is a k, e.g.*

$$k = 2((m+1)4^{K+2} + 4^{K+1} + 4^{K+1} + \cdots + 4),$$

such that

$$\left[E^K\left(\frac{1}{s} A\right), \text{“}G\text{”}(s^k A)\right] \subset E(s^m A).$$

Proof. Since k does not depend on R or A and since

$$E^K\left(\frac{1}{s}\_\_\_\right), \quad \text{“}G\text{”}(s^k \_\_\_) \quad \text{and} \quad E(s^m \_\_\_)$$

commute with direct limits, one can reduce to the case A is finite over R and R is finitely generated as an algebra over its prime ring (i.e. image $(\mathbb{Z} \to R)$) and is, therefore, Noetherian.

I shall show that

$$\left[E^1\left(\frac{1}{s} A\right), \text{“}G\text{”}(s^k A)\right] \subset E(s^{(m+1)4^{K-1} + 4^{K-2} + \cdots + 4} A).$$

It will then follow that $[E^K((1/s)A), \text{“}G\text{”}(s^k(A)] \subset$ (by (2.1))

$${}^{E^{K-1}((1/s)A)}E(s^{(m+1)4^{K-1} + 4^{K-2} + \cdots + 4} A) \subset \text{(by (4.6))}\ E(s^m A).$$

Let

$$\varepsilon_{ij}\left(\frac{a}{s}\right) \in E^1\left(\frac{1}{s}A\right) \quad \text{and} \quad \text{“}\sigma\text{”} \in \text{“}G\text{”}(s^kA).$$

Let $m' = (m+1)4^{K-1} + 4^{K-2} + \cdots + 4$. Suppose it has been shown that for each maximal ideal $\mathfrak{m}$ of R, there is an element $t_{\mathfrak{m}}$ in the multiplicative set $R - \mathfrak{m}$ and a nonnegative integer $l_{\mathfrak{m}}$ such that

$$\left[\varepsilon_{ij}\left(\frac{t_{\mathfrak{m}}^{l_{\mathfrak{m}}}a}{s}\right), \text{“}\sigma\text{”}\right] \in E(s^{(m'+1)4}A)$$

for any $a \in A$. There is a finite set $t_{\mathfrak{m}_1}, \ldots, t_{\mathfrak{m}_r}$ of $t_{\mathfrak{m}}$'s such that the ideal they generate is all of R. Choose $x_1, \ldots, x_r \in R$ such that

$x_1 t_{\mathfrak{m}_1}^{l_{\mathfrak{m}_1}} + \cdots + x_r t_{\mathfrak{m}_r}^{l_{\mathfrak{m}_r}} = 1$. Then

$$\left[e_{ij}\left(\frac{a}{s}\right), \text{“}\sigma\text{”}\right]$$

$$= \left[\varepsilon_{ij}\left(\frac{t_{\mathfrak{m}_1}^{l_{\mathfrak{m}_1}}(x_1 a)}{s}\right) \cdots \varepsilon_{ij}\left(\frac{t_{\mathfrak{m}_r}^{l_{\mathfrak{m}_r}}(x_r a)}{s}\right), \text{“}\sigma\text{”}\right]$$

$$\in (\text{by (2.1) and supposition above}) \ {}^{E^1((1/s)A)}E(s^{(m'+1)4}A)$$

$$\in (\text{by (4.6)}) \ E(s^{m'}A).$$

It remains to verify the supposition above. Let $\sigma \in G(s^kA)$ and let “σ” denote its image in “G”(s^kA). Let $A_{\mathfrak{m}}$ denote the localization of A at the multiplicative set $R - \mathfrak{m}$. Since $R_{\mathfrak{m}}$ is a local ring and $A_{\mathfrak{m}}$ is finite over $R_{\mathfrak{m}}$, it follows that $A_{\mathfrak{m}}$ is semilocal [3, III(2.5), (2.11)]. Over $A_{\mathfrak{m}}$, one can factor by Corollary (4.6)σ as a product $\delta'\varepsilon'$ where $\varepsilon' \in E(s^{k/2}A_{\mathfrak{m}})$ and δ' is a diagonal matrix all of whose diagonal coefficients are 1, except possibly the hth diagonal coefficient, and h can be chosen arbitrarily. There is an element $t \in R - \mathfrak{m}$ such that over $\langle t\rangle^{-1}A$, σ can be factored as a product $\delta''\varepsilon''$ where

$$\varepsilon'' \in E\left(\frac{s^{k/2}}{t}A\right)$$

and δ'' is a diagonal matrix all of whose diagonal coefficients are 1, except possibly the hth coefficient which lies in $(1/t)A$, and h can be chosen arbitrarily. Let $\bar{\sigma}$, $\bar{\delta}$, and $\bar{\varepsilon}$ denote, respectively, the images of σ, δ'', and ε'' in $G(\langle st\rangle^{-1}A)$. Thus, $\bar{\sigma} = \bar{\delta}\bar{\varepsilon}$. Let $\varepsilon_{ij}(a/s) \in E^1((1/s)A)$ and let $\bar{\varepsilon}_{ij}(a/s)$ denote its image in $G(\langle st\rangle^{-1}A)$. Let $p = (m'+1)4$. Choose, using Lemma (4.10), a nonnegative integer q such that $G(t^q\langle s\rangle^{-1}A)$ maps injectively to $G(\langle st\rangle^{-1}A)$. Let $l \geqslant q$. Since $G(t^q\langle s\rangle^{-1}A)$ is normal in $G(\langle s\rangle^{-1}A)$, it follows that

$$\left[\varepsilon_{ij}\left(\frac{t^l a}{s}\right), \text{“}\sigma\text{”}\right] \in G(t^q\langle s\rangle^{-1}A).$$

Thus, it suffices to show that there is an $l \geqslant q$ such that

$$\left[\bar{\varepsilon}_{ij}\left(\frac{t^l a}{s}\right), \bar{\sigma}\right] \in E(s^p t^q A) \quad (\subset G(\langle st\rangle^{-1}A)).$$

Choose $h \neq i, j$, so that $\bar{\delta}$ commutes with $\bar{\varepsilon}_{ij}(t^l a/s)$. Thus,

$$\left[\bar{\varepsilon}_{ij}\left(\frac{t^l a}{s}\right), \bar{\sigma}\right] = {}^{\bar{\delta}}\left[\bar{\varepsilon}_{ij}\left(\frac{t^l a}{s}\right), \bar{\varepsilon}\right].$$

By Lemma (4.7), there is an l such that $[\bar{\varepsilon}_{ij}(t^l a/s), \bar{\varepsilon}] \in E(s^p t^{q+1} A)$, because

$$\bar{\varepsilon} \in E\left(\frac{s^{k/2}}{t} A\right), \quad \frac{k}{2} = (p+1)4^2 + 4 \quad \text{and} \quad K = 1.$$

Thus,

$${}^{\bar{\delta}}\left[\bar{\varepsilon}_{ij}\left(\frac{t^l a}{s}\right), \bar{\varepsilon}\right] \in {}^{\bar{\delta}}E(s^p t^{q+1} A) \subset E(s^p t^q A).$$

A. Suslin [13] has proved the following result when A is commutative and A. Suslin (cf. [14, §1]) and L. Vaserstein [16] have proved it when A is finite over R.

THEOREM (4.12). *If A is quasi-finite, then $E(A)$ is a normal subgroup of $G(A)$.*

Proof. The theorem follows trivially from Lemma (4.11) by setting $s = 1$.

If $s \in R$ and V is an R-module, let $\hat{V}_s = \varprojlim_{p \geqslant 0} V/s^p V$.

Suppose $\{A_i \mid i \in I\}$ is the set of all finite R_i-subalgebras of A, where R_i ranges over all subrings of R which contain s and are finitely generated as algebras over their prime rings, i.e. over image $(\mathbb{Z} \to R_i)$. Each R_i and A_i is Noetherian, $R = \varinjlim_i R_i$, and if A is quasi-finite over R, then $A = \varinjlim_i A_i$. There is a canonical homomorphism $\varinjlim_i (\hat{A}_i)_s \to \hat{A}_s$, but it is, in general, neither injective nor surjective. In order to be able to reduce problems in which the ring $\hat{A}_s$ normally plays a role to the Noetherian situation, I shall need to replace $\hat{A}_s$ by the ring $\varinjlim_i (\hat{A}_i)_s$. For this reason, the following definition is introduced.

DEFINITION (4.13). Suppose A is a quasi-finite R-algebra and let $\{A_i \mid i \in I\}$ be as above. Define $\tilde{R}_s = \varinjlim_i (\hat{R}_i)_s$ and $\tilde{A}_s = \varinjlim_i (\hat{A}_i)_s$.

DEFINITION (4.14). Suppose A is a quasi-finite R-algebra. Define

$$G(s^{-1}, A) = \operatorname{Ker}(G(A) \to G(\langle s\rangle^{-1}A)/E(\langle s\rangle^{-1}A))$$

and

$$G(\hat{s}, A) = \operatorname{Ker}(G(A) \to G(\tilde{A}_s)/E(\tilde{A}_s)).$$

COROLLARY (4.15). *Let* $\{A_i \mid i \in I\}$ *be as in* (4.13). *Then*

$$G(s^{-1}, A) = \varinjlim_i G(s^{-1}, A_i) \quad \text{and} \quad G(\hat{s}, A) = \varinjlim_i G(s, A_i).$$

Proof. The functors G and E commute with direct limits.

THEOREM (4.16). *Suppose A is quasi-finite over R. Then*

$$[G(s^{-1}, A), G(\hat{s}, A)] \subset E(A).$$

Proof. One reduces by (4.15) to the case A is finite over R and R is Noetherian. Let $\sigma \in G(s^{-1}, A)$ and $\rho \in G(\hat{s}, A)$. From (2.2)b), it follows that

$$E(\langle s \rangle^{-1}A) = \bigcup_{K \geqslant 0} E^K\left(\frac{1}{s}A\right).$$

Thus, the value of σ in $G(\langle s \rangle^{-1}A)$ lies in $E^K((1/s)A)$ for some K. By Lemma (4.10), there is an m such that the canonical homomorphism $G(s^m A) \to G(\langle s \rangle^{-1}A)$ is injective. Let

$$k = 2((m+1)4^{K+2} + 4^{K+1} + \cdots + 4).$$

Since

$$\rho \in \operatorname{Ker}(G(A) \to G(A/s^k A)/E(A/s^k A))),$$

there is an $\varepsilon \in E(A)$ such that $\rho\varepsilon \in G(s^k A)$. Since $E(A)$ is normal in $G(A)$ (by Theorem (4.12)), it follows that $[\sigma, \rho] \in E(A)$ if and only if $[\sigma, \rho\varepsilon] \in E(A)$. Since $\rho\varepsilon \in G(s^k A) \subset G(s^m A)$ and $G(s^m A)$ is normal in $G(A)$, one has that $[\sigma, \rho\varepsilon] \in G(s^m A)$. The proof follows now from the fact that the homomorphism $G(s^m A) \to G(\langle s \rangle^{-1}A)$ is injective and $[\sigma, \rho\varepsilon]$ lies in image$(E(s^m A) \to G(\langle s \rangle^{-1}A))$, by Lemma (4.11).

A topological space is called *irreducible* if it is not the union of two nonempty proper closed subsets. A topological space is called *Noetherian* if its closed subsets satisfy the descending chain condition. Trivially, any subspace of a Noetherian space is Noetherian. The *dimension* of a topological space X is the length m of the longest chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_m$ of nonempty closed irreducible subsets X_i of X. If there is no nonnegative integer m as above, then dimension(X) is infinity. Define $\delta(X)$ to be the smallest nonnegative integer d such that X is a finite union of irreducible Noetherian subspaces of dimension $\leqslant d$. If there is no nonnegative integer d as above, then $\delta(X)$ is infinity.

Let Spec(R) denote the space of all prime ideals of R and let Max(R) denote the subspace of all maximal ideals of R. Give Spec(R) its usual topology, i.e. a subset W is closed if and only if there is an ideal $\mathfrak{a}$ in R such that $W = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R), \mathfrak{p} \supset \mathfrak{a}\}$, and give Max$(R)$ the subspace topology. Define $\delta(R) = \delta(\operatorname{Max}(R))$. There are examples where $\delta(R) < \operatorname{dimension}(\operatorname{Max}(R))$ [3, III(3.13)]. We call $\delta(R)$ the *Bass–Serre dimension* of R. If A is a quasi-finite R-algebra then we define the *Bass–Serre dimension* of A (as an R-algebra) to be that of R.

INDUCTION LEMMA (4.17). *Suppose $\delta(R)$ is finite. Let $X_1 \cup \cdots \cup X_r$ be a decomposition of* Max(R) *into irreducible Noetherian subspaces of dimension $\leqslant \delta(R)$. If $s \in R$ such that for each X_k $(1 \leqslant k \leqslant r)$, s does not lie in some member of X_k then $\delta(\tilde{R}_s) < \delta(R)$.*

Proof. I shall show first that $s\tilde{R}_s \subset$ Jacobson radical$(\tilde{R}_s)$. Let the index set I be as in Definition (4.13). Let $a \in \tilde{R}_s$. For some $i \in I$, $a \in (\hat{R}_i)_s$. The element $1 - sa$ is invertible in $(\hat{R}_i)_s$, its inverse being the element $1 + \Sigma_{j=1}^{\infty} (sa)^j$. Thus, $1 - sa$ is invertible in $\tilde{R}_s$. Thus, by Nakayama's lemma [3, III(2.2)], $s\tilde{R}_s \subset$ Jacobson radical $(\tilde{R}_s)$.

Each element of Max$(\tilde{R}_s)$ contains the Jacobson radical$(\tilde{R}_s)$. Thus, Max$(\tilde{R}_s)$ is canonically homeomorphic to Max$(\tilde{R}_s/s\tilde{R}_s)$. But,

$$\tilde{R}_s/s\tilde{R}_s \cong \varinjlim_i\, (\hat{R}_i)_s/s(\hat{R}_i)_s \cong \varinjlim_i\, (R_i/sR_i) \cong R/sR.$$

Thus,

$$\mathrm{Max}(\tilde{R}_s) \cong \mathrm{Max}(\tilde{R}_s/s\tilde{R}_s) \cong \mathrm{Max}(R/sR).$$

Max(R/sR) is canonically homeomorphic to the closed subset $\{\mathfrak{m} \mid \mathfrak{m} \in \mathrm{Max}(R), \mathfrak{m} \supset sR\}$ of Max(R). Let $Y_k = X_k \cap \mathrm{Max}(R/sR)$. The hypotheses of the lemma show that Y_k is a proper closed subset of X_k. Since X_k is Noetherian, so is Y_k. Thus, by [3, III(3.7), (3.8)], Y_k has only finitely many closed irreducible components $Y_{k1}, \ldots, Y_{kc_k}$ and $Y_k = Y_{k1} \cup \cdots \cup Y_{kc_k}$. Since X_k and Y_{kj} are irreducible, and since Y_{kj} is closed in X_k and $Y_{kj} \subsetneqq X_k$, it follows that dimension$(Y_{kj}) <$ dimension$(X_k) \leqslant \delta(R)$. Since

$$\mathrm{Max}(R/sR) = \bigcup_{k=1}^{r} \bigcup_{j=1}^{c_k} Y_{kj},$$

it follows that $\delta(R/sR) < \delta(R)$. Thus, $\delta(\tilde{R}_s) = \delta(R/sR) < \delta(R)$.

Proof of Thoerem (4.1). We must show that if A is quasi-finite over R and $\delta(R)$ is finite then $D^{1+[\delta(R)+2-n]}SG(A) = E(A)$. The proof is by induction on $[\delta(R) + 2 - n]$.

Since $D^{1+[\delta(R)+2-n]}SG$ and E commute with direct limits of quasi-finite algebras over R, one can reduce to the case that A is finite over R.

If $[\delta(R) + 2 - n] = 0$ then by [3, V(3.5)] and [15, (3.5)], $[G(A), G(A)] = E(A)$. Thus, $D^1SG(A) = E(A)$, because $SG(A) \supset E(A)$ and $E(A)$ is perfect.

Suppose $p = [\delta(R) + 2 - n] > 0$. It is a simple exercise using (2.1) to show that $D^{1+p}SG(A) \subset E(A)$ if and only if for each sequence

$$\sigma, \sigma_1, \ldots, \sigma_p \in SG(A), \quad [\sigma, \rho] \in E(A),$$

where

$$\rho = [\sigma_1, \ldots, [\sigma_{p-2}, \sigma_{p-1}, \sigma_p]] \cdots].$$

Let $X_1 \cup \cdots \cup X_r$ be a decomposition of Max(R) into irreducible Noetherian subspaces of dimension $\leqslant \delta(R)$. Let $\mathfrak{m}_i \in X_i$ $(1 \leqslant i \leqslant r)$. Let R' and A' denote, respectively, the localizations of R and A at the multiplicative set $R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$. A' is semilocal, because A' is finite over R' and R' is semilocal [3, III(2.5), (2.11)]. Since $\sigma \in SG(A)$, its value in $G(A')$ lies in the subgroup $E(A')$. There is an element $s \in R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$ such that the value of σ in $G(\langle s\rangle^{-1}A)$ lies in $E(\langle s\rangle^{-1}A)$. Thus, $\sigma \in G(s^{-1}, A)$. Since the element s satisfies the hypotheses of the Induction Lemma (4.17), $\delta(\tilde{R}_s) < \delta(R)$. Thus, by induction on $[\delta(R) + 2 - n]$, the value of ρ in $G(\tilde{A}_s)$ lies in $E(\tilde{A}_s)$. thus, $\rho \in G(\hat{s}, A)$. The assertion of the theorem follows now from Theorem (4.16).

QUESTION (4.18). If A is quasi-finite, is $E(A)$ the largest perfect subgroup of $G(A)$?

5. Super Special Linear Groups $\mathrm{S^dL}_{n\geqslant 2}$ and the Nilpotent Class of K_1

Super special linear groups $\mathrm{S}^d\mathrm{L}_{n\geqslant 2}(A)$ are defined over quasi-finite R-algebras A and it is shown that the sequence

$$\mathrm{SL}_n(A) = \mathrm{S}^0\mathrm{L}_n(A) \supset \mathrm{S}^1\mathrm{L}_n(A) \supset \mathrm{S}^2\mathrm{L}_n(A) \supset \cdots$$

of super special linear groups is a descending central series for $n \geqslant 3$ and $S^{\delta(R)}\mathrm{L}_n(A) = E_n(A)$ whenever $\delta(R)$ is finite and $n \geqslant 3$.

Throughout the section, it will be assumed that A is quasi-finite. $\delta(R)$ is defined as in Section 4.

DEFINITION (5.1). Suppose $n \geqslant 2$ and $d \geqslant 0$. If A is a finite R-algebra, define the dth super special linear group $\mathrm{S}^d\mathrm{L}_n(A) = \{\sigma \mid \sigma \in G(A)\ (=\mathrm{GL}_n(A))$, value of σ under $G(f)\colon G(A) \to G(A')$ lies in $E(A')\ (=E_n(A')), f\colon A \to A'$ a homomorphism of finite algebras, A' a finite R'-algebra, $\delta(R') \leqslant d\}$. Extend the definition of $\mathrm{S}^d\mathrm{L}_n$ to all quasi-finite algebras, via Lemma (3.3). Define $\mathrm{S}^{-1}\mathrm{L}_n = \mathrm{GL}_n$.

In order to bring the notation above in line with that in the previous sections, let $S^dG(A) = \mathrm{S}^d\mathrm{L}_n(A)$.

It is clear that S^dG defines a functor *q-f-algebras* $\to$ *groups*.

COROLLARY (5.2). $S^0G(A) = SG(A)$.

Proof. The assertion follows from Lemma (3.5) and its proof.

COROLLARY (5.3). *If* $n \geqslant 3$ *then* $S^dG(A)$ *is a normal subgroup of* $S^{-1}G(A)$.

Proof. The assertion follows from Theorem (4.12).

COROLLARY (5.4). *If* $n \geqslant 3$ *then a natural transformation* $S^dG \to S^dG$ *induces natural transformations* $S^{d+1}G \to S^{d+1}G$ *and* $E \to E$.

Proof. The proof of Corollary (4.4) shows that a natural transformation $S^dG \to S^dG$ induces a natural transformation $E \to E$. It is straightforward to show that the pair $S^dG \to S^dG$, $E \to E$ of natural transformations induces a natural transformation $S^{d+1}G \to S^{d+1}G$.

THEOREM (5.5). *If* $n \geqslant 3$ *then the sequence* $SG(A) = S^0G(A) \supset S^1G(A) \supset \cdots \supset S^dG(A) \supset \cdots$ *is a descending central series in* $SG(A)$ *and* $S^dG(A) = E(A)$ *whenever* $\delta(R)$ *is finite and* $d \geqslant \delta(R)$.

Proof. The latter assertion is true by definition.

The former assertion for quasi-finite algebras follows from that for finite algebras. Suppose A is a finite R-algebra. Let $\sigma \in SG(A)$ and $\rho \in S^dG(A)$. We want to show that $[\sigma, \rho] \in S^{d+1}G(A)$.

Let A_0 be a finite R_0-algebra such that $\delta(R_0) \leqslant d+1$. If $f\colon A \to A_0$ is an algebra homomorphism, we must show that $[Gf(\sigma), Gf(\rho)] \in E(A_0)$. If suffices to show that if $\delta(R) \leqslant d+1$ then $[\sigma, \rho] \in E(A)$. The proof is similar to that of Theorem (4.1) and goes as follows.

Let $X_1 \cup \cdots \cup X_r$ be a decomposition of $\operatorname{Max}(R)$ into irreducible Noetherian subspaces of dimension $\leqslant d+1$. Let $\mathfrak{m}_k \in X_k$ $(1 \leqslant k \leqslant r)$. Let R' and A' denote, respectively, the localizations of R and A at the multiplicative set $R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$. A' is semilocal. Since $\sigma \in SG(A)$, its value in $G(A')$ lies in $E(A')$. There is an element $s \in R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$ such that the value of σ in $G(\langle s \rangle^{-1}A)$ lies in $E(\langle s \rangle^{-1}A)$. Thus, $\sigma \in G(s^{-1}, A)$. Since the element s satisfies, by construction, the hypotheses of the Induction Lemma (4.17), $\delta(\tilde{R}_s) < \delta(R) \leqslant d+1$. Thus, $\delta(\tilde{R}_s) \leqslant d$. Thus, the value of ρ in $G(\tilde{A}_s)$ lies in $E(\tilde{A}_s)$. Thus, $\rho \in G(\hat{s}, A)$. Thus, $[\sigma, \rho] \in E(A)$, by Theorem (4.16).

Let $G_n(A) = \mathrm{GL}_n(A)$ and $S^dG_n(A) = \mathrm{S}^d\mathrm{L}_n(A)$. The canonical homomorphism

$$G_n(A) \to G_{n+1}(A), \qquad \sigma \mapsto \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix},$$

induces for each d a canonical homomorphism $S^dG_n(A) \to S^dG_{n+1}(A)$. Although for $n \geqslant \delta(R)+2$, the quotient $SG_n(A)/E_n(A)$ is already abelian, the descending sequence $SG_n(A) = S^0G_n(A) \supset S^1G_n(A) \supset \cdots$ of subgroups is still interesting since it gives a functorial filtration of the quotient $SG_n(A)/E_n(A)$. Moreover, the stability theorems of Bass [3, V(4.2), (4.5)] and Vaserstein [15, (3.2), (3.3)] showing that whenever A satisfies Bass' *stable range condition* $SR_N(A)$ [3, V, §3], the canonical homomorphism $G_n(A) \to G_{n+1}(A)/E_{n+1}(A)$ is surjective for $n \geqslant N-1$ and has kernel $E_n(A)$ for $n \geqslant N$, have the following consequences for the homomorphism $S^dG_n(A) \to S^dG_{n+1}(A)/S^{d+i}G_{n+1}(A)$.

THEOREM (5.6). *Let* i *denote an integer* $\geqslant 0$. *If* A *satisfies* $SR_N(A)$ *then the canonical homomorphism* $S^dG_n(A) \to S^dG_{n+1}(A)/S^{d+i}G_{n+1}(A)$ *is surjective for* $n \geqslant \max(N-1, d+2, 2)$ *and has kernel* $S^{d+i}G_n(A)$ whenever $n \geqslant \max(d+i+2, 2)$.

Proof. The proof is a straightforward, formal functorial argument making use of the stability theorems of Bass and Vaserstein. Details are left to the reader.

Remark (5.7). Bass has shown that if A is finite over R and $\delta(R)$ is finite then A satisfies $SR_{\delta(R)+2}(A)$ [3, V(3.5)]. It is easy to show that this result extends to the case A is quasi-finite over R.

Note that Theorem (5.6) above includes stability isomorphisms for $n = N-1$

which is less than the stable range N of A. Without any stable range conditions on A, the quotients $S^dG_n(A)/S^{d+i}G_n(A)$ still enjoy the following injective stability result.

THEOREM (5.8). (a) *If* $n \geqslant \max(d+i+2, 2)$, *then the canonical homomorphism*

$$S^dG_n(A)/S^{d+i}G_n(A) \to S^dG_{n+1}(A)/S^{d+i}G_{n+1}(A)$$

is injective.

Proof. The theorem is deduced easily from (5.7) and the stability results cited above of Bass and Vaserstein.

DEFINITION (5.9). Let $\mathfrak{q}$ denote a two-sided ideal of A. For $d \geqslant -1$, define $S^dG(\mathfrak{q}) = S^dG(A) \cap G(\mathfrak{q})$.

COROLLARY (5.10). *If* $n \geqslant 3$ *then* $S^dG(\mathfrak{q})$ *is a normal subgroup of* $G(A)$.

Proof. $G(\mathfrak{q}) = \mathrm{Ker}(G(A) \to G(A/\mathfrak{q}))$ is obviously normal in $G(A)$ and $S^dG(A)$ is normal in $G(A)$ by Corollary (5.3).

6. Nilpotent Structure of $\mathrm{GL}_{n \geqslant 3}$: Nilpotent Sandwich Classification Theorem

In this section, the results of Sections 4 and 5 will be extended to the relative case and applied to classifying subgroups of G normalized by congruence, elementary, and relative elementary subgroups.

Throughout this section, it will be assumed that A is quasi-finite, that $\mathfrak{q}$ is a two-sided ideal of A and that $n \geqslant 3$. Let $G = \mathrm{GL}_n$, $S^dG = \mathrm{S}^d\mathrm{L}_n$, and $E = E_n$. Define $G(\mathfrak{q})$, $E(\mathfrak{q})$, and $E(A, \mathfrak{q})$ as in Section 2 and $S^dG(\mathfrak{q})$ as in Section 5. Define $\delta(R)$ as in Section 4, and if z is an integer, let $[z]$ denote the smallest nonnegative integer $\geqslant z$.

THEOREM (6.1). *Suppose* A *is quasi-finite over* R. *If* $\delta(R)$ *is finite then* $D_{SG(A)}^{1+[\delta(R)+2-n]}SG(\mathfrak{q}) = E(A, \mathfrak{q})$.

The proof of Theorem (6.1) will be given after Corollary (6.4).

Suppose J denotes a group and H a normal subgroup. H is called *J-perfect* if $[J, H] = H$. H is called *J-nilpotent* if $D_J^d(H) = 1$ for some d. The smallest d such that $D_J^d(H) = 1$ is called the *J-nilpotent class* of H.

COROLLARY (6.2). *Assume the hypotheses of* (6.1). *Then* $E(A, \mathfrak{q})$ *is the largest* $SG(A)$*-perfect subgroup of* $SG(\mathfrak{q})$ *and the quotient* $SG(\mathfrak{q})/E(A, \mathfrak{q})$ *is* $SG(A)$*-nilpotent of class* $\leqslant 1 + [\delta(d) + 2 - n]$.

Proof. The corollary follows directly from (6.1) and the fact, which is easily deduced from (2.2), that $E(A, \mathfrak{q})$ is $E(A)$-perfect.

COROLLARY (6.3). *Assume the hypotheses of* (6.1). *Then* $E(A, \mathfrak{q})$ *is the largest* $G(A)$*-perfect subgroup of* $G(\mathfrak{q})$ *and the quotient* $G(\mathfrak{q})/E(A, \mathfrak{q})$ *is solvable of degree* $\leqslant 2 + [\delta(R) + 2 - n]$.

Proof. By (3.4), $[G(A), G(\mathfrak{q})] \subset SG(A) \cap G(\mathfrak{q}) = SG(\mathfrak{q})$. By (6.2), $D_{SG(A)}^{1+[\delta(R)+2-n]} \times [G(A), G(\mathfrak{q})] = E(A, \mathfrak{q})$. The assertions of the corollary follow.

COROLLARY (6.4). *A natural transformation* $\sigma\colon S^dG \to S^dG$ $(d \geqslant -1)$ *induces for all* $i \geqslant 0$ *natural transformation* $\sigma_i\colon S^{d+i}G \to S^{d+i}G$ *and* $\rho\colon E \to E$ *with the property that* $(\sigma_i)_A(S^{d+i}G(\mathfrak{q})) \subset S^{d+i}G(\mathfrak{q})$ *and* $\rho_A(E(A, \mathfrak{q})) \subset E(A, \mathfrak{q})$ *for all two-sided ideals* $\mathfrak{q}$ *of* A.

Proof. Let $\sigma\colon S^dG \to S^dG$ denote a natural transformation. By (5.4), σ induces a natural transformation $\sigma_i\colon S^{d+i}G \to S^{d+i}G$. From the commutative diagram

$$\begin{tikzcd} S^{d+i}G(A) \arrow[r] \arrow[d, "(\sigma_i)_A"'] & S^{d+i}G(A/\mathfrak{q}) \arrow[d, "(\sigma_i)_{(A/\mathfrak{q})}"] \\ S^{d+i}G(A) \arrow[r] & S^{d+i}G(A/\mathfrak{q}) \end{tikzcd}$$

and the fact that

$$S^{d+i}G(\mathfrak{q}) = \operatorname{Ker}(S^{d+i}G(A) \to S^{d+i}G(A/\mathfrak{q})),$$

it follows that $(\sigma_i)_A(S^{d+i}G(\mathfrak{q})) \subset S^{d+i}G(\mathfrak{q})$.

Let $\sigma\colon S^dG \to S^dG$ denote a natural transformation. By (5.4), σ induces a natural transformation $\rho\colon E \to E$. Let $A \ltimes \mathfrak{q}$ denote the smash product of A and $\mathfrak{q}$. By (3.2), $A \ltimes \mathfrak{q}$ is quasi-finite. Let $f\colon A \ltimes \mathfrak{q} \to A$, $(a, q) \mapsto a$. The homomorphism f is split by the homomorphism $A \to A \ltimes \mathfrak{q}$, $a \mapsto (a, 0)$. Use this homomorphism to identify $E(A)$ with its image in $E(A \ltimes \mathfrak{q})$. The group $E(A \ltimes \mathfrak{q})$ is the semidirect product $E(A) \ltimes \operatorname{Ker}(E(f))$ and $\operatorname{Ker}(E(f)) = E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})$. From the commutative diagram

$$\begin{tikzcd} E(A \ltimes \mathfrak{q}) \arrow[r, "E(f)"] \arrow[d, "\rho_{A \ltimes \mathfrak{q}}"'] & E(A) \arrow[d, "\rho_A"] \\ E(A \ltimes \mathfrak{q}) \arrow[r, "E(f)"'] & E(A) \end{tikzcd}$$

it follows that

$$\rho_{A \ltimes \mathfrak{q}}(E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})) \subset E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q}).$$

Let $g\colon A \ltimes \mathfrak{q} \to A, (a, q) \mapsto a + q$. $E(g)$ maps $E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})$ bijectively onto $E(A, \mathfrak{q})$. Thus, from the commutative diagram

$$\begin{tikzcd} E(A \ltimes \mathfrak{q}) \arrow[r, "E(g)"] \arrow[d, "\rho_{A \ltimes \mathfrak{q}}"'] & E(A) \arrow[d, "\rho_A"] \\ E(A \ltimes \mathfrak{q}) \arrow[r, "E(g)"'] & E(A) \end{tikzcd}$$

it follows that $\rho_A(E(A, \mathfrak{q})) \subset E(A, \mathfrak{q})$.

Proof of Theorem (6.1). Let $p = [\delta(R) + 2 - n]$. The proof is by induction on p and is similar to that of Theorem (4.1).

Suppose $p = 0$. By (5.7), [3, V(4.2)], and [15, (3.2)], $[G(A), G(\mathfrak{q})] = E(A, \mathfrak{q})$. Thus,

$[SG(A), SG(\mathfrak{q})] = E(A, \mathfrak{q})$, because $SG(A) \supset E(A)$, $SG(\mathfrak{q}) \supset E(A, \mathfrak{q})$, and by (2.2) $[E(A), E(A, \mathfrak{q})] = E(A, \mathfrak{q})$.

Suppose $p > 0$. Let $\sigma, \sigma_1, \ldots, \sigma_p \in SG(A)$ and let $\tau \in SG(\mathfrak{q})$. Let $\rho = [\sigma_1, \ldots, [\sigma_{p-1}, [\sigma_p, \tau]] \cdots]$. One must show that $[\sigma, \rho] \in E(A, \mathfrak{q})$. Let $X_1 \cup \cdots \cup X_r$ be a decomposition of $\operatorname{Max}(R)$ into irreducible Noetherian subspaces of dimension $\leqslant \delta(R)$. Let $\mathfrak{m}_k \in X_k$ $(1 \leqslant k \leqslant r)$. Let R' and A' denote, respectively, the localizations of R and A at the multiplicative set $R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$. A' is semilocal. Since $\sigma \in SG(A)$, its value in $G(A')$ lies in the subgroup $E(A')$. There is an element $s \in R - \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r$ such that the value of σ in $G(\langle s \rangle^{-1}A)$ lies in $E(\langle s \rangle^{-1}A)$. Thus, $\sigma \in G(s^{-1}, A)$. Since the element s satisfies by construction the hypotheses of the Induction Lemma (4.17), $\delta(\tilde{R}_s) < \delta(R)$. Thus, by induction on p, the value of ρ in $G(\tilde{A}_s)$ lies in $E(\tilde{A}_s, \tilde{\mathfrak{q}}_s)$ where $\tilde{\mathfrak{q}}_s$ denotes the ideal of $\tilde{A}_s$ generated by the image of $\mathfrak{q}$ in $\tilde{A}_s$. Let $A \ltimes \mathfrak{q}$ be as in the proof of (6.4). Let

$$f\colon A \ltimes \mathfrak{q} \to A,\ (a, q) \mapsto a + q \quad \text{and} \quad \tilde{f}\colon \tilde{A}_s \ltimes \tilde{\mathfrak{q}}_s \to \tilde{A}_s, \quad (\tilde{a}, \tilde{q}) \mapsto \tilde{a} + \tilde{q}.$$

f is split by $\xi\colon A \to A \ltimes \mathfrak{q} \to A$, $a \to (a, 0)$. One shows easily that Gf maps $G(0 \ltimes \mathfrak{q})$ bijectively onto $G(\mathfrak{q})$. Let ξ' denote the inverse of this isomorphism. I shall show that $[(G\xi(\sigma), \xi'(\rho)] \in E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})$. Clearly, $Gg(\sigma) \in G(s^{-1}, A \ltimes \mathfrak{q})$. From the commutative diagram

$$\begin{array}{ccc} G(A \ltimes \mathfrak{q}) & \longrightarrow & G(\tilde{A}_s \ltimes \tilde{\mathfrak{q}}_s) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle G(\tilde{f})} \\ G(A) & \longrightarrow & G(\tilde{A}_s) \end{array}$$

and the fact that $G(\tilde{f})$ maps $E(\tilde{A}_s \ltimes \tilde{\mathfrak{q}}_s, 0 \ltimes \tilde{\mathfrak{q}}_s)$ isomorphically onto $E(\tilde{A}_s, \tilde{\mathfrak{q}}_s)$, it follows that the image of $g'(\rho)$ in $G(\tilde{A}_s \ltimes \tilde{\mathfrak{q}}_s)$ lies in $E(\tilde{A}_s \ltimes \tilde{\mathfrak{q}}_s, 0 \ltimes \tilde{\mathfrak{q}}_s)$. Thus, $g'(\rho) \in G(\hat{s}, A \ltimes \mathfrak{q})$. Now, by Theorem (4.16), $[Gg(\sigma), g'(\rho)] \in E(A \ltimes \mathfrak{q})$. But, $g'(\rho) \in G(0 \ltimes \mathfrak{q})$ and thus,

$$[Gg(\sigma), g'(\rho)] \in G(0 \ltimes \mathfrak{q}) \cap E(A \ltimes \mathfrak{q}) = E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q}).$$

Moreover,

$$(Gf)(Gg)(\sigma) = \sigma, \qquad (Gf)g'(\rho) = \rho \quad \text{and} \quad Gf(E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})) = E(A, \mathfrak{q}).$$

Thus, $[\sigma, \rho] \in E(A, \mathfrak{q})$.

Suppose J denotes a group and H a normal subgroup. Call a sequence $H = H_0 \supset H_1 \supset H_2 \supset \cdots$ of subgroups of H a *descending J-central series* if for each i, $[J, H_i] \subset H_{i+1}$.

THEOREM (6.5). *Suppose A is quasi-finite over R. Then the sequence*

$$SG(\mathfrak{q}) = S^0G(\mathfrak{q}) \supset S^1G(\mathfrak{q}) \supset \cdots \supset S^dG(\mathfrak{q}) \supset \cdots$$

is a descending $SG(A)$-central series in $SG(\mathfrak{q})$ and $[G(\mathfrak{q}), S^dG(\mathfrak{q})] \subset E(A, \mathfrak{q})$ whenever $\delta(R)$ is finite and $d \geqslant \delta(R)$.

Proof. Let $\sigma \in SG(A)$ and $\rho \in S^dG(\mathfrak{q})$. Clearly, $[\sigma, \rho] \in G(\mathfrak{q})$ and by Theorem (5.5), $[\sigma, \rho] \in S^{d+1}G(A)$. Thus, $[\sigma, \rho] \in S^{d+1}G(A) \cap G(\mathfrak{q}) = S^{d+1}G(\mathfrak{q})$.

By Theorem (5.5), $S^dG(\mathfrak{q}) \subset S^dG(A) = E(A)$ whenever $d \geqslant \delta(R)$ and by the next theorem $[G(\mathfrak{q}), E(A)] = E(A, \mathfrak{q})$. Thus, $[G(\mathfrak{q}), S^dG(\mathfrak{q})] \subset E(A, \mathfrak{q})$ whenever $d \geqslant \delta(R)$.

THEOREM (6.6). $[E(A), G(\mathfrak{q})] = E(A, \mathfrak{q})$.

The proof of Theorem (6.6) and other results use the following lemma.

LEMMA (6.7). *Let* $f\colon A \ltimes \mathfrak{q} \to A$, $(a, q) \mapsto a + q$. *Then*:

(a) *The homomorphism* $Gf\colon G(A \ltimes \mathfrak{q}) \to G(A)$ *is split surjective and maps* $G'(0 \ltimes \mathfrak{q})$ *bijectively onto* $G'(\mathfrak{q})$, $G(0 \ltimes \mathfrak{q})$ *bijectively onto* $G(\mathfrak{q})$, *and* $E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})$ *bijectively onto* $E(A, \mathfrak{q})$.

(b) $E(A \ltimes \mathfrak{q}) = E(A \ltimes A, A \ltimes \mathfrak{q})$ *and* $E(A \ltimes \mathfrak{q}) \cap G(0 \ltimes \mathfrak{q}) = E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q})$.

Proof. Straightforward.

Proof of Theorem (6.6). From (2.2) it follows that $E(A, \mathfrak{q})$ is $E(A)$-perfect. Thus,

$$[E(A), G(\mathfrak{q})] \supset [E(A), E(A, \mathfrak{q})] = E(A, \mathfrak{q}).$$

It remains to show that $[E(A), G(\mathfrak{q})] \subset E(A, \mathfrak{q})$. By Theorem (4.12),

$$[E(A \ltimes \mathfrak{q}), G(0 \ltimes \mathfrak{q})] \subset E(A \ltimes \mathfrak{q}) \cap G(0 \ltimes \mathfrak{q}).$$

The theorem follows now from the lemma above.

COROLLARY (6.8). $E(A, \mathfrak{q})$ *is a normal subgroup of* $G(A)$.

Proof. The result follows from Theorems (4.12) and (6.6).

DEFINITION (6.9). Make an exception to the premises of this section by letting $n \geqslant 2$. Let $d \geqslant 0$. If A is a finite R-algebra define $S^dG(A, \mathfrak{q}) = \{\sigma \mid \sigma \in S^dG(\mathfrak{q})$, value of σ under $Gf\colon G(A) \to G(A')$ lies in $E(A', \mathfrak{q}')$, $f\colon A \to A'$ a homomorphism of finite algebras, $\mathfrak{q}' = A'f(\mathfrak{q})A'$, A' quasi-finite over R', $\delta(R')$ finite, $\delta(R') \leqslant d\}$. Extend the definition of $S^dG(A, \mathfrak{q})$ to all quasi-finite algebras via Lemma (3.3). Define $S^{-1}G(A, \mathfrak{q}) = G(\mathfrak{q})$.

COROLLARY (6.10). $S^dG(A, \mathfrak{q})$ *is a normal subgroup of* $G(A)$.

Proof. The assertion for $d = -1$ is clear. For $d \geqslant 0$, the assertion follows from the fact, given in (6.8), that $E(A', \mathfrak{q}')$ is a normal subgroup of $G(A')$ for any quasi-finite algebra A' and any two-sided ideal $\mathfrak{q}'$ of A'

COROLLARY (6.11). *A natural transformation* $\sigma\colon S^dG \to S^dG$ *induces for all* $i \geqslant 0$, *natural transformations* $\sigma_i\colon S^{d+i}G \to S^{d+i}G$ *with the property that* $(\sigma_i)_A S^{d+i}G(A, \mathfrak{q}) \subset S^{d+i}G(A, \mathfrak{q})$ *for all quasi-finite algebras* A *and all two-sided ideals* $\mathfrak{q}$ *of* A.

Proof. The corollary follows from (6.4).

THEOREM (6.12).

$$[G(A), G(\mathfrak{q})] \subset S^{n-2}G(A, \mathfrak{q}) \text{ and } [S^dG(A), G(\mathfrak{q})] \subset S^dG(A, \mathfrak{q}).$$

Proof. Recall that $G = \mathrm{GL}_n$. To prove the first assertion, it sufficies to show that if A is finite over R and $\delta(R) \leqslant n-2$ then $[G(A), G(\mathfrak{q})] = E(A, \mathfrak{q})$. But, this is an immediate consequence of results of Bass [3, V(4.2), (4.5)] and Vaserstein [15, (3.2), (3.3)].

The first assertion of the theorem infers the cases $d = -1, 0$ and 1 of the second assertion, since $n \geqslant 3$. The case $d \geqslant 0$ of the second assertion follows from Theorem (6.6).

THEOREM (6.13). *Suppose A is quasi-finite over R. Then the sequence* $SG(\mathfrak{q}) = S^{\circ}G(\mathfrak{q}) \supset S^{\circ}G(A, \mathfrak{q}) \supset S^1G(A, \mathfrak{q}) \supset \cdots \supset S^dG(A, \mathfrak{q}) \supset \cdots$ *is a descending $SG(A)$-central series and $S^dG(A, \mathfrak{q}) = E(A, \mathfrak{q})$ whenever $\delta(R)$ is finite and $d \geqslant \delta(R)$.*

The proof of Theorem (6.13) will use the following lemma.

LEMMA (6.14). *The homomorphism $A \ltimes \mathfrak{q} \to A$, $(a, q) \mapsto a + q$, induces an isomorphism $S^dG(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q}) \underset{\rightarrow}{\cong} S^dG(A, \mathfrak{q})$.*

Proof. The result follows from (6.8).

Remark (6.15). For $d \geqslant 0$, it is probably not true in general that the homomorphism $S^dG(0 \ltimes \mathfrak{q}) \to S^dG(\mathfrak{q})$ is surjective.

Proof of Thoerem (6.13). By Theorem (6.12), $[SG(A), S^0G(\mathfrak{q})] \subset S^0G(A, \mathfrak{q})$ and by definition, $S^dG(A, \mathfrak{q}) = E(A, \mathfrak{q})$ whenever $\delta(R) \leqslant d$. It remains to prove that for $d \geqslant 0$, $[SG(A), S^dG(A, \mathfrak{q})] \subset S^{d+1}G(A, \mathfrak{q})$. The proof is similar to that of Theorem (5.5).

As in the proof of Theorem (5.5), one reduces to showing that if $\sigma \in SG(A)$, $\rho \in S^dG(A, \mathfrak{q})$, and $\delta(R) \leqslant d+1$ then $[\sigma, \rho] \in E(A, \mathfrak{q})$ and next, one constructs an element $s \in R$ such that $\sigma \in G(s^{-1}, A)$ and the value of $\rho \in G(\tilde{A}_s)$ lies in $E(\tilde{A}_s, \tilde{\mathfrak{q}}_s)$. Let

$$f\colon A \ltimes \mathfrak{q} \to A, \quad (a, q) \mapsto a + q \quad \text{and} \quad g\colon A \to A \ltimes \mathfrak{q}, \quad a \mapsto (a, 0).$$

Let g' denote the inverse of the isomorphism $Gf\colon G(0 \ltimes \mathfrak{q}) \underset{\rightarrow}{\cong} G(\mathfrak{q})$ in (6.7a). It is clear that $Gg(\sigma) \in G(s^{-1}, A \ltimes \mathfrak{q})$ and by (6.7a), that $g'(\rho) \in G(\hat{s}, A \ltimes \mathfrak{q})$. Thus, by Theorem (4.16), $[Gg(\sigma), g'(\rho)] \in E(A \ltimes \mathfrak{q})$. Since $g'(\rho) \in G(0 \ltimes \mathfrak{q})$, $[Gg(\sigma), g'(\rho)] \in G(0 \ltimes \mathfrak{q})$. Thus

$$[Gg(\sigma), g'(\rho)] \in E(A \ltimes \mathfrak{q}) \cap G(0 \ltimes \mathfrak{q}) = (\text{by (6.7b)})\ E(A \ltimes \mathfrak{q}, 0 \ltimes \mathfrak{q}).$$

Thus, by (6.7a), $[\sigma, \rho] = Gf([Gg(\sigma), g'(\rho)]) \in E(A, \mathfrak{q})$.

THEOREM (6.16). *Let i denote an integer $\geqslant 0$. If A satisfies Bass' stable range condition $SR_N(A)$ then the canonical homomorphisms*

$$S^dG_n(\mathfrak{q}) \to S^dG_{n+1}(\mathfrak{q})/S^{d+i}G_{n+1}(\mathfrak{q}), \quad S^dG_n(\mathfrak{q}) \to S^dG_{n+1}(\mathfrak{q})/S^{d+i}G_{n+1}(A, \mathfrak{q}),$$

and

$$S^dG_n(A, \mathfrak{q}) \to S^dG_{n+1}(A, \mathfrak{q})/S^{d+i}G_{n+1}(A, \mathfrak{q})$$

are surjective for $n \geqslant \max(N-1, d+2, 2)$ *and have, respectively, kernels* $S^{d+i}G_n(\mathfrak{q})$, $S^{d+i}G_n(A, \mathfrak{q})$, *and* $S^{d+i}G_n(A, \mathfrak{q})$ *for* $n \geqslant \max(d+i+2, 2)$.

Proof. The proof is similar to that of Theorem (5.6). Details are left to the reader.

Without any stable range conditions on A, one still has the following injective stability result.

THEOREM (6.17). *Let* i *denote an integer* $\geqslant 0$. *If* $n \geqslant \max(d+i+2, 2)$ *then the kernels of the homomorphisms in* (6.16) *are as in the conclusion of* (6.16).

Proof. The proof is similar to that of Theorem (5.8). Details are left to the reader.

I turn now to classifiying subgroups of $G(A)$ normalized by relative elementary subgroups or congruence subgroups. The following definitions are taken from [1, §3].

DEFINITION (6.18). Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{q}$ denote two-sided ideals of A. Define $\mathfrak{a} <_{\mathfrak{q}} \mathfrak{b}$ if for some $i \geqslant 0$, $\mathfrak{q}^i\mathfrak{a} + \mathfrak{a}\mathfrak{q}^i \subset \mathfrak{b}$. Define $\mathfrak{a} \diamond_{\mathfrak{q}} \mathfrak{b}$ if $\mathfrak{a} <_{\mathfrak{q}} \mathfrak{b}$ and $\mathfrak{b} <_{\mathfrak{q}} \mathfrak{a}$. Clearly, $\diamond_{\mathfrak{q}}$ is an equivalence relation on the set of all two-sided ideals of A.

DEFINITION (6.19). Let H and K denote subgroups of $G(A)$ which are normalized by $E(A, \mathfrak{q})$. Define $H <_{\mathfrak{q}} K$ if for some $i \geqslant 0$, $D^i_{E(A, \mathfrak{q})}(H) \subset K$. Define $H \diamond_{\mathfrak{q}} K$ if $H <_{\mathfrak{q}} K$ and $K <_{\mathfrak{q}} H$. Clearly, $\diamond_{\mathfrak{q}}$ is an equivalence relation on the set of all subgroups of $G(A)$ which are normalized by $E(A, \mathfrak{q})$.

Let $(\mathfrak{a} : \mathfrak{q}) = \{x \mid x \in A, x\mathfrak{q} + \mathfrak{q}x \subset \mathfrak{a}\}$.

THEOREM (6.20) (F.-A. Li and M.-L. Liu [8]). *Suppose* A *is commutative and* $\mathfrak{q}$ *is a fixed ideal of* A. *Then*:

(a) *The rule* $\mathfrak{a} \mapsto E(A, \mathfrak{a})$ *induces an order preserving isomorphism of the* $\diamond_{\mathfrak{q}}$-*equivalence classes of ideals of* A *onto the* $\diamond_{\mathfrak{q}}$-*equivalence classes of subgroups of* $G(A)$ *normalized by* $E(A, \mathfrak{q})$.

(b) *For any* $i \geqslant 0$, $E(A, \mathfrak{q}^i\mathfrak{a}) \diamond_{\mathfrak{q}} E(A, \mathfrak{a}) \diamond_{\mathfrak{q}} G(\mathfrak{a}) \diamond_{\mathfrak{q}} G'(\mathfrak{a}) \diamond_{\mathfrak{q}} G'(\mathfrak{a}\colon \mathfrak{q}^i)$. *Furthermore, if* H *is normalized by* $E(A, \mathfrak{q})$ *then* $H \diamond_{\mathfrak{q}} E(A, \mathfrak{a})$ *if and only if for some* $i \geqslant 0$, $E(A, \mathfrak{q}^i\mathfrak{a}) \subset H \subset G'(\mathfrak{a}\colon \mathfrak{q}^i)$.

Instead of considering $\diamond_{\mathfrak{q}}$-equivalence classes of subgroups of $G(A)$ normalized by the relative elementary subgroup $E(A, \mathfrak{q})$, it is sometimes more natural to consider $\diamond^{\mathfrak{q}}$-equivalence classes of subgroups of $G(A)$ normalized by the congruence subgroup $SG(\mathfrak{q})$, where $\diamond^{\mathfrak{q}}$ is defined as follows.

DEFINITION (6.21). Let H and K denote subgroups of $G(A)$ which are normalized by $SG(\mathfrak{q})$. Define $H <^{\mathfrak{q}} K$ if for some $i \geqslant 0$, $D^i_{SG(\mathfrak{q})}(H) \subset K$. Define $H \diamond^{\mathfrak{q}} K$ if $H <^{\mathfrak{q}} K$ and $K <^{\mathfrak{q}} H$.

Clearly, $\diamond^{\mathfrak{q}}$ is an equivalence relation on the set of all subgroups of $G(A)$ which are normalized by $SG(\mathfrak{q})$.

THEOREM (6.22) *Suppose A is commutative and $\mathfrak{q}$ is a fixed ideal of A. If $\delta(A)$ is finite then*:

(a) *The rule $\mathfrak{a} \to SG(\mathfrak{a})$ induces an order preserving isomorphism of the $\diamond_{\mathfrak{q}}$-equivalence classes of ideals of A onto the $\diamond^{\mathfrak{q}}$-equivalence classes of subgroups of $G(A)$ normalized by $SG(\mathfrak{q})$.*

(b) *For any $i \geqslant 0$,*

$$E(A, \mathfrak{q}^i\mathfrak{a}) \diamond^{\mathfrak{q}} E(A, \mathfrak{a}) \diamond^{\mathfrak{q}} SG(\mathfrak{a}) \diamond^{\mathfrak{q}} G(\mathfrak{a}) \diamond^{\mathfrak{q}} G'(\mathfrak{a}) \diamond^{\mathfrak{q}} G'((\mathfrak{a}\colon \mathfrak{q}^i)).$$

Furthermore, if H is normalized by $SG(\mathfrak{q})$ then $H \diamond^{\mathfrak{q}} SG(\mathfrak{a})$ if and only if for some $i \geqslant 0$, $E(A, \mathfrak{q}^i\mathfrak{a}) \subset H \subset G'(\mathfrak{a}\colon \mathfrak{q}^i))$.

Proof. Theorem (6.22) will follow from Theorem (6.21), once it is established that the equivalence relation $\diamond_{\mathfrak{q}}$ restricted to subgroups of $G(A)$ which are normalized by $SG(\mathfrak{q})$ is the same as the equivalence relation $\diamond^{\mathfrak{q}}$. This is the assertion of Lemma (6.24) below.

LEMMA (6.23). *Suppose that A is quasi-finite and that $\mathfrak{a}$ and $\mathfrak{q}$ are two-sided ideals of A. Then $[G(\mathfrak{a}), G(\mathfrak{q})] \subset SG(\mathfrak{a}\mathfrak{q} + \mathfrak{q}\mathfrak{a})$.*

Proof. One shows in a straightforward manner that $[G(\mathfrak{a}), G(\mathfrak{q})] \subset G(\mathfrak{a}\mathfrak{q} + \mathfrak{q}\mathfrak{a})$. The result then follows from (3.6).

LEMMA (6.24). *Suppose A is commutative and $\delta(A)$ is finite. Then the equivalence relation $\diamond_{\mathfrak{q}}$ restricted to subgroups of $G(A)$ normalized by $SG(\mathfrak{q})$ is the same as the equivalence relation $\diamond^{\mathfrak{q}}$.*

Proof. Let H and K be subgroups of $G(A)$ normalized by $SG(\mathfrak{q})$. It is clear that $H <^{\mathfrak{q}} K$ implies $H <_{\mathfrak{q}} K$, because $D^i_{E(A,\mathfrak{q})}(H) \subset D^i_{SG(\mathfrak{q})}(H)$. Thus, it remains to show that $H <_{\mathfrak{q}} K$ implies $H <^{\mathfrak{q}} K$. By Theorem (6.20), there are ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A such that

$$E(A, \mathfrak{q}^i\mathfrak{a}) \subset H \subset G'((\mathfrak{a}\colon \mathfrak{q}^i)) \text{ and } E(A, \mathfrak{q}^i\mathfrak{b}) \subset K \subset G'((\mathfrak{b}\colon \mathfrak{q}^i)).$$

It suffices to show that $G'((\mathfrak{b}\colon \mathfrak{q}^i)) <^{\mathfrak{q}} E(A, \mathfrak{q}^i\mathfrak{a})$. By Theorem (6.20), there is a natural number j such that $\mathfrak{q}^j(\mathfrak{b}\colon \mathfrak{q}^i) \subset \mathfrak{q}^i\mathfrak{a}$. It is enough to prove the following: If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A such that for some j, $\mathfrak{q}^j\mathfrak{b} \subset \mathfrak{a}$, then $G'(\mathfrak{b}) <^{\mathfrak{q}} E(A, \mathfrak{a})$. By induction on j, it suffices to show that $G'(\mathfrak{b}) <^{\mathfrak{q}} E(A, \mathfrak{q}\mathfrak{b})$.

Clearly, $[SG(\mathfrak{q}), G'(\mathfrak{b})] \subset SG(\mathfrak{b})$. By the previous lemma, $[SG(\mathfrak{q}), SG(\mathfrak{b})] \subset SG(\mathfrak{q}\mathfrak{b})$ and by Theorem (6.13), $D^{1+\delta(A)}_{SG(\mathfrak{q})}(SG(\mathfrak{q}\mathfrak{b})) \subset E(A, \mathfrak{q}\mathfrak{b})$.

Let $G'(A, \mathfrak{q}) = \ker(G(A) \to G(A/\mathfrak{q})/\text{center}(G(A/\mathfrak{q}))$.

NILPOTENT SANDWICH CLASSIFICATION THEOREM (6.25). *Let H be a subgroup of $G(A)$. Then H is normalized by $E(A)$ if and only if for a (unique) two-sided ideal $\mathfrak{q}$ of A, H fits into a sandwich $E(A, \mathfrak{q}) \subseteq H \subseteq G'(A, \mathfrak{q})$. Furthermore, the canonical filtration $H = H \cap G'(A, \mathfrak{q}) \supseteq H \cap S^{-1}G(A, \mathfrak{q}) \supseteq H \cap S^0G(A, \mathfrak{q}) \supseteq \cdots \supseteq H \cap E(A, \mathfrak{q}) = E(A, \mathfrak{q})$ has the property that each quotient of consecutive members of the filtration is abelian and any subgroup of $S^0G(A)$, which normalizes H, acts trivially on each of the quotients above via conjugation.*

Proof. It is routine to reduce the proof of the sandwiching assertion to the case of finite algebras, where it follows from [16, Cor. 5, Lemma 8]. The assertions concerning the filtration follow from Theorem (6.13).

COROLLARY (6.26). *Let H be a subgroup of $G(A)$.*

(a) *If $\delta(R)$ is finite then H is normalized by $E(A)$ if and only if for some (unique) two-sided ideal* $\mathfrak{q}$ of A, $D_{SG(A)}^{\delta(R)+2}(H) = E(A, \mathfrak{q})$.

(b) *If $\delta(R)$ is finite then H is normalized by $SG(A)$ if and only if for some (unique) two-sided ideal $\mathfrak{q}$ of A, there is a filtration*

$$H = H \cap G'(A, \mathfrak{q}) \supseteq H \cap S^{-1}G(A, \mathfrak{q}) \supseteq H \cap S^{0}G(A, \mathfrak{q})$$
$$\supseteq \cdots \supseteq H \cap S^{\delta(R)}G(A, \mathfrak{q}) = E(A, \mathfrak{q})$$

such that the action of $SG(A)$ on each quotient of consecutive members of the filtration above is trivial.

Proof. (a) follows straightforward from Theorems (6.13) and (6.25) and the fact, cf. [1, (3.17)], that if N is a subgroup of $G(A)$ then $[E(A), N] \subseteq G(\mathfrak{q})$ if and only if $N \subseteq G'(A, \mathfrak{q})$. (In fact, one can show using Theorem (6.16) that the exponent $\delta(R) + 2$ in the corollary can be replaced by the exponent $2 + [\delta(R) + 2 - n]$.)

(b) is an immediate consequence of (a).

7. Connections to Noncommutative Homotopy Theory: Lower Bounds for the Nilpotent Class of K_1

Whereas the results of Section 4 establish upper bounds for the nilpotent class of $S^0L_n(A)/E_n(A)$, the current section concentrates on lower bounds for this class. The main goal is to provide families of commutative rings R of finite Bass–Serre dimension $\delta(R)$, such that one can compute a nontrivial lower bound to the nilpotent class of $K_1SL_n(R) := SL_n(R)/E_n(R)$. The lower bound will be obtained by exhibiting a topological space X and a canonical homomorphism $K_1SL_n(R) \to [X, SL_n(F)]$, where F denotes the real or complex numbers and $[X, SL_n(F)]$ the group of homotopy classes of continuous maps from X to $SL_n(F)$, such that the nilpotent class of the image of this homomorphism has a computable lower bound. This bound shows that the groups $SL_n(R)/E_n(R)$ can be nonabelian and is used to construct examples of nonnormal subgroups of $SL_n(R)$ which are normalized by $E_n(R)$.

Let G denote an H-Group [12], e.g. a topological group, with a closed, nondegenerate base point [12] representing an identity element of G. Later, G will be specialized to $SL_n(F)$.

DEFINITION (7.1). *If for some nonnegative integer k and for all topological spaces X with a nondegenerate point [12] (e.g. X a CW-complex), the group $[X, G]$ of homotopy classes of continuous maps from X to G has nilpotent class $\leqslant k$ then by definition the global homotopical nilpotent class $\gamma(G)$ of G is the smallest such k; otherwise, $\gamma(G) = \infty$.*

Below, classical results of R. Bott, W. Browder, I. James and E. Thomas on the homotopy commutivity of compact, connected Lie groups will be used to show the following result.

THEOREM (7.2). *Suppose G is a compact, connected Lie group. Then*

$$\gamma(G) = \begin{cases} 0, & \text{if } G = 1, \\ 1, & \text{if } G = \text{a torus}, \\ > 1, & \text{otherwise}. \end{cases}$$

Before proving the theorem, I shall apply it to prove the results for $K_1\mathrm{SL}_n$ described in the first paragraph.

Let A be any commutative topological ring containing the real numbers $\mathbb{R}$ with the usual topology. Let $\mathbb{M}_n(A)$ denote the ring of all $n \times n$-matrices with coefficients in A with the product topology and give $\mathrm{SL}_n(A)$ the subspace topology.

LEMMA (7.3) *$E_n(A)$ is contained in the path component of the identity matrix in $\mathrm{SL}_n(A)$.*

Proof. Let $[0, 1]$ denote the real closed interval $\{t \mid t \in \mathbb{R}, 0 \leqslant t \leqslant 1\}$. If $a_1, \ldots, a_k \in A$ then the function

$$\omega\colon [0, 1] \to E_n(A), \quad t \mapsto \varepsilon_{i_1 j_1}(ta_1) \cdots \varepsilon_{i_k j_k}(ta_k)$$

is a path from 1 to $\varepsilon_{i_1 j_1}(a_1) \ldots \varepsilon_{i_k j_k}(a_k)$.

Let X be a topological space. If F is a commutative topological ring containing $\mathbb{R}$, let $F(X)$ denote the ring of continuous functions f on X with values in F with the compact-open topology. Let $(X, \mathrm{SL}_n(F))$ denote the group of all continuous maps from X to $\mathrm{SL}_n(F)$. There is a canonical isomorphism

$$\mathrm{SL}_n(F(X)) \to (X, \mathrm{SL}_n(F)), \; ((f_{ij}) \mapsto (x \mapsto (f_{ij}(x)))$$

of topological groups and by Lemma (7.3) the kernel of the composite homomorphism

$$\mathrm{SL}_n(F(X)) \to (X, \mathrm{SL}_n(F)) \twoheadrightarrow [X, \mathrm{SL}_n(F)]$$

contains the subgroup $E_n(F(X))$. Thus, for any subring $R \subset F(X)$, there is a canonical homomorphism $K_1\mathrm{SL}_n(R) \to [X, \mathrm{SL}_n(F)]$.

Suppose now that $F = \mathbb{R}$ or $\mathbb{C}$ so that $\mathrm{SL}_n(F)$ is a compact, connected Lie group. By Theorem (7.2), $\gamma(\mathrm{SL}_n(F)) > 1$ providing $n \geqslant 3$. Let k be a natural number such that $1 < k \leqslant \gamma(\mathrm{SL}_n(F))$. Choose X such that the nilpotent class of $[X, \mathrm{SL}_n(F)]$ is $\geqslant k$ and let $\alpha_1, \ldots, \alpha_k$ be matrices in $\mathrm{SL}_n(F(X))$ such that the value of the $(k-1)$-iterated commutator $[\alpha_k, [\alpha_{k-1}, \ldots, [\alpha_2, \alpha_1] \cdots]]$ in $[X, \mathrm{SL}_n(F)]$ is not trivial. If R denotes the subring of $F(X)$ generated over F by the coefficients of $\alpha_1, \ldots, \alpha_k$ then clearly the image of the $(k-1)$-iterated commutator above in $K_1\mathrm{SL}_n(R)$ is not trivial and since R is finitely generated as an F-algebra by kn^2 elements, it follows that $\delta(R) \leqslant kn^2$. Thus, we have shown:

COROLLARY (7.4) *Let n and k be natural numbers such that $n \geqslant 3$ and $1 < k \leqslant \gamma(\mathrm{SL}_n(F))$ ($F = \mathbb{R}$ or $\mathbb{C}$). Then there are commutative rings R of dimension $\delta(R) \leqslant kn^2$ such that the nilpotent class of $K_1\mathrm{SL}_n(R)$ is $\geqslant k$.*

CONJECTURE. *For a compact, connected Lie group G with finite fundamental group, $\gamma(G)$ is finite and $\gamma(G) \to \infty$ as rank $(G) \to \infty$.*

A reader, who is familiar with classical results on homotopy commutativity, will see quickly that Theorem (7.2) is a consequence of a theorem of I. James [6, (1.1)]. The proof below is intended for the noninitiate and will recall some standard constructions and facts that are well known to experts.

Suppose X has a nondegenerate point and let such a point be the base point of X. Let G be a connected H-group and let a homotopy identity element in G be the base point for G. There is a canonical homomorphism $[X, G]_* \to [X, G]$ where $[X, G]_*$ denotes the group of equivalence classes under base point preserving homotopy of base point preserving continuous maps $X \to G$ and $[X, G]$ denotes as above the group of equivalence classes under homotopy of continuous maps $X \to G$. By a well known result, cf. [12, Chap. 7, Sec. 3, Theorem 5], the canonical homomorphism above is an isomorphism, because G is a connected H-space. This isomorphism will be used to identify the two groups above and for the rest of this paper, *it will be assumed that all continuous maps and homotopies are base point preserving.*

Let $\psi_0: G \to G$ denote the identity map $\alpha \mapsto \alpha$ on G, let $\psi_1: G \times G \to G$ denote the commutator map

$$(g_2, g_1) \mapsto [g_2, g_1] = g_2 g_1 g_2^{-1} g_1^{-1},$$

and for any $k > 0$ let

$$\psi_k: \underbrace{G \times \cdots \times G}_{k+1} \to G \text{ denote the } k\text{-iterated commutator map}$$

$$(g_{k+1}, \ldots, g_1) \mapsto [g_{k+1}, [g_k, \ldots, [g_2, g_1] \ldots]].$$

Obviously, ψ_k ($k \geqslant 0$) is an element of the H-group $(\underbrace{G \times \cdots \times G}_{k+1}, G)$ of all base point preserving continuous maps $\underbrace{G \times \cdots \times G}_{k+1} \to G$ where $\underbrace{G \times \cdots \times G}_{k+1}$ has as base point the $(k+1)$-tuple $(e, \ldots, e)$ where e is the base point of G. An important observation is the following: If

$$p_i: \underbrace{G \times \cdots \times G}_{k+1} \to G \ (1 \leqslant i \leqslant k+1)$$

denotes the projection map on the ith coordinate then each p_i is a member of the H-group $(G \times \cdots \times G, G)$ and ψ_k is the k-iterated commutator $[p_{k+1}, [p_k, \ldots, [p_2, p_1] \cdots]]$. If for some k, ψ_k is homotopic to the constant map, let $\gamma'(G)$ denote the smallest such k; otherwise, let $\gamma'(G) = \infty$.

LEMMA (7.5). *Let G be a connected H-group with nondegenerate base point. Then $\gamma(G) = \gamma'(G)$.*

Proof. Trivially, $\gamma(G) \geqslant \gamma'(G)$. Conversely, if $\gamma'(G) = 0$, then the base point of G is a strong deformation retract of G. Thus, $\gamma(G) = 0$. Suppose $\gamma'(G)$ is finite and >0. Let $k = \gamma'(G)$. If $f_{k+1}, \ldots, f_1 : X \to G$ are base point preserving continuous maps, one must show that the k-iterated commutator map

$$c : X \to G, \; x \mapsto [f_{k+1}(x), [f_k(x), \ldots, [f_2(x), f_1(x)] \cdots]]$$

is homotopic to the constant map. But c factors as a product $\psi_k f$ where

$$f : X \to \underbrace{G \times \cdots \times G}_{k+1}, \; x \mapsto (f_{k+1}(x), \ldots, f_1(x)).$$

Since ψ_k is homomotopic to the constant map, so is $\psi_k f$. Thus, $\gamma(G) \leqslant \gamma'(G)$.

Let $k \geqslant 0$. Let $\bigwedge^{k+1} X$ denote the $(k+1)$-fold smash product of X. By definition, $\bigwedge^{k+1} X$ is the quotient of the $(k+1)$-fold product

$$\underbrace{X \times \cdots \times X}_{k+1}$$

obtained by identifying the subspace $\bigvee^{k+1} X = \{(x_{k+1}, x_k, \ldots, x_1) |$ at least one x_j $(1 \leqslant j \leqslant k+1)$ is the base point of $X\}$ to a point. The map

$$\psi_k : \underbrace{G \times \cdots \times G}_{k+1} \to G$$

restricted to $\bigvee^{k+1} G$ is homotopic to the constant map. Since the base point of G is closed and nondegenerate, it follows (cf. [12, Chap. 1, Exercise E7]) that ψ_k is homotopic to a map which induces naturally a map $\phi_k : \bigwedge^{k+1} G \to G$. If for some k, ϕ_k is homotopic to the constant map, let $\gamma''(G)$ denote the smallest such k; otherwise, let $\gamma''(G) = \infty$.

LEMMA (7.6). *Let G be a connected H-group with a nondegenerate, closed base point. Then ϕ_k is homotopic to the constant map if and only if ψ_k is. In particular, $\gamma''(G) = \gamma'(G)$.*

Proof. This is an immediate consequence of the cofibre sequence [10].

Proof of Theorem (7.2). By Lemmas (7.5) and (7.6), $\gamma(G) = \gamma''(G)$ and by *I.* James [6, (1.1)], $\gamma''(G)$ is as in the assertion of Theorem (7.2).

There is only one compact, simply connected Lie group G, namely S^3, whose γ-class $\gamma(G)$ is known. The result is given below and follows essentially from work of P. Hilton [5]. I am indebted to H. Baues for explaining to me Samelson and Whitehead products which play a role in the computation of $\gamma(S^3)$.

PROPOSITION (7.7). *If S^3 denotes the real 3-sphere then $\gamma(S^3) = 3$.*

The demonstration of Proposition (7.7) is made, as in the case of Theorem (7.2), with the nonexpert in mind.

Let X and Y be pointed, topological spaces. Let $\Sigma Y = S^1 \wedge Y$ denote the *reduced suspension* of Y and let $\Omega Y = (S^1, Y)$ denote the *loop space* of Y. Σ and Ω define functors from the category of point topological spaces to the category of pointed topological spaces. The exponential law, cf. [12], establishes a homeomorphism *ex*: $(\Sigma X, \Sigma Y) \xrightarrow{\cong} (X, \Omega\Sigma Y)$ of H-groups such that for any diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "f"] \arrow[dr, "gf"'] & \Sigma Y \arrow[d, "g"] \\ & \Sigma Z \end{tikzcd}$$

the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{ex}(f)"] \arrow[dr, "\mathrm{ex}(gf)"'] & \Omega\Sigma Y \arrow[d, "\Omega(g)"] \\ & \Omega\Sigma Z \end{tikzcd}$$

commutes. In particular,

(7.8) *Any map* $G\colon X \to \Omega\Sigma Y$, $G = \mathrm{ex}(g)$, *factors as a product* $\Omega(g)\,\mathrm{ex}(1_{\Sigma X})$ *where* $\Omega(g)$ *is an* H*-map* (*i.e. map of* H*-spaces*), *and*

(7.9) *the diagram*

$$\begin{tikzcd} (X, Y) \arrow[rr, "{(X, \mathrm{ex}(1_{\Sigma Y}))}"] \arrow[dr, "\Sigma"'] & & (X, \Omega\Sigma Y) \\ & (\Sigma X, \Sigma Y) \arrow[ur, "\mathrm{ex}"'] & \end{tikzcd}$$

commutes.

Let $\mathbb{H}$ denote the quaternions with the usual norm. Thus, S^3 can be identified with the elements of norm 1 in $\mathbb{H}$ and the group operation on S^3 is given by multiplication in $\mathbb{H}$. Let

$$\mathbb{H}E^{k+1} = \underbrace{\mathbb{H} \times \cdots \times \mathbb{H}}_{k+1}$$

with the usual norm and let $\mathbb{H}S^k$ denote the elements of norm 1 in $\mathbb{H}E^{k+1}$. S^3 operates on $\mathbb{H}S^k$ by scalar multiplication and by definition, *quaternionic projective space* $\mathbb{H}P^k$ is the orbit space of S^3 acting on $\mathbb{H}S^k$. The embedding

$$\mathbb{H}E^{k+1} \hookrightarrow \mathbb{H}E^{k+2},\ (a_1, \ldots, a_{k+1}) \mapsto (a_1, \ldots, a_{k+1}, 0),$$

preserves scalar multiplication and induces embeddings

$$\mathbb{H}S^k \hookrightarrow \mathbb{H}S^{k+1} \quad \text{and} \quad \mathbb{H}P^k \hookrightarrow \mathbb{H}P^{k+1}.$$

Let

$$\mathbb{H}S^\infty = \varinjlim_k \mathbb{H}S^k \quad \text{and} \quad \mathbb{H}P^\infty = \varinjlim_k \mathbb{H}P^k.$$

The action of S^3 on $\mathbb{H}S^\infty$ is principal, its orbit space is $\mathbb{H}P^\infty$, and the canonical map $\mathbb{H}S^\infty \to \mathbb{H}P^\infty$ is locally trivial with fibre S^3, i.e. $\mathbb{H}S^\infty \to \mathbb{H}P^\infty$ is a principal S^3-bundle. Since $\mathbb{H}S^\infty$ is contractible, there is by a theorem of Samelson [11, Theorem I], an H-map s: $S^3 \to \Omega\mathbb{H}P^\infty$, which is a weak homotopy equivalence (cf. [12]) and therefore, a homotopy equivalence (cf. [9, Theorem (3, 3)]), since S^3 and $\Omega\mathbb{H}P^\infty$ are CW-spaces. For any k-sphere S^k, let $\iota_k : S^k \to S^k$ denote the identity map. By the cellular approximation theorem (cf. [12] or [9]), one can deform s to a map s': $S^3 \to \Omega S^4 = \Omega\mathbb{H}P^1$ $(\subset \Omega\mathbb{H}P^\infty)$ and by (7.8), s' factors as a product

$$S^3 \xrightarrow{\mathrm{ex}(\iota_4)} \Omega S^4 \xrightarrow{s''} \Omega S^4$$

for some H-map s''. Thus, one obtains the following well known fact:

LEMMA (7.10). *The H-map $h = s^{-1}s''$: $\Omega S^4 \to S^3$ is split* (*but not H-split* [10]) *by the map* $\mathrm{ex}(\iota_4)$: $S^3 \to \Omega S^4 = \Omega\Sigma S^3$.

Next, the definitions of Samelson and Whitehead products are recalled. Let X be a pointed topological space and G an H-group. The *k-iterated Samelson product*

$$S_k : \underbrace{[X, G] \times \cdots \times [X, G]}_{k+1} \to [\underbrace{X \wedge \cdots \wedge X}_{k+1}, G]$$

is defined by

$$S_k(f_{k+1}, \ldots, f_1) = [\bar{f}_{k+1}, [\bar{f}_k, \ldots, [\bar{f}_2, \bar{f}_1] \cdots]]$$

where

$$\bar{f}_i = f_i p_i,\ p_i : \underbrace{X \times \cdots \times X}_{k+1} \to X$$

is the projection on the ith coordinate, and [,] is the standard commutator bracket, cf. Baues [4]. S_k depends on the bracketing, e.g. $[\bar{f}_3, [\bar{f}_2, \bar{f}_1]]$ is not in general equal to $[[\bar{f}_3, \bar{f}_2], \bar{f}_1]$, and if X is the suspension $\Sigma X'$ of some space X' then S_k is $(k + 1)$-multiplicative. Let Y be a pointed topological space. The *k-iterated Whitehead product*

$$W_k : \underbrace{[\Sigma X, Y] \times \cdots \times [\Sigma X, Y]}_{k+1} \to [\Sigma(\underbrace{X \wedge \cdots \wedge X}_{k+1}), Y]$$

is defined as the composite

$$\begin{aligned}[\Sigma X, Y] \times \cdots \times [\Sigma X, Y] &\overset{\mathrm{ex}}{\underset{\cong}{\longrightarrow}} [X, \Omega Y] \times \cdots \times [X, \Omega Y] \xrightarrow{S_k} [X \wedge \cdots \wedge X, \Omega Y] \\ &\overset{\mathrm{ex}}{\underset{\cong}{\longrightarrow}} [\Sigma(X \wedge \cdots \wedge X), Y],\end{aligned}$$

cf. Baues [4]. As above, W_k depends on the bracketing and if X is a suspension then W_k is $(k + 1)$-multiplicative.

Since the map $\mathrm{ex}(\iota_4)$: $S^3 \to \Omega S^4$ is not an H-map, the following result is a bit surprising.

LEMMA (7.11). *For $k \geqslant 2$, the diagram below is commutative*

$$\begin{tikzcd}
\pi_4(S^4) \times \cdots \times \pi_4(S^4) \arrow[r, "W_k"] & \pi_{(k+1)4-k}(S^k) \\
\pi_3(S^3) \times \cdots \times \pi_3(S^3) \arrow[r, "S_k"'] \arrow[u, "\Sigma"] & \pi_{(k+1)3}(S^3) \arrow[u, "\Sigma"']
\end{tikzcd}$$

and the suspension homomorphisms Σ are injective.

Proof. Let $h: \Omega S^4 \to S^3$ denote the H-map in Lemma (7.10) and consider the diagram

$$\begin{tikzcd}
\pi_4(S^4) \times \cdots \times \pi_4(S^4) \arrow[r, "W_k"] \arrow[d, "\mathrm{ex}"', "\cong"] & \pi_{(k+1)4-k}(S^4) \arrow[d, "\cong"', "\mathrm{ex}"] \\
\pi_3(\Omega S^4) \times \cdots \times \pi_3(\Omega S^4) \arrow[r, "S_k"] \arrow[d, "h_*"'] & \pi_{(k+1)3}(\Omega S^4) \arrow[d, "h_*"] \\
\pi_3(S^3) \times \cdots \times \pi_3(S^3) \arrow[r, "S_k"] \arrow[uu, dashed, bend left=60, "\Sigma"] & \pi_{(k+1)3}(S^3) \arrow[uu, dashed, bend right=60, "\Sigma"']
\end{tikzcd}$$

By (7.9) and (7.10), the homomorphism ex Σ splits h_*. Thus, Σ splits h_*ex. Thus, the suspension homomorphisms Σ are injective.

To complete the proof of the lemma, it suffices in view of the commutativity of the solid arrows in the diagram above and the fact that Σ splits h_*ex to show that image$(W_k) \subset$ image(Σ). Since W_k is $(k+1)$-additive and $\pi_4(S^4)$ is generated by ι_4, it suffices to show that $W_k(\iota_4, \ldots, \iota_4) \in$ image(Σ). But, for $k \geqslant 2$, this follows from [5, Corollaries (2.4) and (2.5)].

Proof of Proposition (7.7). By Lemmas (7.5) and (7.6), it suffices to show that $\gamma''(S^3) = 3$. Thus it is enough to show that the homotopy class $[\phi_k]$ of ϕ_k is trivial for $k > 2$ and nontrivial for $k = 2$. Clearly, $[\phi_k] = S_k(\iota_3, \ldots, \iota_3)$. By Lemma (7.17), the diagram below is commutative

$$\begin{tikzcd}
(\iota_4, \ldots, \iota_4) \arrow[r, mapsto] & W_k(\iota_4, \ldots, \iota_4) \\
(\iota_3, \ldots, \iota_3) \arrow[r, mapsto] \arrow[u, "\Sigma"] & S_k(\iota_3, \ldots, \iota_3) = [\varphi_k] \arrow[u, "\Sigma"']
\end{tikzcd}$$

and $S_k(\iota_3, \ldots, \iota_3) = 0$ if and only if $W_k(\iota_4, \ldots, \iota_4) = 0$. But, by [5, Corollaries (2.4) and (2.5)], $W_k(\iota_4, \ldots, \iota_4)$ is trivial or not according to whether $k > 2$ or $= 2$.

COROLLARY (7.12). $\gamma(\mathrm{SL}_4(\mathbb{R})) \geqslant 3$.

Proof. Since $\mathrm{SO}_4(\mathbb{R})$ is a deformation retract of $\mathrm{SL}_n(\mathbb{R})$, cf. [7, §2], it suffices to show that $\gamma(\mathrm{SO}_4(\mathbb{R})) \geqslant 3$. By well known results, the action of $\mathrm{SO}_4(\mathbb{R})$ on S^3 induces a semidirect decomposition $\mathrm{SO}_4(\mathbb{R}) = S^3 \ltimes \mathrm{SO}_3(\mathbb{R})$. Thus, $\gamma(\mathrm{SO}_4(\mathbb{R})) \geqslant \gamma(S^3)$ and by Proposition (7.7), $\gamma(S^3) = 3$.

Remark. It is reasonable to hope that with some work, one can compute $\gamma(\mathrm{SO}_3(\mathbb{R}))$ and then using the semidirect decomposition above, compute $\gamma(\mathrm{SO}_4(\mathbb{R}))$.

This might give one sufficient insight to guess and then prove the value of $\gamma(\mathrm{SO}_n(\mathbb{R}))$ for arbitrary n. In a similar vein, using the observation that $S^3 \cong \mathrm{Sp}_1$, one might try by *ad-hoc* methods to compute $\gamma(\mathrm{SU}_3)$ (resp. $\gamma(\mathrm{Sp}_2)$) and then, assuming one has been successful, apply the experience gained to compute $\gamma(\mathrm{SU}_n)$ (resp. $\gamma(\mathrm{Sp}_n)$) for arbitrary n.

The next corollary answers a question of H. Bass concerning the existence of nonnormal subgroups of $\mathrm{GL}_n(A)$ which are normalized by $E_n(A)$.

COROLLARY (7.13). *There are commutative finitely generated $\mathbb{Z}$-algebras R such that $SL_4(R)$ contains nonnormal subgroups normalized by $E_4(R)$.*

Proof. Suppose for some R as above, there are elements $\sigma, \rho \in \mathrm{SL}_4(R)$ such that the order of σ in $K_1\mathrm{SL}_4(R)$ is infinite and such that the commutator $[\rho, \sigma]$ does not vanish in $K_1\mathrm{SL}_4(R)$. The subgroup H of $\mathrm{SL}_4(R)$ generated by σ and $E_4(R)$ is normalized by $E_4(R)$, because $E_4(R)$ is normal in $\mathrm{SL}_4(R)$. I shall show that H is not normalized by ρ. Each element of H can be written as a product $\sigma^i\varepsilon$ for some $i \in \mathbb{Z}$ and $\varepsilon \in E_4(R)$. If H were normalized by ρ then $[\rho, \sigma] \equiv \sigma^i \bmod E_4(R)$ for some $i \neq 0$. Thus, for any $k \geqslant 1$, the k-iterated commutator $[\rho, [\rho, \ldots, [\rho, \sigma] \cdots] \equiv \sigma^{ki} \bmod E_4(R)$. But, since the Bass–Serre dimension $\delta(R)$ is finite, $K_1\mathrm{SL}_n(R)$ is nilpotent by Theorem (4.1) and thus, for some $k \geqslant 1$, $\sigma^{ki} \equiv 1 \bmod E_4(R)$. This contradicts the fact that σ has infinite order in $K_1\,\mathrm{SL}_4(R)$.

Next, I shall construct R, σ, and ρ satisfying the assumptions above. Let σ'' and $\rho''\colon S^3 \times S^3 \times S^3 \to S^3$ denote the projections on the third and second coordinates, respectively, and let σ' and ρ' denote the composition of σ'' and ρ'', respectively, with the canonical map $S^3 \to \mathrm{SO}_4(\mathbb{R})$ given by the semidirect decomposition $\mathrm{SO}_4(\mathbb{R}) = S^3 \ltimes \mathrm{SO}_3(\mathbb{R})$. The proof of Proposition (7.7) shows that the commutator $[\rho', \sigma'] \in [S^3 \times S^3 \times S^3, \mathrm{SO}_4(\mathbb{R})]$ is nontrival (in fact, if $\tau'\colon S^3 \times S^3 \times S^3 \to \mathrm{SO}_4(\mathbb{R})$ denotes the composite of projection on the first coordinate followed by the canonical map $S^3 \to \mathrm{SO}_4(\mathbb{R})$ then the proof of Proposition (7.7) shows that $[\tau', [\rho', \sigma']]$ is still nontrivial). Furthermore, σ' has infinite order in $[S^3 \times S^3 \times S^3, \mathrm{SO}_4(\mathbb{R})]$, because the canonical map $S^3 \to S^3 \times S^3 \times S^3$, $x \mapsto (1, 1, x)$, and the canonical map $\mathrm{SO}_4(\mathbb{R}) \to S^3$ induce a homomorphism $[S^3 \times S^3 \times S^3, \mathrm{SO}_4(\mathbb{R})] \to [S^3, S^3]$ which takes $\sigma' \mapsto 1_{S^3}$ and it is well known that $[S^3, S^3] = \pi_3(S^3)$ is isomorphic to $\mathbb{Z}$ with generator 1_{S^3}. Using the fact that $\mathrm{SO}_4(\mathbb{R})$ is a deformation retract of $\mathrm{SL}_4(\mathbb{R})$ [7, §2], identify $[S^3 \times S^3 \times S^3, \mathrm{SO}_4(\mathbb{R})] \cong [S^3 \times S^3 \times S^3, \mathrm{SL}_4(\mathbb{R})]$ and identify the elements σ' and ρ' with their images in $[S^3 \times S^3 \times S^3, \mathrm{SL}_4(\mathbb{R})]$. Consider now the canonical surjective homomorphism

$$K_1\mathrm{SL}_4(\mathbb{R}(S^3 \times S^3 \times S^3)) \to [S^3 \times S^3 \times S^3, \mathrm{SL}_4(\mathbb{R})],$$

described prior to (7.4). Let σ and ρ be preimages, respectively, of σ' and ρ'. If R denotes the subring of $\mathbb{R}(S^3 \times S^3 \times S^3)$ generated by the coefficients of σ and ρ then it is clear that R, σ, and ρ satisfy the assumptions in the first paragraph of the proof.

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