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Stability for Hermitian K_1

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Abstract

The general Hermitian group GH_{2n} and its elementary subgroup EH_{2n} are the analogs in the theory of Hermitian forms of the general linear group GL_n and its elementary subgroup E_n. This article proves that the canonical map $GH_{2n}/EH_{2n} \longrightarrow GH_{2(n+1)}/EH_{2(n+1)}$ is an isomorphism whenever n is large with respect to a suitable stable range condition for rings with involution.

1 Introduction

An open question since the 1960's is whether stability theorems for K_0 and K_1 of projective modules and quadratic forms have analogs for Hermitian forms. This paper establishes an analog for K_1 and a companion article [BT] an analog for K_0.

The long time required for demonstrating a K_1-analog is explained by the lack of a notion of elementary subgroup in the general Hermitian group, which is necessary in formulating a K_1-stability result. This subgroup was discovered recently by the second author [T].

The general Hermitian group $GH_{2n}(R,\ a_1, \ldots, a_r)$ is the analog in the theory of Hermitian forms of the general linear group $GL_n(R)$ in the theory of projective modules. It is by definition the group of isomorphisms of an orthogonal sum $\mathbb{M}(a_1) \perp \ldots \perp \mathbb{M}(a_n)$ of metabolic planes $\mathbb{M}(a_i)$

where $a_i = 0$ for all $i > r$. There is an obvious stabilization homomorphism $GH_{2n}(R,\ a_1, \ldots, a_r) \longrightarrow GH_{2(n+1)}(R,\ a_1, \ldots, a_r)$. The elementary Hermitian group $EH_{2n}(R,\ a, \ldots, a_r)$ is a subgroup of $GH_{2n}(R,\ a, \ldots, a_r)$, which is generated by certain functorially defined matrices called elementary Hermitian matrices. Some generators are very complex and require many nonzero off diagonal coefficients. The stabilization homomorphism takes $EH_{2n}(R,\ a_1, \ldots, a_r)$ to $EH_{2(n+1)}(R,\ a_1, \ldots, a_r)$. We define $KH_{1,n}(R,\ a_1, \ldots, a_r) = GH_{2n}(R,\ a_1, \ldots, a_r)/EH_{2n}(R,\ a_1, \ldots, a_r)$. A priori $KH_{1,n}(R,\ a_1, \ldots, a_r)$ is just a coset space. The stabilization homomorphism above induces a stabilization map $KH_{1,n}(R,\ a_1, \ldots, a_r) \longrightarrow KH_{1,n+1}(R,\ a_1, \ldots, a_r)$.

The stability theorem is proved under a stable range condition which is weaker than its predecessors and easier to apply. We describe this condition. Let R be an associative ring with identity 1 and involution $a \mapsto \bar{a}$. Let $\lambda \in center(R)$ such that $\lambda\bar{\lambda} = 1$. Let $\max^{\lambda}(R) = \{a \in R | a = -\lambda\bar{a}\}$. The ring R is said to satisfy the $\max^{\lambda}(R)$-stable range condition $\max^{\lambda}(R)S_m$ of degree m if R satisfies the usual stable range condition SR_m of H. Bass and if given a (right) unimodular vector $(a_1, \cdots, a_{(m+1)}, b_1, \cdots, b_{(m+1)})$ of length $2(m+1)$, there is an $(m+1) \times (m+1)$ $-\bar{\lambda}$-Hermitian matrix γ such that $(a_1, \cdots, a_{m+1}) + (b_1, \cdots, b_{m+1})\gamma$ is a unimodular of length $m+1$.

The main result is as follows.

Theorem 1.1 Let R and $\max^{\lambda}(R)$ be as above. Suppose that R satisfies the stable range condition $\max^{\lambda}(R)S_m$. Then for all $n > m + r$,

$$KH_{1,n}(R,\ a, \ldots, a_r)$$

is a group, the canonical map

$$KH_{1,n-1}(R,\ a_1, \ldots, a_r) \longrightarrow KH_{1,n}(R,\ a_1, \ldots, a_r)$$

is surjective, and the canonical homomorphism

$$KH_{1,n}(R,\ a_1, \ldots, a_r) \longrightarrow KH_{1,n+1}(R,\ a_1, \ldots, a_r)$$

is an isomorphism.

The rest of the article is organized as follows. In §2, we recall in detail the definitions of GH_{2n} and EH_{2n}, and of important subgroups of EH_{2n} which are used in establishing a decomposition of EH_{2n} when stable range conditions are imposed on R. In §3, we define a generalization of the stable range condition above, which uses form parameters, and show that it is weaker than its predecessors, namely the unitary stable range condition and that developed by W. van der Kallen, B. Magurn and L. Vaserstein. In §4, we prove our main result Theorem 1.1. An important tool in the proof is the decomposition theorem for EH_{2n}, which is also proved in the section.

2 Preliminaries on GH and EH

The basic references for the general Hermitian group GH and its elementary subgroup EH are [B] and [T]. The groups GH and EH are the analogs for Hermitian forms of the general quadratic group GQ and its elementary subgroup EQ in the theory of quadratic forms. Whereas the groups GQ and EQ have been known for a long time and their quotient $KQ_1 = GQ/EQ$ intensively studied, the group EH has been only recently discovered. Investigation of the quotient group $KH_1 = GH/EH$ and of the higher Hermitian K-groups KH_i defined using the Volodin construction is only beginning now. The topic K_1-stability for quadratic forms was treated already in the late 1960's by A. Bak, H. Bass, and A. Roy, and in the early 1970's by M. Kolster and L. Vaserstein. The fact that KQ-groups defined with respect to the maximal form parameter agree with KH-groups defined for $r = 0$, by [B, Theorem (1.1) and (1.3)], leads one to conjecture that stability results for KQ_1-groups have analogs for KH_1-groups.

We recall now the definitions of the groups GH and EH and of subgroups of EH which will be used in obtaining in §4 a decomposition of EH under the $\max^\lambda(R)$-stable range condition.

We fix the following notation. Let R be an associative ring with identity 1 and involution $a \mapsto \bar{a}$; thus $\overline{ab} = \bar{b}\bar{a}$ and $\bar{\bar{a}} = a$ for all $a, b \in R$. If $\alpha = (a_{ij})$ denotes an $m \times n$ matrix with coefficients $a_{ij} \in R$, let $\bar{\alpha}$ denote its conjugate transpose; thus $\bar{\alpha} = (a'_{kl})$ is the $n \times m$ matrix such that $a'_{kl} = \bar{a}_{lk}$.

Let r and n be natural numbers such that $n \geq r$. Let $\lambda \in center(R)$ such that $\lambda\bar{\lambda} = 1$. Let $a_1, \ldots, a_n$ be a sequence of elements in R such that $a_i = \lambda\bar{a}_i$ for all $1 \leq i \leq n$ and $a_{r+1} = a_{r+2} = \cdots = a_n = 0$. In the context we are working, it makes sense letting $r = 0$ mean that $a_1 = \cdots = a_n = 0$. So we shall do this. Let

$$
\begin{aligned}
A_1 &= r \times r \text{ diagonal matrix } \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}, \\
A &= n \times n \text{ diagonal matrix } \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \\
I &= \text{ an identity matrix .}
\end{aligned}
$$

Define the n**-th general Hermitian group of the elements** $a_1, \ldots, a_r$ by

$$GH_{2n}(R,\ a_1, \ldots, a_r) = \left\{ \sigma \in GL_{2n}(R) |\ \bar{\sigma} \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \sigma = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \right\}.$$

A typical element of this group is denoted by a $2n \times 2n$ matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where α, β, γ and δ are $n \times n$ block matrices. There is an obvious embedding

$$GH_{2n}(R,\ a_1, \ldots, a_r) \longrightarrow GH_{2(n+1)}(R,\ a_1, \ldots, a_r)$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and one defines

$$GH(R,\ a_1, \ldots, a_r) = \varinjlim_{n \geq r} GH_{2n}(R,\ a_1, \ldots, a_r).$$

Let $\min^{-\lambda}(R) = \{a + \lambda \bar{a} | a \in R\}$. For any $a_1, \ldots, a_r$ as above, let

$$C = \{\ {}^t(x_1, \ldots, x_r) \in {}^t(R^r)\ |\ \sum_{i=1}^{r} \bar{x}_i a_i x_i \in \min{}^{-\lambda}(R)\}.$$

In order to deal effectively with technical difficulties caused by the elements $a_1, \ldots, a_r$, we shall finely partition a typical matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of $GH_{2n}(R,\ a_1, \ldots, a_r)$ into the form

$$\text{(2.1)} \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\ \gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22} \end{pmatrix}$$

where $\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11}$ are $r \times r$ matrices, $\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12}$ are $r \times (n-r)$ matrices, $\alpha_{21}, \beta_{21}, \gamma_{21}, \delta_{21}$ are $(n-r) \times r$ matrices, and $\alpha_{22}, \beta_{22}, \gamma_{22}, \delta_{22}$ are $(n-r) \times (n-r)$ matrices. By [T, 3.4],

(2.2) the columns of $\alpha_{11} - I, \alpha_{12}, \beta_{11}, \beta_{12}, \bar{\beta}_{11}, \bar{\beta}_{21}, \bar{\delta}_{11} - I$ and $\bar{\delta}_{21}$ belong to C.

Letting $GQ_{2n}(R, \max^\lambda(R))$ denote the general quadratic group [B, §3] over R for maximal form parameter $\max^\lambda(R)$, one checks straightforward that the subgroup of $GH_{2n}(R,\ a_1, \ldots, a_r)$ consisting of

$$(2.3)\quad \left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & \beta_{22} \\ 0 & 0 & I & 0 \\ 0 & \gamma_{22} & 0 & \delta_{22} \end{pmatrix} \in GH_{2n}(R,\ a_1, \ldots, a_r) \right\} \cong GQ_{2(n-r)}(R, \max{}^\lambda(R)).$$

We identify now functorically defined elements of GH_{2n}, which will be used to generate EH_{2n}. The first 3 kinds of generators are taken for the most part from $GQ_{2(n-r)}(R, \max^\lambda(R))$ which is embedded as in (2.3) as a subgroup of GH_{2n} and the last 2 kinds are motivated by the result (2.2) concerning the columns of a matrix in GH_{2n}.

Let

$$H\epsilon_{ij}(a) \qquad (a \in R \text{ and } r+1 \le i \le n, 1 \le j \le n, i \ne j)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the (i,j)'th position, $-\bar{a}$ in the $(n+j, n+i)$'th position, and 0 elsewhere. Let

$$r_{ij}(a) \qquad (a \in R \text{ and } r+1 \le i, j \le n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(i, n+j)$'th position, $-\lambda\bar{a}$ in the $(j, n+i)$'th position, and 0 elsewhere. If $i = j$, this forces of course that $a = -\lambda\bar{a}$. Let

$$l_{ij}(a) \qquad (a \in R \text{ and } 1 \le i, j \le n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(n+i, j)$'th position, $-\bar{\lambda}\bar{a}$ in the $(n+j, i)$'th position, and 0 elsewhere. If $i = j$, this forces of course that $a = -\bar{\lambda}\bar{a}$.

For $\zeta = {}^t(x_1, \cdots, x_r) \in C$, let

$$\zeta_f \in R \qquad \text{such that } \zeta_f + \lambda\bar{\zeta}_f = \sum_{i=1}^{r} \bar{x}_i a_i x_i.$$

The element ζ_f is not in general unique. Define

$$Hm_i(\zeta) = \begin{pmatrix} I & \alpha_{12} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -\bar{A}_1\alpha_{12} & I & 0 \\ 0 & \gamma_{22} & -\bar{\alpha}_{12} & I \end{pmatrix} \qquad (\zeta \in C \text{ and } r+1 \le i \le n)$$

to be the $2n \times 2n$ matrix such that α_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$'th column and all other column's zero, and γ_{22} is the $(n-r) \times (n-r)$ matrix with $\bar{\zeta}_f$ in its $(i-r, i-r)$'th position and 0 elsewhere. Define

$$r_i(\zeta) = \begin{pmatrix} I & 0 & 0 & \beta_{12} \\ 0 & I & -\lambda\bar{\beta}_{12} & \beta_{22} \\ 0 & 0 & I & -\bar{A}_1\beta_{12} \\ 0 & 0 & 0 & I \end{pmatrix} \qquad (\zeta \in C \text{ and } r+1 \le i \le n)$$

to be the $2n \times 2n$ matrix such that β_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$'th column and all other columns 0, and β_{22} is the $(n-r) \times (n-r)$ matrix with $\lambda\bar{\zeta}_f$ in its $(i-r, i-r)$'th position and 0 elsewhere.

Each of the matrices above is called an **elementary Hermitian matrix** for the elements $a_1, \ldots, a_r$.

One can show by direct computation as in [T, §4] that each elementary Hermitian matrix is in $GH_{2n}(R,\ a_1, \ldots, a_r)$.

Define the **n'th elementary Hermitian group**

$$EH_{2n}(R,\ a_1, \ldots, a_r)$$

of the elements $a_1, \ldots, a_r$ to be the subgroup of $GH_{2n}(R,\ a_1, \ldots, a_r)$ generated by all elementary Hermitian matrices. It is obvious that the embedding $GH_{2n}(R,\ a_1, \ldots, a_r) \longrightarrow GH_{2(n+1)}(R,\ a_1, \ldots, a_r)$ takes $EH_{2n}(R,\ a_1, \ldots, a_r)$ to $EH_{2(n+1)}(R,\ a_1, \ldots, a_r)$ and one defines

$$EH(R,\ a_1, \ldots, a_r) = \varinjlim_{n \ge r} EH_{2n}(R,\ a_1, \ldots, a_r).$$

It is customary to identify $GH_{2(n-1)}(R,\ a_1, \ldots, a_r)$ and $EH_{2(n-1)}(R,\ a_1, \ldots, a_r)$, respectively, with their images in $GH_{2n}(R,\ a_1, \ldots, a_r)$ and $EH_{2n}(R,\ a_1, \ldots, a_r)$.

The following subgroups of $EH_{2n}(R,\ a_1, \ldots, a_r)$ will be used to establish a decomposition of $EH_{2n}(R,\ a_1, \ldots, a_r)$ under stable range conditions. Let

$$\begin{aligned}
C_n &= < H\epsilon_{in}(a),\ r+1 \le i < n; l_{in}(a), 1 \le i \le n \text{ and } Hm_n(\zeta), a \in R, \zeta \in C > \\
R_n &= < H\epsilon_{nj}(a),\ 1 \le j < n; r_{nj}(a), r+1 \le j \le n \text{ and } r_n(\zeta), a \in R, \zeta \in C > \\
P_n &= \{\sigma\sigma_1 \mid \sigma \in EH_{2(n-1)}(R,\ a_1, \ldots, a_r) \text{ and } \sigma_1 \in C_n\} \\
Q_n &= < Hm_j(\zeta), H\epsilon_{ij}(a),\ r+1 \le i,j \le n, i \ne j; \text{ and } l_{ij}(a), 1 \le i,j \le n, a \in R, \zeta \in C > .
\end{aligned}$$

Lemma 2.4 Suppose $n \ge 2$. Suppose $\sigma \in GH_{2n}(R,\ a_1, \ldots, a_r)$ such that the n'th row and n'th column of σ are identical with the n'th row and n'th column of the $2n \times 2n$ identity matrix, respectively. Then the $2n$'th row and $2n$'th column of σ are identical with the $2n$'th row and $2n$'th column of the $2n \times 2n$ identity matrix, respectively. In particular, if $n > r$ then $\sigma \in GH_{2(n-1)}(R,\ a_1, \ldots, a_r)$.

Proof Let
$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$
By [T, (3.1)]
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} + \bar{\beta}A & \lambda\bar{\beta} \\ \bar{\alpha}\bar{A} - \bar{A}\bar{\beta}A + \bar{\lambda}\bar{\gamma} - \bar{A}\bar{\delta} & \bar{\alpha} - A\bar{\beta} \end{pmatrix}.$$
Using the equation
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{\delta} + \bar{\beta}A & \lambda\bar{\beta} \\ \bar{\alpha}\bar{A} - \bar{A}\bar{\beta}A + \bar{\lambda}\bar{\gamma} - \bar{A}\bar{\delta} & \bar{\alpha} - A\bar{\beta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$
and the fact that A is a diagonal matrix, one deduces routinely the conclusion of the lemma. □

Lemma 2.5 $GH_{2(n-1)}(R,\ a_1, \ldots, a_r)$ normalizes $EH_{2n}(R,\ a_1, \ldots, a_r)$.

Proof By [T, (8.3)], R_n and C_n generates $EH_{2n}(R,\ a_1, \ldots, a_r)$. But it is obvious that R_n and C_n are normalized by $GH_{2(n-1)}(R,\ a_1, \ldots, a_r)$. □

The following corollary obvious.

Corollary 2.6 $EH(R,\ a_1, \ldots, a_r)$ is a normal subgroup of $GH(R,\ a_1, \ldots, a_r)$.

Furthermore according to the Hermitian Whitehead Lemma [T, §5], $EH(R,\ a_1, \ldots, a_r)$ is the commutator subgroup of $GH(R,\ a_1, \ldots, a_r)$. One defines
$$KH_{1,n}(R,\ a_1, \ldots, a_r) = GH_{2n}(R,\ a_1, \ldots, a_r)/EH_{2n}(R,\ a_1, \ldots, a_r)$$
and
$$KH_1(R,\ a_1, \ldots, a_r) = GH(R,\ a_1, \ldots, a_r)/EH(R,\ a_1, \ldots, a_r).$$
Whereas $KH_1(R,\ a_1, \ldots, a_r)$ is an abelian group, $KH_{1,n}(R,\ a_1, \ldots, a_r)$ is in general just a coset space.

3 Λ-stable range condition

Let R be an associative ring with identity. A vector $(a_1, \ldots, a_n)$ with coefficients $a_i \in R$ is called **right unimodular** if there are elements $b_1, \ldots, b_n \in R$ such that $a_1 b_1 + \cdots + a_n b_n = 1$. The **stable range condition** SR_m of A. Bass in the formulation of L. Vaserstein says that if $(a_1, \ldots, a_{m+1})$ is a unimodular vector then there exist elements $b_1, \ldots, b_m \in R$ such that $(a_1 + a_{m+1} b_1, \ldots, a_m + a_{m+1} b_m)$ is unimodular. It follows easily that $SR_m \Rightarrow SR_n$ for any $n \geq m$.

Suppose that R has an involution $a \mapsto \bar{a}$. Let $\lambda \in center(R)$ such that $\lambda\bar{\lambda} = 1$. Let $\min^\lambda(R) = \{a - \lambda\bar{a} | a \in R\}$ and $\max^\lambda(R) = \{a \in R | a = -\lambda\bar{a}\}$. A **form parameter** Λ is an additive subgroup of R such that

1) $a\Lambda\bar{a} \subseteq \Lambda$ for all $a \in R$,

2) $\min^\lambda(R) \subseteq \Lambda \subseteq \max^\lambda(R)$.

Clearly the extremes in (2) satisfy (1) so that they are form parameters. Let

$$\mathbb{M}_m(\Lambda) \quad (\text{ resp. } \mathbb{M}_m(\bar{\Lambda}))$$

denote the set of all $m \times m$ matrices γ such that $\gamma = -\lambda\bar{\gamma}$ and the diagonal coefficients of γ lie in Λ (resp. $\gamma = -\bar{\lambda}\bar{\gamma}$ and the diagonal coefficients of γ lie in $\bar{\Lambda}$).

Definition 3.1 Let Λ be a form parameter on R. R is said to satisfy the **Λ-stable range condition** ΛS_m if it satisfies SR_m and if given any unimodular vector $(a_1, \cdots, a_{m+1}, b_1, \cdots, b_{m+1}) \in R^{2m+2}$ there exists a matrix $\gamma \in \mathbb{M}_{m+1}(\bar{\Lambda})$ such that $(a_1, \cdots, a_{m+1}) + (b_1, \cdots, b_{m+1})\gamma$ is unimodular.

Lemma 3.2 The following conditions are equivalent for a ring R with involution and form parameter $\Lambda \subseteq R$.

(3.2.1) R satisfies ΛS_m,

(3.2.2) R satisfies SR_m and given any unimodular vector $(a_1, \ldots, a_{m+1}, b_1, \ldots, b_{m+1})$ there is a $2(m+1) \times 2(m+1)$ matrix

$$\sigma = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix}$$

where I is the $(m+1) \times (m+1)$ identity matrix and $\gamma \in \mathbb{M}_{m+1}(\bar{\Lambda})$ such that $v\sigma = (a'_1, \ldots, a'_{m+1}, b'_1, \ldots, b'_{m+1})$ and $(a'_1, \ldots, a'_{m+1})$ is unimodular.

(3.2.3) R satisfies SR_m and given any unimodular vector $(a_1, \ldots, a_{m+1}, b_1, \ldots, b_{m+1})$ there is a $2(m+1) \times 2(m+1)$ matrix

$$\sigma = \begin{pmatrix} \epsilon & 0 \\ \gamma & \bar{\epsilon}^{-1} \end{pmatrix}$$

where ϵ an invertible $(m+1)\times(m+1)$ matrix and $\gamma\epsilon^{-1}\in\mathbb{M}_{m+1}(\bar{\Lambda})$ such that $v\sigma=(a'_1,\ldots,a'_{m+1},b'_1,\ldots,b'_{m+1})$ and $(a'_1,\ldots,a'_{m+1})$ is unimodular.

Proof It is clear that (3.2.1) $\iff$ (3.2.2) $\Rightarrow$ (3.2.3). Suppose that (3.2.3) holds. We show that (3.2.2) holds. Let

$$\rho=\begin{pmatrix} I & 0\\ \gamma\epsilon^{-1} & I\end{pmatrix}$$

and $v\rho=(a''_1,\ldots,a''_{m+1},b''_1,\ldots,b''_{m+1})$. Since

$$\sigma=\rho\begin{pmatrix}\epsilon & 0\\ 0 & \bar{\epsilon}^{-1}\end{pmatrix},$$

it is clear that $(a'_1,\ldots,a'_{m+1})=(a''_1,\ldots,a''_{m+1})\epsilon$. Thus $(a'_1,\ldots,a'_{m+1})\epsilon^{-1}=(a''_1,\ldots,a''_{m+1})$. Since$(a'_1,\ldots,a'_{m+1})$ is unimodular and ϵ is invertible, it follows that $(a''_1,\ldots,a''_{m+1})$ is unimodular. □

Lemma 3.3 $\Lambda S_m\Rightarrow\Lambda S_n$ for all $n\geq m$.

Proof We shall use the matrix notation introduced in §2 with $r=0$. Let $n>m$. Clearly SR_n holds. By induction on n, we can assume that R satisfies ΛS_{n-1}. We shall show that R satisfies the formulation of ΛS_n given in (3.2.3).Let $v=(a_1,\cdots,a_{n+1},b_1,\cdots,b_{n+1})$ be a unimodular vector. By SR_n (cf. [K, Chap VI, Remark 1.5.1]), there are elements $x_1,\ldots,x_n\in R$ such that if $\sigma=H\epsilon_{1n+1}(-\bar{x}_1)\cdots H\epsilon_{nn+1}(-\bar{x}_n)$ then $v\sigma=(a'_1,\cdots,a'_{n+1},b'_1,\cdots,b'_{n+1})$ and $(a'_1,\cdots,a'_{n+1},b'_1,\cdots,b'_n)$ is unimodular. Again by SR_n, there exists elements $y_1,\ldots,y_n\in R$ such that if $\rho=H\epsilon_{n+1,1}(y_1)\cdots H\epsilon_{n+1,n}(y_n)$ then $v\sigma\rho=(a''_1,\cdots,a''_{n+1},b''_1,\cdots,b''_{n+1})$ and $(a''_1,\cdots,a''_n,b''_1,\cdots,b''_n)$ is unimodular. By ΛS_{n-1}, there is a $2n\times 2n$ matrix

$$\tau'=\prod_{1\leq i\leq j\leq n} l_{ij}(a_{ij})=\begin{pmatrix} I & 0\\ \gamma' & I\end{pmatrix}$$

where $\gamma'\in\mathbb{M}_n(\bar{\Lambda})$ such that $(a''_1,\cdots,a''_n,b''_1,\cdots,b''_n)\tau'=(c_1,\cdots,c_n,d_1,\cdots,d_n)$ and $(c_1,\cdots,c_n)$ is unimodular. Let τ denote the $2(n+1)\times 2(n+1)$ stabilization of τ'. Thus

$$\tau=\prod_{1\leq i\leq j\leq n} l_{ij}(a_{ij})=\begin{pmatrix} I & 0\\ \gamma & I\end{pmatrix}$$

where

$$\gamma=\begin{pmatrix}\gamma' & 0\\ 0 & 0\end{pmatrix}\in\mathbb{M}_{n+1}(\bar{\Lambda}).$$

Clearly $v\sigma\rho\tau = (c_1, \cdots, c_{n+1}, d_1, \cdots, d_{n+1})$ and $(c_1, \cdots, c_{n+1})$ is unimodular because $(c_1, \cdots, c_n)$ is. $\square$

If S is a set of elements of R, let

$$\mathfrak{J}(S) = \bigcap_{\mathfrak{M} \supseteq S} \mathfrak{M} \bigcap R$$

where $\mathfrak{M}$ runs through all maximal right ideals of R. Note that a vector $(a_1, \ldots, a_n)$ is unimodular $\iff$ $\mathfrak{J}\{a_1, \ldots, a_n\} = R$.

The **absolute stable range condition** AS_m of M. Stein, W. van der Kallen, B. Magurn, and L. Vaserstein says if $(a_1, \cdots, a_{m+1})$ is a vector then there are elements $x_1, \cdots, x_m \in R$ such that $a_{m+1} \in \mathfrak{J}\{a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m\}$, i.e. $\mathfrak{J}\{a_1, \cdots, a_{m+1}\} = \mathfrak{J}\{a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m\}$.

Lemma 3.4 Let R be a ring with involution. Then $AS_m \Rightarrow \Lambda S_m$ for any form parameter Λ on R.

Proof We show first that SR_m holds. Let $(a_1, \cdots, a_{m+1})$ be a unimodular vector. By AS_m, there are elements $x_1, \cdots, x_m$ such that $\mathfrak{J}\{a_1, \cdots, a_{m+1}\} = \mathfrak{J}\{a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m\}$. Since $R = \mathfrak{J}\{a_1, \cdots, a_{m+1}\}$, it follows that $(a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m)$ is unimodular. Thus SR_m holds.

We shall use now the equivalent formulation of ΛS_m given in (3.2.3). Let $v = (a_1, \cdots, a_{2(m+1)})$ be a unimodular vector. Let $p = m + 3$ and $q = 2(m+1)$. By AS_m, there exist elements $x_1, \cdots, x_m \in R$ such that if $\sigma_1 = H\epsilon_{1,m+1}(-\bar{x}_1) \cdots H\epsilon_{m,m+1}(-\bar{x}_m)$ then $v\sigma_1 = (a_1^{(1)}, \cdots, a_q^{(1)})$ and $\mathfrak{J}\{a_{p-1}^{(1)}, \cdots, a_{q-1}^{(1)}\} = \mathfrak{J}\{a_{p-1}^{(1)}, \cdots, a_q^{(1)}\} = \mathfrak{J}\{a_{p-1}, \cdots, a_q\}$. Let $2 \leq n \leq m + 1$ $(= \frac{q}{2})$ and suppose that for each $1 \leq i < n$, we have found a $q \times q$ matrix

$$\sigma_i = \begin{pmatrix} \epsilon_i & 0 \\ \gamma_i & \bar{\epsilon}_i^{-1} \end{pmatrix}$$

as in (3.2.3) such that if $v\sigma_1 \cdots \sigma_i = (a_1^{(i)}, \cdots, a_q^{(i)})$ then $\mathfrak{J}\{a_{p-i}^{(i)}, \cdots, a_{q-i}^{(i)}\} = \mathfrak{J}\{a_{p-i}^{(i)}, \cdots, a_q^{(i)}\}$. We construct now a $q \times q$ matrix σ_n with the same properties. By AS_m, there exist elements $y_1, \cdots, y_m \in R$ such that if

$$\sigma_n = H\epsilon_{1,\frac{q}{2}-n+1}(-\bar{y}_1) \cdots H\epsilon_{\frac{q}{2}-n,\frac{q}{2}-n+1}(-\bar{y}_{\frac{q}{2}-n}) l_{\frac{q}{2}-n+1,\frac{q}{2}-n+2}(y_{\frac{q}{2}-n+1}) \cdots l_{\frac{q}{2}-n+1,\frac{q}{2}}(y_m),$$

i.e.

$$\sigma_n = \left(\begin{array}{ccccccc|ccccccc}
1 & & & -\bar{y}_1 & & & & & & & & & & \\
 & \ddots & & \vdots & & & & & & & & & & \\
 & & 1 & -\bar{y}_{\frac{q}{2}-n} & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
\hline
 & & & & & & & 1 & & & & & & \\
 & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & 1 & & & & \\
 & & & 0 & y_{\frac{q}{2}-n+1} & \cdots & y_m & y_1 & \cdots & y_{\frac{q}{2}-n} & 1 & & & \\
 & & & -\bar{\lambda}\bar{y}_{\frac{q}{2}-n+1} & & & & & & & & 1 & & \\
 & & & \vdots & & & & & & & & & \ddots & \\
 & & & -\bar{\lambda}\bar{y}_m & & & & & & & & & & 1
\end{array}\right)$$

then $v\sigma_1 \cdots \sigma_n = (a_1^{(n)}, \cdots, a_q^{(n)})$ and $\mathfrak{J}\{a_{p-n}^{(n)}, \cdots, a_{q-n}^{(n)}\} = \mathfrak{J}\{a_{p-n}^{(n)}, \cdots, a_{q-n+1}^{(n)}\} = \mathfrak{J}\{a_{p-n}^{(n-1)}, \cdots, a_{q-n+1}^{(n-1)}\}$. By the induction the hypothesis, $\mathfrak{J}\{a_{p-n+1}^{(n-1)}, \cdots, a_{q-n+1}^{(n-1)}\} = \mathfrak{J}\{a_{p-n+1}^{(n-1)}, \cdots, a_q^{(n-1)}\}$ and clearly, $\mathfrak{J}\{a_{p-n+1}^{(n-1)}, \cdots, a_q^{(n-1)}\} = \mathfrak{J}\{a_{p-n+1}^{(n)}, \cdots, a_q^{(n)}\}$. Thus $\mathfrak{J}\{a_{p-n}^{(n)}, \cdots, a_{q-n}^{(n)}\} = \mathfrak{J}\{a_{p-n}^{(n)}, \cdots, a_q^{(n)}\}$. By induction, we can construct a sequence $\sigma_1, \cdots \sigma_{m+1}$ of matrices σ_i as above such that $\mathfrak{J}\{a_2^{(m+1)}, \cdots, a_{m+1}^{(m+1)}\} = \mathfrak{J}\{a_2^{(m+1)}, \cdots, a_{2(m+1)}^{(m+1)}\}$. Thus $\mathfrak{J}\{a_1^{(m+1)}, \cdots, a_{m+1}^{(m+1)}\} = \mathfrak{J}\{a_1^{(m+1)}, \cdots, a_{2(m+1)}^{(m+1)}\} = R$. $\square$

Let Λ be a form parameter on R. The **Λ-unitary stable range condition** ΛUS_m (cf. [HO, p. 526]) says that SR_m holds and that if $(a_1, \ldots, a_m, b_1, \cdots, b_m)$ is a unimodular vector then there is a vector $(x_1, \ldots, x_m, y_1, \cdots, y_m)$ such that $x_1\bar{y}_1 + \ldots + x_m\bar{y}_m \in \Lambda$ and $a_1x_1 + \ldots + a_mx_m + b_1y_1 + \cdots + b_my_m = 1$. It is not difficult to show that $\Lambda US_m \Rightarrow \Lambda US_n$ for any $n \geq m$.

Lemma 3.5 $\Lambda US_m \Rightarrow \Lambda S_m$.

Proof We use the formulation of ΛS_m given in (3.2.3). Let $v = (a_1, \cdots, a_{m+1}, b_1, \cdots, b_{m+1})$ be a unimodular vector. As in the proof of (3.3), we can find a product of hyperbolic elementary matrices $H\epsilon_{ij}(x)$ such that after multiplying v on the right by this product, we can assume $(a_1, \cdots, a_m, b_1, \cdots, b_m)$ is unimodular. By ΛUS_m, there is a vector $(x_1, \ldots, x_m, y_1, \cdots, y_m)$ such that $x_1\bar{y}_1 + \ldots + x_m\bar{y}_m \in \Lambda$ and $a_1x_1 + \ldots + a_mx_m + b_1y_1 + \cdots + b_my_m = 1$. Let $\sigma = H\epsilon_{1,m+1}(x_1') \cdots H\epsilon_{m,m+1}(x_m') l_{1,m+1}(y_1') \cdots l_{m+1,m+1}(y_{m+1}')$ where $x_i' = x_i(1 - a_{m+1})$, $y_i' = y_i(1 - a_{m+1})$ $(1 \leq i \leq m)$ and $y_{m+1}' = -\sum_{i=1}^{m} \overline{(1 - a_{m+1})}\bar{x}_i y_i (1 -$

$a_{m+1})$. Then $v\sigma = (a'_1, \cdots, a'_{m+1}, b'_1, \cdots, b'_{m+1})$ has the property that $a'_{m+1} = 1$. Thus $(a'_1, \cdots, a'_{m+1})$ is unimodular. By (3.2.3), we are finished. □

Lemma 3.6 Suppose that R is module finite over a subring $k \subseteq center(R)$. Let Max(k) denote the maximal ideal spectrum of k in the Zariski topology. Define $d_k(R) = \text{dimension}(\text{Max}(k))$, cf. [Bs, pp. 92-102]. Suppose that Max(k) is Noetherian and $d_k(R)$ is finite. Then R satisfies $\Lambda S_{d_k(R)+1}$ for any form parameter Λ in R.

Proof By [MKV, Theorem (3.1)], R satisfies $AS_{d_k(R)+1}$. Thus by (3.4), we are finished. □

4 Proof of Theorem 1.1

Throughout this section, R denotes an associative ring with identity and involution $a \mapsto \bar{a}$. λ denotes an element in the $center(R)$ such that $\lambda\bar{\lambda} = 1$ and $\max^\lambda(R) = \{a \in R | a = -\lambda\bar{a}\}$. It will be assumed throughout that

$$R \text{ satisfies the stable range condition} \max{}^\lambda(R) S_m.$$

Lemma 4.1 Let $n \geq r + m + 1$. Then for any $\sigma \in GH_{2n}(R,\ a_1, \ldots, a_r)$, there is an element $\tau \in Q_n$ such that $\sigma\tau$ has 1 in its (n, n)'th position.

Proof Let

$$\sigma = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\ \gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22} \end{pmatrix}$$

be the 4×4 block matrix description of σ given in (2.1) and (2.2). Since $\sigma^{-1} \in GH_{2n}(R,\ a_1, \ldots, a_r)$ and therefore also has such a description, there are $r \times (n - r)$ matrices x_1, y_1 and $(n - r) \times (n - r)$ matrices x_2, y_2 such that $\alpha_{21}x_1 + \alpha_{22}x_2 + \beta_{21}y_1 + \beta_{22}y_2 = I$ and the columns of x_1 lie in C. Thus $(\alpha_{21}x_1, \alpha_{22}, \beta_{21}y_1, \beta_{22})$ is a unimodular vector in $(M_{(n-r)}(R))^4$. Let v_i denote the bottom row of α_{2i} $(i = 1, 2)$ and w_i denote the bottom row of β_{2i} $(i = 1, 2)$. Then (v_1x_1, v_2, w_1, w_2) is the bottom row of $(\alpha_{21}x_1, \alpha_{22}, \beta_{21}, \beta_{22})$ and hence is unimodular in $R^{3(n-r)+r}$. Since the stable range condition SR_m holds and $n - r \geq m + 1$, there exists (cf. [K, Chap.VI, Remark 1.5.1]) an $(n - r) \times (n - r)$ matrix z_1 such that $(v_2 + v_1x_1z_1, w_1, w_2)$ is unimodular in $R^{2(n-r)+r}$. Since the columns of x_1 belong to C, it follows straightforward that the columns of x_1z_1 belong to C. Let ζ_i denote the i'th column of x_1z_1 and let z_2 be the $(n-r)\times(n-r)$ matrix defined by

$$\begin{pmatrix} I & x_1z_1 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -\bar{A}_1x_1z_1 & I & 0 \\ 0 & z_2 & -\bar{z}_1\bar{x}_1 & I \end{pmatrix} = \prod_{i=1}^{n-r} Hm_{r+i}(\zeta_i) \in Q_n.$$

Set

$$\tau_1 = \begin{pmatrix} I & x_1z_1 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -\bar{A}_1x_1z_1 & I & 0 \\ 0 & z_2 & -\bar{z}_1\bar{x}_1 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \bar{A}_1x_1z_1 & I & 0 \\ -\bar{\lambda}\bar{z}_1\bar{x}_1A_1 & 0 & 0 & I \end{pmatrix} \in Q_n.$$

Then the n'th row of $\sigma\tau_1$ is

$$\left(v_1 - \bar{\lambda}w_2\bar{z}_1\bar{x}_1A_1, v_2 + v_1x_1z_1 + w_2(z_2 - \bar{z}_1\bar{x}_1\bar{A}_1x_1z_1), w_1 - w_2\bar{z}_1\bar{x}_1, w_2\right).$$

Let (v_1', v_2', w_1', w_2') denote this row. Then (v_2', w_1', w_2') is unimodular in $R^{2(n-r)+r}$, because $(v_2 + v_1x_1z_1, w_1, w_2)$ is. Since R satisfies SR_m and $n - r \geq m+1$, there exists an $r\times(n-r)$ matrix z_3 such that $(v_2' + w_1'z_3, w_2')$ is unimodular in $R^{2(n-r)}$. Set

$$\tau_2 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & z_3 & I & 0 \\ -\bar{\lambda}\bar{z}_3 & 0 & 0 & I \end{pmatrix} \in Q_n.$$

Then the n'th row of $\sigma\tau_1\tau_2$ is $(v_1' - w_2'(\bar{\lambda}\bar{z}_3), v_2' + w_1'z_3, w_1', w_2')$ and $(v_2' + w_1'z_3, w_2')$ is unimodular. Let $(v_1'', v_2'', w_1'', w_2'')$ denote this row. Thus (v_2'', w_2'') is unimodular in $R^{2(n-r)}$. Since R satisfies $\max^\lambda(R)S_m$ and $n-r \geq m+1$, there exists a matrix $\gamma \in \mathbb{M}_{n-r}(\overline{\max^\lambda(R)})$ such that $v_2'' + w_2''\gamma$ is unimodular in R^{n-r}. Set

$$\tau_3 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & \gamma & 0 & I \end{pmatrix} = \prod_{r+1\leq i\leq j\leq n} l_{ij}(a_{ij}) \in Q_n$$

where a_{ij} is the $(i-r, j-r)$'th coefficient of γ. Since R satisfies SR_m and $(n-r)\geq m+1$, there is by [Bs, Theorem 5.3.3] a product ϵ of elementary $(n-r)\times(n-r)$ matrices such that $(v_2'' + w_2''\gamma)\epsilon = (0, \cdots, 0, 1)$. Set

$$\tau_4 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \bar{\epsilon}^{-1} \end{pmatrix} \in Q_n.$$

Then $\sigma\tau_1\tau_2\tau_3\tau_4$ has n'th row $(v_1'', (0, \cdots, 0, 1), w_1'', w_2''')$. $\square$

Recall the subgroups P_n, Q_n, and R_n of $EH_{2n}(R, a_1, \ldots, a_r)$, which are defined in §2.

Definition 4.2 Let $\phi \in EH_{2n}(R,\ a_1, \ldots, a_r)$. A PRQ**-decomposition** of ϕ is a product decomposition $\phi = \sigma\alpha\tau$ where $\sigma \in P_n, \alpha \in R_n$, and $\tau \in Q_n$.

Decomposition Theorem 4.3 Let $n \geq r + m + 2$. Then every element of $EH_{2n}(R,\ a_1, \ldots, a_r)$ has a PRQ-decomposition, i.e. $EH_{2n}(R,\ a_1, \ldots, a_r) = P_n R_n Q_n$.

Proof Let $\phi \in EH_{2n}(R,\ a_1, \ldots, a_r)$. A PRQ-decomposition $\sigma\alpha\tau$ of ϕ will be called **reduced** if the $(n-1, n)$'th coefficient of σ is 0. The strategy of the proof is as follows. First we show that if ϕ has a PRQ-decomposition then it has a reduced one. Then we identify generators θ of $EH_{2n}(R,\ a_1, \ldots, a_r)$ and show using reduced PRQ-decompositions that $\theta P_n R_n Q_n \subseteq P_n R_n Q_n$. It follows trivially that $EH_{2n}(R,\ a_1, \ldots, a_r) = P_n R_n Q_n$.

Let $\sigma\alpha\tau$ be a PRQ-decomposition of ϕ. Write

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 \\ 0 & 1 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & 1 \end{pmatrix}$$

and set

$$\sigma_1 = \begin{pmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{pmatrix}.$$

By definition, $\sigma_1 \in EH_{2(n-1)}(R,\ a_1, \ldots, a_r)$. Since $n \geq r + m + 2$, it follows from (4.1) that there is a $\tau_1 \in Q_{n-1}$ such that the $(n-1, n-1)$'th coefficient of $\sigma_1\tau_1$ is 1. It is obvious that if τ_1 is identified with its image in $EH_{2n}(R,\ a_1, \ldots, a_r)$ (under the stabilization map $EH_{2(n-1)}(R,\ a_1, \ldots, a_r) \longrightarrow EH_{2n}(R,\ a_1, \ldots, a_r)$) then $\tau_1 \in Q_n$ and the $(n-1, n-1)$'th coefficient of $\sigma\tau_1$ is 1. Furthermore $\tau_1 \in P_n \cap Q_n$ and τ_1 normalizes R_n. Thus $(\sigma\tau_1)(\tau_1^{-1}\alpha\tau_1)(\tau_1^{-1}\tau)$ is a PRQ-decomposition of ϕ such that the $(n-1, n-1)$'th coefficient of $\sigma\tau_1$ is 1. Choose $x \in R$ such that the $(n-1, n)$'th coefficient of $\sigma\tau_1 H\epsilon_{n-1,n}(x)$ is 0. Choose $y \in R$ such that the $(n, n-1)$'th coefficient of $\tau_1^{-1}\alpha\tau_1 H\epsilon_{n,n-1}(y)$ is 0. Let $\tau_2 = H\epsilon_{n-1,n}(x)$ and $\tau_3 = H\epsilon_{n,n-1}(y)$. Then $\tau_2^{-1}(\tau_1^{-1}\alpha\tau_1\tau_3)\tau_2 = \sigma_2\alpha_1$ for some $\sigma_2 \in EH_{2(n-1)}(R,\ a_1, \ldots, a_r) \subseteq P_n$ and some $\alpha_1 \in R_n$. Thus $\phi = \sigma\alpha\tau = (\sigma\tau_1\tau_2)(\tau_2^{-1}(\tau_1^{-1}\alpha\tau_1\tau_3)\tau_2)(\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau) = (\sigma\tau_1\tau_2\sigma_2)\alpha_1(\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau)$ which is a reduced PRQ-decomposition of ϕ.

The relations

$$
\begin{aligned}
&H\epsilon_{ni}(a) = [H\epsilon_{n,n-1}(a), H\epsilon_{n-1,i}(1)] \qquad (a \in R \text{ and } r+1 \le i \le n-1),\\
&r_{nj}(a) = [H\epsilon_{nj}(a), r_{ji}(1)] \qquad (a \in R \text{ and } r+1 \le i,j \le n-1),\\
&r_{nn}(a)r_{n-1,n}(-a) = [r_{n-1,n-1}(a), H\epsilon_{n,n-1}(1)], \qquad (a \in \max{}^{\lambda}(R)),\\
&r_n(\zeta) = \cdots [r_j(\zeta), H\epsilon_{nj}(-1)]r_{jn}(\zeta_f), \qquad (\zeta \in C, r+1 \le j \le n-1),
\end{aligned}
$$

show that P_n and the matrices $H\epsilon_{n,n-1}(a)$ $(a \in R)$ generate $EH_{2n}(R,\ a_1,\ldots,a_r)$. Obviously $P_n(P_nR_nQ_n) \subseteq P_nR_nQ_n$. Let $\sigma\alpha\tau$ be a reduced PRQ-decomposition. Since the $(n-1,n)$'th coefficient of σ is 0, σ can be expressed as a product $\sigma = \sigma_3\sigma_4$ where $\sigma_3 \in C_n$ such that the $(n-1,n)$'th coefficient of σ_3 is 0 and $\sigma_4 \in EH_{2(n-1)}(R,\ a_1,\ldots,a_r)$. A straightforward computation shows that $H\epsilon_{n,n-1}(a)\sigma_3 H\epsilon_{n,n-1}(-a) \in P_n$ and it is clear that $EH_{2(n-1)}(R,\ a_1,\ldots,a_r)$ normalizes R_n. Thus $H\epsilon_{n,n-1}(a)\sigma\alpha\tau = (H\epsilon_{n,n-1}(a)\sigma_3 H\epsilon_{n,n-1}(-a)\sigma_4)$ $(\sigma_4^{-1}H\epsilon_{n,n-1}(a)\sigma_4\alpha)\tau$ which is a PRQ-decomposition. □

Proof of Theorem (1.1) Let $\sigma \in GH_{2n}(R,\ a_1,\ldots,a_r)$. By (4.1), there is a $\tau_1 \in Q_n \subseteq EH_{2n}(R,\ a_1,\ldots,a_r)$ such that the (n,n)'th coefficient of $\sigma\tau_1$ is 1. Clearly there is a matrix $\tau_2 = \prod_{i=1}^{n-1} H\epsilon_{ni}(x_i)$ such that $\sigma\tau_1\tau_2$ has 0 in the first $(n-1)$ entries of its n'th row and 1 in the n'th entry of this row. From (2.2), it follows that there is a matrix $\tau_3 = \prod_{i=1}^{n} \ell_{in}(y_i) \prod_{i=r+1}^{n-1} \epsilon_{in}(y_i')Hm_n(\zeta)$ such that $\tau_3\sigma\tau_1\tau_2$ has the same n'th row as $\sigma\tau_1\tau_2$ and the same n'th column as the $2n \times 2n$ identity matrix. For any matrix

$$
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GH_{2n}(A, a_1, \cdots, a_r),
$$

it follows from the identity

$$
\overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix}
$$

that

(i) $\bar{\alpha}A\beta + \bar{\gamma}\beta + \lambda\bar{\alpha}\delta = \lambda I$, and

(ii) $\bar{\beta}A\beta + \bar{\delta}\beta + \lambda\bar{\beta}\delta = 0$.

From (i), we obtain that the $(2n,2n)$'th coefficient of $\tau_3\sigma\tau_1\tau_2$ is 1. From (2.2), it follows now that there is a matrix $\tau_4 = \prod_{i=1}^{n-1} H\epsilon_{ni}(z_i) \prod_{i=r+1}^{n} r_{in}(z_i')r_n(\xi)$ such that $\tau_4\tau_3\sigma\tau_1\tau_2$ has the same n'th row and n'th column as $\tau_3\sigma\tau_1\tau_2$ and the same $2n$'th column as the $2n \times 2n$ identity matrix. It follows now from (ii) that $\tau_4\tau_3\sigma\tau_1\tau_2$ has the same n'th row as the $2n \times 2n$ identity matrix. Thus $\tau_4\tau_3\sigma\tau_1\tau_2 \in GH_{2(n-1)}(R,\ a_1,\ldots,a_r)$, by (2.4). Let $\rho = \tau_4\tau_3\sigma\tau_1\tau_2$. By (2.5), ρ normalizes $EH_{2n}(R,\ a_1,\ldots,a_r)$. Since $\sigma = \tau_3^{-1}\tau_4^{-1}\rho\tau_2^{-1}\tau_1^{-1}$, it follows that σ normalizes $EH_{2n}(R,\ a_1,\ldots,a_r)$. Thus $KH_{1,n}(R,\ a_1,\ldots,a_r)$ is a group and the map $KH_{1,n-1}(R,\ a_1,\ldots,a_r) \longrightarrow KH_{1,n}(R,\ a_1,\ldots,a_r)$ is surjective. By induction on $n-m-r$, we obtain that the map $KH_{1,m+r}(R,\ a_1,\ldots,a_r) \longrightarrow KH_{1,n}(R,\ a_1,\ldots,a_r)$ is surjective.

Let $\phi \in GH_{2n}(R,\ a_1,\ldots,a_r) \cap EH_{2(n+1)}(R,\ a_1,\ldots,a_r)$. Let $\sigma\alpha\tau$ be a $P_{(n+1)}R_{(n+1)}Q_{(n+1)}$-decomposition of ϕ. Since the $(n+1)$'th row of σ coincides with that of the $2(n+1)\times 2(n+1)$ identity matrix, it follows that the $(n+1)$'th row of $\sigma\alpha\tau$ coincides with the $(n+1)$'th row of $\alpha\tau$. Thus the $(n+1)$'th row of $\alpha\tau$ coincides with that of the $2(n+1) \times 2(n+1)$ identity matrix. Write

$$\tau = \begin{pmatrix} \epsilon & 0 \\ \gamma & \bar{\epsilon}^{-1} \end{pmatrix}.$$

If (v,w) denotes the $(n+1)$'th row of α then the $(n+1)$'th row of $\alpha\tau$ is

$$(v,w)\begin{pmatrix} \epsilon & 0 \\ \gamma & \bar{\epsilon}^{-1} \end{pmatrix} = (v\epsilon + w\gamma, w\bar{\epsilon}^{-1}).$$

Thus $w\bar{\epsilon}^{-1} = 0$. Since $\bar{\epsilon}^{-1}$ is invertible, $w = 0$. Thus $\alpha \in Q_{n+1}$. Write $\sigma = \sigma_1\tau_1$ where $\sigma_1 \in EH_{2n}(R,\ a_1,\ldots,a_r)$ and $\tau_1 \in C_{(n+1)} \subseteq Q_{(n+1)}$. Obviously $\phi = \sigma_1(\tau_1\alpha\tau)$ and $\tau_1\alpha\tau \in Q_{(n+1)} \cap GH_{2n}(R,\ a_1,\ldots,a_r)$. It suffices to show that $\tau_1\alpha\tau \in EH_{2n}(R,\ a_1,\ldots,a_r)$. In fact, we shall show that $\tau_1\alpha\tau \in Q_n$.

Write

$$\tau_1\alpha\tau = \begin{pmatrix} \epsilon_1 & 0 \\ \gamma_1 & \bar{\epsilon}_1^{-1} \end{pmatrix}.$$

From the definition of Q_{n+1}, ϵ_1 is an $(n+1)\times(n+1)$ matrix in the elementary group $E_{n+1}(R)$, of the form

$$\epsilon_1 = \begin{pmatrix} I & \alpha_2 \\ 0 & \epsilon_1' \end{pmatrix}$$

where α_2 is an $r \times (n+1-r)$ matrix whose columns lie in C. Furthermore since $\tau_1\alpha\tau$ lies in $GH_{2n}(R,\ a_1,\ldots,a_r)$, the $(n+1-r)$'th column of α_2 is trivial and the last row and column of ϵ_1' are the same as those of the $(n+1-r)$ identity matrix. Let ξ_i denote the i'th column of α_2 $(1 \le i \le n-r)$ and set $\tau_2 = \prod_{i=1}^{n-r} Hm_{r+i}(-\xi_i)$. Then $\tau_2 \in Q_n$ and

$$\tau_2\tau_1\alpha\tau = \begin{pmatrix} \epsilon_2 & 0 \\ \gamma_2 & \bar{\epsilon}_2^{-1} \end{pmatrix}$$

where

$$\epsilon_2 = \begin{pmatrix} I & 0 \\ 0 & \epsilon_2' \end{pmatrix} \in E_{n+1}(R)$$

and $\epsilon_2' \in GL_{n+1-r}(R)$ whose last row and column are the same as those of the $(n+1-r)$ identity matrix. Thus $\epsilon_2' \in E_{n+1}(R) \cap GL_{n-r}(R)$. Since $n - r \geq m + 1$ and A satisfies SR_m, we obtain by stability for K_1 of the general linear group [Bs, Theorem 5.4.2] that $\epsilon_2' \in E_{n-r}(R)$. Set

$$\tau_3 = \begin{pmatrix} \epsilon_2^{-1} & 0 \\ 0 & \bar{\epsilon}_2 \end{pmatrix}.$$

Then $\tau_3 \in Q_n$ and

$$\tau_3\tau_2\tau_1\alpha\tau = \begin{pmatrix} I & 0 \\ \gamma_3 & I \end{pmatrix}.$$

Since the matrix on the right hand side of the equality lies in $GH_{2n}(R,\ a_1, \ldots, a_r)$, it must lie in Q_n. Thus $\tau_1\alpha\tau$ lies in Q_n. □

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