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THE DIMENSION OF SPHERES WITH SMOOTH ONE FIXED POINT ACTIONS

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Abstract. The article proves that there are smooth one fixed point actions of the alternating group of degree 5 on the 8-dimensional sphere. It follows that a sphere has a smooth one fixed point action of some finite group if and only if the dimension of the sphere is greater than or equal to 6.

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1. Introduction

This paper completes research stretching over several decades to determine which spheres have smooth one fixed point actions of finite groups. Except for the 8-dimensional sphere, the answer has been known for about a decade; namely if $n \neq 8$ then the standard n-dimensional sphere S^n has a smooth one fixed point action of some finite group if and only if $n \geq 6$. In the current paper, it will be shown that S^8 has a smooth one fixed point action of the alternating group A_5 of degree 5. Recall that an action is called a *one fixed point* action, if there is exactly one point which is left fixed by every element of the group.

We review some of the major steps of the past.

A fundamental problem in the theory of transformation groups is determining fixed point sets of smooth actions of finite groups on disks and spheres. R. Oliver showed in [28, Theorem 7] that a finite group can act smoothly, without fixed points on some (closed) disk of undetermined dimension if and only if it is not mod p hyperelementary for all primes p. A finite group G is called *mod p hyperelementary*, if it has a normal

p-subgroup P such that the quotient G/P is hyperelementary. The result provides a necessary condition for a finite group to act smoothly, with exactly one fixed point on some sphere of undetermined dimension.

E. Stein was the first to find a smooth one fixed point action on a sphere, namely he showed that S^7 admits a smooth one fixed point action of the special linear group $SL(2, \mathbb{Z}_5)$ [37, Proposition 4.3]. Next T. Petrie proved in [34, Theorem A] that a finite, odd order abelian group having at least three noncyclic Sylow subgroups, e.g. $\mathbb{Z}_{pqr} \times \mathbb{Z}_{pqr}$ where p, q, and r are distinct odd primes, has a smooth one fixed point action on some higher dimensional spheres. The same conclusion holds for any finite group which is not mod p hyperelementary for all primes p, thanks to [13, Theorem A] and [12, Theorem A]. The result is the strongest possible one, in view of Oliver's work on disks.

Aside from Stein's result, all of the above smooth one fixed point theorems tell us only that there exist higher dimensional spheres which have smooth one fixed point actions of finite groups which are not mod p hyperelementary for all primes p. E. Laitinen and P. Traczyk posed in [14] the problem whether or not S^6 has a smooth one fixed point action of A_5. This was answered affirmatively by the second author. In fact he proved in [15, Theorem A], [17, Theorem 0.1], and [19, Theorem A] that if $n \geq 6$ and $n \neq 7, 8$ then S^n has smooth one fixed point actions of A_5. Thereafter it was shown in a joint paper [2, Theorem 7] that S^7 has a smooth one fixed point action of A_5. This left S^8 as the only unsolved case for $n \geq 6$. On the other hand, it has been shown in M. Furuta [11, Theorem 0.1] (cf. [16, Theorem A]), Buchdahl-Kwasik-Schultz [5, Theorem I.1, II.1, and II.4], and DeMichels [6, Theorem 5.1] that if $0 \leqq n \leqq 5$ then S^n does not have a smooth one fixed point action of any finite group.

It is worthwhile remarking that the existence of a smooth one fixed point action of A_5 on S^6 implies that for any $n \geq 7$, there is a continuous one fixed point action of A_5 on S^n, which is everywhere smooth, except possibly at the fixed point. This fact motivated Buchdahl-Kwasik-Schultz to show in [5, Theorem II.5] that for each $n \geq 6$, there is a locally linear, one fixed point action of A_5 on S^n. Their arguments, however, do not provide ways of constructing smooth one fixed point actions on spheres.

The advances above of T. Petrie and the current authors depend on parallel developments in equivariant surgery theory. In Petrie's case, it was his invention [33] (cf. [9], [35]) of equivariant surgery theory under the so-called strong gap hypothesis for singular sets. Here the surgery groups are those of Wall. In the authors' case, it was the second author's replacement of the strong gap hypotheses in Petrie's theory, by the gap hypothesis. Here the surgery groups are those of the first author. They are generalizations of Wall's groups, which are obtained by using form parameters to adjust the quadratic form. The procedure for applying equivariant surgery to one fixed point problems starts with a smooth two fixed point action satisfying one of the gap hypotheses and then uses equivariant surgery to remove one of the fixed points. However, for the current problem of establishing a smooth one fixed point action of A_5 on S^8, we were not able to find a smooth two fixed point action of A_5 on S^8, satisfying the gap hypothesis, but only smooth two fixed point actions of A_5 on S^8, whose singular sets have dimension up to and including 4, which is half of the dimension of S^8. This motivated us to develop in [4] equivariant surgery for smooth actions having dimensions of singular sets up to and including half the dimension of the manifold. Here the surgery groups generalize those above by using two form parameters to adjust simultaneously the quadratic and Hermitian forms, rather than just one parameter for adjusting the quadratic form. The resulting surgery theory is applied in the current paper to prove the theorem below. It played also a central role in establishing the smooth analog of Oliver's theorem in papers [12] and [13] above.

Theorem 1.1. *There are smooth one fixed point actions of* A_5 *on* S^8.

The following corollary follows immediately from the above.

Corollary 1.2. *For a natural number* n*, the following are equivalent:*

(1) *There is a smooth one fixed point action of some finite group on* S^n.
(2) *There is a smooth one fixed point action of* A_5 *on* S^n.
(3) $n \geq 6$.

The proof of Theorem 1.1 is organized as follows. In Section 2, we recall the Burnside ring and certain facts concerning it. This includes showing the existence of a certain idempotent. In Section 3, we recall equivariant framed maps and use the idempotent above to construct a G-equivariant framed map, whenever G is nontrivial and perfect. In Section 4, we prove Theorem 1.1. We start by showing that a certain two fixed point, linear action of A_5 on S^8 has only singular sets of dimension $\leqq 4$. Then the results of Section 3 are used to construct a good A_5-framed map to S^8. In order to show that the surgery obstruction of the map vanishes, we need to know that the map has certain properties. These are isolated in Proposition 4.1 whose proof is postponed to Section 5. Applying the equivariant surgery theory in [4], we convert the map to a homotopy equivalence whose source is a homotopy sphere of dimension 8 with exactly one fixed point. Using Stein's trick of taking the equivariant connected sum of copies of this manifold, we obtain a smooth one fixed point action of A_5 on the standard 8-dimensional sphere S^8. This completes the proof of Theorem 1.1. In section 5, we develop a certain way of performing equivariant surgery which we call the reflection method and apply this to prove Proposition 4.1.

2. The Burnside ring and an idempotent

Let G be a finite group. Let $\Omega(G)$ denote the Burnside ring of G, namely the Grothendieck group of the category of finite G-sets with addition and multiplication induced by disjoint union and Cartesian product, respectively. For any subgroup $H \subset G$, there is a canonical homomorphism $\chi_H : \Omega(G) \to \mathbb{Z}$ defined by

$$\chi_H(\alpha) = |A^H| - |B^H|$$

where A and B are finite G-sets, $\alpha = [A] - [B]$, A^H and B^H denote the sets of elements in A and B respectively which are left pointwise fixed by the action of H, and $|X|$ denotes the number of elements of X. By [8, IV, Theorem 5.7], we obtain

Lemma 2.1. *Suppose G is a finite nontrivial perfect group. Then there exists an idempotent α in $\Omega(G)$ such that $\chi_G(\alpha) = 1$ and $\chi_H(\alpha) = 0$ for any $H \neq G$.*

For a finite G-CW complex X, the n-th equivariant cohomology $\omega_G^n(X)$ of X is defined to be

$$\varinjlim_m [(\mathbb{R}^{n_1} \oplus \mathbb{C}[G]^m)^+ \wedge (X \amalg \{pt\}), (\mathbb{R}^{n_2} \oplus \mathbb{C}[G]^m)^+]_0^G,$$

where $n = n_2 - n_1 \in \mathbb{Z}$ and W^+ denotes the one point compactification of W, $\mathbb{C}[G]$ the complex regular representation of G, and $[A, B]_0^G$ the set of G-homotopy classes of basepoint preserving G-maps from A to B. We refer the reader to [8, pp.140–141] for details concerning $\omega_G^n(X)$, e.g. its group structure. For the special case where X is a point, we obtain

$$\omega_G^0(pt) = \varinjlim_m [(\mathbb{C}[G]^m)^+, (\mathbb{C}[G]^m)^+]_0^G \quad (\cong [S(\mathbb{R} \oplus \mathbb{C}[G]), S(\mathbb{R} \oplus \mathbb{C}[G])]_0^G)$$

and this becomes a ring whose multiplication is induced by taking composition of maps (cf. [25]). Furthermore $\omega_G^n(X)$ is a module over $\omega_G^0(pt)$, with multiplication defined as follows. If $[f : (\mathbb{C}[G]^m)^+ \to (\mathbb{C}[G]^m)^+] \in \omega_G^0(pt)$ and $[h : (\mathbb{R}^{n_1} \oplus \mathbb{C}[G]^m)^+ \wedge (X \amalg \{pt\}) \to (\mathbb{R}^{n_2} \oplus \mathbb{C}[G]^m)^+] \in \omega_G^n(X)$, then the product $[f][h] \in \omega_G^n(X)$ is defined to be the G-homotopy class including the composition $(id_{\mathbb{R}^{n_2+}} \wedge f) \circ h$. For each subgroup H of G, there is a canonical homomorphism $\deg_H : \omega_G^0(pt) \to \mathbb{Z}$ defined by

$$\deg_H([f]) = \deg(f^H : S(\mathbb{R} \oplus \mathbb{C}[G])^H \to S(\mathbb{R} \oplus \mathbb{C}[G])^H).$$

By Segal's theorem [36, Corollary], there is a canonical ring isomorphism $\Omega(G) \to \omega_G^0(pt)$ such that the diagram

$$\begin{array}{ccc} \Omega(G) & \xrightarrow{\cong} & \omega_G^0(pt) \\ & {}_{\chi_H}\searrow \quad \swarrow_{\deg_H} & \\ & \mathbb{Z} & \end{array}$$

commutes for every $H \subset G$ and we identify $\Omega(G)$ with $\omega_G^0(pt)$ via this isomorphism.

Lemma 2.2 ([33, Lemma 1.8], [34, Lemma 1.6])**.** *Let G and α be as in the previous lemma. For any finite G-CW complex X and any integer n, the restriction homomorphism $\alpha^{-1}\omega_G^n(X) \to \alpha^{-1}\omega_G^n(X^G)$ is an isomorphism, where $\alpha^{-1}\omega_G^n(X)$ denotes the localization of $\omega_G^n(X)$ with respect to the mutiplicatively closed set $\{\alpha\}$.*

Lemma 2.3 ([33, Corollary 1.9]). *Let $G = A_5$ and let Y be a finite G-CW complex such that $Y^G = \{y_+, y_-\}$. The $\omega_G^0(pt)$-module $\omega_G^0(Y^G)$ is free of rank 2 and we identify $\omega_G^0(y^G) = \Omega(G) \oplus \Omega(G)$. The assertion is that there exists an element $\omega \in \omega_G^0(Y)$ such that*

$$\omega|_{Y^G} = ([G/G] - \alpha,\ [G/G]) \in \Omega(G) \oplus \Omega(G),$$

where α is the element appearing in Lemma 2.1.

Define the subset $G(2)$ of G by

$$G(2) = \{g \in G \mid g^2 = e,\ g \neq e\}.$$

It is regarded as a G-set via conjugation. Let $SGW_0(\mathbb{Z}, G, G(2))$ denote the special Grothendieck–Witt ring defined in [12, p.509]. By definition, the multiplicative identity element 1_G of $SGW_0(\mathbb{Z}, G, G(2))$ is the equivalence class of $(\mathbb{Z}, B, \alpha)$, where $B : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the canonical multiplication and $\alpha : G(2) \to \mathbb{Z}$ is the map $g \mapsto 1$.

Let $\mathrm{subgp}(G)$ denote the category of subgroups of G with morphisms defined by inclusion and conjugation. For a precise definition, see [1]. Let $M : \mathrm{subgp}(G) \to$ ((abelian groups)) be a Mackey functor. By definition M is a pair of functors consisting of a covariant functor and a contravariant functor, both taking the same value on objects. It is assumed that the pair forms a bifunctor and satisfies the Mackey subgroup property. See [1]. If $K \to H$ is a morphism in $\mathrm{subgp}(G)$ then Ind_K^H denotes as usual the covariant map $\mathrm{Ind}_K^H : M(K) \to M(H)$ and is called *induction*. If $K \to H$ is as above then Res_K^H denotes as usual the contravariant map $\mathrm{Res}_K^H : M(H) \to M(K)$ and is called *restriction*.

Let $R : \mathrm{subgp}(G) \to$ ((rings)) be a Green ring functor [1] (cf. also [7, p.165]). By definition, R is a Mackey functor whose contravariant part delivers ring homomorphisms, but whose covariant part only group homomorphisms, and satisfies the Frobenius reciprocity law. A Mackey functor M is called an *R-module* or a *Green module* over R, if for each $H \subset G$, $M(H)$ is an $R(H)$-module and the following two properties are satisfied:

$$(\mathrm{Ind}_K^H r) \cdot x = \mathrm{Ind}_K^H (r \cdot \mathrm{Res}_K^H x) \quad \text{for all } r \in R(K) \text{ and } x \in M(H). \tag{2.1}$$

$$r \cdot (\mathrm{Ind}_K^H x) = \mathrm{Ind}_K^H ((\mathrm{Res}_K^H r) \cdot x) \quad \text{for all } r \in R(H) \text{ and } x \in M(K). \tag{2.2}$$

Lemma 2.4. *Let G be a nontrivial finite perfect group, α the idempotent in Lemma 2.1, and 1_G the multiplicative identity of the ring $SGW_0(\mathbb{Z}, G, G(2))$. Then, $\alpha 1_G = 0$ and hence*

$$SGW_0(\mathbb{Z}, G, G(2)) = \sum_{H \subsetneq G} \mathrm{Ind}_H^G (SGW_0(\mathbb{Z}, H, H(2))).$$

Consequently, for a module $M(-)$ over $SGW_0(\mathbb{Z}, -, -(2))$ the map

$$\mathrm{Res} : M(G) \to \bigoplus_{H \subsetneq G} M(H)$$

is injective.

Remark. For the lemma above, the property (2.2) is unnecessary.

Proof. By definition, $\mathrm{Res}^G_H([G/G]-\alpha)=0$ for all $H \subsetneq G$. By [12, Proposition 6.4], we get

$$([G/G]-([G/G]-\alpha))^2 1_G = 0,$$

which implies $\alpha 1_G = 0$. Since $[G/G]-\alpha = \sum_{H\subsetneq G} a_H[G/H]$ for some integers a_H, we obtain

$$1_G = ([G/G]-\alpha)1_G = \sum_{H\subsetneq G} a_H \mathrm{Ind}^G_H 1_H.$$

The remainder of the lemma follows from the equality

$$x = \sum_{H\subsetneq G} a_H(\mathrm{Ind}^G_H 1_H)\cdot x = \sum_{H\subsetneq G} a_H \mathrm{Ind}^G_H(1_H \cdot \mathrm{Res}^G_H x)$$

for $x \in M(G)$. □

3. Equivariant framed maps

For a finite group G, a pair (f,b) consisting of a G-map $f: X \to Y$ between G-manifolds and an isomorphism $b: T(X)\oplus \varepsilon_X(\mathbb{R}^m) \to f^*T(Y)\oplus\varepsilon_X(\mathbb{R}^m)$ of real G-vector bundles, for some m, is called a *G-framed map* from X to Y. Here $\varepsilon_X(\mathbb{R}^m)$ denotes the product bundle over X with fiber $\mathbb{R}^m$. If the degree of f is one then (f,b) is also said to be of *degree one*. In the case where X and Y are closed, a *G-framed cobordism* (F,B) between degree one, G-framed maps (f,b) and (f',b') is by definition a pair consisting of a degree one G-map

$$F: (W,\partial W) \to (I\times Y, Y\times \partial I),$$

where $I=[0,1]$, and a G-isomorphism

$$B: T(W)\oplus\varepsilon_W(\mathbb{R}^m) \to F^*(T(I\times Y))\oplus\varepsilon_W(\mathbb{R}^m)$$

satisfying the following conditions:

(1) $\partial W = (-X)\amalg X'$ as oriented G-manifolds,
(2) $F|_{(-X)} = f$ as G-maps $(-X)\to Y\times\{0\}$,
(3) $F|_{X'} = f'$ as G-maps $X'\to Y\times\{1\}$,
(4) $B|_{(-X)} = -id_{\varepsilon_{(-X)}(\mathbb{R})}\oplus b$ as G-isomorphisms

$$\varepsilon_{(-X)}(\mathbb{R})\oplus T(-X)\oplus\varepsilon_{(-X)}(\mathbb{R}^m)\to\varepsilon_{(-X)}(\mathbb{R})\oplus f^*T(Y)\oplus\varepsilon_{(-X)}(\mathbb{R}^m), \text{ and}$$

(5) $B|_{X'} = id_{\varepsilon_{X'}(\mathbb{R})}\oplus b'$ as G-isomorphisms

$$\varepsilon_{X'}(\mathbb{R})\oplus T(X')\oplus\varepsilon_{X'}(\mathbb{R}^m)\to\varepsilon_{X'}(\mathbb{R})\oplus f'^*T(Y)\oplus\varepsilon_{X'}(\mathbb{R}^m)$$

(after stabilizing b and b' if necessary), where $(-X)$ is a copy of X but with opposite orientation.

Let V be a real G-module with G-invariant inner product and set $Y = S(\mathbb{R} \oplus V)$. The north and south poles of Y, namely the points $(1,0)$, $(-1,0) \in \mathbb{R} \oplus V$, will be denoted by y_+ and y_-, respectively. We have the canonical isomorphisms

$$T(Y) \oplus \varepsilon_Y(\mathbb{R}) \cong \varepsilon_Y(\mathbb{R}) \oplus T(Y) \cong \nu(Y, \mathbb{R} \oplus V) \oplus T(Y) \cong T(\mathbb{R} \oplus V)|_Y \cong \varepsilon_Y(\mathbb{R} \oplus V).$$

Define $b_+ : T(Y) \oplus \varepsilon_Y(\mathbb{R}) \to T(Y) \oplus \varepsilon_Y(\mathbb{R})$ to be the identity map and Define $b_- : T(Y) \oplus \varepsilon_Y(\mathbb{R}) \to T(Y) \oplus \varepsilon_Y(\mathbb{R})$ by

$$b_-(y,(r,v)) = (y,(-r,v))$$

for $y \in Y$, $r \in \mathbb{R}$ and $v \in V$. From the above we obtain G-framed maps $\mathbf{1}_{Y,G} := (id_Y, b_+)$ and $-\mathbf{1}_{Y,G} := (id_Y, b_-)$.

Fix small G-invariant disk neighborhoods $U_{Y,+}$ and $U_{Y,-}$ of y_+ and y_- in Y, respectively. Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. For a set $\mathcal{A}$ of subgroups of G and maps φ_+, $\varphi_- : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, define the G-manifold Z_{φ_+,φ_-} by

$$Z_{\varphi_+,\varphi_-} = \coprod_{H \in \mathcal{A}} \left(\coprod_{i=1}^{\varphi_+(H)} (G \times_H U_{Y,+,H,i}) \amalg \coprod_{i=1}^{\varphi_-(H)} (G \times_H U_{Y,-,H,i}) \right),$$

where $U_{Y,+,H,i}$ and $U_{Y,-,H,i}$ are copies of $U_{Y,+}$ and $-U_{Y,+}$ respectively. Define the G-framed map $\boldsymbol{f}_{\varphi_+,\varphi_-}$ from Z_{φ_+,φ_-} to $U_{U,+}$ by

$$\boldsymbol{f}_{\varphi_+,\varphi_-} = \bigcup_{H \in \mathcal{A}} \left(\bigcup_{i=1}^{\varphi_+(H)} (G \times_H (\mathbf{1}_{Y,+,H,i}|_{U_{Y,+,H,i}})) \cup \bigcup_{i=1}^{\varphi_-(H)} (G \times_H (\mathbf{1}_{Y,-,H,i}|_{U_{Y,-,H,i}})) \right),$$

where $\mathbf{1}_{Y,+,H,i}$ and $\mathbf{1}_{Y,-,H,i}$ are copies of $\mathbf{1}_{Y,H}$ and $-\mathbf{1}_{Y,H}$ respectively.

Proposition 3.1. *Let G be a finite nontrivial perfect group and V a real G-module with $V^G = 0$. Let Y denote $S(\mathbb{R} \oplus V)$, α ($\in \Omega(G)$) the element appearing in Lemma 2.1, and ω ($\in \omega_G^0(Y)$) the element appearing in Lemma 2.3. Furthermore let $\mathcal{A}$ be a set of subgroups of G and φ_+, $\varphi_- : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ maps such that*

$$[G/G] - \alpha = \sum_{H \in \mathcal{A}} (\varphi_+(H) - \varphi_-(H))[G/H].$$

Then there exists a G-framed map $\boldsymbol{f} = (f, b)$ consisting of a G-map $f : X \to Y := S(\mathbb{R} \oplus V)$ and a G-vector bundle isomorphism

$$b : T(X) \oplus \varepsilon_X(\mathbb{R}^{m+1}) \to f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^{m+1}),$$

and H-framed cobordisms (F_H, B_H) between $\mathbf{1}_{Y,H}$ and $\mathrm{Res}^G_H \boldsymbol{f}$, consisting of H-maps

$$F_H : (W_H, \partial W_H) \to (I \times Y, \partial(I \times Y))$$

and H-vector bundle isomorphisms

$$B_H : T(W_H) \oplus \varepsilon_{W_H}(\mathbb{R}^{m+1}) \to F_H^* T(I \times Y) \oplus \varepsilon_{W_H}(\mathbb{R}^{m+1})$$

respectively, for all proper subgroups $H \subsetneq G$ such that

(1) $\boldsymbol{f}|_{f^{-1}(U_{Y,-})}$ *is the stabilization of* $\mathbf{1}_{Y,G}|_{U_{Y,-}}$,

(2) $\boldsymbol{f}|_{f^{-1}(U_{Y,+})}$ *is the stabilization of* $\boldsymbol{f}|_{\varphi_+,\varphi_-}$.

Here m is some (large) integer.

Proof. Define Z_+ and Z_- as the subsets $\{y_+\}$ and $\{y_-\}$ of Y respectively, and set

$$\gamma_+ = \sum_{H\in\mathcal{A}} (\varphi_+(H) - \varphi_-(H))[G/H],$$
$$\gamma_- = (1-0)[G/G],$$

where $Y^G = \{y_+, y_-\}$. It follows immediately from Lemma 2.3 that $\omega|_{Z_+} = \gamma_+$ and $\omega|_{Z_-} = \gamma_-$. By [20, Theorem 4.4], we obtain a G-normal map (f, b') consisting of a G-map $f : X \to Y$ and a G-vector bundle isomorphism

$$b' : T(X) \oplus \varepsilon_X(V') \to f^*T(Y) \oplus \varepsilon_X(V'),$$

and, for each $H \subsetneq G$, an H-normal cobordism (F_H, B'_H), between $\mathrm{Res}^G_H(id_Y, id_{T(Y)\oplus\varepsilon_Y(V')})$ and $\mathrm{Res}^G_H(f, b)$, consisting of an H-map $F_H : (W_H, \partial W_H) \to (I \times Y, \partial(I \times Y))$ and an H-vector bundle isomorphism

$$B'_H : T(W_H) \oplus \varepsilon_{W_H}(V') \to F_H^*T(I \times Y) \oplus \varepsilon_{W_H}(V'),$$

where V' is some real G-module. Using [20, Theorem 3.6], one can replace V' by $\mathbb{R}^{m+1}$ for sufficiently large m, and obtain a G-vector bundle isomorphism $b : T(X) \oplus \varepsilon_X(\mathbb{R}^{m+1}) \to f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^{m+1})$ and H-vector bundle isomorphisms $B_H : T(W_H) \oplus \varepsilon_{W_H}(\mathbb{R}^{m+1}) \to F_H^*T(I \times Y) \oplus \varepsilon_{W_H}(\mathbb{R}^{m+1})$ from b' and B'_H above. By [20, Theorem 4.4], we can choose the G-framed map $\boldsymbol{f} := (f, b)$ so as to satisfy the conditions (1) and (2) above. □

We remark that for the G-map f above and any subgroup H of G such that $\dim V^H \geq 1$, the restriction $f^H : X^H \to Y^H$ is of degree 1.

4. Proof of Theorem 1.1

In the current section, we assume throughout that $G = A_5$. We begin by recalling basic properties of A_5, the alternating group of degree 5. For $n = 2$, 3 and 5, choose cyclic subgroups C_n of A_5 of order n so that $C_2 \subset N_{A_5}(C_3)$ and $C_2 \subset N_{A_5}(C_5)$ (see [27, p.339]). Since $N_{A_5}(C_n)$ is dihedral of order $2n$, we denote $N_{A_5}(C_n)$ by D_{2n}. Moreover, $N_{A_5}(D_4)$ is isomorphic to the alternating group of degree 4 and hence we denote $N_{A_5}(D_4)$ by A_4. Any subgroup of A_5 is conjugate to one of the subgroups $\{e\}$, C_n, D_{2n}, A_4 or A_5, where $n = 2$, 3 or 5. The group A_5 has precisely five (mutually nonisomorphic) irreducible real representations, and the dimensions of them are 1, 3, 3, 4 and 5. The two irreducible real A_5-representaions of dimension 3 are Galois conjugate to each other. Let $\mathbb{R}$, $U(3)$, $U(4)$ and $U(5)$ be irreducible real A_5-representation spaces (with G-invariant inner product) of dimension 1, 3, 4 and 5, respectively. The dimensions of the H-fixed point set W^H of $W = U(3)$, $U(4)$ and $U(5)$ are as in Table 1.

H	$\{e\}$	C_2	C_3	C_5	D_4	D_6	D_{10}	A_4	A_5
$\dim U(3)^H$	3	1	1	1	0	0	0	0	0
$\dim U(4)^H$	4	2	2	0	1	1	0	1	0
$\dim U(5)^H$	5	3	1	1	2	1	1	0	0

TABLE 1

Define a real A_5-module V by

(4.1) $$V = U(3) \oplus U(5)$$

and set $Y = S(\mathbb{R} \oplus V)$. Then, the dimensions of the H-fixed point sets Y^H, $H \subset A_5$, are as in Table 2.

H	$\{e\}$	C_2	C_3	C_5	D_4	D_6	D_{10}	A_4	A_5
$\dim Y^H$	8	4	2	2	2	1	1	0	0

TABLE 2

It follows that

$$\mathrm{Iso}(A_5, Y) = \mathcal{S}(A_5) \smallsetminus (A_4),$$

where $\mathrm{Iso}(A_5, Y)$ and $\mathcal{S}(A_5)$ denote respectively the set of all isotropy subgroups of Y in G and the set of all subgroups of G.

Next, recall that the formula

$$\sum_{g \in G} |(G/H)^g| = |G|$$

is valid for any finite group G and any subgroup H of G. Using the formula, we obtain Table 3, whose entries are the numbers of elements of $(G/H)^g$ when $G = A_5$.

H	$\{e\}$	C_2	C_3	C_5	D_6	D_{10}	A_4
$\mathrm{ord}(g) = 5$	0	0	0	2	0	1	0
$\mathrm{ord}(g) = 3$	0	0	2	0	1	0	2
$\mathrm{ord}(g) = 2$	0	2	0	0	2	2	1
$g = e$	60	30	20	12	10	6	5

TABLE 3

It follows from the table that in the case $G = A_5$, the idempotent α in Lemma 2.1 is explicitly given by

(4.2) $$\alpha = [G/G] - [G/A_4] - [G/D_{10}] - [G/D_6] + [G/C_3] + 2[G/C_2] - [G].$$

Now set

$$\mathcal{A} = \{\{e\},\ C_2,\ C_3,\ D_6,\ D_{10},\ A_4\}.$$

From the formula (4.2), it follows that

$$[A_5/A_5] - \alpha = \sum_{H \in \mathcal{A}} (\varphi_+(H) - \varphi_-(H))[A_5/H],$$

for the maps $\varphi_+, \varphi_- : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ defined in Table 4 below.

H	$\{e\}$	C_2	C_3	D_6	D_{10}	A_4
$\varphi_+(H)$	1	0	0	1	1	1
$\varphi_-(H)$	0	2	1	0	0	0

TABLE 4

Proposition 4.1. *Let $G = A_5$ and let V, Y, φ_+ and φ_- be as above. Then, the A_5-framed map $\boldsymbol{f} = (f, b)$ and the H-framed cobordisms $\boldsymbol{F} = (F, B)$, $H \subsetneq A_5$, in Proposition 3.1 can be chosen to satisfy the following additional properties:*

(1) *For each subgroup H such that $\{e\} \subsetneq H \subsetneq A_5$,*

$$\text{the map } f^H : X^H \to Y^H$$

is a homotopy equivalence.

(2) *For each subgroup $H = D_6$, D_{10}, A_4, and for any nontrivial subgroup K of H,*

$$\text{the map } F_H^K : W_H^K \to I \times Y^K$$

is an integral homology equivalence.

We shall prove the proposition above in Section 5.

The next remark follows immediately from Proposition 3.1.

Remark 4.2. The map f in Proposition 4.1 has the following properties:

(1) $f^{-1}(p_+)^{A_5} = \emptyset$,
(2) $f^{-1}(p_+)^{A_4} = \{x_{+,A_4}\}$ (one point),
(3) $\mathrm{Iso}(G, f^{-1}(U_{Y,+})) = \mathcal{S}(G) \smallsetminus (A_5)$, and
(4) $\mathrm{Iso}(G, f^{-1}(U_{Y,-})) = \mathcal{S}(G) \smallsetminus (A_4)$.

Let $\boldsymbol{f} = (f, b)$ and $\boldsymbol{F}_H = (F_H, B_H)$, where $H \subsetneq A_5$, be the A_5-framed map and the H-framed cobordisms described in Proposition 4.1. Note that $\dim Y^H = 4$ if and only if H is a subgroup of order 2, and that $\dim Y^K \neq 3$ for any subgroup K. Since $f^{C_2} : X^{C_2} \to Y^{C_2}$ is a homotopy equivalence, X^{C_2} is a 4-dimensional homotopy sphere and is connected. Define the doubly parametrized form ring $\boldsymbol{A}$ (cf. [4, p.278]) by

$$\boldsymbol{A} = (\mathbb{Z}, A_5, \emptyset, A_5(2), \lambda, w_Y),$$

where

$$A_5(2) = \{g \in A_5 \mid g^2 = e,\ g \neq e\},$$

$\lambda = (-1)^4$ and $w_Y : G \to \{1, -1\}$ is the trivial homomorphism. Then, by [4, Theorem 7.3], the obstruction $\sigma(\boldsymbol{f})$ (to convert $f : X \to Y$ to a homotopy equivalence by equivariant surgery on the free part of X) lies in the abelian group $W_8(\boldsymbol{A}, A_5(2))$ $(= W_8(\mathbb{Z}, A_5, \emptyset, A_5(2), A_5(2))$, according to the notation in [12]).

Applying the framed cobordism invariance theorem [23, Theorem 3.2] to the H-framed cobordism $\boldsymbol{F}_H$, we conclude $\sigma(\mathrm{Res}^G_H \boldsymbol{f}) = 0$ for each $H \subsetneq A_5$. Since $\mathrm{Res}^G_H \sigma(\boldsymbol{f}) = \sigma(\mathrm{Res}^G_H \boldsymbol{f})$, Lemma 2.4 implies $\sigma(\boldsymbol{f}) = 0$. Thus, we can perform equivariant surgery on the free part of X so that the resulting map $f' : X' \to Y$ is a homotopy equivalence. Then, X' is a homotopy sphere with exactly one G-fixed point. Taking the equivariant connected sum of the right number of copies of X', we obtain a smooth action of G on the 8-dimensional standard sphere with exactly one fixed point. This is explained in [13, Proposition 1.3] by letting the $\mathcal{F}$ there denote the set of Sylow subgroups of A_5. □

5. Reflection Method and Proof of Proposition 4.1

We return in this section to the general situation where G is any finite group. We develop a certain way of performing equivariant surgery which we call the reflection method and apply it to prove 4.1.

Let $\boldsymbol{f}$ and $\boldsymbol{F}_H$, $H \subsetneq G$, be as in Proposition 3.1.

For an H-invariant subset Z of W_H, a compact H-submanifold N of W_H is called a *product H-cobordism neighborhood* of Z with respect to Ψ if N is an H-neighborhood of Z and $\Psi : N \to I \times (X \cap N)$ is an H-diffeomorphism such that

(1) $\Psi(X \cap N) = \{0\} \times (X \cap N)$,

(2) the restriction

$$\Psi|_{X \cap N} : X \cap N \to \{0\} \times (X \cap N)$$

coincides with the canonical map, and

(3) $\Psi^{-1}(\{1\} \times (X \cap N)) = Y \cap N$.

Definition 5.1. Let $\boldsymbol{f}$ be a G-framed map, H a subgroup of G, $\boldsymbol{F}_H$ an H-framed cobordism between $\boldsymbol{f}$ and another G-framed map f', and $K \neq \{e\}$ a subgroup of H. We say that the pair $(\boldsymbol{f}, \boldsymbol{F}_H)$ is *adjusted* for (H, K) if

(1) $f^K : X^K \to Y^K$ is a homotopy equivalence, and

(2) $F^K_H : W^K_H \to I \times Y^K$ is an integral homology equivalence.

We provide next a method which enables us to adjust a pair $(\boldsymbol{f}, \boldsymbol{F}_H)$ as in (5.1), for (H, K) under the assumption that $N_G(K) \subset H$. The following notation

$$X^{>K} = \{x \in X \mid G_x \supsetneq K\} \text{ and}$$
$$W^{>K}_H = \{w \in W_H \mid H_w \supsetneq K\}$$

will be used where G_x and H_w denote the isotropy subgroups of G at x and H at w, respectively.

Lemma 5.2 (Reflection Method)**.** *Let H and $K \neq \{e\}$ be proper subgroups of G such that $N_G(K) \subset H$, and let $\boldsymbol{f}$ and $\boldsymbol{F}_H$ be as in Proposition 3.1. Suppose there is a product H-cobordism neighborhood $N_{>K}$ of $HX^{>K} \cup HW_H^{>K}$ in W_H with respect to an H-diffeomorphism $\Psi_{>K} : N_{>K} \to I \times (X \cap N_{>K})$. Then one can perform G-surgery of $\boldsymbol{f}$ and H-surgery of $\boldsymbol{F}_H$ of isotropy type (K) so that the resulting maps $\boldsymbol{f}'$ and $\boldsymbol{F}'_H$ satisfy the conditions:*

(1) *${f'}^K : {X'}^K \to Y^K$ is a homotopy equivalence,*
(2) *${F'}_H^K : {W'}_H^K \to I \times Y^K$ is a homotopy equivalence, and*
(3) *there exists a closed H-regular neighborhood U_K of $H{W'}_H^K$ in W'_H such that $U_K \supset N_{>K}$ and U_K is a product H-cobordism neighborhood of $H{W'}_K^K$ with respect to an H-diffeomorphism $\psi_K : U_K \to I \times (X' \cap U_K)$ with $\psi_K|_{N_{>K}} = \Psi_{>K}$.*

Proof. This follows from [17, Theorem 4.2] and its proof. □

Proof of Proposition 4.1. Let $G = A_5$ and $H \subsetneq A_5$. Let $\boldsymbol{f}$ and $\boldsymbol{F}_H$ denote respectively an A_5-framed map and H-framed cobordism obtained by Proposition 3.1 where φ_+ and φ_- are specified as in Proposition 4.1. Since each maximal subgroup of A_5 is conjugate to A_4, D_6 or D_{10}, it suffices to treat the cases $H = A_4$, D_6 and D_{10}.

Step 1 : $(H, K) = (A_5, A_4)$. By construction, X^{A_4} has precisely two points and $f^{A_4} : X^{A_4} \to Y^{A_4}$ is a homotopy equivalence. It is clear that $W_{A_4}^{A_4}$ has two components diffeomorphic to $[0, 1]$ and the others are diffeomorphic to S^1. Perform 1-dimensional A_4-surgery of $\boldsymbol{F}_{A_4}$ to remove all components diffeomorphic to S^1 and thereby obtain a new $\boldsymbol{F}_{A_4}$ such that $F_{A_4}^{A_4} : W_{A_4}^{A_4} \to I \times Y^{A_4}$ is a homotopy equivalence. Thus the new $W_{A_4}^{A_4}$ is diffeomorphic to $I \times X^{A_4}$. Hence the new pair $(\boldsymbol{f}, \boldsymbol{F}_{A_4})$ is adjusted for (A_4, A_4). Moreover there exists a product A_4-cobordism neighborhood U_{A_4} of $W_{A_4}^{A_4}$ in W_{A_4} with respect to some A_4-diffeomorphism $\psi_{A_4} : U_{A_4} \to I \times (X \cap U_{A_4})$.

Step 2 : $(H, K) = (D_6, D_6)$. Note that $X^{>D_6} = X^{A_5} = f^{-1}(y_-)$, and $W_{D_6}^{>D_6} = \emptyset$. There exists a product D_6-cobordism neighborhood $N_{>D_6}$ of $f^{-1}(y_-)$ $(= D_6X^{>D_6} \cup D_6W_{D_6}^{>D_6})$ in W_{D_6} with respect to some D_6-diffeomorphism $\Psi_{>D_6} : N_{>D_6} \to I \times (X \cap N_{>D_6})$. Since $N_{A_5}(D_6) = D_6$, we can apply the Reflection Method above. Doing this, we obtain a pair $(\boldsymbol{f}, \boldsymbol{F}_{D_6})$ which is adjusted for (D_6, D_6). Moreover there exists a product D_6-cobordism neighborhood U_{D_6} in W_{D_6} with respect to some D_6-diffeomorphism $\varphi_{D_6} : U_{D_6} \to I \times (X \cap U_{D_6})$.

Step 3 : $(H, K) = (D_{10}, D_{10})$. This is similar to Step 2.

Step 4 : $(H, K) = (A_4, D_4)$. Note that $X^{>D_4} = X^{A_4}$ and $W_{A_4}^{>D_4} = W_{A_4}^{A_4}$. Recall that in Step 1, we obtained a product A_4-cobordsim neighborhood $N_{>D_4} = U_{A_4}$ of $W_{A_4}^{A_4}$ $(= A_4X^{>D_4} \cup A_4W_{A_4}^{>D_4})$ in W_{A_4} with respect to $\Psi_{>D_4} = \psi_{A_4}$. Since $N_{A_5}(D_4) = A_4$, we can apply the Reflection Method. Doing this, we obtain a pair $(\boldsymbol{f}, \boldsymbol{F}_{A_4})$ which is adjusted for (A_4, D_4). Moreover $W_{A_4}^{D_4}$ has a product A_4-cobordism neighborhood U_{D_4} in W_{A_4} with respect to some A_4-diffeomorphism $\psi_{D_4} : U_{D_4} \to I \times (X \cap U_{D_4})$.

Step 5 : $(H,K) = (D_{10}, C_5)$. Note that $X^{>C_5} = X^{D_{10}}$ and $W_{D_{10}}^{>C_5} = W_{D_{10}}^{D_{10}}$. Recall that in Step 3, we obtained a product D_{10}-cobordism neighborhood $N_{>C_5} = U_{D_{10}}$ of $W_{D_{10}}^{D_{10}}$ $(= D_{10}X^{>C_5} \cup D_{10}W_{D_{10}}^{>C_5})$ in $W_{D_{10}}$ with respect to some $\Psi_{>C_5} = \psi_{D_{10}} : U_{D_{10}} \to I \times (X \cap U_{D_{10}})$. Since $N_{A_5}(C_5) = D_{10}$, we can apply the Reflection Method and obtain a pair $(\boldsymbol{f}, \boldsymbol{F}_{D_{10}})$ which is adjusted for (D_{10}, C_5). Moreover $W_{D_{10}}^{C_5}$ has a product D_{10}-cobordism neighborhood U_{C_5} in $W_{D_{10}}$ with respect to some D_{10}-diffeomorphim ψ_{C_5} : $U_{C_5} \to I \times (X \cap U_{C_5})$.

Step 6 : $(H,K) = (D_6, C_3)$. Note that $W_{D_6}^{>C_3} = W_{D_6}^{D_6}$, but $X^{>C_3} \supsetneq X^{D_6}$. Recall that in Step 2, we obtained a product D_6-cobordism neighborhood U_{D_6} of $W_{D_6}^{>C_3}$ in W_{D_6}. There are exactly four subgroups of A_5 properly containing C_3. They are A_5, D_6, and two subgroups isomorphic to A_4, say $A(1)$ and $A(2)$. Thus

$$X^{>C_3} = X^{D_6} \cup X^{A(1)} \cup X^{A(2)}.$$

Recall that $\dim X^{C_3} = 2$ and $\dim X^{A(i)} = 0$, hence $\dim W_{D_6} = 3$ and

$$\dim(X^{>C_3} \smallsetminus W_{D_6}^{>C_3}) = 0.$$

Without loss of generality, we can assume that $F_{D_6}^{C_3}$ is 1-connected, consequently $W_{D_6}^{C_3}$ is connected. Then it is easy to find a product D_6-cobordism neighborhood $N_{>C_3}$ of $X^{>C_3} \cup W_{D_6}^{>C_3}$ in W_{D_6} with respect to some D_6-diffeomorphism $\Psi_{>C_3} : N_{>C_3} \to I \times (X \cap N_{>C_3})$. Since $N_{A_5}(C_3) = D_6$, we can apply the Reflection Method and obtain a pair $(\boldsymbol{f}, \boldsymbol{F}_{D_6})$ which is adjusted for (D_6, C_3). Moreover there is a product D_6-cobordism neighborhood U_{C_3} of $W_{D_6}^{C_3}$ in W_{D_6} with respect to some D_6-diffeomorphism $\psi_{C_3} : U_{C_3} \to I \times (X \cap U_{C_3})$.

The next step needs some terminology.

Definition 5.3. Let W be a manifold with boundary ∂W. A submanifold A of W is called a *neat submanifold* of W if the following holds:

(1) $\partial A = A \cap \partial W$.

(2) There exists a collar neighborhood C of ∂W in W such that $C \overset{\varphi}{\cong} \partial W \times I$ with respect to some diffeomorphism φ, and $A \cap C \overset{\varphi|_{A\cap C}}{\cong} \partial A \times I$.

Step 7 : $(H,K) = (A_4, C_2)$. Note that $W_{A_4}^{>C_2} = W_{A_4}^{D_4}$, but $X^{>C_2} \supsetneq X^{D_4}$. There are seven subgroups of A_5 which properly contain D_4. They are A_5, A_4, D_4, two distinct subgroups isomorphic to D_6, say $D(1)$ and $D(2)$, and two distinct subgroups isomorphic to D_{10}, say $D(3)$ and $D(4)$. Thus we have

$$X^{>C_2} = X^{D_4} \cup \bigcup_{i=1}^{4} X^{D(i)}.$$

Recall that $\dim X^{D(i)} = 1$, $\dim X^{D_4} = 2$ and $\dim X^{C_2} = 4$. Hence $\dim W_{A_4}^{D_4} = 3$ and $\dim W_{A_4}^{C_2} = 5$. Without loss of generality, we can suppose that $F_{A_4}^{C_2}$ is 2-connected. Consequently $W_{A_4}^{C_2}$ is 1-connected.

Below we shall find a product A_4-cobordism neighborhood $N_{>C_2}$ of $A_4X^{>C_2} \cup W_{A_4}^{D_4}$ $(= A_4X^{>C_2} \cup A_4W_{A_4}^{>C_2})$. After this is done, since $N_{A_5}(C_2) = D_4 \subset A_4$, we can apply the Reflection Method and obtain a pair $(\boldsymbol{f}, \boldsymbol{F}_{A_4})$ which is adjusted for (A_4, C_2). Moreover there is a product A_4-cobordism neighborhood U_{C_2} of $A_4W_{A_4}^{C_2}$ in W_{A_4} with respect to some A_4-diffeomorphism $\psi_{C_2} : U_{C_2} \to I \times (X \cap U_{C_2})$.

Let U_{D_4} be a product A_4-cobordism neighborhood of $W_{A_4}^{D_4}$ in W_{A_4} with respect to an A_4-diffeomorphims ψ_{D_4} obtained in Step 4. By definition, U_{D_4} is an A_4-tubular neighborhood of $W_{A_4}^{D_4}$ in W_{A_4}. Set

$$\partial_0(U_{D_4}) = U_{D_4} \cap \mathrm{Closure}(W_{A_4} \smallsetminus U_{D_4}), \text{ and}$$
$$\mathrm{Int}_0(U_N) = U_N \smallsetminus \partial_0(U_{D_4}).$$

The restriction

$$\psi_{D_4}^{-1}|_{I\times(X^{>C_2}\cap\partial_0(U_N))} : I \times (X^{>C_2} \cap \partial_0(U_N)) \to W_{A_4}^{C_2} \smallsetminus U_{D_4}$$

is an A_4-embedding. It is easy to see that $X^{>C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$ has four components each of which is diffeomorphic to $[0,1]$. They are

$$I(i) = X^{D(i)} \smallsetminus \mathrm{Int}_0(U_{D_4})$$

for $i = 1, 2, 3, 4$. Consider the inclusion

$$\iota_i : (I(i), \partial I(i)) \to (W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4}), \partial_0(U_{D_4}))$$

for $i = 1, 3$. Since the map

$$\pi_1(Y^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})) \to \pi_1(W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4}))$$

induced by the inclusion is surjective, there exists a map

$$\eta_i : I \times I(i) \to W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$$

satisfying the following properties:

(1) η_i is an embedding and $\mathrm{Im}(\eta_i)$ is a neat submanifold of $\partial W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$,
(2) $\eta_i|_{\{0\}\times I(i)} : \{0\} \times I(i) \to W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$ coincides with ι_i,
(3) $\eta_i(\{1\} \times I(i)) \subset Y^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$, and
(4) the restriction of η_i to $I \times \partial I(i)$ coincides with the restriction of $\psi_{D_4}^{-1}$.

Furthermore, putting η_1 and η_3 in general position, we may suppose that η_1, $h\eta_1$, η_3 and $h\eta_3$ are mutually disjoint for some $h \in D_4 \smallsetminus C_2$. Then, extending the maps η_1 and η_3, we get a D_4-embedding

$$\eta : I \times \coprod_{i=1}^{4} I(i) \to W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$$

whose image is a neat submanifold of $W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$. Clearly

$$A_4X^{>C_2} \smallsetminus X^{D_4} = \coprod \{gD_4(X^{>C_2} \smallsetminus X^{D_4}) \mid gD_4 \in A_4/D_4\}.$$

Thus the map

$$A_4 \times_{D_4} \eta : I \times (A_4 \times_{D_4} \coprod_{i=1}^{4} I(i)) \to A_4 W_{A_4}^{C_2} \smallsetminus \mathrm{Int}_0(U_{D_4})$$

is an A_4-embedding. Moreover, if U' is a closed A_4-tubular neighborhood of $\mathrm{Im}(A_4 \times_{D_4} \eta)$ in $W_{A_4} \smallsetminus \mathrm{Int}_0(U_{D_4})$, then the union

$$N_{>C_2} := U_{D_4} \cup U'$$

is a product A_4-cobordism neighborhood of $A_4 X^{>C_2} \cup A_4 W_{A_4}^{>C_2}$ in W_{A_4}.

Step 8 : $(H, K) = (A_4, C_3)$. By Step 6, f^{C_3} is a homotopy equivalence. Thus it suffices to modify F_{A_4} by A_4-surgery of isotropy type (C_3) (relative to the boundary) so that $F_{A_4}^{C_3}$ becomes an integral homology equivalence. Note that $N_{A_4}(C_3) = C_3$, $\dim W_{A_4}^{C_3} = 3$ and $\dim W_{A_4}^{>C_3} = 1$. Thus by [24, Theorem 1.1], the equivariant surgery obstruction $\sigma(\boldsymbol{F}_{A_4}^{C_3})$ lies in Wall's group $L_3^h(\mathbb{Z}[C_3/C_3], triv)$. But the group is trivial and hence the obstruction vanishes. Thus we can adjust $(\boldsymbol{f}, \boldsymbol{F}_{A_4})$ for (A_4, C_3).

Step 9 : $(H, K) = (D_6, C_2)$. By Step 7, f^{C_2} is a homotopy equivalence. Thus, it suffices to modify F_{D_6} by D_6-surgery of isotropy type (C_2) (relative to the boundary) so that $F_{D_6}^{C_2}$ becomes an integral homology equivalence. Note that $N_{D_6}(C_2) = C_2$, $\dim W_{D_6}^{C_2} = 5$ and $\dim W_{D_6}^{>C_2} = 2$. Thus by [18, Theorem A], the equivariant surgery obstruction $\sigma(\boldsymbol{F}_{D_6}^{C_2})$ lies in Wall's group $L_5^h(\mathbb{Z}[C_2/C_2], triv)$. But the group is trivial and hence the obstruction vanishes. Thus we can adjust $(\boldsymbol{f}, \boldsymbol{F}_{D_6})$ for (D_6, C_2).

Step 10 : $(H, K) = (D_{10}, C_2)$. This is similar to Step 9.

This completes the proof of Proposition 4.1. □

Remark 5.4. We proved in [19] and [2], respectively, that S^6 and S^7 have smooth one fixed point actions of A_5, using the triviality of certain obstruction groups, namely that $W_6(\mathbb{Z}[A_5], \Gamma A_5(2); triv) = 0$ and $W_7(\mathbb{Z}[A_5], \Gamma A_5(2); triv) = 0$, respectively. But replacing the real A_5-module $V = U(4) \oplus U(4)$ in (4.1) by $V = U(3) \oplus U(3)$ and $V = U(3) \oplus U(4)$, respectively, the arguments of the present paper provide alternative proofs of the results above without employing the triviality of obstruction groups.

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