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SPLITTING ALONG SUBMANIFOLDS AND $\mathbb{L}$-SPECTRA

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Abstract. The problem of splitting of homotopy equivalence along a submanifold is closely related to surgeries of submanifolds and exact sequences in surgery theory. We describe possibilities and methods of application of $\mathbb{L}$-spectra for the investigation of the problem of splitting of (simple) homotopy equivalence of manifolds along submanifolds. This approach naturally leads us to commutative diagrams of exact sequences, which play an important role in calculational problems of surgery theory.

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1. Introduction

Let X^n be an n-dimensional, connected, closed topological manifold, $\pi = \pi_1(X)$ be its fundamental group, and $w : \pi \to \{\pm 1\}$ be the orientation homomorphism. The main problem of geometric topology is to describe all closed (smooth, piecewise-linear) topological manifolds that are (simply) homotopy equivalent to X (see [1–3]).

For this, we consider the structural set of equivalence classes of (simple) orientation-preserving homotopy equivalences $h : M \to X$, where M is a closed, connected n-manifold of the corresponding category (O, PL, TOP). We consider in detail the category TOP of topological manifolds and simple homotopy equivalences (see [1, 3]).

Two simple homotopy equivalences $f_i : M_i \to X$, $i = 0, 1$, are equivalent if there exists an orientation-preserving homomorphism of manifolds $g : M_0 \to M_1$ such that $f_1 g$ is homotopic to f_0. The set of equivalence classes is denoted by $\mathcal{S}_n^s(X)$ and is a term of the following exact sequence of the surgery theory (see [3]):

$$\cdots \longrightarrow [\Sigma X, G/TOP] \xrightarrow{\sigma_{n+1}} L_{n+1}(\pi, w) \longrightarrow \mathcal{S}_n^s(X) \longrightarrow [X, G/TOP] \xrightarrow{\sigma_n} L_n(\pi, w). \quad (1)$$

There exists a similar exact sequence for smooth and piecewise-linear structures on a manifold X [1]. Elements of the set $[X, G/TOP]$ are called normal invariants. For $n \geq 5$, this set coincides with concordance classes of normal topological mappings $(f, b) : M \to X$ (see [2, 3]). By definition, an n-dimensional, normal topological mapping $(f, b) : M \to X$ consists of the following ingredients:

(i) an n-dimensional topological manifold M and a normal topological block bundle

$$\nu_M = \nu_{M \subset S^{n+k}} : M \to BTOP(k),$$
$$\rho_M : S^{n+k} \to S^{n+k}/\overline{S^{n+k} - E(\nu_M)} = T(\nu_M);$$

(ii) an n-dimensional topological manifold X and a normal topological block bundle

$$\nu_X : X \to BTOP(k), \quad \rho_X : S^{n+k} \to T(\nu_X);$$

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(iii) a mapping $f : M \to X$ of degree 1;

(iv) a mapping $b : \nu_M \to \nu_X$ of topological block bundles, which covers f and is such that

$$T(b)_*(\rho_M) = \rho_X \in \pi_{n+k}(T(\nu_X)).$$

Surgery obstruction groups $L_n(\pi, w)$ (see [1]) functorially depend on the pair (π, w) and the dimension of the manifold $n \bmod 4$. Surgery obstruction groups are the same for any of the categories O, PL, and TOP. A mapping σ determines an obstruction of the surgery of a normal mapping to a simple homotopy equivalence, i.e., $\sigma(f, b) = 0$, if the class of the normal cobordism (f, b) contains a simple homotopy equivalence. Conversely, if $\sigma(f, b) = 0$ and $n \geq 5$, then the concordance class (f, b) contains a simple homotopy equivalence (see [1, 3]).

Thus, for describing the set $\mathcal{S}^s_n(X)$, we need to know the set of normal invariants, surgery obstruction groups $L_n(\pi, w) = L^s_n(\pi, w)$, and the mapping σ (the assembly mapping). Approaches to these problems are quite distinct. At the present time, there exist effective methods for calculation of L-groups (see [4–8]) based on results in number theory and algebraic K-theory. Methods of algebraic and geometric topology are used for describing normal invariants and assembly mapping (see [1, 3, 9–13]). However, our knowledge about the set $\mathcal{S}^s_n(X)$ in general case is far from complete. The investigation of the assembly mapping is closely related to the Novikov conjecture on higher signatures, the problem on the realization of elements of L-groups by normal mappings of closed manifolds, and the series of other classical problems of surgery theory (see [9–13]).

Let X be a manifold and $Y \subset X$ be its subset of codimension q. Then exact sequence (1) of surgery theory can be included into different commutative diagrams of exact sequences that also contain obstruction groups distinct from surgery obstruction groups. The obtained relationships provide us with abundant additional information and are very useful from both the algebraic and geometrical points of view since many objects and mappings have an explicit geometrical description. The key role in this consideration is played by the problem on the splitting of a simple homotopy equivalence $f : M \to X$ along a submanifold Y.

By definition [1, 3], a simple homotopy equivalence $f : M \to X$ splits along a submanifold Y if the mapping f is homotopic to a mapping g transversal to Y, where $N = g^{-1}(Y)$, and such that

$$g|_N : N \to Y, \quad g|_{(M \setminus N)} : M \setminus N \to X \setminus Y$$

are simple homotopy equivalences. A homomorphism splits along an arbitrary submanifold; this obvious fact is a simple consequence of the definition. Conversely, if a mapping f does not split along a submanifold, then it is not homeomorphic to a homomorphism. There exists well-developed algebraic methods for investigating the splitting problem (see [1–3, 14]).

We denote by ∂U the boundary of a tubular neighborhood U of a submanifold Y in a manifold X and by

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix} \tag{2}$$

the quadrate of fundamental groups with orientation, in which all mappings are induced by natural mappings of manifolds. By the Van Kampen theorem, the quadrate F is a universally repelling quadrate of groups. By [1, 3], there exist splitting obstruction groups $LS_{n-q}(F)$, which functorially depend on the quadrate F and $n - q \bmod 4$. Thus, for a simple homotopy equivalence $f : M \to X$, there exists an obstruction $\Theta(f) \in LS_{n-q}(F)$, which vanishes if the mapping f splits along Y. Conversely, if $\Theta(f) = 0$ and $n - q \geq 5$, then f splits along Y.

If horizontal mappings in the quadrate F are isomorphisms $A \cong C$ and $B \cong D$, then groups $LS_*(F)$ are denoted by $LN_*(A \to B)$ (see [1]). In the case of one-sided submanifolds (codimension $q = 1$), groups $LN_*(A \to B)$ are called Browder–Livesay groups [15] and are used in many problems of geometric topology (see [14–17]).

For a normal mapping $(f, b) : M \to X$, there are also defined obstruction groups $LP_{n-q}(F)$ (see [1–3]) for surgery of a pair (M, N) of manifolds to a simple homotopy equivalence of pairs. These groups functorially depend on the quadrate F and $n - q \bmod 4$. For $q \geq 3$, obstruction groups $LS_{n-q}(F)$ for surgery of a submanifold inside a manifold M coincide with abstract surgery obstruction groups $L_{n-q}(\pi_1(Y))$ and surgery obstruction groups $LP_{n-q}(F)$ for pairs of manifolds are naturally isomorphic to the direct sum $L_{n-q}(\pi_1(Y)) \oplus L_n(\pi_1(X))$ (see [1, 2]). Thus, the most interesting case is the case of a manifold with a submanifold of codimension 1 or 2.

In this case, the exact sequence of surgery theory can be included into the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{S}_n^s(X) & \longrightarrow & [X, G/TOP] & \xrightarrow{\sigma} & L_n(\pi_1(X)) & & \\ & & \downarrow & & \downarrow{\scriptstyle v_\xi} & & \downarrow{\scriptstyle \simeq} & & \\ \cdots & \longrightarrow & LS_{n-q}(F) & \longrightarrow & LP_{n-q}(F) & \xrightarrow{p} & L_n(\pi_1(X)) & \longrightarrow & \\ & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle q} & & \downarrow & & \\ \cdots & \longrightarrow & LS_{n-q}(F) & \longrightarrow & L_{n-q}(\pi_1(Y)) & \longrightarrow & L_n(\pi_1(X \setminus Y) \to \pi_1(X)) & \longrightarrow & \end{array} \tag{3}$$

whose rows are exact sequences.

Commutative diagram (3) establishes deep connections between different obstruction groups and exact sequences of surgery theory. All mappings in this diagram have a geometrical description. The mapping $LP_{n-q}(F) \xrightarrow{p} L_n(\pi_1(X))$ is the neglecting mapping. We treat a normal mapping of a pair of manifolds only as a normal mapping to the target manifold X. The mapping $LP_{n-q}(F) \xrightarrow{q} L_{n-q}(\pi_1(Y))$ is defined similarly, but in this case we consider the restriction of the normal mapping to the transversal preimage Y. Every normal mapping in $[X, G/TOP]$ (see [2, 3]) determines a normal mapping of pairs $(M, N) \to (X, Y)$ and, therefore, determines the mapping v_ξ. By definition, the composition pv_ξ coincides with the mapping σ in exact sequence (1) of surgery theory. Every simple homotopy equivalence $f : M \to X$ determines a normal mapping (f, b) (see [1–3]) and, therefore, a normal mapping of pairs. Thus, we obtain the mapping $LS_{n-q}(F) \to LP_{n-q}(F)$. Elements of the set $\mathcal{S}_n^s(X)$ are equivalence classes of simple homotopy equivalences $f : M \to X$. Assigning a splitting obstruction along the submanifold Y to f, we obtain the mapping $\mathcal{S}_n^s(X) \to LS_{n-q}(F)$. From the geometric standpoint, the mapping $L_{n+1}(\pi_1(X)) \to \mathcal{S}_n^s(X)$ is an action defined as follows [1]. We represent an element $x \in L_{n+1}(\pi_1(X))$ as a normal mapping $F : Z \to X \times [0, 1]$ of manifolds with boundaries such that $\partial_0(Z) = F^{-1}(X \times 0)$, $\partial_1(Z) = F^{-1}(X \times 1)$, and the restriction $F|_{\partial_0(Z)}$ is the given simple homotopy equivalence $(f : M \to X) \in \mathcal{S}_n^s(X)$ (see [1]). Then the restriction class for the upper boundary

$$\left\{F|_{\partial_1(Z)} : \partial_1(Z) \to X \times 1\right\} \in \mathcal{S}_n^s(X)$$

of the mapping F is a simple homotopy equivalence considered as the result of the action of the element x on $(f : M \to X)$. The mapping $L_n(\pi_1(X)) \to L_n(\pi_1(X \setminus Y) \to \pi_1(X))$ is a mapping from the relative exact sequence of L-groups for embedding of manifolds $(X \setminus Y) \to X$ of the same dimension [1].

Algebraic approaches based on spectra in surgery theory allow one to obtain deeper connections between different algebraic objects appearing in the classification of geometric structures on pairs of manifolds (see [1, 14, 18–24]).

2. Splitting Obstruction Groups and $\mathbb{L}$-Spectra

In this section, we assume that all groups are equipped with orientation homomorphisms and group homomorphisms commute with orientation homomorphisms (if the contrary is not assumed). Surgery obstruction groups and natural mappings between them such as inducing and transfer are realized on the spectrum level [1, 3, 18]. In particular, the L-functor from the category $F(\mathbf{2}^n, \mathcal{G}pd)$ (see [1]) to the category of Abelian groups passes through the category of spectra.

First, we recall necessary definitions [1, 3, 18, 25].

A spectrum E consists of a family of cellular spaces $(E_n, *)$, $n \in \mathbb{Z}$, and a family of cellular mappings $(\epsilon_n : SE_n \to E_{n+1})$, where SE_n denotes the suspension of the space E_n.

For any mapping ϵ_n, there is defined the conjugate mapping (see [25])

$$(\epsilon'_n : E_n \to \Omega E_{n+1}).$$

A spectrum E is called an Ω-spectra if, for any $n \in \mathbb{Z}$, conjugate mappings are weak homotopy equivalences [25].

Let $f : \pi \to \pi'$ be a homomorphism of oriented groups. Then there are defined Ω-spectra $\mathbb{L}(\pi)$, $\mathbb{L}(\pi')$, and $\mathbb{L}(f)$ and a cobundle

$$\mathbb{L}(\pi) \longrightarrow \mathbb{L}(\pi') \longrightarrow \mathbb{L}(f). \tag{4}$$

The exact homotopy sequence of cobundle (4) is isomorphic to the relative exact sequence

$$\ldots \longrightarrow L_n(\pi) \longrightarrow L_n(\pi') \longrightarrow L_n(f) \longrightarrow L_{n-1}(\pi) \longrightarrow \ldots \tag{5}$$

of the mapping f (see [1, 18]).

Let F be the commutative quadrate (2) of oriented groups. The definition of L-groups (see [1]) and the existence of cobundle (4) immediately imply the existence of a spectrum $\mathbb{L}(F)$ (see [25]), which comprises the following homotopically commutative diagram of spectra:

$$\begin{array}{ccccc}
\mathbb{L}(A) & \longrightarrow & \mathbb{L}(C) & \longrightarrow & \mathbb{L}(A \to C) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(B) & \longrightarrow & \mathbb{L}(D) & \longrightarrow & \mathbb{L}(B \to D) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(A \to B) & \longrightarrow & \mathbb{L}(C \to D) & \longrightarrow & \mathbb{L}(F);
\end{array} \tag{6}$$

the rows and columns of this diagram are cobundles.

Another important mapping naturally arising in surgery theory is a transfer. We consider a bundle $p : X \to Y$ with fiber m-manifold M^m over an n-manifold X. There exists a transfer mapping $p^* : L_n(\pi_1(Y)) \to L_{n+m}(\pi_1(X))$ (see [1, 26, 27]), which is also realized on the spectrum level by the mapping

$$p^! : \mathbb{L}(\pi_1(Y)) \to \nleq^m \mathbb{L}(\pi_1(X)).$$

In particular, for a (D^q, S^{q-1})-bundle $p : (X, \partial X) \to Y$ with zero section, where D^q is the standard q-dimensional disk, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{L}(\pi_1(Y)) & \xrightarrow{p^!} & \Omega^q \mathbb{L}(\pi_1(\partial X) \to \pi_1(X)) \\
{\scriptstyle p^!_1}\downarrow & & \downarrow \\
\Omega^{q-1}\mathbb{L}(\pi_1(\partial X)) & = = & \Omega^{q-1}\mathbb{L}(\pi_1(\partial X)).
\end{array} \tag{7}$$

We consider a disk bundle $p : (X^n, \partial X^n) \to Y$, where $q = 1$ or $q = 2$. The presence of a zero section allows one to consider Y as a submanifold of X of codimension q. In this case, the quadrate of fundamental groups F (2) becomes

$$\Psi = \begin{pmatrix}
\pi_1(\partial X) & \xrightarrow{\cong} & \pi_1(X \setminus Y) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \xrightarrow{\cong} & \pi_1(X)
\end{pmatrix} = \begin{pmatrix}
A & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & B
\end{pmatrix}. \tag{8}$$

Therefore, in this case splitting obstruction groups for the pair (X, Y) are isomorphic to $LN_{n-q}(A \to B)$.

We introduce the spectrum $\mathbb{L}N(A \to B)$ as a homotopy cofiber of the mapping of spectra

$$\Omega(p^!) : \Omega\mathbb{L}(\pi_1(Y)) \to \Omega^{q+1}\mathbb{L}(\pi_1(\partial X) \to \pi_1(X))$$

and the spectrum $\mathbb{L}P(\Psi)$ as a homotopy cofiber of the mapping

$$\Omega(p_1^!) : \Omega\mathbb{L}(\pi_1(Y)) \to \Omega^q\mathbb{L}(\pi_1(X)).$$

In diagram (7), the mapping $p^!$ realizes the transfer mapping on the spectrum level and the mapping

$$\Omega^q\mathbb{L}(\pi_1(\partial X) \to \pi_1(X)) \to \Omega^{q-1}\mathbb{L}(\pi_1(\partial X))$$

realizes the boundary mapping from the relative exact sequence of L-groups for the mapping $\pi_1(\partial X) \to \pi_1(X)$. The definitions of groups $LN_{n-q}(A \to B)$ and groups $LP_{n-q}(\Psi)$ (see [1–3]) imply that the obtained spectra satisfy the isomorphisms

$$\pi_n(\mathbb{L}N(\pi_1(\partial X) \to \pi_1(X))) \cong LN_n(\pi_1(\partial X) \to \pi_1(X))$$

and $\pi_n(\mathbb{L}P(\Psi)) \cong LP_n(\Psi)$ for any n.

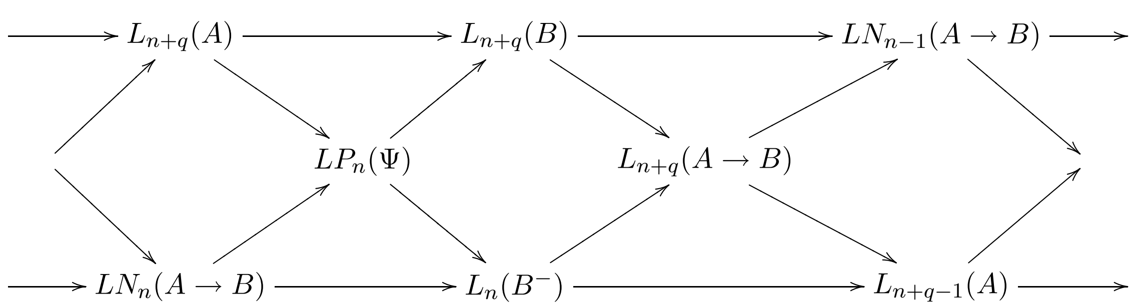
Diagram (7) and the definition of the spectra $\mathbb{L}N(\pi_1(\partial X) \to \pi_1(X))$ and $\mathbb{L}P(\Psi)$ for the splitting problem with quadrate (8) yield the homotopically commutative diagram of spectra

$$\begin{array}{ccccc}
\Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\pi_1(\partial X) \to \pi_1(X)) & \longrightarrow & \mathbb{L}N(\pi_1(\partial X) \to \pi_1(X)) \\
\big\downarrow = & & \big\downarrow & & \big\downarrow \\
\Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^q\mathbb{L}(\partial X) & \longrightarrow & \mathbb{L}P(\Psi),
\end{array} \qquad (9)$$

where the right vertical mapping is defined as in [25].

Recall that in the homotopy category of spectra, the notions of universally repelling quadrate and universally attracting quadrate coincide. In what follows, such quadrates of spectra are said to be universal. A quadrate of spectra is universal if and only if homotopy fibers (cofibers) of parallel mappings are naturally homotopy equivalent (see [25]). Any universal quadrate of spectra yields a commutative diagram of exact sequences (so-called Levin braid). For this, it is necessary to write homotopy long exact sequences of mappings of the quadrate and identify the corresponding homotopy groups.

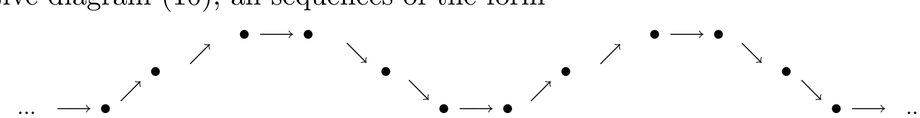
We apply this procedure to the right quadrate of diagram (9), which is universal since homotopy fibers of horizontal mappings coincide. A homotopy fiber of the right vertical mapping is homotopy equivalent to a fiber of the middle vertical mapping $\Omega^{q+1}\mathbb{L}(\pi_1(X))$. Considering homotopy long exact sequences of mappings of the right quadrate of diagram (9) yields the following commutative diagram of exact sequences:



(10)

where the homomorphism $A \to B$ is the homomorphism of oriented groups $\pi_1(\partial X) \to \pi_1(X)$ and B^- is the oriented group $\pi_1(Y)$. The last fact must be taken into account since in the quadrate (8) of fundamental groups, orientations of the right and left group B may be distinct. For example, these orientations are distinct for one-sided submanifolds of codimension 1 and, in particular, for Browder–Livesay groups (see, e.g., [14, 19, 28, 29]). As a simple example, we cite the pair $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$, where $\mathbb{R}P^n$ is the n-dimensional real projective space.

In commutative diagram (10), all sequences of the form



are exact. Using the diagram search, we can show that the top and bottom rows of diagram (10) are chain complexes with isomorphic homologies. We shall see below that this diagram written on the spectrum level allows one to construct different spectral sequences on surgery theory (see [22, 23, 30, 31]).

If $Y \subset X$ is a one-sided submanifold and horizontal mappings in the quadrate Ψ are isomorphisms, then the groups $LN_*(A \to B)$ are Browder–Livesay groups. Diagram (10) for Browder–Livesay groups was initially constructed in [32] for quadratic extensions of rings in the form of two chain complexes. The algebraic construction of diagram (10) for quadratic extensions of anti-structures was given in [33]. In this case, diagram (10) can be efficiently used for calculation of obstruction groups and natural mappings in L-theory (see [7, 8, 14, 34]). Moreover, diagram (10) allows one to obtain deep geometric results on representability of elements of Wall groups by normal mappings of closed manifolds (see [16, 17, 30, 35]). (We discuss this in the following section when investigating spectral sequences in surgery theory.)

Now we consider the problem on splitting of a simple homotopy equivalence $f : M \to X$ along a submanifold $Y \subset X$ of codimension q in the general case. There exists the quadrate F of fundamental groups (2) and we can construct the commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \xrightarrow{p!} & \Omega^q\mathbb{L}(\pi_1(\partial U) \to \pi_1(Y)) & \xrightarrow{\alpha} & \Omega^q\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ & \searrow & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(\partial U)) & \xrightarrow{\beta} & \Omega^{q-1}\mathbb{L}(\pi_1(X \setminus Y)) \end{array} \qquad (11)$$

where the left triangle is implied by diagram (9) and the right quadrate is implied by diagram (6).

We define the spectra

$$\mathbb{L}S(F) = \text{homotopy cofiber } \left[\Omega(\alpha p^!) : \Omega\mathbb{L}(\pi_1(Y)) \to \Omega^{q+1}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X))\right],$$
$$\mathbb{L}P(F) = \text{homotopy cofiber } \left[\Omega(\beta p^!) : \Omega\mathbb{L}(\pi_1(Y)) \to \Omega^{q}\mathbb{L}(\pi_1(X \setminus Y))\right]$$

where α, β, and $p^!$ are the mapping in diagram (11) (see [1, 3, 24]).

Now we can write the homotopically commutative diagram of spectra

$$\begin{array}{ccccc} \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \longrightarrow & \mathbb{L}S(F) \\ {\scriptstyle =}\downarrow & & \downarrow & & \\ \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q}\mathbb{L}(\pi_1(X \setminus Y)) & \longrightarrow & \mathbb{L}P(F) \end{array} \qquad (12)$$

where the horizontal rows are cobundles and the right quadrate is universal. Let $LS_n(F)$ and $LP_n(F)$ be the splitting obstruction groups and surgery obstruction groups for the pair (X, Y) of manifolds, respectively. Then (see, e.g., [24]) the following isomorphisms hold:

$$\pi_n(\mathbb{L}S(F)) \cong LS_n(F), \quad \pi_n(\mathbb{L}P(F)) \cong LP_n(F).$$

This is implied by the fact that mappings are functorial on the spectrum level and the 5-lemma.

Homotopy long exact sequences of the universal quadrate in diagram (12) generate the following diagram of exact sequences (see [12, p. 264]):

$$\begin{array}{ccccccccc} \longrightarrow & L_{n+1}(C) & & \longrightarrow & & L_{n+1}(D) & & \xrightarrow{\Theta} & LS_{n-q}(F) \\ \nearrow & & \searrow & & \nearrow & & \searrow & & \\ & & & LP_{n-q+1}(F) & & & & L_{n+1}(C \to D) & \\ \searrow & & \nearrow & & \searrow & & \nearrow & & \\ \longrightarrow & LS_{n-q+1}(F) & & \longrightarrow & & L_{n-q+1}(B) & & \longrightarrow & L_n(C) \end{array} \qquad (13)$$

where $A = \pi_1(\partial U)$, $B = \pi_1(Y)$, $C = \pi_1(X \setminus Y)$, and $D = \pi_1(X)$.

Note that the mapping Θ in diagram (13) coincides with the composition

$$L_{n+1}(\pi_1(X)) \to \mathcal{S}^s_n(X) \to LS_{n-q}(F)$$

in diagram (3). In particular, this implies that if $\Theta(x) \neq 0$ for some element $x \in L_{n+1}(\pi_1(X))$, then the element x acts nontrivially on the set of homotopy triangulations $\mathcal{S}^s_n(X)$.

Let (X, Y) be a pair of manifolds with universal quadrate F (2) for the splitting problem along the submanifold Y. Then the pair of manifolds (U, Y) is naturally mapped in the pair (X, Y). Denote by

$$\Psi = \begin{pmatrix} A & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B \end{pmatrix}$$

the quadrate of fundamental groups in the problem of splitting of a simple homotopy equivalence $f : M \to U$ along the submanifold Y (see [1]).

A mapping of pairs of manifolds induces the mappings of quadrates of fundamental groups $\Psi \to F$ (see [1, 3]). Recall that we denote the groups $LS_*(\Psi)$ by $LN_*(A \to B)$, where the orientation on the group B corresponds to the orientation of the right group in the quadrate Ψ [1].

The mapping $\Psi \to F$ induces the mapping of the top row of diagram (9) to the top row of diagram (12) and we obtain the universal quadrate of spectra (see [19])

$$\begin{array}{ccc} \Omega^{q+1}\mathbb{L}(A \to B) & \longrightarrow & \Omega^{q+1}\mathbb{L}(C \to D) \\ \big\downarrow & & \big\downarrow \\ \mathbb{L}N(A \to B) & \longrightarrow & \mathbb{L}S(F) \end{array}.$$

For this quadrate, the commutative diagram of exact sequences of homotopy groups has the form

$$\begin{array}{ccccccccccc} \longrightarrow & L_{n+1}(B) & & \longrightarrow & & L_{n+q+1}(C \to D) & & \longrightarrow & & L_{n+q+1}(F) & \longrightarrow \\ \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & L_{n+q+2}(A \to B) & & & & LS_n(F) & & & \\ \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \longrightarrow & L_{n+q+1}(F) & & \longrightarrow & & LN_n(A \to B) & & \longrightarrow & & L_n(B) & \longrightarrow \end{array}. \quad (14)$$

Thus, we have obtained the diagram of exact sequences [12, p. 146].

The mapping $\Psi \to F$ induces the mapping of the bottom row of diagram (9) to the bottom row of diagram (12) and we obtain the universal quadrate of spectra (see [20, 36])

$$\begin{array}{ccc} \Omega^q\mathbb{L}(A) & \longrightarrow & \mathbb{L}P(\Psi) \\ \big\downarrow & & \big\downarrow \\ \Omega^q\mathbb{L}(C) & \longrightarrow & \mathbb{L}P(F) \end{array}.$$

For this quadrate of spectra, the commutative diagram of exact sequences of homotopy groups has the form

$$
\begin{array}{ccccccccccc}
\longrightarrow & L_{n+q+1}(A\to C) & & \longrightarrow & & LP_n(\Psi) & & \longrightarrow & & L_n(B) & \longrightarrow \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & L_{n+q}(A) & & & & LP_n(F) & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\longrightarrow & L_{n+1}(B) & & \longrightarrow & & L_{n+q}(C) & & \longrightarrow & & L_{n+q}(A\to C) & \longrightarrow
\end{array}
\tag{15}
$$

Note that for the Browder–Livesay pair, the groups $LP_n(\Psi)$ are isomorphic to the relative groups $L_{n+1}(p^*)$, where

$$p^* : L_n(\pi_1(Y)) \to L_n(\pi_1(\partial U))$$

is the transfer mapping (see [20, 33]). Thus, if $p : A \to B$ is the left mapping in the quadrate Ψ, then we obtain the following exact sequence for the one-sided submanifold $Y \subset X$ $(q = 1)$:

$$\cdots \longrightarrow L_{n+1}(A \to C) \longrightarrow L_n(p^*) \longrightarrow LP_{n-1}(F) \longrightarrow L_n(A \to C) \longrightarrow \cdots$$

(see [20]).

In the case where the right column of the quadrate F determines the splitting problem with the quadrate of fundamental groups

$$\Phi = \begin{pmatrix} C & \longrightarrow & C \\ \downarrow & & \downarrow \\ D & \longrightarrow & D \end{pmatrix}$$

and there exists a natural morphism $r : F \to \Phi$, additional connections between different splitting obstruction groups and surgery obstruction groups appear (see [19–21, 23, 36]).

Quadrates of such form naturally appear in a wide class of geometric problems (see [23, 28, 29, 37]) and are called *geometric diagrams*. Note that for a geometric diagram, there exists a natural mapping $\Psi \to \Phi$ of quadrates of fundamental groups determined by the composition $\Psi \to F \to \Phi$.

The mapping r induces a mapping of diagrams (11) for the quadrates F and Φ, respectively. Proceeding as above (in the case of mappings induced by the mapping $\Psi \to F$), we obtain the following universal quadrates of spectra:

$$
\begin{array}{ccc}
\mathbb{L}S(F) & \longrightarrow & \mathbb{L}(B) \\
\downarrow & & \downarrow \\
\mathbb{L}N(C \to D) & \longrightarrow & \mathbb{L}(D)
\end{array}, \qquad
\begin{array}{ccc}
\mathbb{L}P(F) & \longrightarrow & \mathbb{L}P(\Phi) \\
\downarrow & & \downarrow \\
\mathbb{L}(B) & \longrightarrow & \mathbb{L}(D)
\end{array},
$$

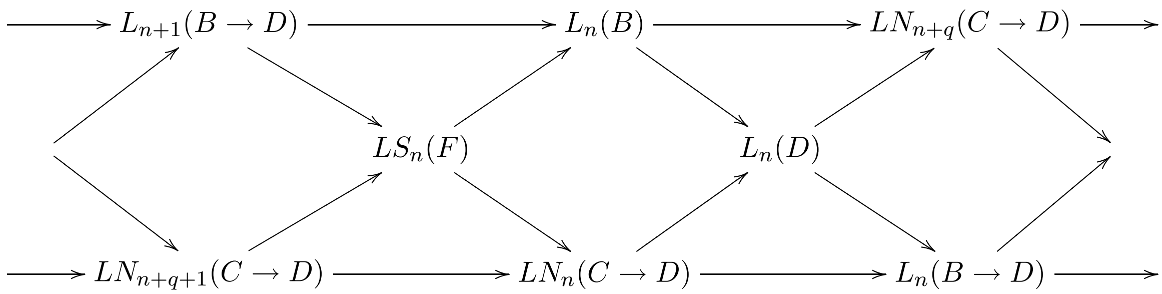
$$
\begin{array}{ccc}
\mathbb{L}S(F) & \longrightarrow & \mathbb{L}P(F) \\
\downarrow & & \downarrow \\
\mathbb{L}N(C \to D) & \longrightarrow & \mathbb{L}P(\Phi)
\end{array}, \qquad
\begin{array}{ccc}
\Omega\mathbb{L}N^{\mathrm{rel}}(F) & \longrightarrow & \Omega\mathbb{L}(B \to D) \\
\downarrow & & \downarrow \\
\mathbb{L}N(A \to B) & \longrightarrow & \mathbb{L}S(F)
\end{array},
$$

where the spectrum $\mathbb{L}N^{\mathrm{rel}}(F)$ is a cofiber of the natural mapping of spectra

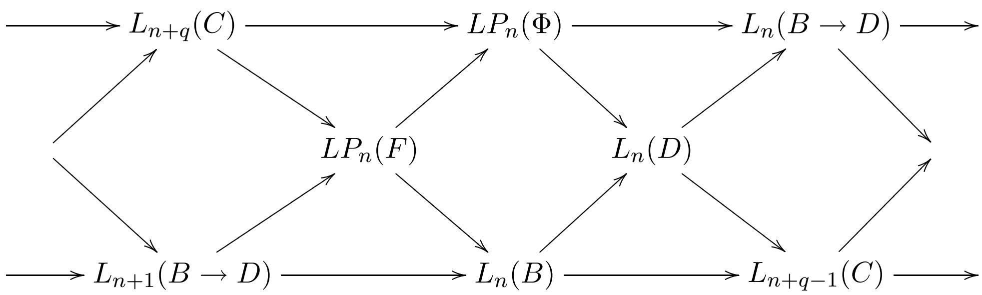
$$\mathbb{L}N(A \to B) \longrightarrow \mathbb{L}N(C \to D)$$

induced by the mapping $\Psi \to \Phi$.

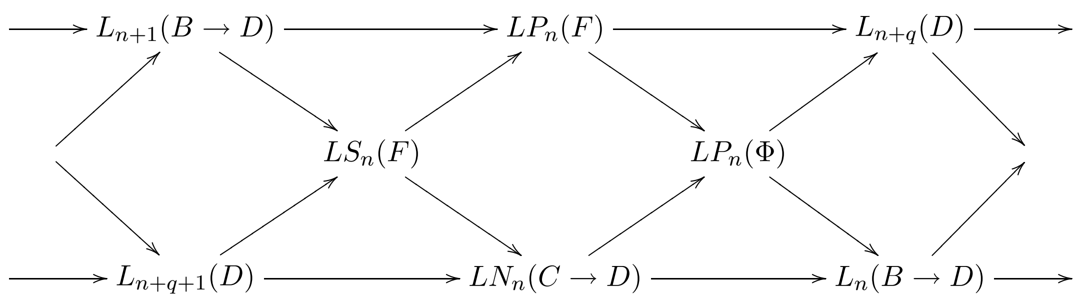
For each of obtained universal quadrates, we can write the commutative diagram of exact sequences (see [19–21, 23, 36]). Thus, we obtain the following diagrams:



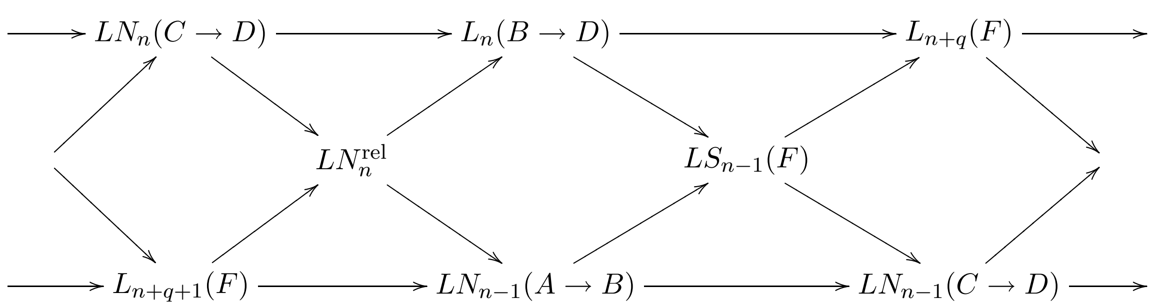
(16)



(17)



(18)



(19)

where $LN_n^{\mathrm{rel}} = LN_n^{\mathrm{rel}}(F)$ are relative splitting obstruction groups comprising the exact sequence

$$\cdots \longrightarrow LN_n(A \to B) \longrightarrow LN_n(C \to D) \longrightarrow LN_n^{\mathrm{rel}}(F) \longrightarrow LN_{n-1}(A \to B) \longrightarrow \cdots.$$

Note that for a geometric diagram of groups in the case of a one-sided submanifold ($q = 1$), the isomorphism $LP_n(\Phi) \cong L_{n+1}(j^*)$ holds, where $j^* : L_n(D) \to L_n(C)$ is the transfer mapping (see [20, 33]). Commutative diagrams (13)–(19) yield rich information on different groups in surgery theory and the natural mappings between them. In particular, they are very efficient in calculations.

Now we discuss the realization of other groups and mappings in diagram (3) on the spectrum level. In [2, p. 571], Ranicki introduced the set $\mathcal{S}_n(X, Y, \xi)$ of s-triangulations of a pair of manifolds (X, Y), where ξ is the normal bundle of the submanifold Y in the manifold X. This set consists of concordance classes of mappings $f : (M, N) \to (X, Y)$ split along Y such that $N = f^{-1}(Y)$. The set $\mathcal{S}_n(X, Y, \xi)$ is a term of the exact sequence

$$\cdots \longrightarrow \mathcal{S}_n(X,Y,\xi) \longrightarrow [X, G/TOP] \xrightarrow{v_\xi} LP_{n-q}(F) \longrightarrow \cdots, \tag{20}$$

which can be constructed algebraically (see [2, 3]). In particular, the set $\mathcal{S}_n(X,Y,\xi)$ possesses a group structure.

The properties of the groups $\mathcal{S}_n(X,Y,\xi)$ and their connections with different obstruction groups for a pair of manifolds (X,Y) are described in [2, Proposition 7.2.6]. These connections can be obtained by using exact sequence (20) on the spectrum level.

Let $\mathbf{L}_\bullet$ be the simple-connected covering of a Ω-spectrum $\mathbf{L}_\bullet(\mathbb{Z})$ [3]. By [3], the homotopy equivalence $\mathbf{L}_0 \simeq G/TOP$ holds. For any topological space X such that $\pi_1(X) = \pi$, there exists the following algebraically exact sequence of surgery theory

$$\cdots \longrightarrow L_{n+1}(\pi) \longrightarrow \mathbb{S}_{n+1}(X) \longrightarrow H_n(X, \mathbf{L}_\bullet) \longrightarrow L_n(\pi) \longrightarrow \cdots. \tag{21}$$

The corresponding part of sequence (21) is naturally isomorphic to exact sequence (1) if X is a topological manifold. Exact sequence (21) is a homotopy long exact sequence of the cobundle of spectra [1, 3]:

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi).$$

Denote by $\mathbf{S}$ the homotopy fiber of the obtained mapping of spectra; then we obtain

$$\pi_n(X_+ \wedge \mathbf{L}_\bullet) = H_n(X, \mathbf{L}_\bullet) = [X, G/TOP], \quad \pi_n(\mathbf{S}) = \mathcal{S}^s_n(X).$$

By [24], there exists the following commutative quadrate of spectra:

$$\begin{array}{ccc} \Omega^q(X_+ \wedge \mathbf{L}_\bullet) & \xrightarrow{\Omega^q \mathbf{A}} & \Omega^q \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^q \mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)), \end{array} \tag{22}$$

where $\mathbf{A}$ is the assembly mapping. Long exact sequences of horizontal mappings of this quadrate yield the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathcal{S}^s_n(X) & \longrightarrow & [X, G/TOP] & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & LS_{n-q}(F) & \longrightarrow & L_{n-q}(\pi_1(Y)) & \longrightarrow & L_n(\pi_1(X \setminus Y) \to \pi_1(X)) & \longrightarrow \cdots \end{array}. \tag{23}$$

Diagram (23) is a part of diagram (3) and is realized on the spectrum level. We also consider the commutative quadrate of spectra, which is implied by diagram (22) (see [24])

$$\begin{array}{ccc} \Omega^q \mathbf{S} & \longrightarrow & \Omega^q(X_+ \wedge \mathbf{L}_\bullet) \\ \downarrow & & \downarrow \\ \mathbb{L}S(F) & \longrightarrow & \mathbb{L}P(F) \end{array}. \tag{24}$$

We define the spectrum $\Omega^q \mathbf{S}(X,Y,\xi)$ as the homotopy fiber of the mapping $\Omega^q \mathbf{S}(X) \to \mathbb{L}S(F)$ in diagram (24).

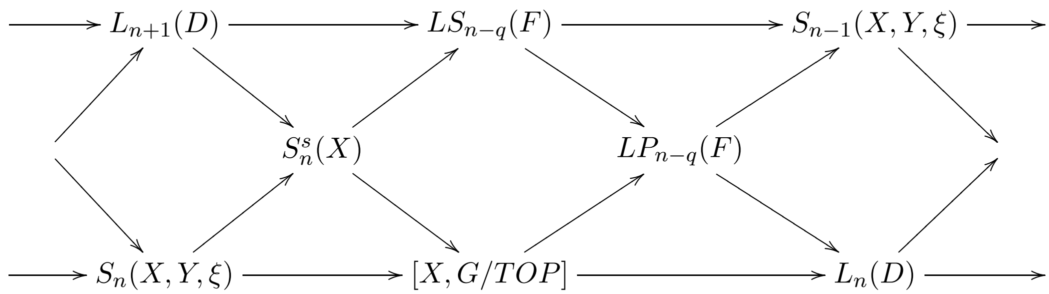
Commutative diagram (3) implies that cofibers of horizontal mappings in diagram (24) are homotopy equivalent, i.e., quadrate (24) is universal and cofibers of vertical mappings are homotopy equivalent. Therefore, the isomorphisms

$$\pi_n(\mathbf{S}(X,Y,\xi)) = \mathcal{S}_n(X,Y,\xi)$$

hold and the groups $\mathcal{S}_n(X,Y,\xi)$ are terms of the exact sequence

$$\cdots \longrightarrow \mathcal{S}_n(X,Y,\xi) \longrightarrow \mathcal{S}^s_n(X) \longrightarrow LS_{n-q}(F) \longrightarrow \cdots.$$

The diagram of long homotopy exact sequences for quadrate (24) has the form



. (25)

Commutative diagram (25) coincides with the diagram in [2, p. 284].

3. Spectral Sequences in Surgery Theory

We consider a topological manifold X of dimension $n \geq 5$. Let the fundamental group $\pi_1(X) = B$ of the manifold X have a subgroup $A \subset B$ of index 2. We consider a mapping $\chi : X \to \mathbb{R}P^m$ of the manifold X to an m-dimensional real projective space of sufficiently large dimension. Let χ induce an epimorphism of fundamental groups with kernel A. Denote by Y the transversal preimage $\chi^{-1}(\mathbb{R}P^{m-1})$; it is a one-sided submanifold in X. Deforming the mapping χ, we can assume that the embedding $Y \subset X$ induces an isomorphism of fundamental groups. Thus, the pair (X, Y) is a Browder–Livesay pair, and we have the splitting problem along Y and commutative diagram (10) of exact sequences with $q = 1$.

The interest in diagram (10) for one-sided submanifolds is stipulated by the following two circumstances. First, all groups and mappings in this diagram have an algebraic description on the level of rings with anti-structures (see [6, 14, 16, 33]). At the present time, there exists a sufficiently complete description of groups and mappings for the case of finite 2-groups (see, e.g., [6, 7, 34]). Further, the possibility of constructing a characteristic submanifold $Y \subset X$ by any subgroup of index 2 allows one to obtain deep results for the problem on the realization of elements of surgery obstruction groups by normal mappings of closed manifolds [14, 16, 17, 30, 35]. We emphasize the following theorem [16].

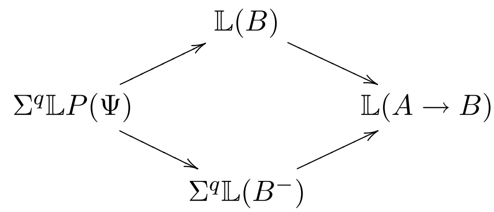
Let $A \to B$ be an embedding of index 2 and $\Theta : L_{n+2}(B) \to LN_n(A \to B)$ be a mapping in diagram (10). If $\Theta(x) \neq 0$ for $x \in L_{n+2}(B)$, then the element x cannot be realized by normal mappings of closed manifolds. Moreover, it acts nontrivially on any element of the set $\mathcal{S}^s_n(X)$ for any connected manifold X with $\pi_1(X) = B$.

The mapping Θ is called the *Browder–Livesay invariant*. Some generalizations and the notion of iterated Browder–Livesay invariant can be found in [17, 29, 30, 35].

A spectral sequence in surgery theory, based on the realization of diagram (10) on the spectrum level, was constructed in [30]. Differentials in this spectral sequence are closely related to iterated Browder–Livesay invariants and the problem on realization of elements of surgery obstruction groups by normal mappings of closed manifolds.

In [30], the construction of the spectral sequence by diagram (10) was performed for Browder–Livesay pairs. From the algebraic standpoint, we construct diagram (10) for an embedding $A \to B$ of oriented groups of index 2. The realization of diagram (10) on the spectrum level plays the crucial role. Since a similar realization is valid for codimension 2 (see the quadrate of spectra for diagram (9)), we can construct a spectral sequence in the general case (see [23]). Recall that there exists the suspension functor Σ in the category of spectra. For any spectrum $\mathbb{E} = \{\mathbb{E}_n\}$, this functor from the category of spectra into itself is defined by the condition $(\Sigma\mathbb{E})_n = \mathbb{E}_{n+1}$.

Let Ψ be the quadrate of fundamental groups (8) in the problem of splitting along a submanifold Y of codimension $q = 1, 2$. We denote by B^- the oriented left group B. We write the universal quadrate of spectra in diagram (9), which determines diagram (10), as follows:

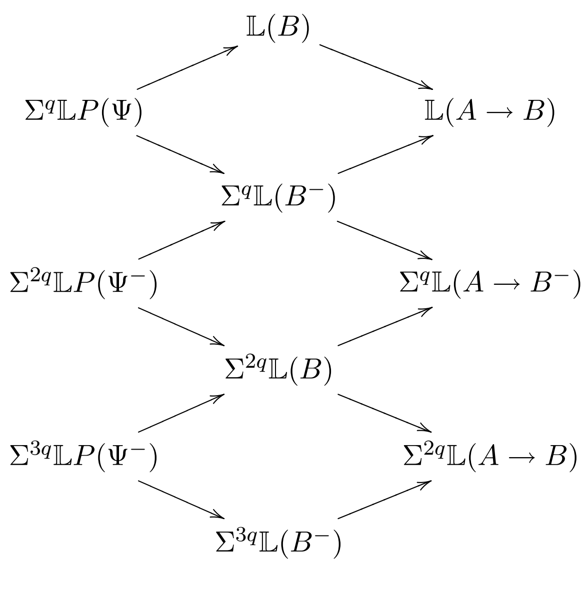


(26)

Let Ψ^- be the universal repelling quadrate of groups with alternated orientation of the group B:

$$\Psi^- = \begin{pmatrix} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B^- \end{pmatrix}. \tag{27}$$

If (X, Y) is a Browder–Livesay pair generated by the quadrate of spectra (26), then quadrate (27) naturally appears for the problem on splitting of a simple homotopy equivalence $f_1 : M_1 \to Y$ along a one-sided submanifold Y_1 for the Browder–Livesay pair (Y, Y_1) (see [35]). As an example, we consider the triple of manifolds $\mathbb{R}P^{n-2} \subset \mathbb{R}P^{n-1} \subset \mathbb{R}P^n$. Now we can construct the diagram of spectra:



(28)

Diagram (28) contains the universal quadrates of spectra (26) constructed for the quadrates of fundamental groups Ψ and Ψ^-. Diagram (28) can be naturally extended downwards; it can also be extended upwards by using the loop functor Ω. We introduce the following notation (see [30, 31]):

$$\begin{array}{llll} X_{0,0} = \mathbb{L}(B), & X_{2,2} = \Sigma^{2q}\mathbb{L}(B), & X_{1,0} = \Sigma^q \mathbb{L}P(\Psi), & X_{3,2} = \Sigma^{3q}\mathbb{L}P(\Psi), \quad \dots, \\ X_{0,1} = \mathbb{L}(A \to B), & X_{2,3} = \Sigma^{2q}\mathbb{L}(A \to B), & X_{1,1} = \Sigma^q \mathbb{L}(B^-), & X_{3,3} = \Sigma^{3q}\mathbb{L}(B^-), \quad \dots, \\ X_{2,1} = \Sigma^{2q}\mathbb{L}P(\Psi^-), & X_{4,3} = \Sigma^{4q}\mathbb{L}P(\Psi^-), & X_{1,2} = \Sigma^q \mathbb{L}(A \to B^\xi), & X_{3,4} = \Sigma^{3q}\mathbb{L}(A \to B^\xi). \end{array}$$

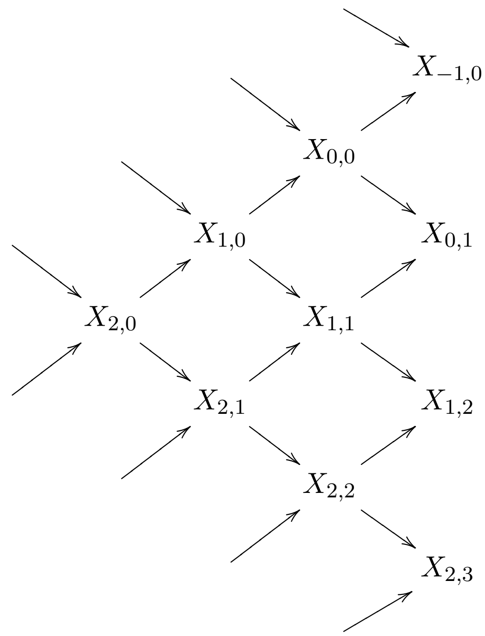
Diagram (28) can be infinitely extended to the left as follows. The spectrum $X_{k,k-2}$ is constructed by the known spectra and mappings

$$X_{k-1,k-2} \longrightarrow X_{k-1,k-1} \longleftarrow X_{k,k-1}$$

such that the quadrate

$$\begin{array}{ccc} X_{k,k-2} & \longrightarrow & X_{k-1,k-2} \\ \downarrow & & \downarrow \\ X_{k,k-1} & \longrightarrow & X_{k-1,k-1} \end{array}$$

is a homotopy universally attracting quadrate. A similar construction with homotopy universally repelling quadrates is used for the construction of the spectrum $X_{k,k+2}$ [30]. We have obtained the infinite homotopically commutative diagram of spectra consisting of universal quadrates of spectra:



(29)

The spectral sequence in surgery theory is constructed by using the filtration

$$\cdots \longrightarrow X_{3,0} \longrightarrow X_{2,0} \longrightarrow X_{1,0} \longrightarrow X_{0,0} \longrightarrow X_{-1,0} \longrightarrow \cdots, \tag{30}$$

contained in diagram (29) (see [10]). By definition (see [10]), we set

$$E_1^{p,s} = \pi_{s-p}(X_{p,0}, X_{p+1,0}) = \pi_{s-p}(X_{p,i}, X_{p+1,i}) \quad \forall i.$$

Since the quadrates in diagram (29) are universal, we see that

$$\begin{aligned} E_1^{p,s} &= \pi_{s-p}(X_{p,p}, X_{p+1,p}) = \pi_{s-p}(X_{p-1,p}) = \pi_{s-p}\left(\Sigma^{(p+1)q+1}\mathbb{L}S(\Psi^{(-)^p})\right) \\ &= \pi_{s-(q+1)(p+1)}\mathbb{L}S(\Psi^{(-)^p}). \end{aligned}$$

Then the differential

$$d_1^{p,s} : E_1^{p,s} \to E_1^{p+1,s}$$

is determined by the natural composition

$$\pi_{s-p}(X_{p,p}, X_{p+1,p}) \xrightarrow{\cong} \pi_{s-p}(X_{p,p+1}, X_{p+1,p+1}) \longrightarrow \pi_{s-p-1}\left(X_{p+1,p+1}, X_{p+2,p}\right)$$

of mappings of homotopy groups.

For one-sided submanifolds ($q = 1$), there exists the isomorphism $E_1^{p,s} = LN_{s+2}(A \to B)$ (see [30]). In this case, the differential $d_1^{p,s}$ coincides with the composition

$$LN_{s-2p-2}(A \to B^{(-)^p}) \longrightarrow L_{s-2p-2}(B^{(-)^{p+1}}) \longrightarrow LN_{s-2p}(A \to B^{(-)^{p+1}})$$

of mappings in diagram (10) for the quadrates Ψ and Ψ^- (see [30] for the discussion in algebraic terms).

For $q = 2$, the first term of the spectral sequence $E_1^{p,s}$ depends on p (see [31]) and there exists the isomorphism

$$E_1^{p,s} = LS_{s+p+1}(\Psi^{(-)^p}) = LN_{s+p+1}(A \to B^{(-)^p}).$$

The first differential is still the composition of mappings in diagram (10)

$$d_1^{p,s} : LS_{s+p+1}(\Psi^{(-)^p}) \longrightarrow L_{s+p+1}(B^{(\xi)^{p+1}}) \longrightarrow LS_{s+p-2}(\Psi^{(-)^{p+1}}).$$

for the quadrate $\Psi^{(-)^p}$ and for the quadrate $\Psi^{(-)^{p+1}}$.

We present the main results of [30, 35] on the connection of the spectral sequence in surgery theory with the realization of elements of Wall groups by normal mappings of closed manifolds. We consider a restricted filtration for (30) in the case of Browder–Livesay pairs

$$X_{\infty,0} \longrightarrow \cdots \longrightarrow X_{3,0} \longrightarrow X_{2,0} \longrightarrow X_{1,0} \longrightarrow X_{0,0} = \mathbb{L}(B). \qquad (31)$$

Denote by D the natural mapping

$$L_n(B) = \pi_n(\mathbb{L}(B)) \longrightarrow \pi_n(X_{0,0}, X_{\infty,0}).$$

Then we obtain (see [30, 35]) that if $D(x) \neq 0$, then the element x cannot be realized by normal mappings of closed manifolds. Another class of elements $x \in L_n(B)$ is determined by the following two conditions:

$$x \in \operatorname{Im}\left[\pi_n(X_{\infty,0}) \to \pi_n(X_{0,0})\right],$$
$$x \notin \operatorname{Ker}\left[\pi_n(X_{0,0}) \to \pi_n(X_{0,\infty})\right];$$

these elements are called *second-type elements* (see [30, 35]). Second-type elements cannot be realized by normal mappings of closed manifolds (see [35]).

From the algebraic standpoint, we can construct the spectral sequence for quadratic extensions of anti-structures and L-groups with decorations (see [22]). Decorations themselves can also be considered as relative L-groups. In this case, the spectral sequence has a sufficiently simple structure (see [31]).

Spectral sequences in surgery theory are still poorly studied. There exists an example of nontrivial second differential (see [30]), but it is unknown whether there exists a spectral sequence with nontrivial higher-order differentials. It is also known that for $q = 1$ and manifolds with finite Abelian fundamental 2-groups, all differentials except for the first differential are trivial (see [30]).

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