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ARF'S THEOREM FOR TRACE NOETHERIAN AND OTHER RINGS

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1. Introduction and statement of main results

The purpose of the paper is to extend Arf's Theorem below to a larger class of rings.

Arf's Theorem [2, Satz 13]. *Let k be a field of characteristic 2 such that the similarity classes of central simple k-algebras of dimension 4 form a group under $\otimes_k$. Then the isomorphism class of a nonsingular quadratic form q over k is determined by three invariants; the rank $r(q)\in\mathbb{Z}$, the Clifford algebra $C(q)\in \mathrm{Br}(k)=$ Brauer group (k), and the Arf invariant $\Delta(q)\in k/\{c+c^2 | c\in k\}$.*

A consequence of the extension will be that one can remove Arf's restriction on the similarity classes of central simple k-algebras. This is accomplished by replacing the invariants $C(q)$ and $\Delta(q)$ by a single invariant $\beta(q)$ with values in $k\otimes_{k^2} k/\{a\otimes b=b\otimes a,\ a\otimes b=a\otimes b^2a\}$.

We describe now the extension. The key idea will be to replace k in the tensor product above by a quotient Γ/Λ of two form parameters.

Let A be a ring with involution $a\mapsto \bar{a}$; thus $\overline{ab}=\bar{b}\bar{a}$ and $\bar{\bar{a}}=a$ for all $a, b\in A$. Let $\lambda\in$ center A such that $\lambda\bar{\lambda}=1$. A *form parameter* Λ is an additive subgroup of A such that

(1) $\{a-\lambda\bar{a} \mid a\in A\}\subset\Lambda\subset\{a \mid a\in A,\ a=-\lambda\bar{a}\}$,

(2) $a\Lambda\bar{a}\subset\Lambda$ for all $a\in A$.

The minimum and maximum choice of the form parameter are denoted respectively by min and max. A *Λ-quadratic module* is a pair (M,ψ) where M is a right A-module and ψ is a sesquilinear form on M. Associated to (M,ψ) are a *Λ-quadratic form* $q_\psi: M\to A/\Lambda,\ \ m\mapsto[\psi(m,m)]$, and an *even λ-hermitian form* $\langle m,n\rangle_\psi=\psi(m,n)+\lambda\overline{\psi(n,m)}$. A *morphism* $(M,\psi)\to(M',\psi')$ of Λ-quadratic modules is an A-linear map $M\to M'$ which preserves the Λ-quadratic and λ-hermitian forms. Define the *product* $(M,\psi)\perp(M',\psi')=(M\oplus M',\psi\oplus\psi')$. Call (M,ψ) *nonsingular* if M is a finitely generated projective A-module and the map $M\to\mathrm{Hom}_A(M,A)$,

$m \mapsto \langle m, \ \rangle_\psi$, is bijective. An example of a nonsingular module is the *hyperbolic module* $\mathbb{H}(P) = (P \oplus \mathrm{Hom}_A(P, A), \psi_P)$ such that P is a finitely generated projective right A-module and $\psi_P(p, f), (q, g)) = f(q)$. $\mathrm{Hom}_A(P, A)$ is given a right A-module structure via the rule $(fa)(p) = \bar{a}f(p)$.

$$\mathbb{H}(A) = \left(A \oplus A, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

is called the *hyperbolic plane* (if (u, v) and $(x, y) \in A \oplus A$, and if

$$\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$\psi((u, v), (x, y)) = {}^{(\bar{u}, \bar{v})}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \bar{v}x).$$

Let Γ be another form parameter such that $\Lambda \subset \Gamma$. If $a, b \in \Gamma$ then the Λ-quadratic modules

$$\left(A \oplus A, \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}\right)$$

are called *quasi hyperbolic planes* and will play an important part in our work. Basic facts concerning Λ-quadratic modules can be found in [13] and [8], and a mini introduction to the subject can be found in [9, Section 2].

If the involution on A is trivial then a 0-quadratic module is the classical definition of a quadratic form. If the involution is arbitrary then it follows from lemma 1 below that a max-quadratic module is the classical definition of an even λ-hermitian form. if A is an integral group ring $\mathbb{Z}\pi$, $\lambda = \pm 1$, and $\Lambda = \min$, then one obtains the kind of form which arises in geometric surgery.

Lemma 1. *Let M and M' be right A-modules and $f: M \to M'$ an A-linear map. Then f defines a homomorphism $(M, \psi) \to (M', \psi')$ of max-quadratic modules $\Leftrightarrow$ f preserves the associated even λ-hermitian forms.*

The proof is given in Section 3.

For fields k of characteristic 2 with trivial involution, we shall show that the isomorphism class of a nonsingular 0-quadratic module (V, ψ), i.e. classical quadratic form q_ψ, is determined by the rank V and an invariant $\beta(V, \psi) \in k \otimes_{k^2} k / \{a \otimes b = b \otimes a, a \otimes b = a \otimes b^2 a\}$. An easy exercise [16; XIV, Section 9] shows that the isomorphism class of the max-quadratic module (V, ψ) is determined by the rank V. Thus, two nonsingular 0-quadratic modules (V, ψ) and (V', ψ') are isomorphic $\Leftrightarrow (V, \psi)$ and (V', ψ') are isomorphic as max-quadratic modules and $\beta(V, \psi) = \beta(V', \psi')$. The extension of Arf's Theorem shall take this form.

Fix two form parameters Λ and Γ such that $\Lambda \subset \Gamma$. If $x \in \Gamma$, then the rule $x \mapsto ax\bar{a}$ (resp. $x \mapsto \bar{a}xa$) induces a left (resp. right) action of A on Γ/Λ. Let

$$S(\Gamma/\Lambda) = \Gamma/\Lambda \otimes_A \Gamma/\Lambda / \{a \otimes b = b \otimes a, a \otimes b = a \otimes ba\bar{b}\}.$$

The letter S is used to remind the reader that $S(\Gamma/\Lambda)$ is a quotient of the symmetric tensor product of Γ/Λ. If k is as above, and if $A = k$, $\Lambda = 0$, and $\Gamma = k$ then $S(\Gamma/\Lambda) = k \otimes_{k^2} k/\{a \otimes b = b \otimes a, a \otimes b = a \otimes b^2 a\}$.

Call two nonsingular Λ-quadratic modules (M, ψ) and (M', ψ') *stably isomorphic* if

$$(M, \psi) \perp \overbrace{\mathbb{H}(A) \perp \cdots \perp \mathbb{H}(A)}^{n} \cong (M', \psi') \perp \overbrace{\mathbb{H}(A) \perp \cdots \perp \mathbb{H}(A)}^{n}$$

for some n.

Let $\boldsymbol{Q}(A, \Lambda)$ = category with product of nonsingular Λ-quadratic modules.

Theorem 1. *Assume that A has a family $0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$ of involution invariant ideals with the following properties. If $A_i = A/\mathfrak{g}_i$, Λ_i = image $\Lambda \to A_i$ and Γ_i = image $\Gamma \to A_i$ then for each i such that $1 \leq i \leq n-1$ either $\mathfrak{g}_{i+1}/\mathfrak{g}_i \subset$ annihilator$_{A_i}(\Gamma_i/\Lambda_i)$ or A_i is $\mathfrak{g}_{i+1}/\mathfrak{g}_i$—adically complete, and A_n is semisimple of characteristic 2 (if $\Gamma_{n-1} = \Lambda_{n-1}$, then $A_n = 0$). Then there is a surjective function*

$$\beta: \boldsymbol{Q}(A, \Lambda) \to S(\Gamma/\Lambda)$$

which is well defined on isomorphism classes, respects products (i.e. $\beta((M, \psi) \perp (M', \psi')) = \beta(M, \psi) + \beta(M', \psi')$), and has the property that two nonsingular Λ-quadratic modules (M, ψ) and (M', ψ') are stably isomorphic $\Leftrightarrow$ (M, ψ) and (M', ψ') are stably isomorphic as Γ-quadratic modules and $\beta(M, \psi) = \beta(M', \psi')$.

Furthermore, without any restriction on A_n, the canonical map below is an isomorphism

$$S(\Gamma/\Lambda) \xrightarrow{\cong} S(\Gamma_n/\Lambda_n).$$

Note that any semisimple ring A with involution satisfies the hypotheses of Theorem 1.

Call A *trace neotherian* if A is a noetherian module over the subring generated additively by 1 and all $c + \bar{c}$ such that $c \in$ center A. For example, any order A over a Dedekind ring of characteristic $\neq 2$ is a trace noetherian. On the other hand, an infinite ring with characteristic 2 and trivial involution is not trace noetherian.

Proposition 1. *Trace noetherian rings satisfy the hypotheses of Theorem* 1.

Remark 1. If two nonsingular Λ-quadratic modules (M, ψ) and (M', ψ') are stably isomorphic, then it turns out that under suitable hypotheses on A and (M, ψ) one can assert that (M, ψ) and (M', ψ') are isomorphic. The phenomenon is called cancellation. If A is a field (resp. local ring), then cancellation holds for all nonsingular Λ-quadratic modules by a theorem of E. Witt [19] (resp. the author [3]). More generally [3] shows the following (a résumé of [3] is found in [4]). If A is finitely generated as a module over its center and if the maximal ideal space of the center is noetherian of finite dimension d then cancellation holds for nonsingular Λ-quadratic modules whose h-rank $> d$; if h-rank $(M, \psi) \leq d$, then (M, ψ) stably isomorphic to (M', ψ') implies that

$$(M, \psi) \perp \overbrace{\mathbb{H}(A) \perp \cdots \perp \mathbb{H}(A)}^{n} \cong (M', \psi') \perp \overbrace{\mathbb{H}(A) \perp \cdots \perp \mathbb{H}(A)}^{n}$$

for any $n \geq d + 1 - (h\text{-rank}(M, \psi))$. If one takes into account the size of Λ, then H. Bass [12] has shown that some technical improvements in the size of n are obtainable in certain circumstances.

Remark 2. If $\mathfrak{g}$ is the ideal of A generated by all $c + \bar{c}$ such that $c \in$ center A then $\mathfrak{g} \subset \text{annihilator}_A(\Gamma/\Lambda)$ and $A/\mathfrak{g}$ has characteristic 2.

Next we record some consequences of Theorem 1.

Let π be a group. Let $\chi : \pi \to \{\pm 1\}$ be an homomorphism and let the integral group ring $\mathbb{Z}\pi$ have the involution $a \mapsto \bar{a}$ such that $\bar{\sigma} = \chi(\sigma)\sigma^{-1}$ for all $\sigma \in \pi$. Let $\lambda = \pm 1$ and let $\pi_\lambda =$ subgroup of π generated by all $\sigma \in \pi$ such that $\sigma = -\lambda\bar{\sigma}$. Note that $\sigma = -\lambda\bar{\sigma} \Rightarrow \sigma^2 = 1$.

Corollary 1. *Let $\pi' \subseteq \pi$ be a normal subgroup of π such that the mixed commutator group $[\pi', \pi_\lambda] = 1$. Let $\Lambda \subseteq \Gamma$ be form parameters on $\mathbb{Z}\pi$ defined with respect to λ and the involution above, and let $\Lambda' \subseteq \Gamma'$ denote respectively their images under the canonical map $\mathbb{Z}\pi \to \mathbb{Z}(\pi/\pi')$. Then the canonical map below is an isomorphism*

$$S(\Gamma/\Lambda) \xrightarrow{\cong} S(\Gamma'/\Lambda').$$

Proof. By Theorem 1 it suffices to show that the kernel $(\mathbb{Z}\pi \to \mathbb{Z}(\pi/\pi')) \subseteq \text{Ann}_{\mathbb{Z}\pi}(\Gamma/\Lambda)$. The kernel is generated as an ideal by all $1 - \sigma$ such that $\sigma \in \pi'$ and Γ/Λ is generated additively by elements $x = \sum_i a_i\sigma_i$ such that $\sigma_i \in \pi_\lambda$ and $a_i \in \mathbb{Z}$. The hypothesis $[\pi', \pi_\lambda] = 1$ implies that σ commutes with x. Thus $(1 - \sigma)x\overline{(1 - \sigma)} = (x + \chi(\sigma)x) - (\sigma + \bar{\sigma})x \equiv (\text{mod } \Lambda)\ 2x + (\sigma + \bar{\sigma})x \in \Lambda$.

The next result was announced in [5, Theorem 6].

Corollary 2. *Let π be a group. If π_λ is nilpotent, then*

$$S(\max(\mathbb{Z}\pi)/\min(\mathbb{Z}\pi)) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ generated by } [1 \otimes 1] & \text{if } \lambda = -1, \\ 0 & \text{if } \lambda = 1. \end{cases}$$

Proof. Consider the canonical commutative diagram.

$$\begin{array}{ccc} S(\max(\mathbb{Z}\pi_\lambda)/\min(\mathbb{Z}\pi_\lambda)) & \xrightarrow{\quad f \quad} & S(\max(\mathbb{Z}\pi)/\min(\mathbb{Z}\pi)) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & S(\max(\mathbb{Z})/\min(\mathbb{Z})). & \end{array}$$

Since π_λ is nilpotent, π_λ has a sequence of normal subgroups $1=\gamma_0\subseteq\gamma_1\subseteq\cdots\subseteq\gamma_n=\pi_\lambda$ such that $\gamma_i/\gamma_{i-1}\subseteq$ center $(\pi_\lambda/\gamma_{i-1})$ for all $1\leq i\leq n$. Applying Corollary 1 n times, one obtains that p_1 is an isomorphism. One must apply the full force of Corollary 1 because the image of $\max(\mathbb{Z}\pi_\lambda)$ in $\mathbb{Z}(\pi_\lambda/\gamma_i)$ is not necessarily $\max(\mathbb{Z}(\pi_\lambda/\gamma_i))$ for $i\neq 0$ or n. The map f is surjective because the map $\max(\mathbb{Z}\pi_\lambda)\to\max(\mathbb{Z}\pi)/\min(\mathbb{Z}\pi)$ is surjective. Thus, from the commutativity of the diagram, one deduces that f and p_2 are isomorphisms. If $\lambda=1$, then $\max(\mathbb{Z})=\min(\mathbb{Z})=0$; thus $S(\max(\mathbb{Z})/\min(\mathbb{Z}))=0$. If $\lambda=-1$, then $\max(\mathbb{Z})=\mathbb{Z}$, $\min(\mathbb{Z})=2\mathbb{Z}$, and one computes easily that $S(\max(\mathbb{Z})/\min(\mathbb{Z}))\cong\mathbb{Z}/2\mathbb{Z}$ and is generated by $[1\otimes 1]$.

Let

$$\mathfrak{p}: A\to A,\quad a\mapsto a+a\bar{a},$$

$$\mathrm{s}: A\to A,\quad a\mapsto a+\bar{a}.$$

Corollary 3. *Let D be a characteristic 2 division ring with involution. Suppose that* max/min *has dimension 1 over D, e.g. $D=k$ is a perfect field of characteristic 2 with trivial involution. If x is a basis element for* max/min, *then the map below is an isomorphism*

$$S(\max/\min)\to D/\{s(D)+\mathfrak{p}(D)\},\quad [xa\otimes bx]\mapsto[ab].$$

The proof of Corollary 3 is an easy exercise.

C. Clauwens [*C*] has some overlap with the following result in the case A is the integral group ring $\mathbb{Z}\pi$ of a finite group π. In fact the number r below is the number of conjugacy classes found in [14, Section 4].

Corollary 4. *Suppose the hypotheses of Theorem 1. Suppose in addition that A_n is finite (e.g. A is a $\mathbb{Z}$-order). Factor the* center (A_n) *into a product* center $(A_n)=\prod_i k_i$ *such that k_i is either an involution invariant field or k_i is a product of two fields exchanged by the involution.*

(a) *Let r = number of fields k_i with trivial involution. If $\lambda=-1$, then*

$$S(\max/\min)\cong(\mathbb{Z}/2\mathbb{Z})^r$$

(b) *Let r_Γ* (resp. r_Λ) = *number of fields k_i with trivial involution such that $k_i\subset$* image *$\Gamma\to A_n$ (resp. $k_i\subset$* image *$\Lambda\to A_n$). Then*

$$S(\Gamma/\Lambda)\cong(\mathbb{Z}/2\mathbb{Z})^{r_\Gamma-r_\Lambda}.$$

Note. If $\lambda=-1$ (resp. $\lambda\neq-1$), then it is necessarily (resp. not necessarily) true that the image in A_n of the maximal form parameter for A is the maximum form parameter for A_n.

A detailed proof of Corollary 4 can be found in [10]. The commutative case is an easy exercise.

Theorem 1 will be proved in the framework of algebraic K-theory. Next we translate Theorem 1 into an equivalent result in algebraic K-theory.

Recall that if $\boldsymbol{C}$ is a category with a commutative associative product $\perp$ then $K_0\boldsymbol{C}=$ the free abelian group on the isomorphism classes $[M]$ of objects M of $\boldsymbol{C}$ modulo the relations $[M\perp N]=[M]+[N]$. One can check easily that two objects M and N have the same class $[M]=[N]\in K_0\boldsymbol{C}\Leftrightarrow$ there is an object P such $M\perp P\cong N\perp P$. Let

$\boldsymbol{Q}(A,\Lambda)=$ category with product of nonsingular Λ-quadratic modules,
$KQ_0(A,\Lambda)=K_0\boldsymbol{Q}(A,\Lambda)$,
$WQ_0(A,\Lambda)=KQ_0(A,\Lambda)/\{\mathbb{H}(P)|P$ finitely generated projective$\}$.

It follows from Lemma 2 (in Section 3) that two nonsingular Λ-quadratic modules have the same class in $KQ_0(A,\Lambda)\Leftrightarrow$ they are stably isomorphic. From this fact it follows that Theorem 2 below implies Theorem 1.

Theorem 2. *Suppose the hypotheses of Theorem* 1 *are satisfied. Then there are split exact sequences*

$$0\to S(\Gamma/\Lambda)\to\left\{\begin{matrix}KQ_0(A,\Lambda)\to KQ_0(A,\Gamma)\\ WQ_0(A,\Lambda)\to WQ_0(A,\Gamma)\end{matrix}\right\}\to 0,$$

$$a\otimes b\mapsto\left[A\oplus A,\begin{pmatrix}a&0\\1&b\end{pmatrix}\right]-[\mathbb{H}(A)]$$

and the canonical map below is an isomorphism

$$S(\Gamma/\Lambda)\xrightarrow{\cong}S(\Gamma_n/\Lambda_n).$$

The next result is an easy consequence of Theorem 2. The proof is left to the reader.

Corollary 5. *If A is a commutative semilocal ring which has characteristic* 2, *trivial involution, and is* (*Jacobson radical* (A))*-adically complete then* $WQ_0^1(A,\Lambda)\cong S(\max/\Lambda)$. *In particular* $WQ_0^1(A,0)\cong A\otimes_{A^2}A/\{a\otimes b=b\otimes a,\ a\otimes b=a\otimes b^2a\}$.

If X is an involution invariant subgroup of $K_1(A)$ then one has the concept of a discr-based-X Λ-quadratic module. For a precise definition see [7] or [9, Section 2]. Let $\boldsymbol{Q}(A,\Lambda)_{\text{discr-based-}X}$ denote the category with product of all such modules. Let

$$KQ_0(A,\Lambda)_{\text{discr-based-}X}=K_0\boldsymbol{Q}(A,\Lambda)_{\text{discr-based-}X}$$

and

$$WQ_0(A,\Lambda)_{\text{discr-based-}X}=KQ_0(A,\Lambda)_{\text{discr-based-}X}/\mathbb{H}(A)_{\text{based}}.$$

It is worth noting that the *surgery obstruction groups* $L^{\mathrm{p}}_{2n}(\pi)$, $L^{\mathrm{h}}_{2n}(\pi)$, and $L^{\mathrm{s}}_{2n}(\pi)$ are defined as follows:

$$L^{\mathrm{p}}_{2n}(\pi) = WQ_0^{(-1)^2}(\mathbb{Z}\pi, \min),$$
$$L^{\mathrm{h}}_{2n}(\pi) = WQ_0^{(-1)^n}(\mathbb{Z}\pi, \min)_{\text{discr-based-}K_1(\mathbb{Z}\pi)},$$
$$L^{\mathrm{s}}_{2n}(\pi) = WQ_0^{(-1)^n}(\mathbb{Z}\pi, \min)_{\text{discr-based-}[\pm\pi]}.$$

Theorem 3. *Suppose the hypotheses of Theorem* 1 *are satisfied. Then there are split exact sequences*

$$0 \to S(\Gamma/\Lambda) \to \left\{ \begin{matrix} KQ_0(A, \Lambda)_{\text{discr-based-}X} \to KQ_0(A, \Gamma)_{\text{discr-based-}X} \\ WQ_0(A, \Lambda)_{\text{discr-based-}X} \to WQ_0(A, \Gamma)_{\text{discr-based-}X} \end{matrix} \right\} \to 0$$

$$a \oplus b \mapsto \left[A \oplus A, \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}\right] - \left[A \oplus A, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]$$

where $A \oplus A$ *has the prescribed basis* $(1, 0)$, $(0, 1)$.

The Proof is analogous to that of Theorem 2 and will be omitted.

Corollary 6. *Recall the notation prior to Corollary* 1. *Let* π *be a group such that* π_λ *is nilpotent. Let* X *be an involution invariant subgroup of* $K_1(\mathbb{Z}\pi)$. *Let* K *denote one of the functors* KQ_0, WQ_0, $KQ_0(\ )_{\text{discr-based-}X}$, $WQ_0(\ )_{\text{discr-based-}X}$. *Then*

$$K(\mathbb{Z}\pi, \min) \xrightarrow{\cong} \mathrm{K}(\mathbb{Z}\pi, \max) \quad \text{if } \lambda = 1,$$

and if $\lambda = -1$ *the sequence below is split exact*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K(\mathbb{Z}\pi, \min) \to K(\mathbb{Z}\pi, \max) \to 0,$$

$$1 \mapsto \left[\mathbb{Z}\pi \oplus \mathbb{Z}\pi, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right] - \left[\mathbb{Z}\pi \oplus \mathbb{Z}\pi, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right].$$

Corollary 6 follows easily from Corollary 2, Theorems 2 and 3, Corollary 7 in Section 2, and the analogy of Corollary 7 for discr-based-X quadratic modules.

2. Proofs

Proof of Proposition 1. Let k denote the subring of A generated additively by 1 and all $c + \bar{c}$ such that the $c \in$ center A. Let $\mathfrak{p}$ denote the ideal of k generated additively by all $c + \bar{c}$ above. Let $\mathfrak{g}_1 = \mathfrak{p}A$ and $\mathfrak{g}_2 =$ inverse image in A of the Jacobson radical of $A/\mathfrak{g}_1$. Since $\mathfrak{p} \subset \mathrm{Ann}_A(\Gamma/\Lambda)$ it follows that $\mathfrak{g}_1 \subset \mathrm{Ann}_A(\Gamma/\Lambda)$. Since $k/\mathfrak{p} = 0$ or $\mathbb{Z}/2\mathbb{Z}$ it follows that $A_1 = A/\mathfrak{g}_1$ is finite. Thus A_1 is $(\mathfrak{g}_2/\mathfrak{g}_1)$-adically complete and $A_2 = A/\mathfrak{g}_2$ is semisimple. A_2 has characteristic 2 because $2 \in \mathfrak{p}$.

Proof of Theorem 2. We recall briefly the group $KQ_1(A, \Lambda)$. If α is a matrix, let $\bar{\alpha}$ = transpose conjugate α. Let $GQ_{2n}(A, \Lambda)$ denote the subgroup of $GL_{2n}(A)$ of all

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \overline{\lambda\gamma} & \bar{\alpha} \end{pmatrix}$$

and the diagonal coefficients of $\bar{\gamma}\alpha$ and $\bar{\delta}\beta$ lie in Λ. Let $EQ_{2n}(A, \Lambda)$ denote the subgroup of $GQ_{2n}(A, \Lambda)$ generated by all

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon}^{-1} \end{pmatrix}$$

such that ε is a product of elementary matrices and by all

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

such that $\beta = -\lambda\bar{\beta}$, $\gamma = -\overline{\lambda\gamma}$, and the diagonal coefficients of β and $\bar{\gamma}$ lie in Λ. There is a natural map $GQ_{2n}(A, \Lambda) \to GQ_{2(n+1)}(A, \Lambda)$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} \alpha & & \beta & \\ & 1 & & 0 \\ \hline \gamma & & \delta & \\ & 0 & & 1 \end{array}\right),$$

and one sets $GQ(A, \Lambda) = \varinjlim GQ_{2n}0A, \Lambda)$ and $EQ(A, \Lambda) = \varinjlim EQ_{2n}(A, \Lambda)$. From Lemma 3 (in Section 3) it follows that $EQ(A, \Lambda)$ is the commutator subgroup of $GQ(A, \Lambda)$. We let $KQ_1(A, \Lambda) = GQ(A, \Lambda)/EQ(A, \Lambda)$.

Let $K'_0(F)$ denote the relative group [12, VII] associated to the cofinal functor $F: \boldsymbol{Q}(A, \Lambda) \to \boldsymbol{Q}(A, \Gamma)$. According to [12; VII, Section 5], there is an exact sequence

$$KQ_1(A, \Lambda) \to KQ_1(A, \Gamma) \xrightarrow{\partial} K'_0(F) \to KQ_0(A, \Lambda) \to KQ_0(A, \Gamma).$$

The surjectivity of $KQ_0(A, \Lambda) \to KQ_0(A, \Gamma)$ follows from the definition of Λ- and Γ-quadratic modules. The rest of the proof has essentially three steps

(i) $K'_0(F) \cong S(\Gamma/\Lambda)$ (valid for arbitrary A),

(ii) $\partial = 0$,

(iii) $S(\Gamma/\Lambda) \xrightarrow{\cong} S(\Gamma_n/\Lambda_n)$ (valid for arbitrary A_n).

(i)–(iii) establish the exactness of $0 \to S(\Gamma/\Lambda) \to KQ_0(A, \Lambda) \to KQ_0(A, \Gamma) \to 0$. From the exactness of $0 \to S(\Gamma_n/\Lambda_n) \to KQ_0(A_n, \Lambda_n) \to KQ_0(A_n, \Gamma_n) \to 0$, one deduces easily the exactness of $0 \to S(\Gamma_n/\Lambda_n) \to WQ_0(A_n, \Lambda_n) \to WQ_0(A_n, \Gamma_n) \to 0$. The exactness of $0 \to S(\Gamma_n/\Lambda_n) \to WQ_0(A, \Lambda) \to WQ_0(A, \Gamma) \to 0$ follows from (iii) and the exactness of the preceding sequence. Since A_n has characteristic 2 it follows by Lemma 2 (in Section 3) that $WQ_0(A_n, \Lambda_n)$ has exponent 2. Thus the sequence $0 \to S(\Gamma_n/\Lambda_n) \to WQ_0(A_n, \Lambda_n) \to WQ_0(A_n, \Gamma_n) \to 0$ is split. The splitting assertions in the theorem follows from (iii) and the splitting assertion for the sequence above.

Proof of (i). Let $a, b \in \Gamma$. Let (a, b) denote the Λ-quadratic module

$$(a, b) = \left(A \oplus A, \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}\right).$$

The Λ-quadratic and λ-hermitian forms associated to (a, b) depend only on the classes of a and b modulo Λ. $(0, 0)$ is the hyperbolic plane $\mathbb{H}(A)$. If $a_i, b_i \in \Gamma$ $(i = 1, \ldots, n)$, let

$$\mathop{\perp}_{i=1}^{n} (a_i, b_i) = \left(A^n \oplus A^n, \left(\begin{array}{ccc|ccc} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_n & & & \\ \hline 1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & b_n \end{array}\right)\right).$$

Let $\boldsymbol{Q}(A, \Lambda, \Gamma)$ denote the category with product whose objects are *symbols*

$$\left(\mathop{\perp}_{i=1}^{n} (a_i, b_i), \mathop{\perp}_{i=1}^{n} (c_i, d_i)\right).$$

The product is defined by $(M, N) \perp (M', N') = (M \perp M', N \perp N')$. A morphism $(M, N) \to (M', N')$ is an A-linear isomorphism $A^n \oplus A^n \to A^n \oplus A^n$ which induces isomorphisms $M \to M'$ and $N \to N'$ of Λ-quadratic modules. Let

$$KQ_0(A, \Lambda, \Gamma) = K_0\boldsymbol{Q}(A, \Lambda, \Gamma)/[M, N] + [N, P] = [M, P].$$

There is a canonical map $KQ_0(A, \Lambda, \Gamma) \to K_0'(F)$, $[M, N] \mapsto [M$, identity map on $A^n \oplus A^n, N]$, and using the proof of [11, 10.2] (see also [9, Section 3]) one can show easily that the above map is an isomorphism. Next one shows straight forward that the rules $((a, b), (c, d)) \mapsto a \otimes b - c \otimes d$ and $a \otimes b \mapsto ((a, b), (0, 0))$ induce mutually inverse homomorphisms $KQ_0(A, \Lambda, \Gamma) \to S(\Gamma/\Lambda)$ and $S(\Gamma/\Lambda) \to KQ_0(A, \Lambda, \Gamma)$.

Proof of (iii). Clearly the map $S(\Gamma/\Lambda) \to S(\Gamma_1/\Lambda_1)$ is surjective. Suppose that $\mathfrak{g}_1 \subset \mathrm{Ann}_A(\Gamma/\Lambda)$. If $a, b \in \Gamma$ and $b \in \mathfrak{g}_1$ then the relation $a \otimes b = a \otimes ba\bar{b}$ shows that $a \otimes b$ represents 0 in $S(\Gamma/\Lambda)$. Thus $S(\Gamma/\Lambda) \to S(\Gamma_1/\Lambda_1)$ is an isomorphism. Suppose that A is $\mathfrak{g}_1$-adically complete. Let

$$WQ_1(A, \Lambda) = KQ_1(A, \Lambda) \Big/ \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix} \,\Big|\, \alpha \in GL(A) \right\},$$

and consider the exact sequences

$$\begin{array}{ccccccccc}
WQ_1(A,\Lambda) & \longrightarrow & WQ_1(A,\Gamma) & \longrightarrow & S(\Gamma/\Lambda) & \longrightarrow & KQ_0(A,\Lambda) & \longrightarrow & KQ_0(A,\Gamma) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle g_0} \\
WQ_1(A_1,\Lambda_1) & \rightarrow & WQ_1(A_1,\Gamma_1) & \rightarrow & S(\Gamma_1/\Lambda_1) & \rightarrow & KQ_0(A_1,\Lambda_1) & \rightarrow & KQ_0(A_1,\Gamma_1)
\end{array}$$

f_j and g_j $(j=0,1)$ are isomorphisms by Lemmas 4 and 5 (in Section 3). Thus $S(\Gamma/\Lambda)\to S(\Gamma_1/\Lambda_1)$ is an isomorphism.

Proof of (ii). (iii) allows one to reduce to the case A is semisimple. Since KQ_1 respects finite products one can reduce to the case A is simple. Using some easy Morita theory, one can reduce to the case $A=D$ is a division ring. $KQ_1(D,\Gamma)$ is generated by 2×2 matrices

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$$

which lift back to $KQ_1(D,\Lambda)$. One could do the simple case directly and avoid the Morita argument.

Corollary 7 below is partly a restatement of Theorem 2 and partly a summing up of certain results obtained in the proof of Theorem 2. To round out the results, we introduce a little more notation. Let γ and β denote matrices such that $\gamma=-\overline{\lambda\gamma}$, $\beta=-\lambda\bar{\beta}$, and the diagonal coefficients of $\bar{\gamma}$ and β lie in Γ. If the diagonal coefficients of $\bar{\gamma}+\bar{\gamma}\beta\gamma$ lie in Λ, define

$$(\gamma,\beta)=\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}.$$

Let $EQ(A,\Lambda,\Gamma)$ denote the multiplicative group generated by all (γ,β) and their transpose conjugates. Let $EQ(A,\Lambda)$ denote the subgroup of $EQ(A,\Lambda,\Gamma)$ generated by all (γ,β) and their transpose conjugates such that the diagonal coefficients of $\bar{\gamma}$ and β lie in Λ. One has $EQ(A,\Lambda)\subset EQ(A,\Lambda,\Gamma)\subset GQ(A,\Lambda)$. In R. Sharpe's paper [17] the group $EQ(A,\min)$ is denoted $EU(A)$. In [17, Section 5] there is a certain normal form of elements of $EU(A)$. If one examines the proof, one sees that it can be used verbatim to establish a normal form for $EQ(A,\Lambda)$. (We give in [8, Section 5] a shorter version of Sharpe's proof for arbitrary form parameter.) The normal form says that every element of $EQ(A,\Lambda)$ can be written as a product

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \gamma' & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon}^{-1} \end{pmatrix}\begin{pmatrix} 0 & \Sigma \\ -\Sigma^{-1} & 0 \end{pmatrix}$$

such that the diagonal coefficients of $\bar{\gamma}$, $\bar{\gamma}'$, and β lie in Λ, ε is a product of elementary matrices, and

$$\Sigma = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \end{pmatrix}$$

Since

$$\begin{pmatrix} 0 & \Sigma \\ -\Sigma^{-1} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon}^{-1} \end{pmatrix}$$

lie in $EQ(A, \Omega)$ for any Ω, one deduces easily an exact sequence $0 \to EQ(A, \Lambda, \Gamma)/EQ(A, \Lambda) \to KQ_1(A, \Lambda) \to KQ_1(A, \Gamma)$.

Proposition 2. *If A is any ring with involution, then $(\gamma, \beta)^8 \in (EQ(A, \Lambda)$. Furthermore, if the diagonal coefficients of β lie in Λ, then $(\gamma, \beta)^4 \in EQ(A, \Lambda)$.*

C. Clauwens has told me that he can prove a result similar to Proposition 2.

Remark 3. If A is a commutative $\mathbb{Z}$-order which has a certain technical condition satisfied for example by group rings and maximal real orders then by arithmetic methods one can show that $(\gamma, \beta)^4 \in EQ(A, \Lambda)$. In the case of a group ring $\mathbb{Z}\pi$, we have computed precisely the group $EQ(\mathbb{Z}\pi, \min, \max)/EQ(\mathbb{Z}\pi, \min)$ in [6, Theorems 10 and 15].

Corollary 7. *If A is any ring with involution, then there is an exact sequence*

$$0 \to EQ(A, \Lambda, \Gamma)/EQ(A, \Lambda) \to KQ_1(A, \Lambda) \to KQ_1(A, \Gamma) \xrightarrow{\partial} S(\Gamma/\Lambda) \xrightarrow{\rho}$$

$$KQ_0(A, \Lambda) \to KQ_0(A, \Gamma) \to 0$$

such that

$$\rho[x \otimes y] = \left[A \oplus A, \begin{pmatrix} x & 0 \\ 1 & y \end{pmatrix}\right] - [\mathbb{H}(A)] \quad \textit{and} \quad \partial\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \sum_{i=1}^{n} [x_i \otimes y_i]$$

where $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_n$) are the diagonal coefficients of $\bar{\gamma}\alpha$ (resp. $\bar{\delta}\beta$). Furthermore, if A satisfies the hypotheses of Theorem 1, then $\partial = 0$.

It was indicated already that Corollary 7 follows from the proof of Theorem 2.

Proof of Proposition 2. Since $(\gamma, \beta)^2 = (\gamma, 2\beta)$ and the diagonal coefficients of 2β lie in Λ, it suffices to prove the second assertion in the proposition. The idea of the proof is as follows. $(\gamma, \beta)^4 = (\gamma, 4\beta)$. Write

$$\begin{pmatrix}1 & 4\beta\\ 0 & 1\end{pmatrix} = \left[\left(\begin{array}{cc|cc}1 & 2 & & \\ 0 & 1 & & \\ \hline & & 1 & 0\\ & & -2 & 1\end{array}\right), \left(\begin{array}{cc|cc}1 & & 0 & \\ & 1 & & \beta\\ \hline & & 1 & \\ & & & 1\end{array}\right)\right] \left(\begin{array}{cc|cc}1 & & 0 & -2\beta\\ & 1 & -2\beta & 0\\ \hline & & 1 & \\ & & & 1\end{array}\right).$$

Conjugate the above by

$$\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}$$

and begin simplifying by multiplying on the left and right by elements of $EQ(A, \Lambda)$. The process shows that

$$(\gamma, \beta)^4 = \left(\begin{array}{cc|cc}1 & 2 & & \\ 0 & 1 & & \\ \hline & & 1 & 0\\ & & -2 & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline 0 & 2\gamma & 1 & \\ 2\gamma & 4\gamma & & \end{array}\right) \left(\begin{array}{cc|cc}1 & -2 & & \\ 0 & 1 & & \\ \hline & & 1 & 0\\ & & 2 & 1\end{array}\right)$$

$$\left(\begin{array}{cc|cc}1 & 0 & & \\ -2\gamma\beta & 1 & & \\ \hline & & 1 & 2\gamma\beta\\ & & 0 & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline 4\gamma\beta\gamma + 16\gamma\beta\gamma\beta\gamma & -8\gamma\beta\gamma & 1 & \\ -8\gamma\beta\gamma & 0 & & 1\end{array}\right)$$

$$\left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline 0 & & 1 & \\ & 4\gamma & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & 0 & \\ & 1 & & -\beta\\ \hline & & 1 & \\ & & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline 0 & & 1 & \\ & -4\gamma & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & 0 & \\ & 1 & & \beta\\ \hline & & 1 & \\ & & & 1\end{array}\right)$$

$$\left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline 0 & & 1 & \\ & 4\gamma & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & 4\beta + 16\beta\gamma\beta & 2\beta\\ & 1 & 2\beta & 0\\ \hline & & 1 & \\ & & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & 8\beta\gamma & & \\ 0 & 1 & & \\ \hline & & 1 & 0\\ & & -8\gamma\beta & 1\end{array}\right)$$

$$\left(\begin{array}{cc|cc}1 & & & \\ & 1 & & \\ \hline & -2\gamma & 1 & \\ -2\gamma & & & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & 0 & & \\ 2\beta\gamma & 1 & & \\ \hline & & 1 & -2\gamma\beta\\ & & 0 & 1\end{array}\right) \left(\begin{array}{cc|cc}1 & & 0 & -2\beta\\ & 1 & -2\beta & 4\beta\gamma\beta\\ \hline & & 1 & \\ & & & 1\end{array}\right).$$

3. Six lemmas

Lemma 1. *Let M and M' be right A-modules and $f: M \to M'$ an A-linear map. Then f defines a homomorphism $(M, \psi) \to (M', \psi')$ of max-quadratic modules $\Leftrightarrow$ f preserves the associated even λ-hermitian forms.*

Proof. The assertion $\Rightarrow$ follows by definition. Conversely, it must be shown that if f preserves the associated λ-hermitian forms, then f preserves the associated quadratic forms as well. Suppose that $\psi(m, n)+\lambda\overline{\psi(n, m)}=\psi'(f(m), f(n))+\lambda\overline{\psi'(f(n), f(m))}$ for all $m, n \in M$. We must show that $\psi(m, m)\equiv\psi'(f(m), f(m))$ mod max for all $m \in M$, i.e.

$$\psi(m, m)-\psi'(f(m), f(m))\in \max,$$

i.e.

$$\psi(m, m)-\psi'(f(m), f(m))=-\lambda\,(\overline{\psi(m, m)-\psi'(f(m), f(m))}),$$

i.e.

$$\psi(m, m)+\lambda\overline{\psi(m, m)}=\psi'(f(m), f(m))+\lambda\overline{\psi'(f(m), f(m))}.$$

But this is the first equation above with $m=n$.

Lemma 2. *If (M, ψ) is a nonsingular Λ-quadratic module then $(M, \psi)\perp(M, -\psi)\cong \mathbb{H}(M)$.*

Proof. Let $(N, \varphi)=(M\oplus M, \psi\oplus-\psi)$. Let $M_1=\{(m, m)\mid m\in M\}$ and $M_1'=\{(m, 0)\mid m\in M\}$. $N=M_1\oplus M_1'$ and M_1 is a totally isotropic subspace of (N, φ), i.e. $q_\varphi(x)=0$ and $\langle x, y\rangle_\varphi=0$ for all $x, y\in M_1$. Since $N\to N^*$, $n\mapsto\langle n, \ \rangle_\varphi$ is bijective, one can define a function $h: M_1'\to M_1$ via $-\varphi(x, y)=\langle hx, y\rangle_\varphi$ for all $x, y\in M_1'$. If $M_2=\{(m, 0)+h(m, 0)\mid m\in M\}$, then M_2 is also totally isotropic and $N=M_1\oplus M_2$. If $f: M_2\to M_1^*$, $x\mapsto\langle x, \ \rangle_\varphi$, then the map $M_1\oplus M_2\to M_1\oplus M_1^*$, $(m_1, m_2)\mapsto(m_1, f(m_2))$, defines an isomorphism $(N, \varphi)\to\mathbb{H}(M_1)$. Since $M_1\cong M$, it follows that $(N, \varphi)\cong \mathbb{H}(M)$.

Lemma 3 (*Quadratic Whitehead Lemma*). *If*

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}A & B\\ C & D\end{pmatrix}\in GQ_{2n}(A, \Lambda),$$

then the following equation holds in $GQ_{4n}(A, \Lambda)$

$$\left(\begin{array}{cc|cc}\alpha & & \beta & \\ & 1 & & \\ \hline \gamma & & \delta & \\ & & & 1\end{array}\right)\left(\begin{array}{cc|cc}A & & B & \\ & 1 & & \\ \hline C & & D & \\ & & & 1\end{array}\right)=\left(\begin{array}{cc|cc}1 & & & -\beta\\ \bar\delta & 1 & \lambda\bar\beta & \\ \hline & & 1 & -\delta\\ & & & 1\end{array}\right)\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1\\ & & -1 & \end{array}\right)$$

$$\left(\begin{array}{cc|cc}A & & B & \\ & \alpha & & \beta\\ \hline C & & D & \\ & \gamma & & \delta\end{array}\right)\left(\begin{array}{cc|cc}1 & & & -\lambda\bar B\\ A & 1 & B & \\ \hline & & 1 & -\bar A\\ & & & 1\end{array}\right)\left(\begin{array}{cc|cc}1 & -\bar D & & \\ & 1 & & \\ \hline & -\lambda\overline{C} & 1 & \\ C & & D & 1\end{array}\right).$$

Proof. By direct computation.

Lemma 4. *If $\mathfrak{g}$ is an involution invariant ideal of A such that A is $\mathfrak{g}$-adically complete then the canonical functor $\boldsymbol{Q}(A, \Lambda) \to \boldsymbol{Q}(A/\mathfrak{g}, \Lambda/\mathfrak{g} \cap \Lambda)$ induces a bijection from the isomorphism classes of $\boldsymbol{Q}(A, \Lambda)$ to the isomorphism classes of $\boldsymbol{Q}(A/\mathfrak{g}, \Lambda/\mathfrak{g} \cap \Lambda)$.*

The proof can be read verbatim from the proof given by C. T. C. Wall [18] Lemma 1 and Theorem 2 for the special case $\Lambda = \min$. Walls' remark that the result is "far from being true for hermitian forms" can be disregarded. A proof of Lemma 4 and a related result for not necessarily even hermitian forms will appear in [8, Section 3].

Lemma 5. *If $\mathfrak{g}$ is an involution invariant ideal of A such that A is $\mathfrak{g}$-adically complete, then the canonical map $KQ_1(A, \Lambda) \to KQ_1(A/\mathfrak{g}, \Lambda/\mathfrak{g} \cap \Lambda)$ is surjective, and the canonical map $WQ_1(A, \Lambda) \to WQ_1(A/\mathfrak{g}, \Lambda/\mathfrak{g} \cap \Lambda)$ is an isomorphism.*

Proof. The proof of Lemma 4 above shows that if (M, ψ) and $(N, \varphi) \in \boldsymbol{Q}(A, \Lambda)$ then any isomorphism $(M/\mathfrak{g}M, \psi) \to (N/\mathfrak{g}N, \varphi)$ can be lifted to an isomorphism $(M, \psi) \to (N, \varphi)$. The first assertion of Lemma 5 follows from the special case $(M, \psi) = \mathbb{H}(A^n) = (N, \varphi)$ and the fact that $GQ_{2n}(A, \Lambda) = \mathrm{Aut}(\mathbb{H}(A^n))$. By definition

$$WQ_1(A, \Lambda) = KQ_1(A, \Lambda) \Big/ \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix} \Big| \alpha \in GL(A) \right\}.$$

Thus, it is clear that $WQ_1(A, \Lambda) \to WQ_1(A/\mathfrak{g}, \Lambda/\mathfrak{g} \cap \Lambda)$ is surjective. An element in the kernel can be represented by a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GQ(A, \Lambda)$$

such that α and $\delta \equiv 1 \bmod \mathfrak{g}$ and β and $\gamma \equiv 0 \bmod \mathfrak{g}$. Thus α is invertible and one deduces easily that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\alpha}\gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} \equiv (\bmod\, EQ(A, \Lambda)) \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix}$$

which vanishes in $WQ_1(A, \Lambda)$.

The following lemma is not needed in the paper, but is useful in applying the results of the paper.

Lemma 6. *Let $\Lambda \subset \Gamma$ be two form parameters in A. Assume that A, Λ, and Γ satisfy the hypotheses in Theorem 1. If $\Lambda = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_s = \Gamma$ is a sequence of form parameters, then $S(\Gamma/\Lambda) \cong S(\Gamma_1/\Gamma_0) \oplus \cdots \oplus S(\Gamma_s/\Gamma_{s-1})$.*

Proof. It suffices to consider the case $s = 2$. From Theorem 2 it follows that all the maps in the commutative diagram

$$\begin{array}{ccc} S(\Gamma_2/\Gamma_0) & \longrightarrow & KQ_0(A, \Gamma_0) \\ \uparrow & & \uparrow \\ S(\Gamma_1/\Gamma_0) & \longrightarrow & KQ_0(A, \Gamma_0) \end{array}$$

are injective. Thus there is an exact sequence

$$0 \to S(\Gamma_1/\Gamma_0) \to S(\Gamma_2/\Gamma_0) \to S(\Gamma_2/\Gamma_1) \to 0.$$

Since all the groups have exponent 2, the sequence splits.

4. Construction of β

The purpose of this section is to construct $\beta: \boldsymbol{Q}(A, \Lambda) \to S(\Gamma/\Lambda)$ first under the hypotheses of Theorem 1, and then in the special case that A is a commutative, characteristic 2, complete local ring with trivial involution and $\Gamma = \max$. In the later case a particularly nice construction is obtained.

We begin with the general situation. Here β was constructed already in the proof of Theorem 2, but the details were a little sketchy. Below β is constructed as the composite of a number of functors and maps. Begin by taking the canonical functor $\boldsymbol{Q}(A, \Lambda) \to \boldsymbol{Q}(A_n, \Lambda_n)$. Factor A_n as a product $A_n = A_n^1 \times \cdots \times A_n^t$ of rings A_n^i such that A_n^i is either a simple ring with involution or a product $A_n^i = B \times B^{\mathrm{o}}$ of simple rings B and $B^{\mathrm{o}} = B^{\text{opposite}}$ such that the involution takes $(x, y) \mapsto (y, x)$. The form parameter Λ_n has a corresponding decomposition $\Lambda_n = \Lambda_n^1 \times \cdots \times \Lambda_n^t$ where $\Lambda_n^i = e^i \Lambda_n e^i$ and e^i is the central idempotent which defines A_n^i. Γ_n has an analogous decomposition $\Gamma_n = \Gamma_n^1 \times \cdots \times \Gamma_n^t$. Let $A_n^1 \times \cdots \times A_n^s$ denote the product of all the A_n^i such that A_n^i is simple and the involution on the center A_n^i is trivial. Next take the canonical functor $\boldsymbol{Q}(A_n, \Lambda_n) \to \boldsymbol{Q}(A_n^1, \Lambda_n^1) \times \cdots \times \boldsymbol{Q}(A_n^s, \Lambda_n^s)$. Write A_n^i as a matrix ring $A_n^i = \mathbb{M}_{m_i}(D_i)$ over the division ring D_i. Let $\sim$ denote the involution on A_n^i. By a theorem of A. Albert [1; X, Theorem 12] D_i has an involution $d \mapsto \bar{d}$ which is trivial on the center D_i, and from the Skolem–Noether theorem it follows that there is an element $\alpha_i \in \mathbb{M}_{m_i}(D_i)$ such that for all $x \in \mathbb{M}_{m_i}(D_i)$ $\alpha_i \tilde{x} \alpha_i^{-1} = \bar{x}$ where $\bar{x}$ = transpose $(\overline{x_{kl}})$ and x_{kl} is the (k, l)'th coefficient of x. If $A_n^{i\sim}$ and A_n^{i-} denote A_n with respectively the involution $\sim$ and $-$ then the functor $\boldsymbol{Q}(A_n^{i\sim}, \Lambda_n) \to \boldsymbol{Q}(A_n^{i-}, \alpha_i \Lambda_n^i)$ is an equivalence of categories with product. If $\Lambda_i = \alpha_i \Lambda_n^i \cap D_i$, then there is a Morita equivalence [15], [8, Section 7] $\boldsymbol{Q}(A_n^{i-}, \alpha_i \Lambda_n^i) \to \boldsymbol{Q}(D_i, \Lambda_i)$. Stringing together the functors above, one obtains a product preserving functor $\boldsymbol{Q}(A, \Lambda) \to \boldsymbol{Q}(D_1, \Lambda_1) \times \cdots \times \boldsymbol{Q}(D_s, \Lambda_s)$. Similarly, one obtains a product preserving functor $\boldsymbol{Q}(A, \Gamma) \to \boldsymbol{Q}(D_1, \Gamma_1) \times \cdots \times \boldsymbol{Q}(D_s, \Gamma_s)$. Next, take the canonical map $\boldsymbol{Q}(D_i, \Lambda_i) \to WQ_0(D_i, \Lambda_i)$, $(M, \psi) \mapsto [M, \psi]$. Since D_i has characteristic 2, it follows from Lemma 2 that $WQ_0(D_i, \Lambda_i)$ has exponent 2, and hence the exact sequence $0 \to S(\Gamma_i/\Lambda_i) \to WQ_0(D_i, \Lambda_i) \to WQ_0(D_i, \Gamma_i) \to 0$ has a retract β_i. Thus, one obtains a diagram

$$\begin{array}{ccc} S(\Gamma/\Lambda) & & \mathbf{Q}(A, \Lambda) \\ \downarrow p_1 & & \downarrow p \\ S(\Gamma_1/\Lambda_1)\times\cdots\times S(\Gamma_s/\Lambda_s) & \xleftarrow{\beta_1\times\cdots\times\beta_s} & WQ_0(D_1, \Lambda_1)\times\cdots\times WQ_0(D_s, \Lambda_s). \end{array}$$

The map p_1 is an isomorphism by Theorem 2, and β is constructed as the composite $\beta = p_1^{-1}(\beta_1 \times \cdots \times \beta_s)p$.

Next is a construction of β in the special circumstances indicated at the beginning of the section.

Proposition 3. *Suppose that A is a commutative, characteristic 2, (Jacobson radical A)-adically complete, local ring with trivial involution. If $(M, \psi) \in \mathbf{Q}(A, \Lambda)$, then M is a free A-module and has a basis $e_1, \cdots, e_{2m}$ such that the $2m \times 2m$ matrix*

$$(\psi(e_i, e_j)) = \begin{pmatrix} \begin{pmatrix} a_1 & 0 \\ 1 & b_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a_m & 0 \\ 1 & b_m \end{pmatrix} \end{pmatrix}.$$

The class $[\sum_{i=1}^m a_i \otimes b_i] \in S(A/\Lambda)$ does not depend on the choice of $e_1, \ldots, e_{2m}$ and one can define β by

$$\beta: \mathbf{Q}(A, \Lambda) \to S(A/\Lambda), \qquad [M, \psi] \mapsto \left[\sum_{i=1}^m a_i \otimes b_i\right].$$

Proof. Any finitely generated projective module over a local ring is free. See for example [12, III (2.13)]. Any nonsingular max-quadratic module over a characteristic 2 complete local ring with trivial involution is a product of hyperbolic planes. For the case of a field see [16; XIV, Section 9]. Then lift the result to A via Lemma 4. Let (M, ψ) be as in the proposition. One can always replace (M, ψ) by $(M, \psi + \varphi - \lambda\bar{\varphi})$ where $\bar{\varphi}(m, n) = \overline{\varphi(n, m)}$, because the Λ-quadratic (or λ-hermitian) forms associated to ψ and $\psi + \varphi - \lambda\bar{\varphi}$ are the same. Let $e_1, \ldots, e_n$ be a basis for M and let (a_{kl}) denote the matrix whose (k, l)'th coefficient is $\psi(e_k, e_l)$. After replacing ψ by $\psi + \varphi - \lambda\bar{\varphi}$ for a suitable φ, one can assume that $a_{kl} = 0$ for all (k, l) such that $k < l$. Since all nonsingular max-quadratic modules are a product of hyperbolic planes one can choose at the outset $e_1, \ldots, e_n$ such that

$$(a_{kl}) + \lambda\,(\text{transpose}(\overline{a_{kl}})) = \begin{pmatrix} \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} \end{pmatrix}.$$

It follows that

$$(a_{kl}) = \begin{pmatrix} \begin{pmatrix} a_1 & 0 \\ 1 & b_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a_m & 0 \\ 1 & b_m \end{pmatrix} \end{pmatrix}.$$

The homomorphism

$$\alpha : S(A/\Lambda) \to WQ_0(A, \Lambda), \qquad [a \otimes b] \mapsto \left[A \oplus A, \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \right]$$

is injective by Theorem 2, and bijective by the above. It follows that the element $[\sum_{i=1}^{m} a_i \otimes b_i] \in S(A/\Lambda)$ is independent of the choice of $e_1, \cdots, e_{2m=n}$. Clearly β defines an inverse to α.

5. Cancellation for quasi hyperbolic planes

Let A be a ring with involution. Let Λ and Δ be two form parameters in A such that $\Lambda \subset \Gamma$. Recall that a *quasi Λ-hyperbolic plane of level Γ* is a Λ-quadratic module

$$(a, b) = \left(A \oplus A, \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \right)$$

such that $a, b \in \Gamma$. If $a_i, b_i \in \Gamma$ $(1 \leq i \leq r)$ let

$$\perp_{i=1}^{r} (a_i, b_i) = (a_1, b_1) \perp \cdots \perp (a_r, b_r).$$

Theorem 4. *Let A be a ring with involution. Let $\Lambda \subset \Gamma$ be two form parameters in A. Assume that A has a family of involution invariant ideals $0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$ which satisfy the hypotheses of Theorem* 1.

(a) *Let (M, ψ) be a nonsingular Λ-quadratic module. If*

$$\perp_{i=1}^{r} (a_i, b_i) \perp (M, \psi) \cong \perp_{i=1}^{r} (c_i, d_i) \perp (M, \psi),$$

then

$$\perp_{i=1}^{r} (a_i, b_i) \perp (0,0) \cong \perp_{i=1}^{r} (c_i, d_i) \perp (0,0).$$

(b) *Let $A_n = A/\mathfrak{g}_n$. Factor A_n as a product $A_n = A_n^1 \times \cdots \times A_n^t$ of rings such that A_n^i is either a simple ring with involution or A_n^i is a product $A_n^i = B \times B^{\text{opposite}}$ of two simple rings B and B^{opposite} such that the involution takes $(x, y) \mapsto (y, x)$. Assume that if A_n^i is simple and if the involution on $k^i =$* center(A_n^i) *is trivial, then k^i is a perfect field and A_n^i is a matrix ring over k^i. Let (M, ψ) be a nonsingular Λ-quadratic module. If*

$$\perp_{i=1}^{r} (a_i, b_i) \perp (M, \psi) \cong \perp_{i=1}^{r} (c_i, d_i) \perp (M, \psi),$$

then

$$\perp_{i=1}^{r} (a_i, b_i) \cong \perp_{i=1}^{r} (c_i, d_i).$$

Proof. (b) Identify

$$\perp_{i=1}^{r} (a_i, b_i) = \left(A^r \perp A^r, \left(\begin{array}{ccc|ccc} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ \hline 1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & b_r \end{array}\right)\right).$$

Let F and G denote the images of $\perp_{i=1}^{r} (a_i, b_i)$ and $\perp_{i=1}^{r} (c_i, d_i)$ in $\boldsymbol{Q}(A_n, \Lambda_n)$ ($\Lambda_n =$ image $\Lambda \to A_n$). Since

$$\perp_{i=1}^{r} (a_i, b_i) \perp (M, \psi) \cong \perp_{i=1}^{r} (c_i, d_i) \perp (M, \psi),$$

it follows from Witt cancellation [19] (plus a Morita equivalence if some of the (A_n^i)'s are matrix rings of rank >1) that $F \cong G$. The idea of the rest of the proof is to pick an isomorphism $F \to G$ which one can lift to an isomorphism $\perp_{i=1}^{r} (a_i, b_i) \to \perp_{i=1}^{r} (c_i, d_i)$.

Corresponding to the decomposition $A_n = A_n^1 \times \cdots \times A_n^t$ there are decompositions $\Lambda_n = \Lambda_n^1 \times \cdots \times \Lambda_n^t$ and $\Gamma_n^n = \Gamma_n^1 \times \cdots \times \Gamma_n^t$ such that $\Lambda_n^i = e^i \Lambda_n e^i$, $\Gamma_n = e^i \Gamma_n e^i$, and e^i is the central idempotent which defines A_n^i. Let F^i and G^i denote the images of F and G in $\boldsymbol{Q}(A_n^i, \Lambda_n^i)$. If $\Lambda_n^i = \Gamma_n^i$, then the identity map on $\underbrace{A_n^i \oplus \cdots \oplus A_n^i}_{2r}$ defines an isomorphism $F^i \to G^i$. If $\Lambda_n^i \neq \Gamma_n^i$ then we know that there exists an isomorphism $\rho^i : F^i \to G^i$. ρ^i defines an element of $GQ_{2r}(A_n^i, \Gamma_n^i)$ because as Γ_n^i-quadratic modules $F^i = G^i = \mathbb{H}((A_n^i)^r)$. Since $\Lambda_n^i \neq \Gamma_n^i$ it is necessary that A_n^i is a simple factor such that the involution on $k^i =$ center(A_n^i) is trivial. Thus, by the hypotheses in (b) k^i is perfect, and A_n^i is a matrix ring over k^i. Since $\Gamma_n^i \neq$ min, it follows that $k^i \subset \Gamma_n^i$. From this it follows that $GQ_{2r}(A_n^i, \Gamma_n^i) = EQ_{2r}(A_n^i, \Gamma_n^i)$ (try first the case $A_n^i = k^i$). Thus, there is an isomorphism $F \to G$ which defines an element $\rho \in EQ_{2r}(A_n, \Gamma_n)$.

It suffices now to assume that $n = 1$ and to show that if $\rho \in EQ_{2r}(A_1, \Gamma_1)$ such that ρ defines an isomorphism $F \to G$, then there is a $\sigma \in EQ_{2r}(A, \Gamma)$ covering σ such that σ defines an isomorphism $\perp_{i=1}^r (a_i, b_i) \to \perp_{i=1}^r (c_i, d_i)$. The canonical map $EQ_{2r}(A, \Gamma) \to EQ_{2r}(A_1, \Gamma_1)$ is surjective. In order to use this fact we have insisted that $\rho \in EQ_{2r}(A_1, \Gamma_1)$ rather than in $GQ_{2r}(A_1, \Gamma_1)$.) After picking a representative $\tau \in EQ_{2r}(A, \Gamma)$ for ρ and applying τ to $\perp_{i=1}^r (a_i, b_i)$, one can assume that $\rho = 1$. Choose p_i and q_i in $\mathfrak{g}_1$ such that $a_i = c_i + p_i$ and $b_i = d_i + q_i$. Let $\mathfrak{g} = \mathfrak{g}_1$.

Suppose that $\mathfrak{g} \subset \mathrm{Ann}_A(\Gamma/\Lambda)$. Then

$$\begin{pmatrix} 1 & 0 \\ -\bar{q}_i & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -p_i \\ 0 & 1 \end{pmatrix} \in EQ_2(A, \Gamma)$$

define respectively isomorphisms $(a_i, b_i) \to (a_i, d_i)$ and $(a_i, d_i) \to (c_i, d_i)$ (remember that the Λ-quadratic and λ-hermitian forms associated to (c, d) depend only on the classes of c and d in Γ/Λ). It follows that there is a $\sigma \in EQ_{2r}(A, \Gamma)$ such that $\sigma \equiv 1 \bmod \mathfrak{g}$ and σ defines an isomorphism $\perp_{i=1}^r (a_i, b_i) \to \perp_{i=1}^r (c_i, d_i)$.

Suppose that A is $\mathfrak{g}$-adically complete. let $q = q_i$. Let $y_1 = \bar{q}aq$ and define inductively $y_i (i > 1)$ by $y_i = \bar{y}_{i-1} a y_{i-1}$. Since A is $\mathfrak{g}$-adically complete, it makes sense to define $\alpha = \sum_{i=1}^{\infty} \bar{y}_i a y_i$ and $y = \sum_{i=1}^{\infty} y_i$. Since $\bar{y}ay \in \Gamma$ and since it is checked easily that $\alpha - \bar{y}ay \in \min$, it follows that $\alpha \in \Gamma$. Since $q = b_i - d_i$, it is also true that $q \in \Gamma$. The matrix

$$\begin{pmatrix} 1 & 0 \\ -\bar{q} - \bar{\alpha} & 1 \end{pmatrix} \in EQ_2(A, \Gamma)$$

and defines an isomorphism $(a_i, b_i) \to (a_i, d_i)$. Similarly one can find a

$$\begin{pmatrix} 1 & -p - \beta \\ 0 & 1 \end{pmatrix} \in EQ_2(A, \Gamma)$$

which defines an isomorphism $(a_i, d_i) \to (c_i, d_i)$. It follows that there is a $\sigma \in EQ_{2r}(A, \Gamma)$ such that $\sigma \equiv 1 \bmod \mathfrak{g}$ and σ defines an isomorphism $\perp_{i=1}^r (a_i, b_i) \to \perp_{i=1}^r (c_i, d_i)$.

(a) is proved similarly to (b). One uses the extra hyperbolic plane $(0, 0)$ in the following way. Let F and G be as in (b). One knows that there is an isomorphism $F \to G$ and that this isomorphism can be represented by an element $\rho_{2r} \in GQ_{2r}(A_n, \Gamma_n)$, By stability [3], one can write $\rho_{2r} = \tau\varepsilon$ such that $\tau \in GQ_2(A_n, \Gamma_n)$ and $\varepsilon \in EQ_{2r}(A_n, \Gamma_n)$. If

$$\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \tau_1 = \left(\begin{array}{cc|cc} I_r & & 0 & \\ & a & & b \\ \hline 0 & & I_r & \\ & c & & d \end{array}\right),$$

then by Lemma 3 the element $\rho = \tau\varepsilon\tau_1^{-1} \in GQ_{2(r+1)}(A_n, \Gamma_n)$ actually lies in $EQ_{2(r+1)}(A_n, \Gamma_n)$. ρ defines an isomorphism $F \perp (0,0) \to G \perp (0,0)$ and one can lift as in (b) ρ to an isomorphism

$$\sigma: \perp_{i=1}^{r} (a_i, b_i) \perp (0,0) \to \perp_{i=1}^{r} (c_i, d_i) \perp (0,0).$$

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