Global actions, K-theory and unimodular rows

Anthony Bak & Anuradha S. Garge

Fakulät für Mathematik, Universität Bielefeld,

Bielefeld - 33501, Germany.

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Department of Mathematics,

University of Mumbai, Mumbai- 400098, India.

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Abstract

Global actions were introduced by Bak [1] in order to have a homotopy theory in a purely algebraic setting. In this paper we apply his techniques in a particular case: the (single domain) unimodular row global action. More precisely, we compute the the path connected component and fundamental group for the unimodular row global action. An explicit computation of the fundamental group of the (connected component of) unimodular row global action is closely related to stability questions in K-theory. This will be shown by constructing an exact sequence with the fundamental group functor as the middle term and having surjective stability for the functor K_2 on the left and injective stability for the functor K_1 on the right.

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1 Introduction

Global actions were introduced by Bak, (see [1]) in order to have the flexibility of combining algebraic and topological ideas. In this paper, we will be concentrating on single domain global actions. A single domain global action consists of the following data: a set together with several groups acting on it such that these group actions satisfy a certain compatibility condition. There is a well-defined notion of homotopy in this setting and here we show that this circle of ideas can be applied very effectively in the following situation: given an associative ring R with unity, a unimodular row of length n over R is by definition, an n-tuple of the form $v = (v_1, \ldots, v_n)$, with $v_i \in R$ such that there exists $w = (w_1, \ldots, w_n)$, $w_i \in R$ with $v \cdot w^t = \langle v, w \rangle := \sum_i v_i w_i = 1$. The set of all unimodular rows of length n over R is denoted by $\mathrm{Um}_n(R)$. The action of the general linear group $\mathrm{GL}_n(R)$ (and hence its elementary subgroup $\mathrm{E}_n(R)$) on $\mathrm{Um}_n(R)$ allows one to define in a natural way a single domain global action structure on $\mathrm{Um}_n(R)$.

The aim of this paper is to investigate π_0 and π_1 of this single domain global action, and to show that both objects are closely related to stability questions in algebraic K-theory. An algebraic description for π_0 is easy to formulate and prove:

$$\pi_0(Um_n(R)) = Um_n(R)/E_n(R),$$

where the object on the right denotes the orbit space of the action of $\mathrm{E}_n(R)$ on $\mathrm{Um}_n(R)$ with base point the orbit of $e_1 = (1, 0, ..., 0)$ in $\mathrm{Um}_n(R)$.

An algebraic description of $\pi_1(Um_n(R))$ is more difficult and can defined for the connected component of the base point of this global action, which we denote by $\mathrm{EUm}_n(R)$. Let $\mathrm{St}_n(R)$ denote the Steinberg group and let $\theta_n : \mathrm{St}_n(R) \to \mathrm{GL}_n(R)$ denote the standard group homomorphism defined by sending generators $X_{ij}(r)$ $(i \neq j)$ of the Steinberg group to the elementary matrices $E_{ij}(r)$ $(i \neq j)$. (Note here that for $i \neq j$ if e_{ij} denotes the $n \times n$ matrix whose (i, j)-th entry is 1 and all other entries are 0, then for $r \in R$, let $E_{ij}(r) = I_n + re_{ij}$, where I_n denotes the $n \times n$ identity matrix.) Set:

1. $\mathrm{P}_n(R) = \{\sigma \in \mathrm{GL}_n(R) | e_1\sigma = e_1\}$.

2. $\mathrm{EP}_n(R) = P_n(R) \cap \mathrm{E}_n(R)$.

3. $\widetilde{P_n}(R)$ preimage of $\mathrm{EP}_n(R)$ in $\mathrm{St}_n(R) := \theta_n^{-1}(\mathrm{EP}_n(R))$.

4. $\mathrm{B}_n(R)$ is a certain normal subgroup of $\widetilde{P_n}(R)$ to be defined in Section §3.

Then, one has
$$\pi_1(\mathrm{EUm}_n(R)) = \widetilde{P_n}(R)/\mathrm{B}_n(R).$$

The relationship of π_0 and π_1 of $\mathrm{Um}_n(R)$ to stability in algebraic K-theory is expressed by two short exact sequences, one with π_1 as its middle term and the other with π_0 as its middle term. The sequence with π_1 has surjective stability for the functor K_2 on the left and injective stability for the functor K_1 on the right. The sequence with π_0 has surjective stabilty for the functor K_1 on the left and injective stability for the functor K_0 on the right. Together,these short exact sequences are equivalent to the 8-term exact sequence:

$$(\mathrm{K}_{2,n}(R))_2 \to \mathrm{K}_{2,n}(R) \to \pi_1(\mathrm{EUm}_n(R)) \to \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2 \to$$
$$\mathrm{K}_{1,n}(R) \to \pi_0(\mathrm{Um}_n(R)) \to \mathrm{K}^s_{0,n-1}(R) \to \mathrm{K}^s_{0,n}(R).$$

of pointed sets. By definition,

$$(\mathrm{K}_{2,n}(R))_2 = \mathrm{K}_{2,n}(R) \cap (\widetilde{P_n}(R))_2$$

and contains the
$$\mathrm{image}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))$$
(see (§5));

$$(\mathrm{K}_{1,n-1}(R))_2 = (\mathrm{GL}_{n-1}(R) \cap (\mathrm{EP}_n(R))_2)/\mathrm{E}_{n-1}(R),$$

where $(\mathrm{EP}_n(R))_2$ is a normal subgroup of $\mathrm{EP}_n(R)$ (see (§5)), which contains $\mathrm{E}_{n-1}(R)$. Let $\mathrm{K}^s_{0,m}(R)$ be the set of all isomorphism classes of finitely generated projective modules P such that for some r (depending on P) $P \oplus R^r = R^{m+r}$. The base point of $K^s_{0,m}(R)$ is the isomorphism class of R^m.

The first 3 terms of the exact sequence above, starting from the left, come equipped with group structures and the maps between them are group homomorphisms. So this much of the sequence is an exact sequence of groups. Suppose that $\mathrm{E}_{n-1}(R)$ and $\mathrm{E}_n(R)$ are normal in $\mathrm{GL}_{n-1}(R)$ and $\mathrm{GL}_n(R)$ respectively. Then $\mathrm{K}_{1,n-1}(R)$ and $\mathrm{K}_{1,n}(R)$ are groups and it turns out that $(\mathrm{K}_{1,n-1}(R))_2$ is a normal subgroup of $\ker(\mathrm{K}_{1,\mathrm{n-1}}(\mathrm{R}) \to \mathrm{K}_{1,\mathrm{n}}(\mathrm{R}))$ and that the map $\pi_1(\mathrm{Um}_n(R)) \to \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2$ has as image the group $[\ker(\mathrm{K}_{1,n-1}(R) \to \mathrm{K}_{1,n}(R))]/(\mathrm{K}_{1,n-1}(R))_2$ and is a group homomorphism to this group. So in this case, the first 5 terms behave like an exact sequence of groups. It is an interesting problem to find group structures on the remaining objects so that the entire sequence behaves like an exact sequence of groups.

Assuming the ring R is commutative and noetherian of finite Krull dimension d and n is sufficiently large relative to d, van der Kallen [14], [15] has found a group structure on $\pi_0(\mathrm{Um}_n(R))$, but has shown that the map $\mathrm{GL}_n(R) \to \pi_0(\mathrm{Um}_n(R))$ is not always a group homomorphism. On the other hand, Ravi Rao and van der Kallen [9] have found (nontrivial) examples where it is a group homomorphism. In these examples, we get a 6-term sequence which behaves like an exact sequence of groups. An interesting problem is to find group structures on the $\mathrm{K}^s_{0,i}(R)$ such that the maps involving these groups in the sequence are group homomorphisms.

The rest of the paper is organized as follows: Section 2 gives basic defintions and many relevant examples of global actions. Section 3 describes the notion of homotopy for global actions and in Section 4 we give the details on simply connected coverings of global actions: first a global-action theoretic construction and then an algebraic one. Universality of the simply connected covering then implies that these two constructions are isomorphic. With this one computes the fundamental group of the elementary unimodular row global action. Section 5 constructs the exact sequence mentioned in the introduction and deduces an interesting corollary on the vanishing of the fundamental group, as predicted by algebra.

2 Preliminaries

2.1 Global actions

In this section, we recall from [1] the definition of a global action, a single domain global action and their morphisms, and provide some examples. We begin with the definition of a group acting on a set.

Definition 2.1. If G is a group and X is a set, then a (right) *group action* of G on X is a function $X \times G \to X$, denoted by $(x, g) \mapsto x \cdot g$, such that:

1. $x \cdot e = x$, for all $x \in X$, where e is the identity of the group G.

2. $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$, for all $x \in X$ and $g_1, g_2 \in G$.

Such a group action will be denoted by $X \curvearrowleft G$.

Definition 2.2. Let X, Y be sets with groups G, H acting on them respectively. A *morphism of group actions*, $(\psi, \varphi) : X \curvearrowleft G \to Y \curvearrowleft H$, consists of a function $\psi : X \to Y$ and a homomorphism of groups $\varphi : G \to H$ such that $\psi(x \cdot g) = \psi(x) \cdot \varphi(g)$.

Definition 2.3. A *global action* A consists of a set X_A (called the underlying set of A) together with:

1. An indexing set Φ_A, having a reflexive relation $\leq$ on it.

2. A family $\{(X_A)_\alpha \curvearrowleft (G_A)_\alpha \mid \alpha \in \Phi_A\}$ of group actions on subsets $(X_A)_\alpha$ of X_A. The $(G_A)_\alpha$ are called the local groups of the global action.

3. For each pair $\alpha \leq \beta$ in Φ_A, a group homomorphism,

$$(\theta_A)_{\alpha\beta} : (G_A)_\alpha \to (G_A)_\beta,$$

called a structure homomorphism such that:

   (a) The groups $(G_A)_\alpha$ leave $(X_A)_\alpha \cap (X_A)_\beta$ invariant.

   (b) The pair

$$(\text{inclusion}, (\theta_A)_{\alpha\beta}) : ((X_A)_\alpha \cap (X_A)_\beta) \curvearrowleft (G_A)_\alpha \to (X_A)_\beta \curvearrowleft (G_A)_\beta$$

   is a morphism of group actions. (This will be called the compatibility condition).

Definition 2.4. A global action A is said to be a *single domain global action* if $(X_A)_\alpha = X_A$, for all $\alpha \in \Phi_A$.

Remark 2.5. *For simplicity of notation whenever only one global action is involved, we shall drop the suffix A everywhere in the definition and write $X, G_\alpha, X_\alpha, \theta_{\alpha\beta}$ instead.*

Definition 2.6. Let G be a group and let Φ be an index set (equipped with a reflexive relation) for a family G_α $(\alpha \in \Phi)$ of subgroups of G. One defines a single domain global action A from this data, by letting $X = G$ and letting each G_α act on X by right multiplication. If H denotes a subgroup of G, then one can make the space G/H of right cosets Hg of H in G into a single domain global action by letting each G_α act on G/H in the obvious way, i.e. $(Hg)g_\alpha = Hgg_\alpha$, for all $g_\alpha \in G_\alpha$.

We recall the definition of a morphism between global actions from [1]. To do this one requires the notion of a local frame, which is defined below.

Definition 2.7. Let A be a global action. Let $x \in X_\alpha$ be some point in a local set of A. A *local frame* at x in α or an α-frame at x is a finite subset, say $\{x = x_0, \ldots, x_p\}$ of X_α such that G_α-action on X_α is transitive on $x_0, \ldots, x_p$ i.e., for each j, $1 \leq j \leq p$, there exists $g_j \in G_\alpha$ such that $x_0 \cdot g_j = x_j$.

Definition 2.8. If A and B are global actions, with underlying sets X, Y and index sets Φ, Ψ respectively, a *morphism of global actions* is a function $f : X \to Y$ which preserves local frames. We shall denote such a morphism by $f : A \to B$. More precisely, if $x_0, \ldots, x_p$ is an α-frame at $x = x_0$, then $f(x_0), \ldots, f(x_p)$ is a β-frame at $f(x) = f(x_0)$ for some $\beta \in \Psi$.

Example 2.9. *Let A be a global action. Then the identity function from the underlying set of A to itself is a morphism of global actions.*

2.2 Important examples of global actions

We give below some examples of global actions by describing their underlying set, indexing set, local sets and local groups. It is easy to check that the compatibility condition holds. (See [1]).

- **The line action:** The *line action*, denoted by L is a global action with underlying set $X = \mathbb{Z}$ and indexing set $\Phi = \mathbb{Z} \cup \{*\}$. Let the only relations in Φ be $* \leq n$, for all $n \in \mathbb{Z}$ and $n \leq n$ for all $n \in \mathbb{Z}$. The local sets are $X_n = \{n, n+1\}$ if $n \in \mathbb{Z}$ and $X_* = \mathbb{Z}$. Let the local groups be $G_n = \mathbb{Z}/2\mathbb{Z}$, if $n \in \mathbb{Z}$, $G_* = 1$ and let $\{n, n+1\} \curvearrowleft G_n$ be the group action such that the non-trivial element of G_n exchanges the elements $n, n+1$. Let $\theta_{*\leq n} : \{1\} \to G_n$ denote the unique group homomorphism.

- **The general linear global action:**

  Given $n \geq 3$, let $J_n = ([1, n] \times [1, n]) \setminus \{(i, i) \mid 1 \leq i \leq n\}$ i.e. the cartesian product of the set $\{1, 2, \ldots, n\}$ with itself with the diagonal removed.

  A subset $\alpha \in J_n$ is called *nilpotent* if the following conditions hold:

  - If $(i, j) \in \alpha$, then $(j, i) \notin \alpha$.
  - If $(i, j), (j, k) \in \alpha$, then $(i, k) \in \alpha$.

Note that the empty set is a nilpotent subset and that the intersection of nilpotent subsets is nilpotent. Let R denote an associative ring with unity. The *general linear global action*, which we denote by $\mathrm{GL}_n(R)$ has underlying set $\mathrm{GL}_n(R)$, the general linear group. The index set Φ_n is the set of all nilpotent subsets α of J_n. We give Φ_n the partial ordering defined by $\alpha \le \beta \Leftrightarrow \alpha \subseteq \beta$. For all $\alpha \in \Phi_n$, let the local set $(X_{\mathrm{GL}_n(R)})_\alpha = \mathrm{GL}_n(R)$. For all $\alpha \in \Phi$, let the local group $\mathrm{GL}_n(R)_\alpha$ be the subgroup of $\mathrm{GL}_n(R)$ consisting of all matrices whose diagonal coefficients are 1, whose nondiagonal coefficients are 0 for coordinates $(i,j) \notin \alpha$ and arbitrary for coordinates $(i,j) \in \alpha$. This means that the empty subset of Φ_n is assigned the trivial subgroup of $\mathrm{GL}_n(R)$. Clearly $(\mathrm{GL}_n(R))_\alpha \cap (\mathrm{GL}_n(R))_\beta = (\mathrm{GL}_n(R))_{\alpha\cap\beta}$. Thus the assignment

$$\Phi_n \to \text{ subgroups of } \mathrm{GL_n(R)},$$

sending

$$\alpha \to (\mathrm{GL}_n(R))_\alpha$$

preserves not only partial orderings i.e. $\alpha \le \beta \implies (\mathrm{GL}_n(R))_\alpha \subset (\mathrm{GL}_n(R))_\beta$, but also intersections i.e. $(\mathrm{GL}_n(R))_{\alpha\cap\beta} = (\mathrm{GL}_n(R))_\alpha \cap (\mathrm{GL}_n(R))_\beta$. It is straight forward and easy to verify that if we assign to each pair $\alpha \le \beta \in \Phi_n$ the natural inclusion $\varphi_{\alpha\beta} : (\mathrm{GL}_n(R))_\alpha \to (\mathrm{GL}_n(R))_\beta$ then we get a (single domain) global action. The intersection property is not needed here. It will be used later to establish the covering property in the sense of the Steinberg extension.

It is not difficult to show that $(\mathrm{GL}_n(R))_\alpha$ is generated by all elementary matrices $E_{ij}(r)$, where $(i,j) \in \alpha$ and $r \in R$. We recall the definition of an elementary matrix. If $(i,j) \in J_n$, let e_{ij} denote the $n \times n$ matrix whose (i,j)-th entry is 1 and all other entries are 0. For $r \in R$, let $E_{ij}(r) = I_n + re_{ij}$, where I_n denotes the $n \times n$ identity matrix.

The subgroups $(\mathrm{GL}_n(R))_\alpha, \alpha \in \Phi$ are known in the literature as the *standard unipotent subgroups* of $\mathrm{GL}_n(R)$. Since any elementary matrix is contained in some $(\mathrm{GL}_n(R))_\alpha$ and since each $(\mathrm{GL}_n(R))_\alpha$ is generated by elementary matrices, it follows by definition that the $(\mathrm{GL}_n(R))_\alpha$ generate the elementary subgroup $\mathrm{E}_n(R)$ of $\mathrm{GL}_n(R)$.

- **The elementary global action:** The *elementary global action* has underlying set $\mathrm{E}_n(R)$, the elementary group. The indexing set as well as the local groups are the same as those for the general linear global action. Each local set is the whole $\mathrm{E}_n(R)$.

- **The special linear global action:** Suppose R is commutative. The *special linear global action* has underlying set $\mathrm{SL}_n(R)$, the special linear group. The indexing set as well as the local groups are the same as those for the general linear global action. Each local set is the whole $\mathrm{SL}_n(R)$.

Abusing notation, we shall let $\mathrm{GL}_n(R), \mathrm{E}_n(R)$ and $\mathrm{SL}_n(R)$ denote repectively the global actions defined above. Clearly, the canonical inclusions $\mathrm{E}_n(R) \to \mathrm{GL}_n(R)$ and when R is commutative, $\mathrm{E}_n(R) \to \mathrm{SL}_n(R) \to \mathrm{GL}_n(R)$ are morphisms of global actions.

Before we begin describing the Steinberg global action, we recall the definition of the Steinberg group itself from [8], §5.

Recall that elementary matrices satisfy the property

- $E_{ij}(r)E_{ij}(s) = E_{ij}(r+s)$, for all $r, s \in R$,

and that the following commutator formulae hold:

- $[E_{ij}(r) \quad E_{kl}(s)] = 1$, if $j \neq k, i \neq l$, $r, s \in R$.
- $[E_{ij}(r) \quad E_{jl}(s)] = E_{il}(rs)$, if $i \neq l$, $r, s \in R$.

The *Steinberg group* $\mathrm{St}_n(R)$, associated to a ring R is the free group defined by the generators $X_{ij}(r)$, $r \in R, (1 \leq i, j \leq n, i \neq j)$ subject to exactly the same relations above with E_{ij} replaced by X_{ij}. Thus, the Steinberg group is defined as a quotient $\mathfrak{F}/\mathfrak{N}$, where $\mathfrak{F}$ denotes the free group generated by the symbols $X_{ij}(r), r \in R$ and $\mathfrak{N}$ denotes the smallest normal subgroup of $\mathfrak{F}$ modulo which the above relations are valid. The assignment $X_{ij}(r) \to E_{ij}(r)$ sends the relations between the generators of $\mathrm{St}_n(R)$ into valid identities between elementary matrices.

- **The Steinberg global action**

  The Steinberg global action has underlying set $\mathrm{St}_n(R)$, the Steinberg group. The indexing set Φ_n is the same as that of the general linear global action. For all $\alpha \in \Phi_n$ the local set $(X_{\mathrm{St}_n(R)})_\alpha = \mathrm{St}_n(R)$ and the local group $(\mathrm{St}_n(R))_\alpha = \langle X_{ij}(r) \mid (i,j) \in \alpha, r \in R \rangle$. If we assign to each pair $\alpha \leq \beta \in \Phi_n$ the canonical inclusion $\varphi_{\alpha\beta} : (\mathrm{St}_n(R))_\alpha \to (\mathrm{St}_n(R))_\beta$ then it is straight forward and easy to show that we get a (single domain) global action.

Abusing notation, we let $\mathrm{St}_n(R)$ denote this global action. Clearly the canonical homomorphism of groups $\mathrm{St}_n(R) \to \mathrm{E}_n(R)$ described above is a morphism of global actions.

The next proposition provides the algebraic facts about the Steinberg group, which will be needed in the (algebraic) homotopy theory of $\mathrm{GL}_n(R)$ and $\mathrm{Um}_n(R)$ (to be defined in the next section).

Proposition 2.10. *Let $\theta : \mathrm{St}_n(R) \to \mathrm{E}_n(R)$ denote the canonical homomorphism. Let $\theta_E : \Phi_n \to$ subgroups of $\mathrm{E_n(R)}$, $\alpha \to (\mathrm{E}_n(R))_\alpha$ and let $\theta_{\mathrm{St}} : \Phi_n \to$ subgroups of $\mathrm{St_n(R)}$, $\alpha \to (\mathrm{St}_n(R))_\alpha$. Clearly, θ_E and θ_{St} are partial order preserving maps and thus functors. With these notations, one has:*

[1.] *The maps θ_E and θ_{St} preserve intersections and the commutative diagram*

$$\begin{tikzcd}
\Phi_n \arrow[r, "\theta_{\mathrm{St}}"] \arrow[dr, "\theta_E"'] & \text{Subgrps. of } \mathrm{St_n(R)} \arrow[d, "\theta"] \\
& \text{Subgrps. of } \mathrm{E_n(R)}
\end{tikzcd}$$

defines a natural isomorphism $\theta_{\mathrm{St}} \to \theta_E$ of functors.

[2.] *The canonical homomorphisms*

$$\mathrm{colim}(\mathrm{E}_n(R))_\alpha \longleftarrow \mathrm{colim}(\mathrm{St}_n(R))_\alpha \longrightarrow \mathrm{St}_n(R)$$

are isomorphisms.

[3.] *The canonical map*

$$\bigcup_{\alpha\in\Phi_n} (\mathrm{St}_n(R))_\alpha \to \bigcup_{\alpha\in\Phi_n} (\mathrm{E}_n(R))_\alpha$$

is bijective.

Proof. [1.] Let S_n denote the group of permutations of n elements. Let $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix} \in S_n$. To each element π, we associate the permutation matrix M_π, whose π_i-the column has zeroes in all entries except i-th where it has 1.

The groups S_n acts on $\mathrm{GL}_n(R)$ on the right by conjugation by permutation matrices. It is easy to chek that $E_{ij}(r)^\pi = M_{\pi^{-1}} E_{ij}(r) M_\pi = E_{(i\pi)(j\pi)}$ and the resulting action of S_n on $\mathrm{E}_n(R)$ preserves the 3 relations above for elementary matrices. Thus the action of S_n on $\mathrm{E}_n(R)$ lifts to an action of S_n on $\mathrm{St}_n(R)$ such that the homomorphism θ is S_n equivariant. The group S_n acts on J_n in the obvious way namely, $(i,j)\pi = (i\pi, j\pi)$ and there is an induced action of S_n on Φ_n. It is obvious that the maps θ_E and θ_{St} are S_n equivariant. Let δ denote the nilpotent set

$$\{(i,j) \mid i < j, 1 \leq i, j \leq n\} \subset \Phi_n.$$

The set δ is a maximal nilpotent subset. It is easy to check that any nilpotent subset is contained in a maximal nilpotent subset and that any maximal nilpotent subset is conjugate under the action of S_n to δ.

To prove that θ defines a natural isomorphism of θ_E and θ_{St}, we must show that for any $\alpha \in \Phi_n$, the surjective canonical homomorphism $\mathrm{St}_n(R)_\alpha \to \mathrm{E}_n(R)_\alpha$ is injective as well. By the previous paragraph, it suffices to consider the maximal nilpotent set δ. But here the result follows immediately from [8], Lemma 9.14.

It was shown, following the definition of the global action $\mathrm{GL}_n(R)$ that θ_E preserves intersections. θ_{St} preserves intersections because of the following facts: θ_E preserves intersections, each canonical homomorphism $(\mathrm{St}_n(R))_\alpha \to (\mathrm{E}_n(R))_\alpha$ is bijective and $(\mathrm{E}_n(R))_{\alpha\cap\beta} = (\mathrm{E}_n(R))_\alpha \cap (\mathrm{E}_n(R))_\beta$.

[2.] The left hand isomorphism follows immediately from [1.] above. The right hand isomorphism is defined and is obviously surjective. Using the definition of the Steinberg group by generators and relations, one can construct straightforward an inverse to this homomorphism, since any relation is contained in some local subgroup $(\mathrm{St}_n(R))_\alpha$.

[3.] Let $x \in (\mathrm{St}_n(R))_\alpha$ and $y \in (\mathrm{St}_n(R))_\beta$. Let $\gamma = \alpha \cap \beta$. Suppose $\theta(x) = \theta(y)$. We must show $x = y$. Clearly, $\theta(x) = \theta(y)$ in $(\mathrm{E}_n(R))_\gamma$. Let

$z \in (\mathrm{St}_n(R))_\gamma$ be such that $\theta(z) = \theta(x)$. Since $(\mathrm{St}_n(R))_\gamma \subset (\mathrm{St}_n(R))_\alpha$, it follows that $x = z$, since the homomorphism $(\mathrm{St}_n(R))_\alpha \to (\mathrm{E}_n(R))_\alpha$ is bijective. Similarly, $y = z$.

□

3 Elementary homotopy theory of global actions

This section summarizes in a convenient form the constructions and results we need from the homotopy theory of global actions, in particular of single domain global actions. They are due to the first named author.

3.1 The notion of homotopy

The most natural notion of homotopy is the following.

To begin we recall the notion of product for global actions. Suppose A and B are global actions with underlying sets X and Y and indexing sets Φ_A and Φ_B, respectively. Define the *product global action* $A \times B$ as follows. Its underlying set is the Cartesian product $X \times Y$ of sets and its index set is also the Cartesian product $\Phi_A \times \Phi_B$ with quasi-ordering defined by $(a, b) \leq (a', b')$ if and only if $a \leq a'$ and $b \leq b'$. The local set $(X \times Y)_{(a,b)}$ is the Cartesian product $X_a \times X_b$ and the local group $G_{(a,b)}$ is the product group $G_a \times G_b$. Its action on $(X \times Y)_{(a,b)}$ is the obvious one, namely coordinatewise.

Let $f, g : A \to B$ denote morphisms of global actions. Let L denote the line action, cf. Section 2, with underlying set $\mathbb{Z}$. For $n \in \mathbb{Z}$ let $\iota_n : X \to X \times L$, $x \to (x, n)$. It clearly defines a morphism $\iota_n : A \to A \times L$ of global actions. The morphisms f and g are called *homotopic* if there is a morphism $H : A \times L \to B$ of global actions and integers $n_- \leq n_+$ such that for all $n \leq n_-$, $fH\iota_n = fH\iota_{n_-}$ and for all $n_+ \leq n$, $gH\iota_n = gH\iota_{n_+}$. The morphism H is called, as in topology, a *homotopy* from f to g.

In some situations such as that of paths, a variant of the above concept is needed. We shall call the one needed for paths, stable homotopy, and define it in the next subsection. (In lecture notes distributed in the past, it was called end-point homotopy or end-point stable homotopy.)

3.2 Stable homotopy of paths and the fundamental group

The goal of this section is to define the notion of stable homotopy for paths and to define the fundamental group functor π_1. In passing we define the path connected component functor π_0.

Throughout this section A and B denote global actions with underlying sets X and Y, respectively, and L the line action.

The easiest and most natural way to define a *path* in A is as a finite sequence $x_1, \ldots, x_n$ of points $x_i \in X$ such that for each $i < n$ there is an element g in some local group of A such that x_i is in the domain of the action of g and $x_i g = x_{i+1}$. The following equivalent definition is better for the stable homotopy theory we need and shall develop.

Definition 3.1. Let $\omega : L \to A$ denote a morphism. We say that it is *stable on the left* or simply *left stable* if there is an integer n_- such that for all $n \leq n_-$, $\omega(n) = \omega(n_-)$. In this case we say that ω *stabilizes on the left* to $x = \omega(n_-)$. Similarly we say that ω is *stable on the right* or simply *right stable* if there is an integer n_+ such that for all $n \geq n_+$, $\omega(n) = \omega(n_+)$. In this case we say that ω *stabilizes on the right* to $x = \omega(n_+)$. We say that ω is *left-right stable* if it is stable both on the left and on the right. In this case we can clearly assume that $n_- \leq n_+$. A *path* is a left-right stable morphism $\omega : L \to A$. A *loop* is a path which stabilizes on the left and on the right to the same element of X.

A path $\omega : L \to A$ is *constant* if $\omega(n) = x$ for all $n \in \mathbb{Z}$ and some fixed $x \in X$. If ω is not constant then it is always the case that $n_- < n_+$. On the other hand, if ω is constant then n_- and n_+ can be any integers. For this reason, we exclude constant paths from the following definition.

Definition 3.2. Let ω denote a nonconstant path. The *lower* or *left degree* of ω is defined by

$$\mathrm{ld}(\omega) = \sup\{n_- \in \mathbb{Z} \mid \omega(n) = \omega(n_-) \text{ for all } n \leq n_-\}.$$

The *upper* or *right degree* of ω is defined by

$$\mathrm{ud}(\omega) = \inf\{n_+ \in \mathbb{Z} \mid \omega(n) = \omega(n_+) \text{ for all } n \geq n_+\}.$$

Next we define the notion of composition for paths.

Definition 3.3. Let ω and ω' denote two paths. The *initial point* (in X) of a nonconstant path ω is defined by

$$\mathrm{in}(\omega) = \omega(\mathrm{ld}(\omega)).$$

The *terminal point* of a nonconstant path ω is defined by

$$\mathrm{ter}(\omega) = \omega(\mathrm{ud}(\omega)).$$

The *initial* and *terminal* points of a constant path ω taking the constant value $x \in X$ is defined by

$$\mathrm{in}(\omega) = \mathrm{ter}(\omega) = x.$$

The *composition* $\omega \cdot \omega'$ of paths ω and ω' exists if $\mathrm{ter}(\omega) = \mathrm{in}(\omega')$ and is defined as follows:

$$(\omega \cdot \omega') = \begin{cases} \omega & \text{if } \omega' \text{ is constant} \\ \omega' & \text{if } \omega \text{ is constant.} \end{cases}$$

If ω and ω' are nonconstant then

$$(\omega \cdot \omega')(n) = \begin{cases} \omega'(n) & \text{for all } n \leq \mathrm{ud}\ \omega', \\ \omega(n - \mathrm{ud}\ \omega' + \mathrm{ld}\ \omega) & \text{for all } n \geq \mathrm{ud}\ \omega'. \end{cases}$$

It is clear that the composition law $\cdot$ on paths is associative.

We turn now to the notion of stable homotopy for paths.

Definition 3.4. A homotopy $H : L \times L \to A$ is called a *stable* (or *end-point stable*) homotopy of paths if for any $n \in \mathbb{Z}$, $H\iota_n$ is a path and if for any pair $m, n \in \mathbb{Z}$, $\mathrm{in}(Hi_m) = \mathrm{in}(Hi_n)$ and $\mathrm{ter}(H\iota_m) = \mathrm{ter}(H\iota_n)$.

Suppose there is a homotopy $H : L \times L \to A$ and there exist integers n_-, n_+ such that ω is the unique path with the property $\omega = H\iota_n$ for all $n \leq n_-$ and if ω' is the unique path such that there is an integer n_+ with the property $\omega' = H\iota_n$ for all $n \geq n_+$ then we say that ω is *stably homotopic* to ω' and write $\omega \simeq \omega'$.

The notion of homotopy is a generalization of the notion of path and has a notion of composition such that the compostion of two end-point stable homotopies of paths is again an end-point stable homotopy of paths. We shall take the time to explain this systematically, by replacing the line action L in Definitions 3.1, 3.2 and 3.3 above, by any global action $B \times L$ where B is an arbitrary global action. In other words, we are replacing the trivial global action, consisting of one point being acted on by the trivial group, by an arbitrary global action B. Thus instead of moving a point through the space A, as in the case of a path, we are moving one space B through another space A. In this setting the notion of *constant homotopy* becomes a morphism $H : B \times L \to A$ such that for any pair $m, n \in \mathbb{Z}$ $H\iota_m = H\iota_n$.

We give now the analogues of Definitions 3.1, 3.2 and 3.3.

Definition 3.5. Let $H : B \times L \to A$ denote a morphism. We say that it is *negatively stable* (or lower stable) if there is an integer n_- such that for all $n \leq n_-$, $H\iota_n = H\iota_{n_-}$. In this case we say that ω *stabilizes negatively* (or below) to $f = H\iota_{n_-}$. Similarly we say that ω is *positively stable* (or upper stable) (italex) if there is an integer n_+ such that for all $n \geq n_+$, $H\iota_n = H\iota_{n_+}$. In this case we say that ω *stabilizes positively* (or above) to $g = H\iota_{n_+}$. We say that ω is a *homotopy* if it is both negatively and positively stable. In this case we say that f is *homotopic* to g. Clearly this definition of homotopy for morphisms $B \to A$ is identical with that in Section §3.1.

Definition 3.6. Let H denote a homotopy. The *negative or lower degree* of H is defined by

$$\mathrm{ld}(H) = \sup\{n_- \in \mathbb{Z} \mid H\iota_n = H\iota_{n_-} \text{for all } n \leq n_-\}.$$

The *positive* or *upper degree* of H is defined by

$$\mathrm{ud}(H) = \inf\{n_+ \in \mathbb{Z} \mid H\iota_n = H\iota_{n_+} \text{for all } n \geq n_+\}.$$

Next we define the notion of composition for homotopies.

Definition 3.7. Let H and H' denote homotopies $B \times L \to A$. The *initial morphism* in $Mor(B, A)$ of a nonconstant homotopy is defined by

$$\mathrm{in}(H) = Hi_{\mathrm{ld}(H)}.$$

The *terminal morphism* in $Mor(B, A)$ of a nonconstant H is defined by

$$\mathrm{ter}(H) = Hi_{\mathrm{ud}(H)}.$$

The *initial* and *terminal* morphism of a constant homotopy H taking the constant value f in $Mor(B, A)$ is defined by

$$\mathrm{in}(H) = \mathrm{ter}(H) = f.$$

If H is a stable homotopy of paths then $\mathrm{in}(H)$ is called the *initial path* and $\mathrm{ter}(H)$ the *terminal path*. The *composition* $H \cdot H'$ of homotopies H and H' exists if $\mathrm{ter}(H) = \mathrm{in}(H')$ and is defined as follows:

$$(H \cdot H') = \begin{cases} H & \text{if } H' \text{ is constant} \\ H' & \text{if } H \text{ is constant.} \end{cases}$$

If H and H' are nonconstant then

$$(H \cdot H')(n) = \begin{cases} H'(n) & \text{for all } n \leq \mathrm{ud}\ H', \\ H(n - \mathrm{ud}\ H' + \mathrm{ld}\ H) & \text{for all } n \geq \mathrm{ud}\ H'. \end{cases}$$

Clearly if $f' = \mathrm{in}(H')$ and $f = \mathrm{ter}(H)$ then $\mathrm{in}(H.H') = f'$ and $\mathrm{ter}(H.H') = f$. Thus the relation of homotopy on morphisms in $Mor(B, A)$ is an equivalence relation. Furthermore if H and H' are composable and are at the same time stable homotopies of paths then the composition $H \cdot H'$ is also a stable homotopy of paths. This shows that the relation of stable homotopy on paths is an equivalence relation. It is clear that composition law $\cdot$ is associative, although we won't need this fact.

There is another important way to compose stable homotopies, but not arbitrary homotopies, which goes as follows. This kind of composition will be denoted by a square $\square$.

Definition 3.8. Let $H : L \times L \to A$ be a stable homotopy of paths. By definition the elements $\mathrm{in}(Hi_n)$ and $\mathrm{ter}(Hi_n)$ do not depend on the choice of n. Define the *initial point* (as opposed to intial path) of H by

$$\mathrm{inp}(H) = \mathrm{in}(Hi_n) \text{ for any n} \in \mathbb{Z}.$$

Define the *terminal point* (as opposed to terminal path) of H by

$$\mathrm{terp}(H) = \mathrm{ter}(Hi_n) \text{ for any n} \in \mathbb{Z}.$$

If H and $H' : L \times L \to A$ are stable homotopies of paths such that $\mathrm{terp}(H) = \mathrm{inp}(H')$ then the composition $H \Box H'$ is defined and has the property that $\mathrm{in}(H \Box H') = \mathrm{in}(H) \cdot \mathrm{in}(H')$ and $\mathrm{ter}(H \Box H') = \mathrm{ter}(H) \cdot \mathrm{ter}(H')$. Moreover the composition law $\Box$ is associative, although we won't need this fact.

We leave the construction of $\Box$ to the interested reader. Theorem 3.11 below says that elementary stable homotopies are the only tools one needs to construct $H \Box H'$.

We are now in a position to construct the fundamental monoid $\Pi_1(A)$ and fundamental group $\pi_1(A)$ of a pointed global action A.

Definition 3.9. Let A denote a pointed global action with base point $\circ$ in X. The *fundamental monoid*

$$\Pi_1(A) = \Pi_1(A, \circ)$$

is the set of all loops at $\circ$, with composition given by the composition law of Definition 3.3 and identity the constant loop at $\circ$.

We want to construct the fundamental group π_1 from Π_1. For this we need a definition and a result.

Definition 3.10. A *1-step stable homotopy* $H : L \times L \to A$ is either a constant stable homotopy or a nonconstant homotopy such that $\mathrm{ud}(H) - \mathrm{ld}(H) = 1$. Clearly every stable homotopy of paths is a composition of a finite number of 1-step stable homotopies. Let $n = \mathrm{ld}(H), \omega = Hi_n$, and $\omega' = Hi_{n+1}$. A 1-step homotopy H is called *elementary*, if

(3.9.1) it is constant,

or the following holds. There is an $i \in \mathbb{Z}$ such that for all $j \leq i$ and all $j \geq i+2$, $\omega(j) = \omega'(j)$ and there are elements $x, y \in X$ satisfying one of the following:

(3.9.2) $$\begin{pmatrix} w'(i) & w'(i+1) & w'(i+2) \\ w(i) & w(i+1) & w(i+2) \end{pmatrix} = \begin{pmatrix} x & y & y \\ x & x & y \end{pmatrix}$$

(3.9.3) $$\begin{pmatrix} w'(i) & w'(i+1) & w'(i+2) \\ w(i) & w(i+1) & w(i+2) \end{pmatrix} = \begin{pmatrix} x, x, y \\ x, y, y \end{pmatrix}$$

$$(3.9.4)\quad \begin{pmatrix} w'(i) & w'(i+1) & w'(i+2) \\ w(i) & w(i+1) & w(i+2) \end{pmatrix} = \begin{pmatrix} x, y, x \\ x, x, x \end{pmatrix}$$

$$(3.9.5)\quad \begin{pmatrix} w'(i) & w'(i+1) & w'(i+2) \\ w(i) & w(i+1) & w(i+2) \end{pmatrix} = \begin{pmatrix} x, x, x \\ x, y, x \end{pmatrix}$$

Theorem 3.11. *Every* 1*-step stable homotopy is a composition of elementary homotopies and thus every stable homotopy is a composition of elementary homotopies.*

The proof is not very difficult and is left to the reader. However, in the current paper, we do not use the fact that elementary homotopies generate all stable homotopies, rather we use them, as in the proof of Corollary 3.13 below, to show directly that certain homotopies exist.

Definition 3.12. If ω is a path, define the *inverse path* ω^{-1} by

$$\omega^{-1}(n) = \omega(-n).$$

Corollary 3.13. *If* ω *is a path then* $\omega \cdot \omega^{-1}$ *is stably homotopic to the constant path at* $\mathrm{in}(\omega)$*.*

The corollary follows by an easy application of elementary homotopies.

Definition 3.14. Let A denote a pointed global action. By 3.9 stable homotopy respects composition in $\Pi_1(A)$. Thus the stable homotopy classes of loops in $\Pi_1(A)$ form a monoid with identity the stable homotopy class of the constant loop at the base point. By Corollary 3.13 every loop $\omega \in \Pi_1(A)$ has up to stable homotopy an inverse ω^{-1}. Thus the stable homotopy classes of loops in $\Pi_1(A)$ form a group which we denote by

$$\pi_1(A)$$

and call the *(algebraic) fundamental group* of A.

Two points $x, x' \in X$ are called *path connected* if there is a path ω such that $\mathrm{in}(\omega) = x$ and $\mathrm{ter}(\omega) = x'$. The composition law for paths shows that the relation path connected is transitive, the construction of the inverse path ω^{-1} shows that the relation is symmetric, and the existence of the constant path at any point shows that the relation is reflexive. Thus the relation path connected is an equivalence relation on X.

Definition 3.15. Let

$$\pi_0(A)$$

denote the equivalence classes of the relation path connected on X. It is called the set of *path connected* components of A. If A has a base point then $\pi_0(A)$ is usually given as base point, the equivalence class of the base point of X.

3.3 Path connected component of the unimodular row global action

We now decribe the unimodular row global action and compute its path connected component.

The unimodular global action: The *unimodular global action* has as underlying set $\mathrm{Um}_n(R)$, the set of all R-unimodular row vectors $v = (v_1, v_2, \ldots, v_n)$ of length n, with coefficients $v_i \in R$. Recall that unimodular means there is a row vector $w = (w_1, \ldots, w_n)$ such that $v({}^t w) = \sum_i v_i w_i = 1$, where t denotes the transpose operator on (not necessarily square) matrices. (The row w is automatically unimodular, because $1 =^t 1 =^t (v({}^t w)) = w({}^t v)$.) The general linear group $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$ on the right, in the usual way. The indexing set Φ_n as well as the local groups $(\mathrm{E}_n(R))_\alpha$ are the same as for the global action $\mathrm{GL}_n(R)$. Each local set is the whole $\mathrm{Um}_n(R)$ and the action of each local group $\mathrm{E}_n(R)$ on $\mathrm{Um}_n(R)$ is via that of $\mathrm{GL}_n(R)$ on $\mathrm{Um}_n(R)$. Abusing notation, we shall let $\mathrm{Um}_n(R)$ denote also this (single domain) global action. We give the underlying set of $\mathrm{Um}_n(R)$ the distinguished point $e = (1, 0, \ldots, 0)$.

Proposition 3.16. $\pi_0(\mathrm{Um}_n(R)) = \mathrm{Um}_n(R)/\mathrm{E}_n(R)$. *Give this coset space the base point* $e\mathrm{E}_n(R)$. *Then, the connected component of* e *in* $\mathrm{Um}_n(R)$ *is the coset space* $e\mathrm{E}_n(R)$.

Proof. We prove that v, w belong to the same path component in $\mathrm{Um}_n(R)$ if and only if there exists $\varepsilon \in \mathrm{E}_n(R)$ such that $v\varepsilon = w$ i.e., if and only if $v\mathrm{E}_n(R) = w\mathrm{E}_n(R)$.

Let $v, w \in \mathrm{Um}_n(R)$ belong to the same path component and let ω be a path from v to w. As ω is a morphism of global actions, there exist $\varepsilon_i \in \mathrm{GL}_n(R)_{\alpha_i}$, $1 \leq i \leq N$ such that $v\varepsilon_1 \cdots \varepsilon_N = w$. Then $\varepsilon := \prod_i \varepsilon_i \in \mathrm{E}_n(R)$ has the required property.

Conversely suppose $w = v\varepsilon$, for some $\varepsilon \in \mathrm{E}_n(R)$. Hence there exist $\mathrm{E}_{ij}(\lambda)$, $\lambda \in R$ such that $\varepsilon = \prod \mathrm{E}_{ij}(\lambda)$. As each $\mathrm{E}_{ij}(\lambda)$ lies in some local set, we can easily define a path from v to w. Thus, $\pi_0(\mathrm{Um}_n(R)) = \mathrm{Um}_n(R)/\mathrm{E}_n(R)$.

From this also follows that the path component of the base point e is $e\mathrm{E}_n(R)$.

□

We now introduce a global actions structure on the coset space above. We will introduce another important global action which is a certain coset space of the Steinberg group. These global actions are crucial in computing the fundamental group of the unimodular row global action.

- **The elementary unimodular global action:** The *elementary unimodular global action* has as underlying set $\mathrm{EUm}_n(R) = eE_n(R)$, *the path connected component of the base point in* $\mathrm{Um}_n(R)$. The index set as well as the local groups are the same as those for $\mathrm{Um}_n(R)$. Each local set is the whole of $\mathrm{EUm}_n(R)$. Abusing notation, as usual, we let $\mathrm{EUm}_n(R)$ denote this global action. We give it the base point e.

- **The Steinberg unimodular global action:** Let $\mathrm{P}_n(R)$ denote the subgroup of $\mathrm{GL}_n(R)$ which leaves e fixed. Clearly each matrix in $\mathrm{P}_n(R)$ takes the form $\begin{pmatrix} 1 & 0 \\ v & \tau \end{pmatrix}$, for some $v \in M_{n-1,1}(R)$, $\tau \in \mathrm{GL}_{n-1}(R)$.

  Let $\mathrm{EP}_n(R) = \mathrm{P}_n(R) \cap \mathrm{E}_n(R)$. There is an obvious canonical identification
  $$\mathrm{EUm}_n(R) = \mathrm{E}_n(R)/\mathrm{EP}_n(R)$$
  of global actions, which is induced by sending each element $e\varepsilon$ in $\mathrm{EUm}_n(R)$ to $\varepsilon\mathrm{EP}_n(R)$. The global action on the right is the obvious one: the underlying set is the right coset space $\mathrm{E}_n(R)/\mathrm{EP}_n(R)$ and the index set Φ_n and local groups $\mathrm{E}_n(R)_\alpha$ are the same as for $\mathrm{Um}_n(R)$. The local sets are all of $\mathrm{E}_n(R)/\mathrm{EP}_n(R)$ and the action of each local group on $\mathrm{E}_n(R)/\mathrm{EP}_n(R)$ is induced by the natural right action of the group $\mathrm{E}_n(R)$ on it. Let $\theta : \mathrm{St}_n(R) \to \mathrm{E}_n(R)$ denote the canonical homomorphism. Let
  $$\mathrm{B}_n(R) = \langle x^{-1}abx \in \theta^{-1}(\mathrm{EP}_n(R)) \mid x \in \mathrm{St}_n(R), a \in \mathrm{St}_n(R)_\alpha, b \in St_n(R)_\beta,$$
  $$\text{for some } \alpha, \beta \text{ in } \Phi_\mathrm{n}\rangle.$$

Clearly $\mathrm{B}_n(R)$ is a normal subgroup of $\theta^{-1}(\mathrm{EP}_n(R))$.

The *Steinberg unimodular global action* $\mathrm{StUm}_n(R)$ has underlying set the right coset space $\mathrm{St}_n(R)/\mathrm{B}_n(R)$. The indexing set Φ_n and local groups $\mathrm{St}_n(R)_\alpha$ are the same as those of the Steinberg global action $\mathrm{St}_n(R)$. Each local set is all of $\mathrm{St}_n(R)/\mathrm{B}_n(R)$ and the action of each $\mathrm{St}_n(R)_\alpha$ on $\mathrm{St}_n(R)/\mathrm{B}_n(R)$ is induced by the natural right action of the group $\mathrm{St}_n(R)$ on it. Abusing notation, we shall denote the Steinberg unimodular action also by $\mathrm{St}_n(R)/\mathrm{B}_n(R)$. We give it the distinguished point $e\mathrm{B}_n(R)$. It is easy to check that this is a path-connected global action.

There is a canonical base point preserving morphism

$$\mathrm{St}_n(R)/\mathrm{B}_n(R) \to \mathrm{E}_n(R)/\mathrm{EP}_n(R)$$

of coset spaces and global actions, which is induced by the map

$$\mathrm{St}_n(R) \to \mathrm{EUm}_n(R)$$

sending each x in $\mathrm{St}_n(R)$ to $e\theta(x)$.

We recall the definition of a covering morphism (see [1]) by introducing another important global action: the star global action.

Definition 3.17. Let A be a path-connected global action with underlying set X, index set Φ and local groups $X_\alpha \curvearrowleft G_\alpha$, for $\alpha \in \Phi$. Given $\alpha \in \Phi$ and $x \in X$, let $\mathrm{star}(x)$ denote the following global action: the underlying set $X_{\mathrm{star}(x)}$ is the union of all xG_α where G_α is a local group which acts on x i.e.,

$$X_{\mathrm{star}(x)} = \bigcup_{\{\alpha\in\Phi \mid x\in X_\alpha\}} x \cdot G_\alpha$$

The index set $\Phi_{\mathrm{star}(x)}$ consists of all $\alpha \in \Phi$ such that G_α acts on x. $\Phi_{\mathrm{star}(x)}$ inherits its ordering from Φ. If $\alpha \in \Phi$, then $(X_{\mathrm{star}(x)})_\alpha = xG_\alpha$ and $(G_{\mathrm{star}(x)})_\alpha = G_\alpha$.

Definition 3.18. A morphism $f : B \to A$ of global actions is *surjective*, if it is surjective map on the underlying sets. A surjective morphism $f : B \to A$ of global actions is called a *covering morphism* if for every $b \in X_B$, the induced map $f : \mathrm{star}(b) \to \mathrm{star}(f(b))$ is an isomorphism of global actions.

The next proposition records some facts which are needed for the (algebraic) homotopy theory of $\mathrm{Um}_n(R)$.

Corollary 3.19. *The canonical homomorphism* $\mathrm{St}_n(R) \to \mathrm{E}_n(R)$ *of path-connected global actions is a covering morphism in the sense of [1], §10.*

Corollary 3.20. *The canonical morphism* $\mathrm{StUm}_n(R) \to \mathrm{EUm}_n(R)$ *is a covering morphism of path-connected global actions.*

4 Coverings, fundamental group and elementary unimodular row global action

In this section we state (without proof) results for homotopy theory in the framework of global actions. The interested reader should refer to [1], §11. These will be applied in the concrete case of the elementary unimodular row global action to compute its fundamental group explicitly.

We give some basic definitions and then outline the construction of a connected, simply connected covering of the path-connected single domain global action $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$. The checking of details is not difficult and is left to the reader.

Definition 4.1. A path-connected global action A with base-point a_0 is said to be simply connected, if the fundamental group of A at a_0 is trivial i.e., $\pi_1(A, a_0) = e$.

Theorem 4.2. *There exists a path-connected, simply connected base point preserving covering of the path-connected global action* $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$. *Moreover, it is of the form* $\mathrm{E}_n(R)/H_B$, *where* H_B *is a normal subgroup of* $\mathrm{EP}_n(R)$.

Proof. Follows from "Structure theorem" for single domain global actions (see [1], Definition 3.3, Theorem 11.1, Proposition 11.3.) □

We now would like to prove that the path-connected, simply connected covering of $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ is also universal. For this, we state without proof the "Lifting criterion" in the context of global actions.

Lemma 4.3. *Let* $q : (B, b_0) \to (\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ *be a pointed covering morphism of path-connected global actions Let* C *be a path connected global* *action with base point c_0. Let $f : (C, c_0) \to (\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ be a pointed morphism. Then, a morphism $\widetilde{f} : (C, c_0) \to (B, b_0)$ lifting f exists if and only if $f_*(\pi_1(C, c_0)) \subset q_*(\pi_1(B, b_0))$. Moreover, if f exists, then it is unique.*

Corollary 4.4. *Every path-connected, simply-connected covering of the connected single domain global action $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ is universal. In particular, any two path-connected, simple connected coverings of the single domain global action $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ are isomorphic.*

Proof. Let $f : (C, c_0) \to (\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ be a morphism from a path-connected, simply connected covering (C, c_0). The lifting criterion ensures that (C, c_0) is universal, as $f_*(\pi_1(C, c_0)) = e$. Universality then implies that any two path-connected, simple connected coverings of $(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R))$ are isomorphic. □

Corollary 4.5. *Let $(\mathrm{E}_n(R)/H_B, eH_B)$ be the path-connected, simple connected of $(\mathrm{E}_n(R)/\mathrm{EP}_n(R), e\mathrm{EP}_n(R))$ with pointed covering morphism $p : (\mathrm{E}_n(R)/H_B, eH_B) \to (\mathrm{E}_n(R)/\mathrm{EP}_n(R), e\mathrm{EP}_n(R))$. Then,*

$$\pi_1(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R)) \simeq p^{-1}(e\mathrm{EP}_n(R)).$$

Proof. See [1], Theorem 10.17. □

We can view the fault line between algebra and topology more clearly. This helps us to compute the fundamental group of $\mathrm{EUm}_n(R)$ explicitly. The notion of universal cover is clear: Given $\mathrm{EUm}_n(R)$, a universal cover $X \to \mathrm{EUm}_n(R)$ is a cover such that given any other cover $X' \to \mathrm{EUm}_n(R)$ there is unique base point preserving map $X \to X'$ making the usual diagram commute. The existence of a universal cover for a single domain action has two distinct proofs, one algebraic and the other topological. The algebraic proof shows that $\mathrm{StUm}_n(R) \to \mathrm{EUm}_n(R)$ is a universal cover. The topological proof seen above, shows that any connected simply connected cover Y is universal and explicitly constructs one. The universality of the topological and algebraic constructions yields a unique isomorphism $\mathrm{StUm}_n(R) \to Y$ making the usual diagram commute.

We now record the key observation regarding coverings in coset spaces. This leads us to explicitly computing the fundamental group of $\mathrm{EUm}_n(R)$.

Proposition 4.6. *Let $n \geq 3$ be an integer and let $K \subset H$ be subgroups of $\mathrm{E}_n(R)$. Then, $p : \mathrm{E}_n(R)/K \to \mathrm{E}_n(R)/H$ defined by $p(K\varepsilon) = H\varepsilon$ is a* *covering morphism of global actions if and only if $H_2 \subset K$, where $H_2 = \langle H \cap x^{-1}(\mathrm{E}_n(R))_\alpha(\mathrm{E}_n(R))_\beta x\rangle$ i.e., H_2 is the subgroup of H generated by all elements in H which are also of the form $x^{-1}\varepsilon_\alpha\varepsilon_\beta x$, for some $x \in \mathrm{E}_n(R)$ and some local group elements $\varepsilon_\alpha, \varepsilon_\beta$.*

Proof. It is easy to check that p is a surjective morphism of global actions. We now prove that p is injective on stars if $H_2 \subset K$ i.e., we prove that $p : \text{star }(K\varepsilon) \to \text{star }(H\varepsilon)$ is injective if $H_2 \subset K$. Let $K\varepsilon_1, K\varepsilon_2 \in \text{star }(K\varepsilon)$ with $p(K\varepsilon_1) = p(K\varepsilon_2)$ i.e., $\varepsilon_2\varepsilon_1^{-1} \in H$.

As $K\varepsilon_1, K\varepsilon_2 \in$ star $(K\varepsilon)$, there exist local group elements $\varepsilon_\alpha, \varepsilon_\beta \in (\mathrm{E}_n(R))_\alpha, (\mathrm{E}_n(R))_\beta$ respectively such that $K\varepsilon_1 = (K\varepsilon)\varepsilon_\alpha$ and $K\varepsilon_2 = (K\varepsilon)\varepsilon_\beta$. Now $p(K\varepsilon_1) = p(K\varepsilon_2)$ implies $\varepsilon\varepsilon_\alpha\varepsilon_\beta^{-1}\varepsilon^{-1} \in H$ i.e., $\varepsilon\varepsilon_\alpha\varepsilon_\beta^{-1}\varepsilon^{-1} \in H_2$. Thus $\varepsilon\varepsilon_\alpha\varepsilon_\beta^{-1}\varepsilon^{-1} \in K$, as $H_2 \subset K$. This proves $\varepsilon_2(\varepsilon_1)^{-1} \in K$ i.e., p is injective, if $H_2 \subset K$.

Conversely suppose that $\mathrm{E}_n(R)/K \to \mathrm{E}_n(R)/H$ is a covering morphism. It is enough to prove that every generator of H_2 lies in K, as K is a subgroup. Let $x\varepsilon_\alpha\varepsilon_\beta x^{-1} \in H$. Then $(Kx)\varepsilon_\alpha^{-1}, (Kx)\varepsilon_\beta \in \text{star}(Kx)$, with $p((Kx)\varepsilon_\alpha^{-1}) = p((Kx)\varepsilon_\beta)$. Injectivity of p on $\text{star}(Kx)$ implies $Kx\varepsilon_\alpha^{-1} = Kx\varepsilon_\beta$, i.e., $x\varepsilon_\alpha\varepsilon_\beta x^{-1} \in K$, proving that every generator of H_2 lies in K. $\square$

Theorem 4.7. *Let $\mathrm{EUm}_n(R)$ be the connected single domain global action with base point $e\mathrm{EP}_n(R)$. Then, the simply connected (universal) covering of $\mathrm{EUm}_n(R)$ is $\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2$. Thus*

$$\pi_1(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R)) \simeq \mathrm{EP}_n(R)/(\mathrm{EP}_n(R))_2$$

$$\simeq \widetilde{P_n}(R)/B_n(R) \simeq \widetilde{P_n}(R)/(\widetilde{P_n}(R))_2$$

under the isomorphism induced by the homomorphism $\theta : \mathrm{St}_n(R) \to \mathrm{E}_n(R)$. (see Proposition 2.10). Here $(\mathrm{EP}_n(R))_2$ is defined as in Proposition 4.6 above and the same Proposition also shows that $(\widetilde{P_n}(R))_2 \simeq B_n(R)$.

Proof. Let the simply connected covering of $\mathrm{EUm}_n(R)$ be given by $\mathrm{E}_n(R)/H_B$, where H_B is a normal subgroup of $\mathrm{EP}_n(R)$. Using Proposition 4.6, it is easy to check that $\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2$ is a covering of $\mathrm{E}_n(R)/H_B$, by observing that $(H_B)_2 \subset (\mathrm{EP}_n(R))_2$.

We now prove that $\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2$ is a simply connected covering of $\mathrm{E}_n(R)/H_B$. For this note that $p_*(\pi_1(\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2)) \subset \pi_1(\mathrm{E}_n(R)/H_B)$, which is trivial as $\mathrm{E}_n(R)/H_B$ is a simply connected covering of $\mathrm{EUm}_n(R)$.

Now as p_* is injective, one has $\pi_1(\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2)$ is trivial, thus showing that $\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2$ is another simply connected covering of the global action $\mathrm{EUm}_n(R)$. By Corollary 4.4 we have $\mathrm{E}_n(R)/(\mathrm{EP}_n(R))_2 \simeq \mathrm{E}_n(R)/H_B$ is the universal, path-connected, simply-connected covering of the single domain global action $\mathrm{EUm}_n(R)$. Using Corollary 4.5, we see that

$$\pi_1(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R)) \simeq p^{-1}(e\mathrm{EP}_n(R)) = \mathrm{EP}_n(R)/(\mathrm{EP}_n(R))_2$$
$$\simeq \widetilde{P_n}(R)/B_n(R) \simeq \widetilde{P_n}(R)/(\widetilde{P_n}(R))_2.$$

□

5 Stability in K-theory and fundamental group of unimodular row global action

In this section we construct certain exact sequences of pointed sets. Under suitable conditions on the stable rank of the ring under consideration these exact sequences of pointed sets turn out to be exact sequences of groups. The sandwiching of $\pi_1(\mathrm{EUm}_n(R))$ in this exact sequence of groups helps us to conclude (in certain situations) about vanishing of $\pi_1(\mathrm{EUm}_n(R))$. This matches with the well-known algebraic results.

5.1 Exact sequences for path-connected and fundamental group functors of unimodular row global action

In this section we construct exact sequences for path-connected and fundamental group functors of the unimodular row global action. Let $\mathrm{K}^s_{0,m}(R)$ be the set of all isomorphism classes of finitely generated projective modules P such that for some r (depending on P) $P \oplus R^r \simeq R^{m+r}$. The base point of $\mathrm{K}^s_{0,m}(R)$ is the isomorphism class of R^m.

Proposition 5.1. *Let R be a ring and $n \geq 3$ be an integer. Then, the following exact sequences of pointed sets exist:*

1. $$\mathrm{K}_{1,n}(R) \xrightarrow{\alpha} \pi_0(\mathrm{Um}_n(R)) \xrightarrow{\beta} \mathrm{K}^s_{0,n-1}(R) \xrightarrow{\gamma} \mathrm{K}^s_{0,n}(R)$$
   where the base point of $\mathrm{K}_{1,n}(R)$ *is* $[I_n]$ *and the base point of* $\pi_0(\mathrm{Um}_n(R))$ *is* $[e]$.

2.

$$(\mathrm{K}_{2,n}(R))_2 \xrightarrow{\delta} \mathrm{K}_{2,n}(R) \xrightarrow{\eta} \pi_1(\mathrm{EUm}_n(R)) \xrightarrow{\mu} \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2 \xrightarrow{\lambda}$$

$$\mathrm{K}_{1,n}(R)$$

By definition,

$$(\mathrm{K}_{2,n}(R))_2 = \mathrm{K}_{2,n}(R) \cap (\widetilde{P_n}(R))_2;$$

$$(\mathrm{K}_{1,n-1}(R))_2 = (\mathrm{GL}_{n-1}(R) \cap (\mathrm{EP}_n(R))_2)/\mathrm{E}_{n-1}(R),$$

where $(\widetilde{P_n}(R))_2, (\mathrm{EP}_n(R))_2$ *are defined analogous to* H_2 *in Proposition 4.6.*

Proof. 1. Define $\alpha : \mathrm{K}_{1,n}(R) \to \pi_0(\mathrm{Um}_n(R))$ by $\sigma \mapsto (e\sigma)\mathrm{E}_n(R)$. Clearly this map preserves base points. For defining $\beta : \pi_0(\mathrm{Um}_n(R)) \to \mathrm{K}^s_{0,n-1}(R)$ note that given $v \in \mathrm{Um}_n(R)$, there exists a natural surjective map $\beta_v : R^n \to R$ defined by $\beta_v(w) = w \cdot v^t$ with $\ker\beta_v \oplus R \simeq R^n$ i.e., $[\ker\beta_v] \in \mathrm{K}^s_{0,n-1}(R)$. Define $\beta : \pi_0(\mathrm{Um}_n(R)) \to \mathrm{K}^s_{0,n-1}(R)$ by $\beta([v]) = [\ker\beta_v]$. This is a well-defined base-point preserving map. (For details see [11].) That it is an exact sequence of pointed sets follows using ideas as in Lemma 1.3 in [11].

2. We first define the maps $\delta, \eta, \mu, \lambda$.

2(a) The map δ is the natural inclusion map.

2(b) Define $\eta : \mathrm{K}_{2,n}(R) \to \pi_1(\mathrm{EUm}_n(R))$ by $\eta(Y) = Y(\widetilde{P_n}(R))_2$. Note that $Y \in \mathrm{K}_{2,n}(R)$ implies that $\theta_n(Y) = I_n$; hence $\theta_n(Y) \in \mathrm{EP}_n(R)$ and so $Y \in \widetilde{P_n}(R)$. Hence $\eta(Y)$ is defined.

2(c) To define the map $\mu : \pi_1(\mathrm{EUm}_n(R)) \to \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2$, recall that $\pi_1(\mathrm{EUm}_n(R)) \simeq \mathrm{EP}_n(R)/(\mathrm{EP}_n(R))_2$, via the standard homomorphism $\theta_n : \mathrm{St}_n(R) \to \mathrm{GL}_n(R)$. Given $\sigma \in \mathrm{EP}_n(R)$, there exists $\tau \in \mathrm{GL}_{n-1}(R)$ such that $\sigma = \begin{pmatrix} 1 & 0 \\ v & \tau \end{pmatrix} \in \mathrm{E}_n(R)$. Note that $\tau \in \mathrm{GL}_{n-1}(R) \cap \mathrm{E}_n(R)$ defines an element $\tau\mathrm{E}_{n-1}(R)$ of $K_{1,n-1}(R)$, which we denote by $[\sigma_{rd}]$, the class of the right diagonal of σ. It is easy to check that this map is well-defined.
Define $\mu(Y(\widetilde{P_n}(R))_2) = [(\theta_n(Y))_{rd}](\mathrm{K}_{1,n-1}(R))_2$. As $\mu(I_n(\widetilde{P_n}(R))_2) = [(\theta(I_n))_{rd}](\mathrm{K}_{1,n-1}(R))_2 = [I_{n-1}](\mathrm{K}_{1,n-1}(R))_2$, we have that $\mu(\widetilde{P_n}(R))_2 \subset (\mathrm{K}_{1,n-1}(R))_2$.

2(d) The map $\lambda : \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2 \to \mathrm{K}_{1,n}(R)$ is the natural one induced by the right diagonal inclusion of $\mathrm{GL}_{n-1}(R)$ inside $\mathrm{GL}_n(R)$ i.e., the map given by $[\tau](\mathrm{K}_{1,n-1}(R))_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \mathrm{E}_n(R)$. This map is well-defined: for if $[\tau] = [\tau']$, then $\tau\tau'^{-1} \in (\mathrm{K}_{1,n-1}(R))_2$ i.e., $\tau\tau'^{-1} \in \mathrm{GL}_{n-1}(R)$ and $\begin{pmatrix} 1 & 0 \\ 0 & \tau\tau'^{-1} \end{pmatrix} \in (\mathrm{EP}_n(R))_2 \subset \mathrm{E}_n(R)$. Thus, $\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \mathrm{E}_n(R) = \begin{pmatrix} 1 & 0 \\ 0 & \tau' \end{pmatrix} \mathrm{E}_n(R)$.

3. Having defined the maps, we first check that we get a complex.

3(a) Note that for $Y \in (\mathrm{K}_{2,n}(R))_2$, $(\eta \circ \delta)(Y) = \eta(Y) = Y(\widetilde{P_n}(R))_2) = e(\widetilde{P_n}(R))_2)$, as $Y \in (\widetilde{P_n}(R))_2)$. Thus, $\eta \circ \delta = e$.

3(b) For $Z \in \mathrm{K}_{2,n}(R)$, consider $(\mu \circ \eta)(Z) = \mu(Z(\widetilde{P_n}(R))_2)) = [\theta_n(Z)_{rd}] = [I_n]$, as $Z \in \mathrm{K}_{2,n}(R)$. This proves that $\mu \circ \eta = e$.

3(c) One also has that $\lambda \circ \mu = e$, as $(\lambda \circ \mu)Z(\widetilde{P_n}(R))_2 = \lambda(\mu(Y(\mathrm{EP}_n(R))_2)) = \lambda([Y_{rd}])$ for some $Y \in \mathrm{EP}_n(R)$ via the identification of $\pi_1(\mathrm{EUm}_n(R))$ with the orbit space $\mathrm{EP}_n(R)/(\mathrm{EP}_n(R))_2$. Now $\lambda([Y_{rd}]) = [\begin{pmatrix} 1 & 0 \\ 0 & Y_{rd} \end{pmatrix}]$ $= [I_n] \in \mathrm{K}_{1,n}(R)$, as $Y \in \mathrm{EP}_n(R)$.

4. Now we check exactness at each place:

4(a) We check $\ker\, \eta \subset \mathrm{Im}\delta$. Let $Y \in \mathrm{K}_{2,n}(R) \in \ker\, \eta$. Hence, $Y \in \widetilde{P_n}(R)_2 \cap \mathrm{K}_{2,n}(R)$, which by definition is $(\mathrm{K}_{2,n}(R))_2$. Thus, Y belongs to $\mathrm{Im}\delta$.

4(b) We now check that $\ker\, \mu \subset \mathrm{Im}\eta$. Let $Y(\widetilde{P_n}(R))_2 \in \ker\, \mu$. Then, $\mu(Y(\widetilde{P_n}(R))_2) = [(\theta_n Y)_{\mathrm{rd}}](\mathrm{K}_{1,n-1}(R))_2 = [I_n](\mathrm{K}_{1,n-1}(R))_2$, which implies $(\theta_n Y)_{\mathrm{rd}} \in (\mathrm{K}_{1,n-1}(R))_2$. Now write $\begin{pmatrix} 1 & 0 \\ 0 & (\theta_n Y)_{\mathrm{rd}} \end{pmatrix} = \prod_{(i,j)} \varepsilon_{ij}\varepsilon'$, where $\varepsilon_{ij} \in \mathrm{E}_n(R)$ are elementary generators and $\varepsilon' \in \mathrm{E}_{n-1}(R)$. Breaking up ε' further into a product of elementary generators, we get that $\begin{pmatrix} 1 & 0 \\ 0 & (\theta_n Y)_{\mathrm{rd}} \end{pmatrix} = \prod_{(i',j')} \varepsilon_{i'j'}$, with $\varepsilon_{i'j'} \in \mathrm{E}_n(R)$. Thus, $X := \theta_n^{-1} \begin{pmatrix} 1 & 0 \\ 0 & (\theta_n Y)_{\mathrm{rd}} \end{pmatrix}$ makes sense as an element of $\mathrm{St}_n(R)$.

Let $X' = \theta_n^{-1}((\prod_{(i',j')} \varepsilon_{i'j'})^{-1}) \cdot X$ Then, clearly

$$\theta_n(X') = ((\prod_{(i',j')} \varepsilon_{i'j'})^{-1}) \cdot \theta_n(X) = I_n,$$

i.e., $\theta_n(X') \in \mathrm{K}_{2,n}(R)$. We claim that $\eta(X') = Y(\widetilde{P_n}(R))_2$ i.e., $X'(\widetilde{P_n}(R))_2 = Y(\widetilde{P_n}(R))_2$ i.e., $X'Y^{-1} \in (\widetilde{P_n}(R))_2$. For this we prove $\theta_n(X'Y^{-1}) \in (\mathrm{EP}_n(R))_2$, from which it follows that $X'Y^{-1} \in (\widetilde{P_n}(R))_2$, proving that $X'(\widetilde{P_n}(R))_2 = Y(\widetilde{P_n}(R))_2$, as required. For this note that $\theta_n(Y) \in \mathrm{EP}_n(R)$ and write

$$\theta_n(Y) = \begin{pmatrix} 1 & 0 \\ v & (\theta_n(Y))_{\mathrm{rd}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\theta_n(Y))_{\mathrm{rd}} \end{pmatrix}$$

Then, $\theta_n(X'Y^{-1})$

$$= (\prod_{(i',j')} \varepsilon_{i'j'})^{-1}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & (\theta_n Y)_{\mathrm{rd}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & ((\theta_n(Y))_{\mathrm{rd}})^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -v & I_{n-1} \end{pmatrix}.$$

Thus, $\theta_n(X'Y^{-1}) = \begin{pmatrix} 1 & 0 \\ -v & I_{n-1} \end{pmatrix}$. That this lies in $(\mathrm{EP}_n(R))_2$, can be clearly seen by writing it as a product of elementary generators of the type $\mathrm{E}_{i1}(v_i)$, where $v = (v_2, \ldots, v_n)^t$.

4(c) We check $\ker \lambda \subset \mathrm{Im}\mu$. Let $[Z](\mathrm{K}_{1,n-1}(R))_2 \in \ker \lambda$, i.e., $\begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} \in \mathrm{E}_n(R)$, with $Z \in \mathrm{K}_{1,n-1}(R)$. Then, clearly $\mu(\theta_n^{-1}\widetilde{P_n}(R))_2 = \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}$, proving that $\ker \lambda \subset \mathrm{Im}\mu$.

□

Proposition 5.2. *Let R be a ring and let $n \geq 3$ be an integer. Then,*

$$(\mathrm{K}_{2,n}(R))_2 \xrightarrow{\delta} \mathrm{K}_{2,n}(R) \xrightarrow{\eta} \pi_1(\mathrm{EUm}_n(R)) \xrightarrow{\mu} \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2 \xrightarrow{\lambda}$$

$$\mathrm{K}_{1,n}(R) \xrightarrow{\alpha} \pi_0(\mathrm{Um}_n(R)) \xrightarrow{\beta} \mathrm{K}^s_{0,n-1}(R) \xrightarrow{\gamma} \mathrm{K}^s_{0,n}(R).$$

is an exact sequence of pointed sets.

Proof. Define $\alpha : \mathrm{K}_{1,n}(R) \to \pi_0(\mathrm{Um}_n(R))$ by $\alpha([\sigma]) = (e_1\sigma)\mathrm{E}_n(R)$, i.e., α takes a matrix in $\mathrm{GL}_n(R)$ to its first row. Using the fact that for $n \geq 3$, $\mathrm{E}_n(R)$ is generated by elementary matrices of the form $\mathrm{E}_{1i}(\lambda)$ and $\mathrm{E}_{i1}(\lambda')$, with $\lambda, \lambda' \in R$ and $i \geq 2$, we can check that this map is well-defined.

It remains to check exactness at $\mathrm{K}_{1,n}(R)$. For this first consider $(\alpha \circ \lambda)([\sigma](\mathrm{K}_{1,n-1}(R))_2)$, with $[\sigma] \in \mathrm{K}_{1,n-1}(R)$. This equals $\alpha(\lambda([\sigma]\mathrm{K}_{1,n-1}(R))_2) = \alpha\left(\begin{pmatrix}1 & 0\\ 0 & \sigma\end{pmatrix}\mathrm{E}_n(R)\right) = e\mathrm{E}_n(R)$, proving that $\mathrm{Im}(\lambda) \subset \mathrm{Ker}(\alpha)$.

Let $[\sigma] \in \mathrm{Ker}(\alpha)$. Hence, $(e_1\sigma)\mathrm{E}_n(R) = e\mathrm{E}_n(R)$ i.e., σ is of the form $\begin{pmatrix}1 & 0\\ v & \tau\end{pmatrix}\cdot\varepsilon$, for some $\varepsilon \in \mathrm{E}_n(R)$. Rewrite $\sigma = \begin{pmatrix}1 & 0\\ 0 & \tau\end{pmatrix}\cdot\varepsilon'$, with $\tau \in \mathrm{GL}_{n-1}(R), \varepsilon' \in \mathrm{E}_n(R)$. Then clearly we get that $\lambda([\tau]\mathrm{K}_{1,n-1}(R))_2) = [\sigma]$. This completes the proof of exactness at $\mathrm{K}_{1,n}(R)$.

□

We deduce an important corollary of the result above when certain terms of the above sequence are groups.

Corollary 5.3. *Let $d \geq 2$ be an integer. Let R be a commutative and associative ring such that the maximal ideal space of R is a noetherian space of dimension $\leq d$ (e.g. R is a noetherian ring of Krull dimension atmost d). Then, $\pi_1(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R)) = e$, for all $n \geq d+3$.*

Proof. Note that $n \geq d+3$ implies that $\pi_0(\mathrm{Um}_n(R)) = e$. This gives the following exact sequence of groups:

$$(\mathrm{K}_{2,n}(R))_2 \to \mathrm{K}_{2,n}(R) \to \pi_1(\mathrm{EUm}_n(R)) \to \mathrm{K}_{1,n-1}(R)/(\mathrm{K}_{1,n-1}(R))_2 \to$$
$$\mathrm{K}_{1,n}(R) \to 1.$$

Noting that $(\mathrm{K}_{2,n}(R))_2$ contains the image$(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))$ (see (§5)); we have

$$\frac{\mathrm{K}_{2,n}(R)}{(\mathrm{K}_{2,n}(R))_2} \simeq \frac{\mathrm{K}_{2,n}(R)/\mathrm{image}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))}{(\mathrm{K}_{2,n}(R))_2/\mathrm{image}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))}.$$

Thus,

$$\frac{\mathrm{K}_{2,n}(R)}{(\mathrm{K}_{2,n}(R))_2} \simeq \frac{\mathrm{cokernel}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))}{(\mathrm{K}_{2,n}(R))_2/\mathrm{image}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))}.$$

This now gives the following exact sequence of groups:

$$e \to \frac{\text{cokernel}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))}{(\mathrm{K}_{2,n}(R))_2/\text{image}(\mathrm{K}_{2,n-1}(R) \to \mathrm{K}_{2,n}(R))} \to \pi_1(\mathrm{EUm}_n(R)) \to$$

$$\frac{\ker(\mathrm{K}_{1,n-1}(R) \to \mathrm{K}_{1,n}(R))}{(\mathrm{K}_{1,n-1}(R))_2} \to e.$$

That $\pi_1(\mathrm{EUm}_n(R)) = e$ for all $n \geq d+3$ now follows from results on injective and surjective stability for $\mathrm{K}_1(R), \mathrm{K}_2(R)$. (See [4], [12], [10], [5], [6], [7], [13].) □

Corollary 5.4. *Let A be a Dedekind ring of arithmetic type with infinitely many units. Then, $\pi_1(\mathrm{EUm}_n(R), e\mathrm{EP}_n(R)) = e$ for all $n \geq 3$.*

Proof. Note that in this case $d = 1$. The result then follows from the corresponding stability results for Dedekind rings of arithmetic type with infinitely many units. (See [7], [16], [17], [10].) □

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References

[1] Bak, A.; Brown, R.; Minian, G.; Porter, T. Global actions, groupoid atlases and applications. J. Homotopy Relat. Struct. 1 (2006), no. 1, 101–167 (electronic).

[2] Bass, H. Algebraic K-theory. W. A. Benjamin, Inc., New York-Amsterdam 1968.

[3] Bass, H. K-theory and stable algebra. Publ. I.H.E.S., 22, 489–544 (1964).

[4] Bass, H., Milnor J., Serre J.-P., Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$). Publications mathématiques de I.H.E.S, tome 33 (1967), 59–137.

[5] Dennis, R. K. Stability for K_2. Proceedings of the Conference on Orders and Groups Rings held at Ohio State University, Columbus, Ohio, May 12-15, (1972), Lecture Notes in Math. 353, Springer, Berlin, 1973, 85–94.

[6] Dennis, R. K., Stein M. R., The functor K_2: A survey of computations and problems, Algebraic K-theory II, Lecture Notes in Math. 342, Springer, Berlin, (1973), 243-280.

[7] Kolster, M. Improvement of K_2 stability under transitive actions of elementary groups. Journal of Pure and Applied Algebra 24 (1982) 277–282.

[8] Milnor, J. Introduction to algebraic K-theory.

[9] Rao, Ravi A., van der Kallen, W.; Improved stability for SK_1 and WMS_d of a non-singular affine algebra. K-theory (Strasbourg, 1992). Astérisque No. 226 (1994), 11, 411–420.

[10] Suslin A. A., Tulenbayev, M. S.; A theorem on stabilization for Milnor's K_2 functor. Rings and Modules, Zap. Nau. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 64, 131–152 (1976) (Russian).

[11] Suslin, A. A., Vaserstein, L. N.; Serre's problem on Projective Modules over Polynomial Rings and Algebraic K-theory, Math. USSR Izvestija 10, 937–1001, 1976.

[12] Vaserstein, L. N. On the stabilization of the general linear group over a ring, Math. USSR Sbornik, 8, 1969, 383–400.

[13] Vaserstein, L. N. Stabilization for Milnor's K_2 functor. Uspehi Mat. Nauk 30, 224 (1975) (Russian).

[14] van der Kallen, W.; A group structure on certain orbit sets of unimodular rows, J. Algebra 82, no. 2, (1983), 363–397.

[15] van der Kallen, W.; A module structure on certain orbit sets of unimodular rows, J. Pure Appl. Algebra 57, (1975), 657–663.

[16] van der Kallen, W.; Injective stability for K_2, Algebraic K-theory, Evanston 1976, Lecture Notes in Math. 551, Springer, Berlin, (1976), 85–94.

[17] van der Kallen, W.; Stability for K_2 of Dedekind rings of arithmetic type, Algebraic K-theory, Evanston 1980, Lecture Notes in Math. 854, Springer, Berlin, (1981), 217–248.