Global Actions and Related Objects
Foundations

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1 Introduction

This article introduces a general algebraic concept of space with motion. The spaces consist of a set X together with a collection of group actions $G_\alpha \curvearrowright X_\alpha$ where G_α is a group acting on a subset $X_\alpha \subseteqq X$. It is possible that a given subset of X is acted on by several different groups. The group actions are tabulated by letting α above run through an index set Φ called a coordinate system. We structure the set $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ of group actions by equipping Φ with a reflexive relation $\leqq$ and imposing the condition that G defines a functor $\Phi \to ((\text{groups}))$, $\alpha \mapsto G_\alpha$, such that if $\sigma \in G_\alpha$ and ρ its image in G_β under the homomorphism $G_{\alpha \leqq \beta}$ then for any $\mathrm{x} \in X_\alpha \cap X_\beta, \sigma \mathrm{x} = \rho \mathrm{x}$. The resulting triple (Φ, G, X) is called a global action. Motion is provided by the concept of path. A path is a sequence $\mathrm{x}_0, \cdots, \mathrm{x}_p$ of points in X such that for each $0 \leqq i \leqq p-1$, there is a coordinate $\alpha_i \in \Phi$ with the property that $\mathrm{x}_i, \mathrm{x}_{i+1} \in X_{\alpha_i}$ and $\sigma_i \mathrm{x}_i = \mathrm{x}_{i+1}$ for some $\sigma_i \in G_{\alpha_i}$. It turns out that there is a global action L called a line such that paths in a given global action A are determined by morphisms from L to A. We shall use this natural and intuitive construction of paths to carry over to algebra all of the experience we have with paths in topological spaces. One consequence of this program will be a homotopy theory for algebraic structures which includes a natural, intuitive, as well as rigorous concept of algebraic deformation of morphisms. There are two prerequisites for realizing this goal and they are supplied in the current article. First one must show how to make the set $Mor(A, B)$ of all morphisms from a global action A to a global action B, into a global action. This allows one to deform a morphism $f : A \to B$ to a morphism $g : A \to B$ by a path from f to g. The second is to formulate a general condition for global actions, which guarentees that the exponential map $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$

is an isomorphism of global actions. This implies that the cylinder method for deforming a morphism f to a morphism g is equivalent to the path procedure above. With this equivalence, one develops fundamental constructions and principles of algebraic homotopy theory along the lines of their topological precedents. This will be done in a sequel to the current paper and used to present higher Volodin K-groups of stable and nonstable classical-like groups in terms of their canonical unipotent subgroups.

In order to apply the algebraic homotopy theory above to developing a theory of deformation for morphisms in arbitrary categories, the notion of global object will be used. This concept is introduced also in the current article. It is a generalization to arbitrary categories of the notion of global action and serves in the current article to provide depth and perspective for the notion of global action.

The remainder of the article is organized as follows. §2 introduces the notion of global action. We drop the annoying condition in [1] that the relation on the coordinate system is transitive. The section provides numerous examples. These include the line action L mentioned above, other related, geometrically inspired examples, and global actions which we christen Volodin models. The Volodin models will be used in a future paper to provide an algebraic definition of higher Volodin K-groups and algebraic foundations of algebraic K-theory. Next the concept of global object is introduced, as well as the concept of a representation of a global object by a global action. Examples of both concepts are given and it turns out that the geometrically inspired global actions at the beginning of the section are representations of global simplicial complexes, i.e. of global objects in the category of (abstract) simplicial complexes. The section closes with two results showing how to functorially construct global objects from primitive data. These constructions will be enormously important in constructing a deformation theory for morphisms in arbitrary categories.

§3 studies the concept of morphism for global actions. There is a general notion of morphism and two important special kinds of morphisms, namely normal morphisms and regular morphisms. The regular morphisms provide the strongest notion of morphism and preserve all the structural concepts in the definition of a global action. The general notion takes individually into account, the group actions making up a global action, but does not reflect the coherence among the actions, given by the transitive reflexive relation on the coordinate system Φ and the functoriality of the global group functor G. Normal morphisms lie somewhere between regular and general. All regular morphisms are normal, but not conversely.

We define first the general notion of morphism and then that of regular morphism. The notion of chart is introduced and used to define a global structure on the set $Mor(A, B)$ of all morphisms from a global action A to a global action B. As a global action, $Mor(A, B)$ is a contravariant functor in the first variable, but is not defined over all morphisms in the second variable. The notion of normal morphism is introduced so that $Mor(A, B)$ becomes a covariant functor in the first variable over all normal morphisms. This result will be very important for algebraic homotopy theory, since it will imply that algebraic homotopy groups are functorial over a large class of normal morphisms called $\infty-$L-morphisms. Next the notions of infimum and strong infimum global action are introduced. Volodin models and the geometrically inspired global actions in §2 are examples of strong infimum actions. It is shown that any morphism whose target is an infimum or strong infimum action is $\infty-$normal and that the exponential morphism $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ is an $\infty-$ normal isomorphism if C is an infimum action and a regular isomorphism if C is a strong infimum action. These results will be also required in developing algebraic homotopy theory.

§4 introduces the notion of subaction of a global action and the notion of relative action. A relative action is a pair consisting of a global action and a subaction. Relative actions are required in the homotopy theory of global actions. §4 repeats the entire program of §3, with relative actions replacing global actions. The details are not routine, as in the case of topological spaces. The added complications arise from the notion of relative chart which is needed to put a relative global structure on the set of all morphisms $Mor(A, B)$ from a relative action A to a relative action B. Relative charts are subtler than their absolute counterparts and this added subtlety has to be followed up throughout the entire section. This done, one gets the same results as in §3.

The article is written in an elementary and selfcontained style.

2 Global actions

A global action is an algebraic object which is formed by fitting or gluing together various group actions. The construction resembles that of several well known mathematical objects which are formed by fitting certain building blocks together, in our case group actions, to form more complicated structures. Examples include simplicial complexes where the building blocks are simplices, CW-Complexes where the building blocks are closed disks, manifolds where the building blocks are open disks of a fixed dimension, and varieties (resp. schemes) where the building blocks are affine varieties (resp. affine schemes). Furthermore the homotopy theory of global actions resembles that of the topological examples above in so far as the building blocks turn out to be homotopically trivial, being n-connected for all n > 0.

DEFINITION 2.1 A global action is a set $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ of groups G_α acting on subsets X_α of some set $|X|$, subject to the following conditions.

(2.1.1) Φ is equipped with a reflexive relation $\leqq$.

If the relation $\leqq$ is also transitive then Φ can be considered as category whose objects are the elements α of Φ and whose morphisms are the relations $\alpha \leqq \beta$ between elements of Φ. If Φ is a category then the set $Mor(\alpha, \beta)$ of all morphisms from an object α to an object β has at most one member. Thus if $\alpha \leqq \beta$ and $\beta \leqq \alpha$ then these morphisms are mutually inverse. If it is always the case that $\alpha \leqq \beta$ and $\beta \leqq \alpha$ implies $\alpha = \beta$ then the relation $\leqq$ is by definition a partial ordering on Φ.

(2.1.2) The function $X : \Phi \to$ subsets $|X|$, $\alpha \mapsto X_\alpha$, has in general no special properties.

If Φ is a category and subsets $|X|$ is viewed as a category whose objects are the subsets of $|X|$ and whose morphisms are all natural inclusions between subsets of $|X|$ then the global action is called **contravariant** if the function $X : \Phi \to$ subsets$|X|$ is a contravariant functor, i.e. if $\alpha \leqq \beta$ then $X_\alpha \supseteqq X_\beta$.

(2.1.3) The function $G : \Phi \to ((groups)), \alpha \mapsto G_\alpha$, assigns also to each $\alpha \leqq \beta$ a unique homomorphism $G_{\alpha \leqq \beta} : G_\alpha \to G_\beta$.

If Φ is a category and G defines a covariant functor to groups then the global action is called **covariant**.

(2.1.4) **Compatibility condition.** If $\alpha \leqq \beta$ then G_α leaves $X_\alpha \cap X_\beta$ invariant and for all $\sigma \in G_\alpha$ and all x $\in X_\alpha \cap X_\beta$, $\sigma x = G_{\alpha \leqq \beta}(\sigma)$x. (Note that if $X_\alpha \cap X_\beta$ is empty, the compatibility condition is automatically satisfied.)

The global action is called **bivariant** if it is both covariant and contravariant.

Φ is called the **coordinate system** of the action and each element of Φ is called a **coordinate**. The function G is called the **group function** of the action and the function X the **set function**. $|X|$ is called the **enveloping set**. Its elements are called the **points** of the action. The group function is called **covariant** if the relation on Φ is transitive and G is a covariant functor to groups. The set function is called **contravariant** if the relation on Φ is transitive and X is a contravariant functor to subsets $|X|$. If α is a coordinate then G_α is called the **local group** at α and X_α the **local set** at α. Let x $\in |X|$. The local group G_α or an element $\sigma \in G_\alpha$ is said to be defined at x whenever x $\in X_\alpha$. A **group element** of a global action is an element of some local group G_α.

To handle linguistically the concepts used in defining a global action, it is convenient to give some names to them. Let $\mathcal{B}$ be a collection of objects and of arrows $A \to B$ between objects A and B of $\mathcal{B}$. Thus $\mathcal{B}$ is like a category, except there is no composition rule for arrows. Let $\mathcal{C}$ be a category. A rule F which associates to each object A of $\mathcal{B}$ a unique object $F(A)$ of $\mathcal{C}$ and to each arrow $\mathrm{f} : A \to B$ of $\mathcal{B}$ a unique morphism $F(\mathrm{f}) : F(A) \to F(B)$ of $\mathcal{C}$ will be called a **(covariant) generalized functor**. A **contravariant generalized functor is defined in the obvious way**. A **natural transformation** $\eta : F \to G$ of generalized functors F and $G : \mathcal{B} \to \mathcal{C}$ is a rule associating to each object A of $\mathcal{B}$ a unique morphism $\eta(A) : F(A) \to G(A)$ such that if $\mathrm{f} : A \to B$ is an arrow in $\mathcal{B}$ then the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta(A)"] \arrow[d, "F(\mathrm{f})"'] & G(A) \arrow[d, "G(\mathrm{f})"] \\ F(B) \arrow[r, "\eta(B)"'] & G(B) \end{tikzcd}$$

commutes. Thus in the definition of a global action (Φ, G, X), G is a generalized functor $\Phi \to$ ((groups)). Natural transformations will be used in the definition of a regular morphism in §3.

Remark In a general global action, Φ is not necessarily a category and G is not necessarily a functor on Φ even if Φ is a category. Suppose now that $\alpha_0, \cdots, \alpha_p$ are elements in Φ such that $\alpha_0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_p$. Of course it is not necessarily true that $\alpha_0 \leqq \alpha_p$. It follows easily from the compatibility condition (2.1.4) that for any element $\mathrm{x} \in \bigcap_{i=0}^{p} X_{\alpha_i}$ and any $\sigma \in G_{\alpha_0}, \sigma(\mathrm{x}) = G_{\alpha_0 \leqq \alpha_1}(\sigma)(\mathrm{x}) = (G_{\alpha_1 \leqq \alpha_2}\ G_{\alpha_0 \leqq \alpha_1})(\sigma)(\mathrm{x}) = (G_{\alpha_{p-1} \leqq \alpha_p} \cdots G_{\alpha_1 \leqq \alpha_2}\ G_{\alpha_1 \leqq \alpha_0})(\sigma)(\mathrm{x})$. Moreover if $\alpha_0 \leqq \alpha_p$ then it follows also that $\sigma(\mathrm{x}) = G_{\alpha_0 \leqq \alpha_p}(\sigma)(\mathrm{x})$. Thus $G_{\alpha_0 \leqq \alpha_p}(\sigma)$ and $(G_{\alpha_{p-1} \leqq \alpha_p} \cdots G_{\alpha_1 \leqq \alpha_2}\ G_{\alpha_0 \leqq \alpha_1})(\sigma)$ agree on $\bigcap_{i=0}^{p} X_{\alpha_i}$. This functorial like property of G is used subtly several times in this article. Moreover if Φ is a category and if for each $\alpha \in \Phi, X_\alpha = |X|$ and G_α acts fixed point free on $|X|$ then G is necessarily a functor on Φ.

Remark Many global actions satisfy the additional property that $|X| = \cup_{\alpha \in \Phi} X_\alpha$ or even the property that $|X| = X_\alpha$ for some $\alpha \in \Phi$. If a global action doesn't have this property one can introduce it by enlarging Φ with an element $*$ such that $* \leqq \alpha$ for all $\alpha \in \Phi$ and then setting $X_* = |X|$ and $G_* = \{1\}$. However, this will change subtly the structure of the global action, as we shall see for example in the definition of an $\infty-$ exponential action in (3.18) and in Theorem 3.23 and its proof.

It is allowed that $X_\alpha = \phi$. This will be convenient when making certain constructions, such as that of a standard subspace.

The examples below illustrate the concept of a global action.

Example 2.2 Let G be a group acting on a set $|X|$. Let Φ be a set which indexes a set $\{G_\alpha|\alpha \in \Phi\}$ of subgroups of G. Assume that $G_\alpha = G_\beta \Leftrightarrow \alpha = \beta$. Partially order $\{G_\alpha|\alpha \in \Phi\}$ by inclusion and give Φ the induced partial ordering. Clearly the rule $\alpha \mapsto G_\alpha$ defines a functor $\Phi \to ((\text{groups}))$. Define the function $X : \Phi \to$ subsets $|X|$, $\alpha \mapsto X_\alpha$, by $X_\alpha = |X|$ for all $\alpha \in \Phi$. Then one obtains a global action (Φ, G, X) which is bivariant.

Definition 2.3 Suppose that in (2.2), $G = |X|$ and the action of G on $|X|$ is by multiplication. Suppose that $G_* = \{1\}$ for some $* \in \Phi$, that $\{G_\alpha|\alpha \in \Phi\}$ is closed under arbitrary intersections, and that the following condition is satisfied: If G_α and G_β are contained in a subgroup $G_{\gamma'}$ then the subgroup $\langle G_\alpha, G_\beta\rangle$ of G generated by G_α and G_β is identical with some subgroup G_γ. Then (Φ, G, X) is called a **Volodin model**. (It turns out that the Volodin K-groups of rings or of rings with extra structure such as an involution are algebraic homotopy groups of certain Volodin models. The intersection property of Volodin models is needed to show that the algebraic homotopy groups of a Volodin model agree with the ordinary homotopy groups used by Volodin of a related topological space.)

If U is a set, let

$$\begin{aligned} \mathrm{Perm}(U) =& \text{Group of all bijections of } U \text{ onto itself.} \\ f\mathrm{Perm}(U) =& \{\sigma \in \mathrm{Perm}(U) | \sigma \text{ fixes all but a finite number of elements of } U\}. \end{aligned}$$

If U is a well ordered nonempty finite set, let

$$\begin{aligned} \mathrm{cPerm(U)} = \ & \text{cyclic subgroups of } \mathrm{Perm}(U) \text{ generated by the cyclic} \\ & \text{permutation which sends each element of } U, \\ & \text{except for the last, to its successor and} \\ & \text{sends the last element to the first.} \end{aligned}$$

Example 2.4 This example is called the **line action** and is important for the homotopy theory of global actions. Let $\Phi = \mathbb{Z} \cup \{*\}$. Give Φ the partial ordering such that there is no relation between elements of $\mathbb{Z}$ and such that $* < n$ for all $n \in \mathbb{Z}$. Let $|X| = \mathbb{Z}$ and define $X : \Phi \to$ subsets $|X|, \alpha = n \mapsto \{n, n+1\}$ and $\alpha = * \mapsto |X|$. Define

$G : \Phi \mapsto ((\text{groups}))$,$\alpha = n \mapsto G_\alpha = \mathrm{Perm}(\{n, n+1\})$ and $\alpha = * \mapsto G_\alpha = \{1\}$. Then the triple (Φ, G, X) is a bivariant action.

The next example generalizes the one above.

EXAMPLE 2.5 Let S denote an abstract simplicial complex and let $|X|$ denote the set of its vertices. If α is a subcomplex of S, let X_α denote the set of its vertices. Call a subcomplex α **simple**, if X_α has a partition into subsets U such that any finite subset of U is a simplex in α and such that any simplex of α is a subset of some U. Clearly if α is simple then the partition above of (X_α) is unique; let $\mathrm{Part}(X_\alpha)$ denote this partition. Let Φ denote the set of all simple subcomplexes α of S. Define $\alpha \leqq \beta \Leftrightarrow$each member $U \in Part(X_\alpha)$ has the property that either $U \cap X_\beta = \varnothing$ or there is a $V \in Part(X_\beta)$ such that $U \subseteqq V$. Clearly the subcomplex of S whose vertices are $|X|$ and whose simplices are the singleton subsets of $|X|$ is the smallest element of Φ. For $\alpha \in \Phi$, define

$$G_\alpha = \prod_{U \in \mathrm{Part}(X_\alpha)} \mathrm{Perm}(U)$$
$$fG_\alpha = \prod_{U \in \mathrm{Part}(X_\alpha)} f\mathrm{Perm}(U).$$

There is a canonical action of G_α (resp. fG_α) on X_α defined by the action of each permutation group Perm(U) (resp. fPerm (U)) on U. If $\alpha \leqq \beta$ and $X_\alpha \cap X_\beta \neq \varnothing$, there is an obvious group homomorphism $G_\alpha \rightarrow G_\beta$ such that (2.1.4) holds. If $X_\alpha \cap X_\beta = \varnothing$ then one could take any homomorphism $G_\alpha \rightarrow G_\beta$ and (2.1.4) would hold, but for the sake of having a concrete definition, we take the trivial homomorphism. Define

$$gl(S) = (\Phi, G, X)$$
$$fgl(S) = (\Phi, fG, X).$$

Then $gl(S)$ and $fgl(S)$ are global actions called **simplicial actions**. They are not in general bivariant.

Well order now the vertices $|X|$ of S and let $c\Phi$ denote the subset of Φ of all simple subcomplexes α such that $Part(X_\alpha)$ contains only finite sets. The smallest element of Φ, say $*$, clearly lies in $c\Phi$. Give $c\Phi$ a new partial ordering such that $\alpha \leqq \beta \Leftrightarrow$ each member U of $Part(X_\alpha)$ has the property that either $U \cap X_\beta = \varnothing, U$ has exactly one element, or $U = V$ for some $V \in Part(X_\beta)$. For $\alpha \in c\Phi$, define $(cX)_\alpha = X_\alpha$ and

$$cG_\alpha = \prod_{U \in Part(X_\alpha)} cPerm(U).$$

There is a canonical action of cG_α on cX_α defined by the action of each cyclic group $cPerm(U)$ on U. If $\alpha \leqq \beta$ and $cX_\alpha \cap cX_\beta \neq \varnothing$ there is an obvious group homomorphism $cG_\alpha \to cG_\beta$ such that (2.1.4) holds. If $cX_\alpha \cap cX_\beta = \varnothing$, we define $cG_\beta \to cG_\beta$ equal to the trivial homomorphism. Define

$$cgl(S) = (c\Phi, cG, cX).$$

Then $cgl(S)$ is a global action called a **cyclic simplicial action**. It is not in general bivariant.

Remark There are two important variations of the above, each obtained by eliminating certain relations between coordinates. The first is to eliminate in Φ or $c\Phi$ any relation $\alpha \leqq \beta$ where $X_\alpha \cap X_\beta = \varnothing$. The second is to eliminate any relation $\alpha \leqq \beta$ such that $X_\alpha \not\supseteqq X_\beta$. In this case, we get a global action which is bivariant.

To prepare for further examples, a few concepts from category theory are recalled.

Let $\mathcal{C}$ be a category. Let O be an object in $\mathcal{C}$. Let $O' \rightarrowtail O$ be a subobject of O. If $\sigma \in Aut_{\mathcal{C}}(O)$ then one says that σ **leaves** O' **invariant** or **stabilizes** O', if there is a $\rho \in Aut_{\mathcal{C}}(O')$ such that the diagram

$$\begin{tikzcd}
O' \arrow[r, tail] \arrow[d, "\rho"'] & O \arrow[d, "\sigma"] \\
O' \arrow[r, tail] & O
\end{tikzcd}$$

commutes. Clearly if ρ exists, it is unique. The set of all automorphisms of O which stabilize O' form a subgroup

$$Stab_O(O')$$

of $Aut_{\mathcal{C}}(O)$ called the **stabilizer** of O' in O. There is a canonical group homomorphism

$$\begin{array}{ccc} Stab_O(O') & \to & Aut_{\mathcal{C}}(O') \\ \sigma & \mapsto & \rho \end{array} .$$

Let P be an object of $\mathcal{C}$. A **P-point** of $\mathcal{C}$ is an element of $Mor_{\mathcal{C}}(P,O)$ where O is any object of $\mathcal{C}$. For a fixed P, the concept of P-point allows one to associate to an arbitrary object O of $\mathcal{C}$, an **underlying set**, namely the set $Mor_{\mathcal{C}}(P,O)$ of all **P-points in** O. Moreover given any set $\mathcal{P}$ of objects of $\mathcal{C}$ (for the current purposes, it can be assumed that no two distinct objects in $\mathcal{P}$ are isomorphic), it makes sense to define the **underlying set of $\mathcal{P}$-points of** O as the set $\bigcup_{P\in\mathcal{P}} Mor_{\mathcal{C}}(P,O)$. If $O' \rightarrowtail O$ is a subobject then there is a canonical injection

$$\mathcal{P}\text{-points } (O') \subseteq \mathcal{P}\text{-points } (O)$$

of sets which will be frequently used to identify the former set with a subset of the latter.

Next we generalize the concept of global action to arbitrary categories by the concept of global object. Then we define the notion of representing a global object by a global action. After that a canonical method of constructing global objects from simple data is developed. All of this provides a wealth of examples of global objects and global actions and paves the way for applying global action methods to many different kinds of problems.

Definition 2.6 Let $\mathcal{C}$ be a category. A **global object** in $\mathcal{C}$ consists of a set $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ of subobjects $O_\alpha \rightarrowtail O$ of an object O of $\mathcal{C}$ and a set $\{G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha) | \alpha \in \Phi\}$ of groups G_α and group homomorphisms $G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha)$ satisfying the following conditions.

(2.6.1) Φ is equipped with a reflexive relation $\leqq$.

(2.6.2) The function $O : \Phi \longrightarrow \text{subobjects}(O), \alpha \mapsto (O_\alpha \rightarrowtail O)$, has in general no special properties. If Φ is a category and subobjects (O) is viewed as a category in the usual way then the global object is called **contravariant** if the function $O : \Phi \to$ subobjects (O) is a contravariant functor.

(2.6.3) The function $G : \Phi \longrightarrow ((groups)), \alpha \mapsto G_\alpha$, assigns also to each $\alpha \leqq \beta$ a unique homomorphism $G_{\alpha\leqq\beta} : G_\alpha \to G_\beta$. Thus G is a generalized functor. If Φ is a category and G defines a covariant functor to groups then the global object is called covariant.

(2.6.4) **Compatibility condition**. If $\alpha \leqq \beta$ then the pullback diagram

$$\begin{tikzcd} O_\alpha \cap O_\beta \arrow[r, tail, "\varphi_\alpha"] \arrow[d, tail, "\varphi_\beta"'] & O_\alpha \arrow[d, tail] \\ O_\beta \arrow[r, tail] & O \end{tikzcd}$$

exists in $\mathcal{C}$ and there is a (necessarily unique) group homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\alpha \cap O_\beta)$ such that φ_α and φ_β are G_α -equivariant.

The global object is called **bivariant** if it is both covariant and contravariant.

Clearly the concept global action is identical with that of global set.

DEFINITION 2.7 Let (Φ, G, O) be a global object in the category $\mathcal{C}$. Let P be a set of objects in $\mathcal{C}$. A P-**representation** of (Φ, G, O) is a set $\{X_\alpha | \alpha \in \Phi, X_\alpha \subseteqq \text{P-points}(O_\alpha), X_\alpha$ is G_α-invariant$\}$. **The** P-representation of (Φ, G, O) is the set $\{\text{P-points}O_\alpha | \alpha \in \Phi\}$. It is easy to check that (Φ, G, X) is a global action. The P-representation of a contravariant global object is a contravariant action, but the same is not true of an arbitrary P-representation of a contravariant global object. Obviously any P-representation of a covariant global object is a covariant action.

EXAMPLE 2.8 Let $\mathcal{C}$ denote the category of abstract simplicial complexes. Let S be an object of $\mathcal{C}$. Let Φ be as in (2.5) and for each $\alpha \in \Phi$, set $S_\alpha = \alpha$. Thus S_α is a simple subcomplex of S. Let G_α and fG_α be defined as in (2.5). Thus G_α and fG_α are subgroups of $Aut_{\mathcal{C}}(S_\alpha)$. One checks routinely that (Φ, G, S) and (Φ, fG, S) are global simplicial complexes. They are not in general bivariant. Moreover if P denotes the simplicial complex with precisely one vertex then the P-representation of (Φ, G, S) $(resp.(\Phi, fG, S))$ is the global action $\text{gl}(S)$ $(resp.fgl(S))$ defined in (2.5).

Let $c\Phi$ be as in (2.5) and for each $\alpha \in c\Phi$, set $(cS)_\alpha = \alpha$ and let $(cG)_\alpha$ be as in (2.5). Then $(c\Phi, cG, cS)$ is a global simplicial complex and the P-representation of $(c\Phi, cG, cS)$ is the global action $cgl(S)$ defined in (2.5). $(c\Phi, cG, cS)$ is not in general bivariant.

REMARK There are two important variations of the above, just as in Example 2.5, each obtained by eliminating certain relations between coordinates. The first is to eliminate in Φ or $c\Phi$ any relation $\alpha \leqq \beta$ where vertices $(\alpha) \cap$ vertices $(\beta) = \phi$. The second is to eliminate any relation $\alpha \leqq \beta$ such that vertices $(\alpha) \not\supseteqq$ vertices (β). In this case, we get a global simplicial complex which is bivariant.

The following method of constructing global objects from data generalizes Example 2.2, even in the case of sets, and is very useful.

CONSTRUCTION-LEMMA 2.9 Let $\mathcal{C}$ be a category and O an object in $\mathcal{C}$.

(2.9.1) **Global data** for O consists of a set $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ of subobjects $O_\alpha \rightarrowtail O$ of O and a set $\{G_\alpha \subseteqq Aut_{\mathcal{C}}(O_\alpha) | \alpha \in \Phi\}$ of subgroups $G_\alpha \subseteqq Aut_{\mathcal{C}}(O_\alpha)$.

(2.9.2) Given global data, define a transitive reflexive relation $\leqq_{ct}$ on Φ, called the **canonical transitive relation** , as follows: $\alpha \leqq_{ct} \beta \Leftrightarrow$ there is a commutative diagram

$$\begin{tikzcd}
O_\alpha \arrow[r, tail] & O \\
O_\beta \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

such that $G_\alpha \subseteqq Stab_{O_\alpha}(O_\beta)$ and the canonical homomorphism $G_\alpha \longrightarrow Aut_{\mathcal{C}}(O_\beta)$ has its image in G_β. One checks straightforward that (Φ, G, O) is a bivariant global object. Moreover, if $\leqq$ is any transitive reflexive relation on Φ such that $((\Phi, \leqq), G, O)$ is a bivariant global object then the identity map $\Phi \to \Phi$ defines a functor $(\Phi, \leqq) \longrightarrow (\Phi, \leqq_{ct})$ of categories.

(2.9.3) Given global data, define a reflexive relation $\leqq_{cr}$ on Φ, called the **canonical relation** , as follows: $\alpha \leqq_{cr} \beta \Longleftrightarrow$ there is a pullback diagram

$$\begin{tikzcd}
O_\beta \cap O_\alpha \arrow[r, tail] \arrow[d, tail] & O_\alpha \arrow[d, tail] \\
O_\beta \arrow[r, tail] & O
\end{tikzcd}$$

in $\mathcal{C}$ such that $G_\alpha \subseteqq Stab_{O_\alpha}(O_\beta \cap O_\alpha), G_\beta \subseteqq Stab_{O_\beta}(O_\beta \cap O_\alpha)$, the canonical homomorphism $G_\beta \longrightarrow Aut_{\mathcal{C}}(O_\beta \cap O_\alpha)$ is injective, and the image $(G_\alpha \longrightarrow Aut_{\mathcal{C}}(O_\beta \cap O_\alpha))$ is contained in the image of the previous homomorphism. It follows that if $\alpha \leqq_{cr} \beta$ then there is a unique homomorphism $G_\alpha \longrightarrow G_\beta$ such that the morphism $O_\beta \cap O_\alpha \longrightarrow O_\beta$ is G_α-equivariant. Obviously (Φ, G, O) is a global object. It is in general neither covariant nor contravariant. Moreover if $\leqq$ is any reflexive relation on Φ such that the triple $((\Phi, \leqq), G, O)$ is a global object with the property that $\alpha \leqq \beta \Rightarrow G_\beta \subseteqq Stab_{O_\beta}(O_\beta \cap O_\alpha)$ and the canonical homomorphism $G_\beta \to Aut_{\mathcal{C}}(O_\beta \cap O_\alpha)$ is injective then the identity map $\Phi \to \Phi$ is a morphism $(\Phi, \leqq) \longrightarrow (\Phi, \leqq_{cr})$ of relations. In particular $(\Phi, \leqq_{ct}) \longrightarrow (\Phi, \leqq_{cr})$ is a morphism of relations.

PROOF The only assertions left to prove are those concerning the universality of $\leqq_{ct}$ and $\leqq_{cr}$. The proofs are similar and we carry out only that for $\leqq_{ct}$.

Let $\leqq$ be a transitive reflexive relation on Φ such that $((\Phi, \leqq), G, O)$ is a bivariant global object. Suppose $\alpha \leqq \beta$. By contravariantness, there is a commutative diagram

$$\begin{tikzcd} O_\alpha \arrow[r, tail] & O \ . \\ O_\beta \arrow[u, tail] \arrow[ur, tail] & \end{tikzcd}$$

From the compatibility condition (2.6.4), it follows that the morphism $O_\beta \rightarrowtail O_\alpha$ is G_α-equivariant. This says that G_α leaves O_β invariant. Furthermore it is clear that the canonical homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\beta)$ must have its image in G_β and that the resulting homomorphism $G_{\alpha \leqq_{ct} \beta} : G_\alpha \to G_\beta$ must be the homomorphism $G_{\alpha \leqq \beta} : G_\alpha \to G_\beta$, because there is exactly one homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\beta)$ namely the canonical one above which makes $O_\beta \rightarrowtail O_\alpha$ $G_\alpha-$equivariant and the homomorphism $G_{\alpha \leqq \beta} : G_\alpha \longrightarrow G_\beta$ makes $O_\beta \rightarrowtail O_\alpha$ $G_\alpha-$equivariant. Thus $\alpha \leqq_{ct} \beta$, by definition. $\square$

Let Ψ be an index set. Let O be an object in a category $\mathcal{C}$ and let $O : \Psi \to$ subobjects(O), $\alpha \mapsto (O_\alpha \rightarrowtail O)$, be a function. Let

$$Sub_\Psi(O)$$

denote the category whose objects are $\{O_\alpha | \alpha \in \Psi\}$ and whose morphisms are the unique morphisms $O_\alpha \rightarrowtail O_\beta$ such that the diagram

$$\begin{tikzcd} O_\alpha \arrow[rr, tail] \arrow[dr, tail] & & O_\beta \arrow[dl, tail] \\ & O & \end{tikzcd}$$

commutes. If $S \subsetneqq Sub_\Psi(O)$ is a subcategory and if $colimS$ exists in $\mathcal{C}$ then there is a canonical morphism $colimS \to O$.

The following method of constructing global objects generalizes Example 2.8.

Construction-Lemma 2.10 Let $\mathcal{C}$ be a category and O an object in $\mathcal{C}$.

(2.10.1) Let $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ be a set of subobjects $O_\alpha \rightarrowtail O$ of O. For each $\alpha \in \Phi$, let $\{O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha | (\alpha, \mathrm{i}) \in \Phi_\alpha\}$ be a set of subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha$ of O_α such that there is a subcategory $S_\alpha \subsetneqq Sub_{\Phi_\alpha}(O_\alpha)$ with the property that the colim (S_α) exists in $\mathcal{C}$ and the canonical morphism colim $(S_\alpha) \to O_\alpha$ is an isomorphism. Let $G_\alpha = \{\sigma \in$

$\mathrm{Aut}_{\mathcal{C}}(O_\alpha)|\sigma \in \mathrm{Stab}_{O_\alpha}(O_{\alpha,\mathrm{i}})\forall(\alpha,\mathrm{i}) \in \Phi_\alpha\}$. Obviously the sets $\{O_\alpha \rightarrowtail O|\alpha \in \Phi\}$ and $\{G_\alpha \subsetneqq \mathrm{Aut}_{\mathcal{C}}(O_\alpha)|\alpha \in \Phi\}$ are global data in the sense of (2.9.1).

(2.10.2) Given the data above, define a transitive reflexive relation $\leqq_c$ on Φ as follows: $\alpha \leqq_c \beta \Leftrightarrow$ there is a commutative diagram

$$\begin{tikzcd} O_\alpha \arrow[r, tail] & O \\ O_\beta \arrow[u, tail] \arrow[ur, tail] & \end{tikzcd}$$

such that for each $(\beta, j) \in \Phi_\beta$, the object $O_{\beta,j}$ is a colimit of not necessarily all subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_{\beta,j}$ for which there is a commutative diagram

$$\begin{tikzcd} O_{\alpha,i} \arrow[r, tail] \arrow[d, tail] & O_\alpha \\ O_{\beta,j} \arrow[r, tail] & O_\beta. \arrow[u, tail] \end{tikzcd}$$

This implies $G_\alpha \subsetneqq \mathrm{Stab}_{O_\alpha}(O_\beta)$ and the canonical homomorphism $G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\beta)$ takes its image in G_β. Applying (2.9.2), one obtains that (Φ, G, O) is a bivariant global object.

(2.10.3) Given the data above, define a reflexive relation $\leqq_r$ on Φ as follows: $\alpha \leqq_r \beta \Leftrightarrow$ there is a pullback diagram

$$\begin{tikzcd} O_\alpha \cap O_\beta \arrow[r, tail] \arrow[d, tail] & O_\alpha \arrow[d, tail] \\ O_\beta \arrow[r, tail] & O \end{tikzcd}$$

in $\mathcal{C}$ such that $O_\alpha \cap O_\beta$ is a colimit of not necessarily all subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha \cap O_\beta(\mathrm{resp.}O_{\beta,j} \rightarrowtail O_\alpha \cap O_\beta)$ for which there is a commutative diagram

$$\begin{tikzcd} O_{\alpha,i} \arrow[d, tail] \arrow[dr, tail] & \\ O_\alpha \cap O_\beta \arrow[r, tail] & O_\alpha \end{tikzcd}$$

(resp.

$$\begin{tikzcd} O_{\beta,j} \arrow[d, tail] \arrow[dr, tail] & \\ O_\alpha \cap O_\beta \arrow[r, tail] & O_\beta \ ), \end{tikzcd}$$

the canonical homomorphism $G_\beta \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha \cap O_\beta)$ is injective, and the image $(G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha \cap O_\beta))$ is contained in the image of the previous homomorphism. Applying (2.9.3), one obtains that (Φ, G, O) is a global object. Moreover, the identity map $\Phi \to \Phi$ defines a morphism $(\Phi, \leqq_c) \to (\Phi, \leqq_r)$ of relations.

3 Morphisms and morphism spaces

There is a general notion of morphism for global actions and two important special kinds of morphisms, namely normal morphisms and regular morphisms. The set $Mor(A, B)$ of all morphisms from a global action A to a global action B will be given the structure of a global action such that $Mor(,)$ defines a contravariant functor with values in global actions with respect to the first variable over all morphisms and a contravariant functor with values in global actions with respect to the second variable over all normal morphisms.

Regular morphisms provide the strongest notion of morphism and preserve all the essential structural concepts in the definition of a global action. Two global actions which are regularly isomorphic are essentially the same. On the other hand, two global actions which are only isomorphic can behave very differently, since their structures are not necessarily in $1-1$ correspondence. For example, they can have different higher algebraic homotopy groups because the construction of such groups is functorial only over a certain class of morphisms containing the regular morphisms. This class is called the ∞ -L-normal morphisms and will also be defined below.

The notion of morphism depends on the concepts of path, local path, and local frame. The concepts local path and local frame are really the same, but the notion local frame suggests possible directions for movement rather than a definite direction of movement and this will be helpful in developing the notion of normal morphism.

If A is a global action, let

$$\begin{aligned}\Phi_A &= \text{coordinate system of } A\\ G_A &= \text{group function of } A\\ X_A &= \text{set function of } A\\ |A| &= \text{enveloping set of } A.\end{aligned}$$

Definition 3.1 Let A be a global action.

(3.1.1) A **path** in A is a sequence $\mathrm{x}_0, \cdots, \mathrm{x}_p$ of points in $|A|$ such that for each i $(0 \leqq i \leqq p-1)$, there is a group element g_i defined at x_i with the property that $g_i \mathrm{x}_i = \mathrm{x}_{i+1}$. If $0 = p$, it is assumed that x_0 lies in some local set $(X_A)_\alpha$.

(3.1.2) A **local path** at $\alpha \in \Phi_A$ is a path $\mathrm{x}_0, \cdots, \mathrm{x}_p$ in A such that each $\mathrm{x}_i \in (\mathrm{X}_A)_\alpha$ and each $g_{\mathrm{i}} \in (G_A)_\alpha$. (Clearly if $\mathrm{x}_0, \cdots, \mathrm{x}_p$ is a local path then so is $\mathrm{x}_{\pi(\mathrm{x}_0)}, \cdots, \mathrm{x}_{\pi(p)}$ where π is any permutation of $(p+1)$ letters.)

(3.1.3) Let $\mathrm{x} \in (X_A)_\alpha$. A **local frame** at x in α or simply an α-**frame** at x is a sequence $\mathrm{x} = \mathrm{x}_0, \cdots, \mathrm{x}_p$ of points in $(X_A)_\alpha$ such that for each i $(1 \leqq i \leqq p)$ there is a $g_i \in (G_A)_\alpha$ such that $g_i \mathrm{x} = \mathrm{x}_i$. (Clearly $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is an α-frame at x $\Leftrightarrow$ $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is a local path at α.)

Definition 3.2 A **morphism** f :$A \to B$ of global actions is a function f :$|A| \to |B|$ which preserves local frames or equivalently local paths. Specifically if $\mathrm{x}_0, \cdots, \mathrm{x}_p$ is an α-frame at x_0 then $f(\mathrm{x}_0), \cdots, f(\mathrm{x}_p)$ is an β-frame at $f(\mathrm{x}_0)$ for some $\beta \in \Phi_B$.

Definition 3.3 A **regular** morphism $\eta : A \to B$ of global actions is a triple $(\eta_\Phi, \eta_G, \eta_X)$ satisfying the following conditions.

(3.3.1) $\eta_\Phi : \Phi_A \to \Phi_B$ is a relation preserving function, i.e. if $\alpha \leqq \beta$ then $\eta_\Phi(\alpha) \leqq \eta_\Phi(\beta)$.

(3.3.2) $\eta_G : G_A \to (G_B)_{\eta_\Phi(\,)}$ is a natural transformation , as defined following (2.1.4), of group valued generalized functors on Φ_A where $(G_B)_{\eta_\Phi(\,)}$ denotes the composition of η_Φ with G_B.

(3.3.3) $\eta_X : |A| \to |B|$ is a function such that $\eta_X((X_A)_\alpha) \subseteqq (X_B)_{\eta_\Phi(\alpha)}$ for all $\alpha \in \Phi_A$.

(3.3.4) For each $\alpha \in \Phi_A$, the pair $(\eta_G, \eta_X) : (G_A)_\alpha \curvearrowright (X_A)_\alpha \to (G_B)_{\eta_\Phi(\alpha)} \curvearrowright (X_B)_{\eta_\Phi(\alpha)}$ is a morphism of group actions, i.e. for $\sigma \in (G_A)_\alpha$ and $\mathrm{x} \in (X_A)_\alpha, \eta_X(\alpha)(g\mathrm{x}) = \eta_G(\alpha)(\sigma)(\eta_X(\alpha)(\mathrm{x}))$. (This implies that a regular morphism is one in the usual sense).

If $\mathrm{f} : A \to B$ is a morphism of global actions then a regular morphism $\eta = (\eta_\Phi, \eta_G, \eta_X) : A \to B$ is called an **extension** of f if $\eta_X = \mathrm{f}$.

A **regular isomorphism** $\eta : A \to B$ is a regular morphism such that there is a regular morphism $\eta' : B \to A$ called the **regular inverse** of η with the property that η'_Φ is inverse to η_Φ, η'_X is inverse to η_X, and for each $\alpha \in \Phi_A$ $\eta'_G(\eta_\Phi(\alpha))$ is inverse to $\eta_G(\alpha)$.

It is of course not true in general that a regular morphism which is an isomorphism in the general sense is a regular isomorphism.

Remark There is a concept of regular isomorphism which is not identical with that above, but is also useful. It starts off with an ordinary isomorphism $\mathrm{f} : A \to B$ of global actions with inverse $g : B \to A$ and then requires that f and g extend to regular morphisms η and μ respectively, but does not assume that η and μ are regular isomorphisms in the sense above.

The notion of chart, to be introduced next, will be used to put a global action structure on the set $Mor(A, B)$ of all morphisms from a global action A to be a global action B.

Definition 3.4 Let A and B be global actions. An A-**chart** in B is a morphism $f : A \to B$ of global actions and a function $\beta : |A| \to \Phi_B$ such that the following conditions are satisfied.

(3.4.1) $\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$ for all $\mathrm{x} \in |A|$.

(3.4.2) If $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is an a-frame at $\mathrm{x} \in |A|$ then $\mathrm{f}(\mathrm{x}), \mathrm{f}(\mathrm{x}_1), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\mathrm{f}(\mathrm{x})$ for some b such that $b \geqq \beta(\mathrm{x}_i) \quad (0 \leqq i \leqq p)$.

Definition-Lemma 3.5 Let (f, β) be an A-chart in B.

If

$$\sigma = (\sigma_\mathrm{x}) \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$$

define

$$\begin{aligned} \sigma\mathrm{f} : |A| &\to |B|. \\ \mathrm{x} &\mapsto \sigma_\mathrm{x}\mathrm{f}(\mathrm{x}) \end{aligned}$$

Then σf is a morphism $A \to B$ of global actions and $(\sigma\mathrm{f}, \beta)$ is an A-chart in B.

PROOF Since $\sigma_{\mathrm{x}} \in (G_B)_{\beta(\mathrm{x})}$, it follows that $\sigma\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$. Thus the pair $(\sigma\mathrm{f}, \beta)$ satisfies (3.4.1). To show that σf is a morphism of global actions and that $(\sigma\mathrm{f}, \beta)$ is an A-chart in B, it suffices to show that (3.4.2) is satisfied. Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame at $\mathrm{x}_0 \in |A|$. By definition $\mathrm{f}(\mathrm{x}_0), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\mathrm{f}(\mathrm{x}_0)$ for some $b \geqq \beta(\mathrm{x}_i)$ $(0 \leqq i \leqq p)$. Let $\rho_{\mathrm{x}_0}, \cdots, \rho_{\mathrm{x}_p}$ denote respectively the images of $\sigma_{\mathrm{x}_0}, \cdots, \sigma_{\mathrm{x}_p}$ in $(G_B)_{\mathrm{b}}$ under the homomorphisms $(G_B)_{\beta(\mathrm{x}_i)} \to (G_B)_{\mathrm{b}}$ $(0 \leqq \mathrm{i} \leqq \mathrm{p})$. Since $\sigma_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i) = \rho_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i)$by (2.1.4), it follows that $\sigma_{\mathrm{x}_0}\mathrm{f}(\mathrm{x}_0), \cdots, \sigma_{\mathrm{x}_p}\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\sigma_{\mathrm{x}_0}\mathrm{f}(\mathrm{x}_0)$. Thus $\sigma\mathrm{f}(\mathrm{x}_0), \cdots, \sigma\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\sigma\mathrm{f}(\mathrm{x}_0)$ and $b \geqq \beta(\mathrm{x}_i)$ $(0 \leqq i \leqq p)$. □

DEFINITION 3.6 Let (f, β) be an A-chart in B. An A-**frame at** f **on** (f, β) is a set $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p : A \to B$ of morphisms for which there are elements $\sigma_1, \cdots, \sigma_p \in \prod\limits_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}$ such that $\sigma_i\mathrm{f} = \mathrm{f}_i$ $(1 \leqq i \leqq \mathrm{p})$. (In view of Lemma (3.5), $\mathrm{f} = \mathrm{f}_0, f_1, \cdots, \mathrm{f}_{\mathrm{p}}$ is also an A-frame at f_i on (f_i, β) for any i $(0 \leqq \mathrm{i} \leqq \mathrm{p})$.)

The next lemma will be very useful.

LOCAL-GLOBAL LEMMA 3.7 Let (f, β) be an A-chart in B. Then $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is an A-frame at f on $(\mathrm{f}, \beta) \Leftrightarrow$ for each $\mathrm{x} \in |A|, \mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local frame at $\mathrm{f}(\mathrm{x})$ in $\beta(\mathrm{x})$.

PROOF The assertions are trivial consequences of Lemma (3.5).

DEFINITION 3.8 An A-**normal** morphism $\mathrm{g} : B \to C$ of global actions is one which preserves A-frames, i.e. if $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_{\mathrm{p}}$ is an A-frame at f on (f, β) then $\mathrm{gf}, \mathrm{gf}_1, \cdots, \mathrm{gf}_{\mathrm{p}}$ is an A-frame at gf on (gf, γ) for some A-chart (gf, γ) in C. A **normal** morphism $\mathrm{g} : B \to C$ is one which preserves A-frames for any global action A. An A-**normal** (resp. **normal**) **isomorphism** is an A-normal (resp. normal) morphism which has an A-normal (resp. normal) inverse.

It is not true in general that an A-normal (resp. normal) morphism which is an isomorphism in the usual sense is an A-normal (resp. normal) isomorphism.

LEMMA 3.9 A regular morphism is normal.

PROOF Let $\eta : B \to C$ be a regular morphism. If (f, β) is an A-chart in B then it follows straightforward that $(\eta_X\mathrm{f}, \eta_\Phi\beta)$ is an A-chart in C. Let $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at f on (f, β) and let $\sigma_1, \cdots, \sigma_p \in \prod\limits_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}$ such that $\sigma_{\mathrm{i}}\mathrm{f} = \mathrm{f}_{\mathrm{i}}$ $(1 \leqq \mathrm{i} \leqq \mathrm{p})$. If $\sigma = (\sigma_{\mathrm{x}}) \in \prod\limits_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}$, define $\eta_G(\sigma) = (\eta_G(\beta(\mathrm{x}))(\sigma_{\mathrm{x}})) \in \prod\limits_{\mathrm{x}\in|A|} (G_C)_{\eta_\Phi(\beta(\mathrm{x}))}$. Then $\eta_G(\sigma_i)(\eta_X\mathrm{f}) = \eta_X\mathrm{f}_{\mathrm{i}}$ $(1 \leqq i \leqq \mathrm{p})$, by (3.3.4). Thus $\eta_X\mathrm{f}, \eta_X\mathrm{f}_1, \cdots, \eta_X\mathrm{f}_p$ is an A-frame at η_Xf on $(\eta_X\mathrm{f}, \eta_\Phi\beta)$. □

Next the set $Mor(A,B)$ of all morphisms from a global action A to a global action B is given the structure of a global action.

DEFINITION 3.10 Let A and B be global actions. Let $|Mor(A,B)|$ denote the set of all morphisms from A to B. Define a global action

$$Mor(A,B) = (\Phi_{(A,B)}, G_{(A,B)}, X_{(A,B)})$$

whose enveloping set is $|Mor(A,B)|$ as follows. Define

$$\Phi_{(A,B)} = \{\beta : |A| \to \Phi_B\}.$$

Give $\Phi_{(A,B)}$ the reflexive relation defined by $\beta \leqq \beta' \Leftrightarrow \beta(\mathrm{x}) \leqq \beta'(\mathrm{x})\ \forall \mathrm{x} \in |A|$. For $\beta \in \Phi_{(A,B)}$, define

$$(G_{(A,B)})_\beta = \prod_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}.$$

If $\beta \leqq \beta'$, there is for each $\mathrm{x} \in |A|$ a canonically defined homomorphism $(G_B)_{\beta(\mathrm{x})} \to (G_B)_{\beta'(\mathrm{x})}$ and therefore a canonically defined homomorphism $(G_{(A,B)})_\beta \longrightarrow (G_{(A,B)})_{\beta'}$. For $\beta \in \Phi_{(A,B)}$, define

$$(X_{(A,B)})_\beta = \{\mathrm{f} : |A| \to |B| | (\mathrm{f}, \beta)\ A-\text{chart in } B\}.$$

By (3.5), if $\sigma \in (G_{(A,B)})_\beta$ and $\mathrm{f} \in (X_{(A,B)})_\beta$ then $\sigma\mathrm{f} \in (X_{(A,B)})_\beta$ and so there is an action of $(G_{(A,B)})_\beta$ on $(X_{(A,B)})_\beta$. All the conditions for a global action are obviously satisfied except possibly the compatibility condition (2.1.4) which is straightforward to verify. Moreover if B is covariant, contravariant, or bivariant, respectively the so is $Mor(A,B)$.

PROPOSITION 3.11 As a functor taking values in global actions, $Mor(,)$ is contravariant and regular over all morphisms in the first variable and covariant over all normal morphisms in the second variable. More precisely the following holds.

(3.11.1) Let C be a global action and let $\mathrm{f} : A \to B$ be a morphism of global actions. Then f defines a regular morphism

$$\eta = Mor(\mathrm{f}, 1_C) : Mor(B,C) \to Mor(A,C)$$

as follows. Define the relation preserving morphism

$$\begin{aligned} \eta_\Phi : \Phi_{(B,C)} &\to \Phi_{(A,C)}. \\ \beta &\mapsto \beta \mathrm{f} \end{aligned}$$

Define the natural transformation

$$\eta_G : G_{(B,C)} \to G_{(A,C)}$$

such that for each $\beta \in \Phi_{(B,C)}$, the group homomorphism

$$\begin{array}{ccc} \eta_G(\beta) : (G_{(B,C)})_\beta & \longrightarrow & (G_{(A,C)})_{\eta_\Phi(\beta)} \\ \| & & \| \\ \prod_{y \in |B|} (G_C)_{\beta(y)} & & \prod_{\mathrm{x} \in |A|} (G_C)_{\beta \mathrm{f}(\mathrm{x})} \end{array}$$

is defined by the property that

$$\eta_G(\beta)|_{(G_C)_{\beta(y)}} \text{ is the diagonal homomorphism}$$
$$(G_C)_{\beta(y)} \to \prod_{\mathrm{x} \in |A|,\ f(\mathrm{x}) = y} (G_C)_{\beta \mathrm{f}(\mathrm{x})},$$

under the convention that the empty product of groups, which can occur on the right hand side of the arrow above, is the trivial group. Define

$$\begin{aligned} \eta_X : |Mor(B,C)| &\to |Mor(A,C)|. \\ \mathrm{g} &\mapsto \mathrm{gf} \end{aligned}$$

Then $\eta = (\eta_\Phi, \eta_G, \eta_X)$ is a morphism of global actions.

(3.11.2) Let A be a global action and let $\mathrm{g} : B \to C$ be a morphism of global actions. Then the function

$$Mor(1_A, \mathrm{g}) : |Mor(A,B)| \to |Mor(A,C)|$$

is a morphism $Mor(A,B) \to Mor(A,C)$ of global actions $\Leftrightarrow$ g is A-normal.

PROOF (3.11.1) Straightforward and routine. Details are left to the reader.

(3.11.2) Let (f, β) be an A-chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ be an A-frame on (f, β). By definition of the term local frame, $\mathrm{f}_0, \cdots, \mathrm{f}_\mathrm{p}$ is also a local β-frame in the global action $Mor(A, B)$ and conversely, any local frame in $Mor(A, B)$ is an A-frame on some A-chart in B. Thus the function $Mor(1_A, \mathrm{g}) : |Mor(A, B)| \to |Mor(A, C)|$ is a morphism of global actions $\Leftrightarrow$ it preserves A-frames $\Leftrightarrow$ g is A-normal.□

REMARK If B is a global action then letting $\Phi_{B'}$ denote a subrelation of Φ_B whose elements exhaust those of Φ_B, one obtains a global action $B' = (\Phi_{B'}, G_B, X_B)$ which at first glance looks very much like B, in fact the identity map $|B| \to |B|$ defines a regular morphism $B' \to B$ which is an isomorphism of global actions, but not in general a regular isomorphism. Consequences of the structural difference between B' and B can be observed by comparing the global action $Mor(A, B')$ with the global action $Mor(A, B)$, via the canonical morphism $Mor(A, B') \to Mor(A, B)$. The set of A-charts in B' is in general smaller than the set of A-charts in B, which has the consequence that the domain of a local group $(G_{(A,B')})_\beta$ is in general smaller than the domain of the corresponding group $(G_{(A,B)})_\beta$, i.e. $(X_{(A,B')})_\beta \subsetneqq (X_{(A,B)})_\beta$. Of course the corresponding comparison between the domain of the local group $(G_{B'})_b$ and that of $(G_B)_b$ is equality, i.e. $(X_{B'})_b = (X_B)_b$. It is worth noting that if B satisfies the condition that for each coordinate b, the canonical homomorphism $(G_B)_b \to \mathrm{Perm}((X_B)_b)$ is injective then the construction in (2.9.3) shows how to enlarge the set of arrows in Φ_B to an absolute maximum for the data (see (2.9.1)) provided by B.

DEFINITION 3.12 Let $g : B \to C$ be a morphism of global actions. A sequence $A_n, \cdots, A_1$ of global actions is called a **normal chain** of length n for g if g is A_1-normal and if for each i $(1 \leqq i \leqq n-1)$, the morphism $Mor(1_{A_{i-1}}, \cdots, Mor(1_{A_1}, g)) \cdots) : Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, B)) \cdots) \to Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, C)) \cdots)$ is A_{i+1}-normal. Let $\mathcal{N}$ be a class of global actions. The morphism g is called ***n*-$\boldsymbol{\mathcal{N}}$**-normal if every sequence of n objects from $\mathcal{N}$ forms a normal chain for g. The morphism g is called **$\boldsymbol{\mathcal{N}}$-normal** (resp. **$\boldsymbol{\infty}$-$\boldsymbol{\mathcal{N}}$-normal**) if it is 1-$\mathcal{N}$-normal (resp. n-$\mathcal{N}$-normal for all $n > 0$). If $\mathcal{N} = \{A\}$ (resp. $\mathcal{N}$ = all global actions), we shall write **$\boldsymbol{\infty}$-*A*-normal** (resp. **$\boldsymbol{\infty}$-normal**) in place of ∞-$\mathcal{N}$-normal.

If the expression t-morphism denotes anyone of the notions of normality above or the notion of regularity then a t-isomorphism is a t-morphism which has a t-morphism as its inverse.

In order to associate to a morphism g : $A \to B$ of global actions a long exact sequence of algebraic homotopy groups, we shall need that g is ∞-L-normal where L is the line action defined in Example (2.4).

LEMMA 3.13 If g : $B \to C$ is a regular morphism then for any global action A, the morphism $Mor(1_A, \mathrm{g}) : Mor(A, B) \to Mor(A, C)$ exists and extends to a regular morphism. Thus g is ∞-normal.

PROOF By (3.9)and (3.11.2), the morphism $Mor(1_A, \mathrm{g}) : Mor(A, B) \to Mor(A, C)$ exists. Let $(\eta_\Phi, \eta_G, \eta_X = \mathrm{g})$ be the regular structure of g. We define a regular structure $(\mu_\Phi, \mu_g, \mu_X = Mor(1_A, \mathrm{g}))$ for $Mor(1_A, \mathrm{g})$ as follows.

Define the coordinate morphism

$$\begin{aligned} \mu_\Phi : \Phi_{(A,B)} &\to \Phi_{(A,C)}. \\ \beta &\mapsto \eta_\Phi \beta \end{aligned}$$

Define the natural transformation

$$\mu_G : G_{(A,B)} \to G_{(A,C)}$$

such that for each $\beta \in \Phi_{(A,B)}$, the group homomorphism

$$\mu_G(\beta) : (G_{(A,B)})_\beta \to (G_{(A,C)})_{\mu_\Phi(\beta)}$$

is defined by the commutative diagram

$$\begin{tikzcd} (G_{(A,B)})_\beta \arrow[r, "\mu_G(\beta)"] \arrow[d, equal] & (G_{(A,C)})_{\mu_\Phi(\beta)} \arrow[d, equal] \\ \prod\limits_{\mathrm{x} \in |A|} (G_{(A,B)})_{\beta(\mathrm{x})} \arrow[r, "\prod\limits_{\mathrm{x} \in |A|} \eta_G(\beta(\mathrm{x}))"'] & \prod\limits_{\mathrm{x} \in |A|} (G_{(A,C)})_{\eta_\Phi(\beta(\mathrm{x}))} \end{tikzcd}$$

One checks straightforward that $(\mu_\Phi, \mu_G, Mor(1_A, \mathrm{g}))$ is a regular morphism.

That g is ∞-normal follows by a trivial induction argument from the result just proved. □

DEFINITION 3.14 Let N denote the name of a kind of morphism defined in (3.12). A global action is called an **N action** if it has the property that every morphism to it is an N morphism. For example an ∞ **-normal action** has the property that every morphism to it is ∞-normal.

For the results below on the exponential law, the notion of product is needed. We construct this next.

Definition-Lemma 3.15 Let A and B be global actions. Their **product** $A \times B$ is constructed as follows.

$$\Phi_{A\times B} = \Phi_A \times \Phi_B$$

and $(\alpha, \beta) \leqq (\alpha', \beta') \Leftrightarrow \alpha \leqq \alpha' \text{and} \beta \leqq \beta'$.

$$\begin{aligned} G_{A\times B} &= G_A \times G_B \\ |A \times B| &= |A| \times |B| \\ X_{A\times B} &= X_A \times X_B. \end{aligned}$$

For any coordinate $(\alpha, \beta) \in \Phi_{A\times B}$, there is an obvious action of $(G_{A\times B})_{(\alpha,\beta)}$ on $(X_{A\times B})_{(\alpha,\beta)}$, namely the one defined coordinatewise. One checks easily that $A\times B$ satisfies the universal property of a product.

The following notation will be used below. If S and T are sets, let

$$(S, T) = Mor_{((sets))}(S, T).$$

If U is also a set then there is a canonical isomorphism

$$\begin{aligned} E : (U, (S, T)) &\xrightarrow{\cong} (U \times S, T) \\ f &\longmapsto Ef \end{aligned} \tag{3.16}$$

of sets such that $Ef(u, s) = f(u)(s)$. Its inverse is obviously the function

$$\begin{aligned} E' : (U \times S, T) &\longrightarrow (U, (S, T)) \\ f &\longmapsto E'f \end{aligned}$$

where $(E'f(u))(s) = f(u, s)$.

Definition 3.17 Let A, B and C be global actions. We define a regular morphism

$$E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$$

as follows. Denote the structural components of the global action $Mor(A, Mor(B, C))$ by $(\Phi_{(A,(B,C))}, G_{(A,(B,C))}, X_{(A,(B,C))})$. Define

$$\begin{array}{ccc} E_\Phi : \Phi_{(A,(B,C))} & \longrightarrow & \Phi_{(A\times B,C)} \\ \| & & \| \\ (|A|, (|B|, \Phi_C)) & & (|A| \times |B|, \Phi_C) \end{array}$$

to be the set theoretic exponential isomorphism (3.16). Clearly E_Φ is relation preserving. Define the natural transformation

$$E_G : G_{(A,(B,C))} \to (G_{(A\times B,C)})_{E_\Phi(\ )}$$

such that for each $\alpha \in \Phi_{(A,(B,C))}$, the group homomorphism

$$\begin{array}{ccc} E_G(\alpha) : (G_{(A,(B,C))})_\alpha & \longrightarrow & (G_{(A\times B,C)})_{E_\Phi(\alpha)} \\ \| & & \| \\ \prod\limits_{\mathrm{x}\in|A|} (\prod\limits_{y\in|B|} (G_C)_{\alpha(\mathrm{x})(y)}) & & \prod\limits_{(\mathrm{x},y)\in|A|\times|B|} (G_C)_{(E_\Phi\alpha)(\mathrm{x},y)} \end{array}$$

maps the factor $(G_C)_{\alpha(\mathrm{x})(y)}$ via the identity map onto the factor $(G_C)_{(E_\Phi\alpha)(\mathrm{x},y)} = (G_C)_{\alpha(\mathrm{x})(y)}$. One verifies easily that the composite mapping $|Mor(A, Mor(B, C))| \to (|A|, (|B|, |C|))$ @ > (3.16) >> $(|A| \times |B|, |C|)$ takes its image in $|Mor(A \times B, C)|$ and we define

$$E_X : |Mor(A, Mor(B, C))| \to |Mor(A \times B, C)|$$

to be the resulting mapping. One checks straightforward that

$$E = (E_\Phi, E_G, E_X)$$

is a regular morphism. (It fails in general to be an isomorphism (resp. regular isomorphism) because E_X is not necessarily surjective (resp. $E_X((X_{(A,(B,C))})_\alpha)$ is not necessarily all of $(X_{(A\times B,C)})_{E_\Phi(\alpha)}$).

Let $A_n, \cdots, A_1$ be an arbitrary sequence of global actions. Iterating the procedure above, one defines for any $n \geq 2$ a regular morphism

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C)) \cdots) \to Mor(A_n \times \cdots \times A_1, C)$$

as follows. For $n = 2$, the morphism is defined above. Suppose $n > 2$ and that the morphism has been defined for every natural number N where $2 \leqq N \leqq n-1$. Let E_{n-1} denote the morphism for the sequence $A_{n-1}, \cdots, A_1$. Define E_n for the sequence $A_n, A_{n-1}, \cdots, A_1$ as the composite of the regular morphism $Mor(1_{A_n}, E_{n-1})$ (see (3.13)) and the regular morphism $E_2 : Mor(A_n, Mor(A_{n-1} \times \cdots \times A_1, B)) \to Mor(A_n \times \cdots \times A_1, B)$.

The next definition is made to cope with the problem of finding an inverse to the morphism E_n above.

Definition 3.18 Let $\mathcal{P}$ be a class of global actions closed under finite products. A global action C is called **∞-$\mathcal{P}$-exponential** if the morphism $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ is an ∞-$\mathcal{P}$-normal isomorphism for all pairs $A, B \in \mathcal{P}$. C is called **regularly ∞-$\mathcal{P}$-exponential** if E is a regular isomorphism for all pairs $A, B \in \mathcal{P}$. If $\mathcal{P}$ = all finite products of A (resp. $\mathcal{P}$ = all global actions A such that $|A| = \cup_{\alpha \in \Phi_A} X_\alpha$) then C is called **$\infty$-$A$-exponential** (resp. **$\infty$-exponential**) if it is ∞-$\mathcal{P}$-exponential.

Lemma 3.19 Suppose the global action C is ∞-$\mathcal{P}$-exponential (resp. regularly ∞-$\mathcal{P}$-exponential). Then for any sequence $A_n, \cdots, A_1 \in \mathcal{P}$ such that $n \geq 2$, the morphism E_n in (3.17) is an ∞-$\mathcal{P}$-normal (resp. regular) isomorphism.

PROOF For $n = 2$, the conclusion holds by hypothesis. Proceeding by induction on n, we can assume that the result holds for $n - 1$. By definition $E_n = E_2 Mor(1, E_{n-1})$. By induction E_2 and $Mor(1, E_{n-1})$ are ∞-$\mathcal{P}-$isomorphisms (resp. regular isomorphisms). The conclusion of the lemma follows. □

The next condition provides a useful criterion for guaranteeing that a global action is ∞-normal and either ∞-exponential or regularly ∞-exponential.

Definition 3.20 Let A be a global action. Let $\Delta \subsetneqq \Phi_A$ be a finite possibly empty subset and let $\Phi_A^{\geqq\Delta} = \{\alpha \in \Phi_A \mid \alpha \geqq \beta \; \forall \; \beta \in \Delta\}$. Let $U \subsetneqq |A|$ be a finite nonempty subset such that for any $\beta \in \Delta, U \cap (X_A)_\beta \neq \varnothing$. The **strong infimum condition** for A says that for any Δ and U as above, the set $\{\alpha \in \Phi_A^{\geqq\Delta} | U \text{ an } \alpha\text{-frame } \}$ is either empty or contains an initial element.

A is called an **infimum action** if it satisfies the condition above at least for $\Delta = \varnothing$.

The next lemma provides a condition guaranteeing that a global action is a strong infimum action and the lemma thereafter proves the important result that if the target object in a morphism space is an infimum (resp. strong infimum) action then the morphism space inherits this property.

Any global action A has the property that if α and β are coordinates such that $\alpha \leqq \beta$ then for each by $\mathrm{x} \in (X_A)_\alpha \cap (X_A)_\beta, (G_A)_\alpha(\mathrm{x}) \subseteqq (G_A)_\beta(\mathrm{x})$. The next lemma shows that the reverse implication coupled with a certain intersection property is a sufficient condition for A to satisfy the strong infimum condition.

Lemma 3.21 Let A be a global action. Consider the following conditions.

(3.21.1) Let $\alpha, \beta \in \Phi_A$. Then $\alpha \leqq \beta \Leftrightarrow \exists \mathrm{x} \in (X_A)_\alpha \cap (X_A)_\beta$ such that $(G_A)_\alpha(\mathrm{x}) \subseteqq (G_A)_\beta(\mathrm{x})$.

(3.21.2) Let $\Psi \subseteqq \Phi_A$. Then for any $\mathrm{x} \in \cap_{\alpha\in\Psi}(X_A)_\alpha,\ \cap_{\alpha\in\Psi}\ (G_A)_\alpha(\mathrm{x}) = (G_A)_\beta(\mathrm{x})$ for some $\beta \in \Phi_A$.

The assertion of the lemma is that if A satisfies (3.21.1) and (3.21.2) then it is a strong infimum action.

PROOF Let U be a local frame. Let $\Delta \subseteqq \Phi_A$ be a finite set such that for each $\delta \in \Delta, X_\delta \cap U \neq \varnothing$. Let $\Psi = \{\alpha \in \Phi^{\geqq\Delta} \mid U$ local α-frame $\}$ and assume $\Psi \neq \varnothing$. We must show that Ψ has an initial element. If $u \in U$ then $U \subseteqq \cap_{\alpha\in\Psi}(G_A)_\alpha(u) = (G_A)_\beta(u)$ for some $\beta \in \Phi_A$, by (3.21.2). Clearly U is a local $\beta-$frame. Since $(G_A)_\beta(u) \subseteqq (G_A)_\alpha(u)$, it follows from (3.21.1) that $\beta \leqq \alpha$. This holds of course for all $\alpha \in \Psi$. Thus we are finished if $\Delta = \varnothing$. If $\Delta \neq \varnothing$, we must show that $\delta \leqq \beta$ for any $\delta \in \Delta$. Let $u \in X_\delta \cap U$. Since $\delta \leqq \alpha$ for any $\alpha \in \Psi$, it follows from (2.1.4) that $(G_A)_\delta(u) \subseteqq \cap_{\alpha\in\Psi}(G_A)_\alpha(u) = (G_A)_\beta(u)$. Thus $\delta \leqq \beta$, by (3.21.2). □

Remark Whereas the simplicial actions $gl(S)$ and $fgl(S)$ in (2.5) and any Volodin model (2.3) satisfy the strong infimum condition, only the Volodin model satisfies the conditions in the lemma above.

Lemma 3.22 If B is an infimum action (resp. strong infimum action and the relation on Φ_B is transitive) then for any global action $A, Mor(A,B)$ is an infimum action (resp. strong infimum action and the relation on $\Phi_{(A,B)}$ is transitive).

PROOF Let $U \subseteqq |Mor(A,B)|$ be a finite nonempty subset. Let $\Delta \subseteqq \Phi_{(A,B)}$ be a finite subset such that for each $\delta \in \Delta, (X_{(A,B)})_\delta \cap U \neq \varnothing$. Let $\Psi = \{\beta \in \Phi^{\geqq\Delta}_{(A,B)} | U$ a β-frame$\}$ and assume $\Psi \neq \varnothing$. We must show that Ψ has an initial element. For each $\mathrm{x} \in |A|$, let $U(\mathrm{x}) = \{\mathrm{f}(\mathrm{x})|\ \mathrm{f} \in U\}, \Delta(\mathrm{x}) = \{\delta(\mathrm{x})|\ \delta \in \Delta\}$, and $\Psi(\mathrm{x}) = \{\beta(\mathrm{x})|\ \beta \in \Psi\}$. Set $\Psi'(\mathrm{x}) = \{b \in \Phi_B^{\geqq\Delta(\mathrm{x})} \mid U(\mathrm{x})$ a b-frame$\}$. Obviously $\varnothing \subseteqq \Psi'(\mathrm{x})$ and $(X_B)_{\delta(\mathrm{x})} \cap U(\mathrm{x}) \neq$

$\varnothing$ for each $\delta(\mathrm{x}) \in \Delta(\mathrm{x})$. If $\Delta = \varnothing$ (resp.$\Delta \neq \varnothing$) then by the infimum (resp. strong infimum) condition for B, the set $\Psi'(\mathrm{x})$ has an initial element c_{x}. Let $\gamma : |A| \to \Phi_B, \mathrm{x} \mapsto c_{\mathrm{x}}$. We shall show that if $u \in U$ then (u, γ) is an A-chart B. This done, it will follow from the Local-Global Lemma 3.7 that U is a γ-frame. Thus $\gamma \in \Psi$. Since $\gamma(\mathrm{x})$ is an initial element of $\Psi'(\mathrm{x})$, and hence also of $\Psi(\mathrm{x})$, for each $\mathrm{x} \in |A|$, it follows that γ is an initial element of Ψ. Finally we note that the transitivity of the relation on Φ_B implies the transitivity of the relation on $\Phi_{(A,B)}$.

It remains to show that (u, γ) is an A-chart in B. Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame in A. Let $\beta \in \Psi$. Since (u, β) is an A-chart in B, there is a coordinate $b \in \Phi_B$ such that $b \geqq \beta(\mathrm{x}_0), \cdots, \beta(\mathrm{x}_p)$ and $u(\mathrm{x}_0), \cdots, u(\mathrm{x}_p)$ is a b-frame. Since U is a β-frame, it follows (see the proof of (3.24)) that $U(\mathrm{x}_0) \cup \cdots \cup U(\mathrm{x}_p)$ is a b-frame. Suppose $\Delta = \varnothing$. Since $U(\mathrm{x}_i)$ is a b-frame, it follows from the definition of $\gamma(\mathrm{x}_i)$ that $b \geqq \gamma(\mathrm{x}_i)$. This holds of course for all $0 \leqq i \leqq p$. Thus (u, γ) is an A-chart in B. Suppose $\Delta \neq \varnothing$ and the relation on Φ_B is transitive. By the definition of $b, b \geqq \beta(\mathrm{x}_i)$ and since $\beta \in \Psi, \beta(\mathrm{x}_i) \geqq \delta(\mathrm{x}_i)$ for any $\delta \in \Delta$. Thus by the transitivity of the relation on $\Phi_B, b \geqq \delta(\mathrm{x}_i)$ for all $\delta \in \Delta$. Since $U(\mathrm{x}_i)$ is a b-frame, it follows from the definition of $\gamma(\mathrm{x}_i)$ that $b \geqq \gamma(\mathrm{x}_i)$. This holds of course for all $0 \leqq i \leqq p$. Thus (u, γ) is an A-chart in B.

The next theorem is a main result.

THEOREM 3.23 An infimum action is $\infty-$normal and $\infty-$exponential. A strong infimum action such that the relation on its coordinate system is transitive is $\infty-$normal and regularly $\infty-$exponential.

The proof of Theorem 3.23 will use the next lemma several times.

LEMMA 3.24 Let A and B be global actions. Let (f, β) be an $A-$chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, f_p$ be an $A-$frame at (f, β). If $\mathrm{x}_0, \cdots, \mathrm{x}_q \in |A|$ is a local frame in A then $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local $b-$frame for some $b \in \Phi_B$ such that $b \geqq \beta_{\mathrm{x}_0}, \cdots, \beta_{\mathrm{x}_q}$.

PROOF Since (f_0, β) is an A-chart in B, it follows by definition that $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_0(\mathrm{x}_q)$ is a local b-frame for some $b \in \Phi_B$ such that $b \geqq \beta_{\mathrm{x}_0}, \cdots, \beta_{\mathrm{x}_q}$. Thus $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_0(\mathrm{x}_q)$. To complete the proof, it suffices to show that $(G_B)_b$ acts transitively on $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$. Since $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A-$frame at $(\mathrm{f}, \beta), (G_{(A,B)})_\beta$ acts transitively on $\mathrm{f}_0, \cdots, \mathrm{f}_p$. Thus for any $\mathrm{x} \in |A|, (G_B)_{\beta_{\mathrm{x}}}$ acts transitively on $\mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$. Using the canonical homomorphism $(G_B)_{\beta_{\mathrm{x}_j}} \to (G_B)_b$ and the observation that $\mathrm{f}_0(\mathrm{x}_j) \in (X_B)_b$, one concludes that $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_j), \cdots, \mathrm{f}_p(\mathrm{x}_j)$. Since this holds for each j such that $0 \leqq j \leqq q$ and since $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_p(\mathrm{x}_q)$, it follows that $(G_B)_b$ acts transitively on $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$. $\square$

PROOF of (3.23) Let C be an infimum action. We shall show that C is $\infty-$normal. Lemma 3.22 reduces the proof to showing that C is $A-$normal for any global action A. Let $g : B \to C$ be a morphism of global actions. Let (f, β) be an $A-$ chart in B. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an $A-$frame on (f, β). We must show that $g\mathrm{f}_0, \cdots, g\mathrm{f}_p$ is an $A-$frame in C. We construct first a coordinate $(\gamma : |A| \to \Phi_C) \in \Phi_{(A,C)}$ such that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C.

For $\mathrm{x} \in |A|$, let $U(\mathrm{x}) = \{g\mathrm{f}_0(\mathrm{x}), \cdots, g\mathrm{f}_p(\mathrm{x})\}$. By the Local-Global Lemma 3.7, $\mathrm{f}_0(\mathrm{x}), \cdots,$ $\mathrm{f}_p(\mathrm{x})$ is a local frame in B. Since g is a morphism, it follows that $U(\mathrm{x})$ is a local frame in C. By the infimum condition for C, the set $\Psi(\mathrm{x}) = \{c \in \Phi_C | U(\mathrm{x}) \text{ a } c-\text{ frame}\}$ has an initial element c_{x}. Define $\gamma : |A| \to \Phi_C, \mathrm{x} \mapsto c_{\mathrm{x}}$. We show that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C. Let $\mathrm{x}_0, \cdots, \mathrm{x}_q$ be a local frame in A. By (3.24), $\{\mathrm{f}_i(\mathrm{x}_j) | \ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local frame in B. Thus $\{g\mathrm{f}_i(\mathrm{x}_j) | \ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local $c-$frame for some $c \in \Phi_C$. Clearly $\gamma(\mathrm{x}_j) = c_{\mathrm{x}_j} \leqq c$, because c_{x_j} is initial in $\Psi(\mathrm{x}_j)$. This shows that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C. By the Local-Global Lemma 3.7, $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A-$frame on $(\mathrm{f}, \gamma) \Leftrightarrow \mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local $\gamma(\mathrm{x})-$frame for all $\mathrm{x} \in |A|$. But the right hand side of the equivalence holds by definition of $\gamma(\mathrm{x})$. This completes the proof that C is $A-$normal.

Let C denote again an infimum action. We shall show that C is $\infty-$exponential. Let A and B be global actions such that $|A| = \cup_{\alpha \in \Phi_A} (X_A)_\alpha$ and $|B| = \cup_{\beta \in \Phi_B} (X_B)_\beta$. Let $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ be the morphism in (3.17). We shall prove that E has an $\infty-$normal inverse. By (3.22), $Mor(A, Mor(B, C))$ is an infimum action and thus by the first assertion of the current theorem, it must be $\infty-$normal. Thus if an inverse to E exists, it must be $\infty-$normal. So it suffices to show that E has an inverse. There is an obvious candidate for an inverse, namely the set theoretic map $E' : |Mor(A \times B, C)| \to (A, (B, C)), \mathrm{f} \mapsto E'\mathrm{f}$, where $(E'\mathrm{f}(\mathrm{x}))(y) = \mathrm{f}(\mathrm{x}, y)$. We shall show that $E'\mathrm{f} \in |Mor(A, Mor(B, C))|$ and that the resulting map $E' : |Mor(A \times B, C)| \to |Mor(A, Mor(B, C))|$ is a morphism $Mor(A \times B, C) \to Mor(A, Mor(B, C))$ of global actions. From the set theoretic definition of E', it is obvious that E' will be inverse to E.

We prove that $E'\mathrm{f} : |A| \to (B, C)$ is a morphism $A \to Mor(B, C)$ of global actions. There are two properties to verify. First, if $\mathrm{x} \in |A|$ then $E'\mathrm{f}(\mathrm{x}) : |B| \to |C|,\ y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$, is a morphism $B \to C$ of global actions. Second, the resulting map $E'\mathrm{f} : |A| \to |Mor(B, C)|, \mathrm{x}$ $\mapsto E'\mathrm{f}(\mathrm{x})$, is a morphism $A \to Mor(B, C)$ of global actions.

Let $\mathrm{x} \in |A|$ and let $y_0, \cdots, y_q$ be a local frame in B. Then x is a local frame in A because $|A| = \cup_{\alpha \in \Phi_A} (X_A)_\alpha$ and so $(\mathrm{x}, y_0), \cdots, (\mathrm{x}, y_q)$ is a local frame in $A \times B$. Thus $\mathrm{f}(\mathrm{x}, y_0), \cdots, \mathrm{f}(\mathrm{x}, y_q)$ is a local frame in C. But $\mathrm{f}(\mathrm{x}, y_j) = (E'\mathrm{f}(\mathrm{x}))(y_j)$ $(0 \leqq j \leqq q)$. Thus $(E'\mathrm{f}(\mathrm{x}))(y_0), \cdots, (E'\mathrm{f}(\mathrm{x}))(y_q)$ is a local frame in C. Thus $E'\mathrm{f}(\mathrm{x}) : B \to C$ is a morphism of global actions.

Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame in A. We shall verify that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B, C)$. For each element $y \in |B|, y$ is a local frame in B because $|B| = \cup_{\beta \in \Phi_B}(X_B)_\beta$. Thus $(\mathrm{x}_0, y), \cdots, (\mathrm{x}_p, y)$ is a local frame in $A \times B$. Thus $\mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ is a local frame in C. By the infimum condition for C, we know that the set $\{c \in \Phi_C | \ \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y) \text{ a } c\text{−frame}\}$ has an initial element c_y. Define $\gamma : |B| \to \Phi_C, y \mapsto c_y$. We shall show that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a $B-$ chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 3.7 that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a $B-$frame on $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$. But then by definition, $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B, C)$, which is what we have to verify.

We show now that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a $B-$chart in C. Let $y_0, \cdots, y_q$ be a local frame in B. We must show that $(E'\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (E'\mathrm{f}(\mathrm{x}_0))(y_q)$ is a local $c-$frame for some $c \in \Phi_C$ such that $c \geq \gamma(y_0), \cdots, \gamma(y_q)$. Since $\mathrm{x}_0, \cdots, \mathrm{x}_p$ is a local frame in A and $y_0, \cdots, y_q$ a local frame in $B, \{(\mathrm{x}_i, y_j) | \ 0 \leqq i \leqq p, \ 0 \leqq j \leqq q\}$ is a local frame in $A \times B$. Thus $\{\mathrm{f}(\mathrm{x}_i, y_j) | \ 0 \leqq i \leqq p, \ 0 \leqq j \leqq q\}$ is a local $c-$frame for some $c \in \Phi_C$. But by definition of $\gamma, c \geq \gamma(y_j)$ for all j such that $0 \leqq j \leqq q$.

Next we show that the function $E' : |Mor(A \times B, C)| \to |Mor(A, Mor(B, C))|$ is a morphism $Mor(A \times B, C) \to Mor(A, Mor(B, C))$ of global actions. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p \in |Mor(A \times B, C)|$ be a local frame in $Mor(A \times B, C)$. We must show that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B, C))$. For each element (x, y) in $A \times B$, $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is a local frame in C, by the Local-Global Lemma 3.7. By the infimum condition for C, the set $\{c \in \Phi_C | \ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y) \text{ a } c-\text{frame}\}$ has an initial element $c_{(\mathrm{x}, y)}$. Define $\gamma : |A| \to (|B|, \Phi_C), \mathrm{x} \mapsto c_{(\mathrm{x}, -)}$. We claim that $(E'\mathrm{f}_0, \gamma)$ is an $A-$ chart in $Mor(B, C)$. It will follow then from the definition of γ and the Local-Global Lemma 3.7 that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is an $A-$ frame on $(E'\mathrm{f}, \gamma)$. But this says by definition that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B, C))$ and we are finished.

We show now that $(E'\mathrm{f}, \gamma)$ is an $A-$chart. Let $\mathrm{x}_0, \cdots, \mathrm{x}_q$ be a local frame in A. We must show that $E'\mathrm{f}(\mathrm{x}_0), \cdots E'\mathrm{f}(\mathrm{x}_q)$ is a local $\delta-$frame in $Mor(B, C)$ for some $\delta : |B| \to \Phi_C$ such that $\delta \geqq \gamma(\mathrm{x}_0), \cdots, \gamma(\mathrm{x}_q)$. Since $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is a local frame in $Mor(A \times B, C)$, there is an $A \times B-$ chart $(\mathrm{f}_0, \varepsilon)$ in C such that $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A \times B-$ frame on $(\mathrm{f}_0, \varepsilon)$. For any fixed $y \in |B|, (\mathrm{f}_0(\ , y), \varepsilon(\ , y))$ is an $A-$chart in C and $\mathrm{f}_0(\ , y), \cdots, \mathrm{f}_p(\ , y)$ is an $A-$ frame on $(\mathrm{f}_0(\ , y), \varepsilon(\ , y))$. Since $\mathrm{x}_0, \cdots, \mathrm{x}_q$ is a local frame in A, it follows from (3.24) that the set $\{\mathrm{f}_i(\mathrm{x}_j, y) | \ 0 \leqq i \leqq p, \ 0 \leqq j \leqq q\}$ is a local frame in C. Since C satisfies the infimum condition, the set $\{c \in \Phi_C | \{\mathrm{f}_i(\mathrm{x}_j, y) | \ 0 \leqq i \leqq p, \ 0 \leqq j \leqq q\}$ a local frame in $C\}$ has an initial element d_y. Clearly $d_y \geqq (\gamma(\mathrm{x}_j))(y)$ $(0 \leqq j \leqq q)$. Define $\delta : |B| \to \Phi_C, y \mapsto d(y)$. Thus $\delta \geqq \gamma(\mathrm{x}_j)$ $(0 \leqq j \leqq q)$. Since $(F\mathrm{f}(\mathrm{x}_j))(y) = \mathrm{f}_0(\mathrm{x}_j, y)$ $(0 \leqq j \leqq q)$ and $\{\mathrm{f}_i(\mathrm{x}_j, y) | \ 0 \leqq i \leqq p, \ 0 \leqq j \leqq q\}$ is a $\delta(y)-$frame, it is clear that $(F\mathrm{f}(\mathrm{x}_0))(y), \cdots,$ $(F\mathrm{f}(\mathrm{x}_q))(y)$ is a $\delta(y)-$frame. By the Local-Global Lemma 3.7, $F\mathrm{f}(\mathrm{x}_0), \cdots, F\mathrm{f}(\mathrm{x}_q)$ is a $B-$

frame on $(F\mathrm{f}(\mathrm{x}_0), \delta)$ provided that $(F\mathrm{f}(\mathrm{x}_0), \delta)$ is a $B-$chart in C. We show this next.

Let $y_0, \cdots, y_r$ be a local frame in B. We must show that $(F\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (F\mathrm{f}(\mathrm{x}_0))(y_r)$ is a $c-$frame for some $c \in \Phi_C$ such that $c \geqq \delta(y_0), \cdots, \delta(y_r)$. Since the set $\{(\mathrm{x}_j, y_k)| \; 0 \leqq j \leqq q,$ $0 \leqq k \leqq r\}$ is a local frame in $A \times B$ and $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A \times B-$frame in C, it follows from (3.24) that $\{\mathrm{f}_i(\mathrm{x}_j, y_k)| \; 0 \leqq i \leqq p, \; 0 \leqq j \leqq q, \; 0 \leqq k \leqq r\}$ is a c-frame for some $c \in \Phi_C$. From the definition of δ, it is obvious that $c \geqq \delta(y_0), \cdots, \delta(y_r)$. Since $(F\mathrm{f}(\mathrm{x}_0))(y_k) = \mathrm{f}_0(\mathrm{x}_0, y_k)$ $(0 \leqq k \leqq r)$ and $\{\mathrm{f}_i(\mathrm{x}_j, y_k)| \; 0 \leqq i \leqq p, \; 0 \leqq j \leqq q, \; 0 \leqq k \leqq r\}$ is a c-frame, it is clear that $(F\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (F\mathrm{f}(\mathrm{x}_0))(y_r)$ is a $c-$frame. This completes the proof that C is $\infty-$exponential.

Suppose finally that C is a strong infimum action. We shall show that C is regularly $\infty-$exponential. Our task is to show that the morphism $E : Mor(A, Mor(B, C)) \rightarrow Mor(A \times B, C)$ above has a regular inverse E'. There are obvious candidates for the components (E'_Φ, E'_G, E'_X) of E'. Define

$$(3.25) \qquad E'_X : Mor(A \times B, C)| \longrightarrow |Mor(A, Mor(B, C))| \qquad \mathrm{f} \longmapsto E'\mathrm{f}$$

where $\mathrm{f} \mapsto E'\mathrm{f}$ is the map constructed above. Define

$$E'_\Phi : \Phi_{(A \times B, C)} \longrightarrow \Phi_{(A,(B,C))}$$

as the set theoretic inverse (see (3.16)) of E_Φ. Define the natural transformation

$$E'_G : G_{(A \times B, C)} \longrightarrow (G_{(A,(B,C))})_{E'_\Phi(\;)}$$

such that

$$\begin{array}{ccc} E'_G(\alpha) : (G_{(A\times B,C)})_\alpha & \longrightarrow & (G_{(A,(B,C))})_{E'_\Phi(\alpha)} \\ \| & & \| \\ \prod_{(\mathrm{x},y) \in |A| \times |B|} (G_C)_{\alpha(\mathrm{x},y)} & & \prod_{\mathrm{x} \in |A|} (\prod_{y \in |B|} (G_C)_{E'_\Phi(\alpha)(\mathrm{x})(y)} \end{array}$$

maps the factor $(G_C)_{\alpha(\mathrm{x},y)}$ via the identity map to the factor $(G_C)_{E'_\Phi(\alpha)(\mathrm{x})(y)} = (G_C)_{\alpha(\mathrm{x},y)}$.

If E' is a regular morphism then it is obvious from its construction that it is the regular inverse to the regular morphism E.

All the properties for E' to be a regular morphism are obvious, except the one that $E'_X(X_{(A\times B,C)})_\alpha \subseteqq (X_{(A,(B,C))})_{E'_\Phi(\alpha)}$ for any $\alpha \in \Phi_{(A\times B,C)}$. To establish this, it is enough to show that if (f,α) is an $(A\times B)$-chart in C then $(E'_X(\mathrm{f}), E'_\Phi(\alpha))$ is an $A-$chart in $Mor(B,C)$. Let $\mathrm{x}_0,\cdots,\mathrm{x}_p$ be a local frame in A. We must show that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})$ (x_p) is a $\gamma-$ frame in $Mor(B,C)$ for some $\gamma : |B| \to \Phi_C$ such that $\gamma \geqq E'_\Phi(\alpha)(\mathrm{x}_i)$ $(0\leqq i\leqq p)$. For each $y\in |B|$, the elements $(\mathrm{x}_0,y),\cdots,(\mathrm{x}_p,y)$ form a local frame in $A\times B$. Thus $\mathrm{f}(\mathrm{x}_0,y),\cdots,\mathrm{f}(\mathrm{x}_p,y)$ is a local frame in C. By the strong infimum condition for C, the set $\{c\in\Phi_C|\ \mathrm{f}(\mathrm{x}_0,y),\cdots,\mathrm{f}(\mathrm{x}_p,y)\text{ a }c-\text{frame}, c\geqq E'_\Phi(\alpha)(\mathrm{x}_i)(y)\ (0\leqq i\leqq p)\}$ has an initial element c_y. Define $\gamma : |B|\to\Phi_C,\ y\mapsto c_y$. Clearly $\gamma\geqq E'_\Phi(\alpha)(\mathrm{x}_i)$ $(0\leqq i\leqq p)$. We shall show that $(E'_X(\mathrm{f})(\mathrm{x}_0),\gamma)$ is a $B-$chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 3.7 and the fact that $E'_X(\mathrm{f})(\mathrm{x}_0)(y),\cdots,E'_X(\mathrm{f})(\mathrm{x}_p)(y)$ is a $\gamma_(y)-$frame for each $y\in|B|$ that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})$ (x_p) is a $B-$frame on $(E'_X(\mathrm{f})(\mathrm{x}_0)$, $\gamma)$. But this says by definition that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})(\mathrm{x}_p)$ is a $\gamma-$frame in $Mor(B,C)$. This would complete the proof of the theorem.

We show now that $(E'_X(\mathrm{f})(\mathrm{x}_0),\gamma)$ is a $B-$chart in C. Let $y_0,\cdots,y_q$ be a local frame in B. Then $\{(\mathrm{x}_i,y_j)|\ 0\leqq i\leqq p,\ 0\leqq j\leqq q\}$ is a local frame in $A\times B$. Thus $\{\mathrm{f}(\mathrm{x}_i,y_j)|\ 0\leqq i\leqq p,$ $0\leqq j\leqq q\}$ is a local $c-$frame for some $c\in\Phi_C$ such that $c\geqq\alpha(\mathrm{x}_i,y_j)=E'_\Phi(\alpha)(\mathrm{x}_i)(y_j)$ $(0\leqq i\leqq p,\ 0\leqq j\leqq q)$. Since $E'_X(\mathrm{f})(\mathrm{x}_0)(y_j) = \mathrm{f}(\mathrm{x}_0,\mathrm{x}_j)$ $(0\leqq j\leqq q)$, it is clear that $E'_X(\mathrm{f})(\mathrm{x}_0)(y_0),\cdots,E'_X(\mathrm{f})(\mathrm{x}_0)(y_q)$ is a c-frame and $c\geqq\gamma(y_j)$ $(0\leqq j\leqq q)$. $\square$

4 Relative actions and their morphism spaces

The homotopy theory of global actions will require pointed actions and more generally relative actions. These concepts will be introduced next. They are subtler than their topological counterparts and more care must be taken to define and develop them. The main result of the section is a relative version of the exponential law proved in the previous section.

The organization and development of the current section will follow roughly that of the previous.

Definition 4.1 Let A be a global action. A **subaction** of A is a global action B such that $|B|\subseteqq|A|$ and the inclusion above is a morphism $B\to A$ of a global actions. If B is a subaction of A then we write $B\subseteqq A$. Let $n\in\mathbb{N}\cup\{\infty\}$ and let $\mathcal{N}$ be a class of global actions. A subaction $B\subseteqq A$ is called $\boldsymbol{n-\mathcal{N}}$**−normal** (resp. **regular**) if the canonical morphism $B\to A$ is $n-\mathcal{N}-$normal (resp. extends to a regular morphism).

If G is a group acting on a set X and if $Y\subseteq X$, define

$$Stab_G(Y) = \{\sigma \in G \mid \sigma(Y) = Y\}.$$

Definition 4.2 Let A be a global action and let $Y \subseteqq |A|$. Let $\boldsymbol{Y}$ denote the set of all global actions (Φ, H, X) such that $\Phi = \Phi_A, X$ is the function $X : \Phi_A \to subsets(Y), \alpha \mapsto Y \cap (X_A)_\alpha$, and H is a generalized functor $H : \Phi_A \to ((groups)), \alpha \mapsto H_\alpha,$ such that $H_\alpha \subseteqq Stab_{(G_A)_\alpha}(Y \cap (X_A)_\alpha)$ and such if $\alpha \leqq \beta$ then the homomorphism $H_{\alpha \leqq \beta} : H_\alpha \to H_\beta$ is induced by the homomorphim $(G_A)_{\alpha \leqq \beta} : (G_A)_\alpha \to (G_A)_\beta$. Clearly $\boldsymbol{Y} \neq \varnothing$, since the constant function $H_c : \Phi_A \to ((groups)), \alpha \mapsto (H_C)_\alpha = \{1\}$ makes (Φ, H_c, X) a global action. Define the generalized functor $G : \Phi_A \to ((groups))$, such that $G_\alpha = \langle H_\alpha \mid (\Phi, H, X) \in \boldsymbol{Y} \rangle$ and $G_{\alpha \leqq \beta} : G_\alpha \to G_\beta$ is the restriction of $(G_A)_{\alpha \leqq \beta}$ to G_α. Then (Φ, G, X) is a global action on Y called the **standard subaction** on Y. Clearly if A is covariant, contravariant, or bivariant then so is (Φ, G, X).

Standard subactions satisfy the following universal property.

Lemma 4.3 Let $B \subseteqq A$ be a standard subaction. Then the triple $\eta = (\eta_{\Phi_B}, \eta_{G_B}, \eta_{X_B}) : B \to A$ is a regular morphism of global actions where $\eta_B : \Phi_B \to \Phi_A$ is the canonical identification of Φ_B with $\Phi_A, \eta_{G_B} : G_B \to G_A$ is defined by the natural inclusions $(G_B)_\beta \subseteqq (G_A)_{\eta_{\Phi_B}(\beta)}$ for each $\beta \in \Phi_B,$ and $\eta_{X_B} : |B| \to |A|$ is the natural inclusion $|B| \subseteqq |A|$. Thus B is a regular subaction of A. Moreover $\eta : B \to A$ has the following universal property: Let $\nu = (\nu_\Phi, \nu_G, \nu_X) : C \to A$ be a regular morphism such that image $(\nu_X) = |B|$. Suppose either $\nu_\Phi : \Phi_C \to \Phi_A$ is surjective on elements and relations (i.e. if $\nu_\Phi(\gamma) \leqq \nu_\Phi(\gamma')$ then $\gamma \leqq \gamma'$) or that for each $\gamma \in \Phi_C$ and sequence of relations $\nu_\Phi(\gamma) \leqq \alpha_1 \leqq \cdots \leqq \alpha_n$ in Φ_A, the group $((G_A)_{\alpha_{n-1} \leqq \alpha_n})((G_A)_{\alpha_{n-2} \leqq \alpha_{n-1}}) \cdots ((G_A)_{\alpha_1 \leqq \alpha_2})(\nu_G(\gamma)(G_C)_\gamma)$ leaves $|B| \cap (X_A)_{\alpha_n}$ invariant. Then there is a unique regular morphism $\tau : C \to B$, namely $\tau = (\nu_\Phi, \nu_G, \nu_X)$, such that the diagram

$$\begin{tikzcd} & C \arrow[dl, "\tau"'] \arrow[dr, "\nu"] & \\ B \arrow[rr, "\eta"'] & & A \end{tikzcd}$$

commutes as one of regular morphisms.

Proof Straightforward.

Remark Let A be a global action and let $Y' \subseteqq Y \subseteqq |A|$. Let B' and B denote the standard subactions on Y' and Y, respectively. Then the inclusion $Y' \subseteqq Y$ is not in general a morphism $B' \to B$ of global actions. In fact the inclusion $Y' \subseteqq Y$ is a morphism

$B' \to B$ of global actions $\Leftrightarrow$ for each $\alpha \in \Phi_A, (G_{B'})_\alpha \subseteqq (G_B)_\alpha$. In this case the morphism $B' \to B$ has an obvious extension to a regular morphism.

We want to develop below subactions called defining subactions which are more plentiful than regular subactions. The defining subactions are the most general kind of subactions one can use to put a global structure on morphism spaces of pairs of global actions. We prepare for the definition of defining subaction and the construction of the global action on morphism spaces of pairs of global actions consisting of a global action and a defining subaction.

Definition 4.4 A **pair** A of global actions is by definition an ordered pair $A = (A^{(1)}, A^{(2)})$ of global actions $A^{(1)}$ and $A^{(2)}$ such that $A^{(2)} \subseteqq A^{(1)}$. The pair is called **standard** if $A^{(2)}$ has the standard subaction. The pair is called **regular** if $A^{(2)}$ is a regular subaction of $A^{(1)}$. The pair will be called **defining** if $A^{(2)}$ is a defining subaction of $A^{(1)}$. The concept of a defining subaction is defined in (4.14). The pair is called **covariant, contravariant**, or **bivariant**, respectively, if $A^{(i)}(i = 1, 2)$ is covariant, contravariant, or bivariant, respectively.

Definition 4.5 A **pointed global action** is a standard pair $A = (A^{(1)}, A^{(2)})$ of global actions such that $A^{(2)}$ consists precisely of one point $*$ which is called the **base point** of A. It will be assumed that $* \in (X_{A^{(1)}})_\alpha$ for some $\alpha \in \Phi_{A^{(1)}}$.

Definition 4.6 A **morphism** $f : A \to B$ of pairs is a morphism $f : A^{(1)} \to B^{(1)}$ of global actions such that f takes $|A^{(2)}|$ to $|B^{(2)}|$ and is a morphism $A^{(2)} \to B^{(2)}$ of global actions. A **morphism** $f : A \to B$ of pairs is called **regular** if the morphisms $f : A^{(i)} \to B^{(i)}$ $(i = 1, 2)$ of global actions carry fixed regular structures. A morphism $\mathrm{f} : A \to B$ of pairs is said to **extend** to a regular morphism if the morphisms $\mathrm{f} : A^{(i)} \to B^{(i)}$ $(i = 1, 2)$ of global actions extend to regular morphisms.

Definition 4.7 Let A and B be pairs of global actions. A **relative A-chart**, or simply **A-chart**, in B is a pair (f, β) consisting of a morphism $\mathrm{f} : A \to B$ of pairs and a function $\beta : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}$ such that the following conditions are satisfied. Let

$$|A^{(1)\backslash(2)}| = |A^{(1)}| \backslash |A^{(2)}|.$$

If U is a subset of $A^{(1)}$, set

$$\begin{aligned} U^{(1)\backslash(2)} &= U \cap |A^{(1)\backslash(2)}| \\ U^{(i)} &= U \cap |A^{(i)}| \ (i = 1, 2). \end{aligned}$$

The conditions are as follows.

(4.7.0) β takes $|A^{(1)\backslash(2)}|$ to $\Phi_{B^{(1)}}$ and $|A^{(2)}|$ to $\Phi_{B^{(2)}}$.

(4.7.1) If $\mathrm{x} \in |A^{(1)\backslash(2)}|$ then $\mathrm{f}(\mathrm{x}) \in (X_{B^{(1)}})_{\beta(\mathrm{x})}$. If $\mathrm{x} \in |A^{(2)}|$ then $\mathrm{f}(\mathrm{x}) \in (X_{B^{(2)}})_{\beta(\mathrm{x})}$.

(4.7.2) If U is a local frame in $A^{(1)}$ then there is a $b^{(1)} \in \Phi_{B^{(1)}}$ such that $\mathrm{f}(U) \subseteqq (X_{B^{(1)}})_{b^{(1)}}, (G_{B^{(1)}})_{b^{(1)}}$ acts transitively on $\mathrm{f}(U), \beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)\backslash(2)}$, and $(G_{B^{(2)}})_{\beta(v)}\mathrm{f}(v) \subseteqq (G_{B^{(1)}})_{b^{(1)}}\mathrm{f}(v)$ for all $v \in U^{(2)}$.

(4.7.3) If U is a local frame in $A^{(2)}$ then there is a $b^{(2)} \in \Phi_{B^{(2)}}$ such that $\mathrm{f}(U)$ is a local frame at $b^{(2)}$ and $\beta(u) \leqq b^{(2)}$ for all $u \in U$.

There are additional conditions which one could impose on charts of pairs of global actions and under which one can still prove our main result on exponentiation. Two such conditions are given below.

Condition 4.8 Let A and B be pairs of global actions. Let (f, β) be an A-chart in B. Below are two conditions for (f, β).

(4.8.1) If U is a local frame in $A^{(1)}$ such that $U^{(2)}$ is a local frame in $A^{(2)}$ then there are coordinates $b^{(i)} \in \Phi_{B^{(i)}}(i = 1, 2)$ such that $\mathrm{f}(U) \subseteqq (X_{B^{(1)}})_{b^{(1)}}, (G_{B^{(1)}})_{b^{(1)}}$ acts transitively on $\mathrm{f}(U), \beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)\backslash(2)}, \mathrm{f}(U^{(2)}) \subseteqq (X_{B^{(2)}})_{b^{(2)}}, (G_{B^{(2)}})_{b^{(2)}}$ acts transitively on $\mathrm{f}(U^{(2)}), \beta(v) \leqq b^{(2)}$ for all $v \in U^{(2)}$, and $(G_{B^{(2)}})_{b^{(2)}}\mathrm{f}(v) \subseteqq (G_{B^{(1)}})_{b^{(1)}}\mathrm{f}(v)$ for all $v \in U^{(2)}$.

(4.8.2) If U is a local frame in $A^{(1)}$ then there is a coordinate $b^{(1)} \in \Phi_{B^{(1)}}$ such that $\mathrm{f}(U) \subseteqq (X_{B^{(1)}})_{b^{(1)}}, (G_{B^{(1)}})_{b^{(1)}}$ acts transitively on $\mathrm{f}(U), \beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)\backslash(2)}$ and the following holds: Given a local frame U_2 in $A^{(2)}$ such that $U_2 \subseteqq U$, there is a coordinate $b^{(2)} \in \Phi_{B^{(2)}}$ such that $\mathrm{f}(U_2) \subseteqq (X_{B^{(2)}})_{b^{(2)}}, (G_{B^{(2)}})_{b^{(2)}}$, acts transitively on $\mathrm{f}(U_2)$, $\beta(v) \leqq b^{(2)}$ for all $v \in U_2$, and $(G_{B^{(2)}})_{b^{(2)}}\mathrm{f}(v) \subseteqq (G_{B^{(1)}})_{b^{(1)}}\mathrm{f}(v)$ for any $v \in U_2$. (Thus (4.8.2) is a uniformization of (4.8.1)).

Definition-Lemma 4.9 Let A and B be pairs of global actions. Let (f, β) be an A-chart in B.

If

$$\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$$

define

$$\begin{aligned} \sigma\mathrm{f} : |A^{(1)}| &\to |B^{(1)}|. \\ \mathrm{x} &\mapsto \sigma_{\mathrm{x}}\mathrm{f}(\mathrm{x}) \end{aligned}$$

Then σf is a morphism $A \to B$ of pairs of global actions and $(\sigma\mathrm{f}, \beta)$ is an A-chart in B.

PROOF It is clear that $(\sigma\mathrm{f}, \beta)$ satisfies (4.7.0) and (4.7.1). To show that σf is a morphism of pairs and that $(\sigma\mathrm{f}, \beta)$ is an A-frame in B, it suffices to show that (4.7.2) is satisfied. Let U be a local frame in $U^{(1)}$. By definition $\mathrm{f}(U)$ is a $b^{(1)}$-frame for some $b^{(1)} \in \Phi_{B^{(1)}}$ such that $\beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)\backslash(2)}$ and $(G_{B^{(2)}})_{\beta(v)}\mathrm{f}(v) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}\mathrm{f}(v)$ for all $v \in U^{(2)}$. It follows that for any $\sigma, (\sigma\mathrm{f})(U)$ is a $b^{(1)}$-frame and obviously $\beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)\backslash(2)}$. It remains to show that $(G_{B^{(2)}})_{\beta(v)}(\sigma\mathrm{f})(v) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}(\sigma\mathrm{f})(v)$ for all $v \in U^{(2)}$. Since $(\sigma\mathrm{f})(v) = \sigma_v(\mathrm{f}(v)) \in (X_{B^{(2)}})_{\beta(v)}$ and $(G_{B^{(2)}})_{\beta(v)} = (G_{B^{(2)}})_{\beta(v)}\sigma_v^{-1}$, we obtain that $(G_{B^{(2)}})_{\beta(v)}(\sigma\mathrm{f})(v) = (G_{B^{(2)}})_{\beta(v)}(\sigma_v(\mathrm{f}(v))) = ((G_{B^{(2)}})_{\beta(v)}\sigma_v^{-1})(\sigma_v(\mathrm{f}(v))) = (G_{B^{(2)}})_{\beta(v)}(\mathrm{f}(v)) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}(\mathrm{f}(v)) = (G_{B^{(1)}})_{b^{(1)}}\sigma_v)(\mathrm{f}(v)) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}(\sigma_v(\mathrm{f}(v))) = (G_{B^{(1)}})_{b^{(1)}}((\sigma\mathrm{f})(v)).\square$

DEFINITION 4.10 Let A and B be pairs of global actions. Let (f, β) be an A-chart in B. A **relative *A*-frame**, or simply ***A*-frame**, **at** f **on** (f, β) is a set $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p : A \to B$ of morphisms for which there are elements $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x}\in|A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x}\in|A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$ such that $\sigma_i\mathrm{f} = \mathrm{f}_i$ $(1 \leqq i \leqq \mathrm{p})$. (In view of Lemma 4.9, $\mathrm{f} = \mathrm{f}_0, f_1, \cdots, \mathrm{f}_\mathrm{p}$ is also an A-frame at f_i on (f_i, β) for any i $(0 \leqq \mathrm{i} \leqq \mathrm{p})$.)

The next lemma will be very useful, just as the analogous lemma in §3.

LOCAL-GLOBAL LEMMA 4.11 Let A and B be pairs of global actions. Let (f, β) be an A-chart in B. Then $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is an A-frame at f on $(\mathrm{f}, \beta) \Leftrightarrow$ for each $\mathrm{x} \in |A^{(1)}|, \mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local frame at $\mathrm{f}(\mathrm{x})$ in $\beta(\mathrm{x})$.

PROOF The assertions are trivial consequences of Lemma 4.9.

DEFINITION 4.12 Let A, B and C be pairs of global actions. An A- **normal** morphism $\mathrm{g} : B \to C$ of pairs actions is one such that $g : B^{(2)} \to C^{(2)}$ preserves $A^{(1)}-$ frames and $g : B \to C$ preserves A-frames, i.e. if $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ is an A-frame at f on (f, β) then $\mathrm{gf}, \mathrm{gf}_1, \cdots, \mathrm{gf}_\mathrm{p}$ is an A-frame at gf on (gf, γ) for some A-chart (gf, γ) in C. A **normal** morphism $\mathrm{g} : B \to C$ of pairs is one which is A-normal for any pair A. An A-**normal** (resp. **normal**) **isomorphism** is an A-normal (resp. normal) morphism which has an A-normal (resp. normal) inverse.

It is not true in general that an A-normal (resp. normal) morphism which is an isomorphism in the usual sense is an A-normal (resp. normal) isomorphism.

LEMMA 4.13 A regular morphism of pairs of global actions is normal.

PROOF Let $\eta : B \to C$ be a regular morphism of pairs of global actions. If (f, β) is an A-chart in B then it follows straightforward that $(\eta_X\mathrm{f}, \eta_\Phi\beta)$ is an A-chart in C.

Let $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at f on (f, β) and let $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$ such that $\sigma_{\mathrm{i}}\mathrm{f} = \mathrm{f}_{\mathrm{i}}$ $(1 \leqq \mathrm{i} \leqq \mathrm{p})$. If $\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$, define $\eta_G(\sigma) = (\eta_G(\beta(\mathrm{x}))(\sigma_{\mathrm{x}})) \in \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{C^{(1)}})_{\eta_{\Phi^{(1)}}\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{C^{(2)}})_{\eta_{\Phi^{(2)}}\beta(\mathrm{x})}$ where

$$\eta_G(\beta(\mathrm{x})) = \begin{cases} \eta_{G^{(1)}}(\beta(\mathrm{x})) \text{ if } \mathrm{x} \in |A^{(1)\backslash(2)}| \\ \eta_{G^{(2)}}(\beta(\mathrm{x})) \text{ if } \mathrm{x} \in |A^{(2)}|. \end{cases}$$

Then $\eta_G(\sigma_i)(\eta_X \mathrm{f}) = \eta_X \mathrm{f}_{\mathrm{i}}$ $(1 \leqq i \leqq \mathrm{p})$, by (3.3.4). Thus $\eta_X\mathrm{f}, \eta_X\mathrm{f}_1, \cdots, \eta_X\mathrm{f}_p$ is an A-frame at $\eta_X\mathrm{f}$ on $(\eta_X\mathrm{f}, \eta_\Phi\beta)$. $\square$

Next the set $Mor(A, B)$ of all morphisms from a pair A of global actions to a pair B of global actions is given the structure of a pair of global actions. This construction contains the definition of a defining subaction of a global action and of a defining pair of global actions.

Definition 4.14 Let A and B be a pair of global actions. Define a pair of global actions

$$Mor(A, B) = (Mor(A, B)^{(1)}, Mor(A, B)^{(2)})$$

as follows. $Mor(A, B)^{(2)} = Mor(A^{(1)}, B^{(2)})$. $Mor(A, B)^{(1)}$ is the global action whose enveloping set is $|Mor(A, B)|$ and whose global structure

$$(\Phi_{(A,B)^{(1)}}, G_{(A,B)^{(1)}}, X_{(A,B)^{(1)}})$$

is defined as follows. Define

$$\Phi_{(A,B)^{(1)}} = \{\beta : |A| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}} | (4.7.0) \text{ satisfied}\}.$$

Give $\Phi_{(A,B)^{(1)}}$ the reflexive relation defined by $\beta \leqq \beta' \Leftrightarrow \beta(\mathrm{x}) \leqq \beta'(\mathrm{x})\ \forall \mathrm{x} \in |A^{(1)}|$. For $\beta \in \Phi_{(A,B)^{(1)}}$, define

$$(G_{(A,B)^{(1)}})_\beta = . \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}.$$

If $\beta \leqq \beta'$, there is for each $\mathrm{x} \in |A^{(1)\setminus(2)}|$ a canonically defined homomorphism $(G_{B^{(1)}})_{\beta(\mathrm{x})} \to (G_{B^{(1)}})_{\beta'(\mathrm{x})}$ and for each $\mathrm{x} \in |A^{(2)}|$ a canonically defined homomorphism $(G_{B^{(2)}})_{\beta(\mathrm{x})} \to (G_{B^{(2)}})_{\beta'(x)}$ and therefore a canonically defined homomorphism $(G_{(A,B)})_\beta \longrightarrow (G_{(A,B)})_{\beta'}$. For $\beta \in \Phi_{(A,B)^{(1)}}$, define

$$(X_{(A,B)^{(1)}})_\beta = \{\mathrm{f} : |A^{(1)}| \to |B^{(1)}| \mid (\mathrm{f}, \beta)\ A-\text{chart in } B\}.$$

By (4.9), if $\sigma \in (G_{(A,B)^{(1)}})_\beta$ and $\mathrm{f} \in (X_{(A,B)^{(1)}})_\beta$ then $\sigma\mathrm{f} \in (X_{(A,B)^{(1)}})_\beta$ and so there is an action of $(G_{(A,B)^{(1)}})_\beta$ on $(X_{(A,B)^{(1)}})_\beta$. One sees easily that $Mor(A,B)^{(1)}$ is a global action. If the inclusion morphism $B^{(2)} \subseteqq B^{(1)}$ induces a morphism $Mor(A,B)^{(2)} \to Mor(A,B)^{(1)}$ of global actions then $Mor(A,B)^{(2)}$ is a subaction of $Mor(A,B)^{(1)}$ because the map $|Mor(A,B)^{(2)}| \to |Mor(A,B)^{(1)}|$ is obviously an inclusion of sets. In this case $Mor(A,B) = (Mor(A,B)^{(1)}, Mor(A,B)^{(2)})$ is a pair of global actions. We define now the subaction $B^{(2)}$ of $B^{(1)}$ to be an ***A*-defining subaction** if the canonical inclusion $B^{(2)} \subseteqq B^{(1)}$ induces a morphism $Mor(A,B)^{(2)} \to Mor(A,B)^{(1)}$ of global actions. We say that B is an ***A*-defining pair** of global actions if $B^{(2)}$ is an ***A*-defining subaction** of $B^{(1)}$. Clearly B is an A-defining pair $\Leftrightarrow Mor(A,B)^{(2)}$ is a subaction of $Mor(A,B)^{(1)} \Leftrightarrow Mor(A,B) = (Mor(A,B)^{(1)}, Mor(A,B)^{(2)})$ is a pair of global actions. Let $\mathcal{P}$ be a class of global actions. We say that $B^{(2)}$ is a **$\boldsymbol{\mathcal{P}}$-defining subaction** of $B^{(1)}$ and that B is a **$\boldsymbol{\mathcal{P}}$-defining pair** of global actions if B is an ***A*-defining pair** for all $A \in \mathcal{P}$. Let N be a natural number. We say that $B^{(2)}$ is an ***N*-$\boldsymbol{\mathcal{P}}$-defining subaction** of $B^{(1)}$ and that B is an ***N*-$\boldsymbol{\mathcal{P}}$-defining pair** of global actions if B is a **$\boldsymbol{\mathcal{P}}$-defining pair** and if for all natural numbers $n < N$ and all sequences $A_1, \cdots, A_n$ of pairs in $\mathcal{P}$, the pair $Mor(A_n, \cdots, Mor(A_1, B)) \cdots)$ is defined and is an A-defining pair for all $A \in \mathcal{P}$. We say that $B^{(2)}$ is an **$\boldsymbol{\infty}$-$\boldsymbol{\mathcal{P}}$-defining subaction** of $B^{(1)}$ and that B is an **$\boldsymbol{\infty}$-$\boldsymbol{\mathcal{P}}$-defining pair** if B is an N-$\mathcal{P}$-defining pair for all natural numbers N. We say that $B^{(2)}$ is an **$\boldsymbol{\infty}$-*A*-defining subaction** of $B^{(1)}$ and that B is an **$\boldsymbol{\infty}$-*A*-defining pair** of global actions if B is an ∞-$\{A\}$-defining pair.

If B is an A-defining pair of global actions then clearly the pair $Mor(A,B)$ is covariant, contravariant, or bivariant, respectively, wherever the pair B is so.

Remark Let B be a pair of global actions. If $B^{(1)}$ satisfies the infimum condition (3.20) then by Theorem 3.23, B is an A-defining pair for any pair A of global actions.

Lemma 4.15 If B is a regular pair of global actions then B is ∞-$\mathcal{P}$-defining where $\mathcal{P}$ is the class of all pairs of global actions. Furthermore if $A_1, \cdots, A_n$ is any sequence of pairs of global actions then $Mor(A_n, (\cdots, Mor(A_1, B)) \cdots)$ is a regular pair.

Proof By a simple induction argument on n, one reduces to showing that a regular structure on the canonical morphism $B^{(2)} \to B^{(1)}$ induces a regular morphism $Mor(A,B)^{(2)} \to$

$Mor(A,B)^{(1)}$.

Let $\eta = (\eta_{\Phi^{(2)}}, \eta_{G_{B^{(2)}}}, \eta_{X_{B^{(2)}}})$ be a regular structure on $B^{(2)} \to B^{(1)}$. Define a regular morphism

$$\nu = (\nu_{\Phi_{(A,B)^{(2)}}}, \nu_{G_{(A,B)^{(2)}}}, \nu_{X_{(A,B)^{(2)}}}) : Mor(A,B)^{(2)} \to Mor(A,B)^{(1)}$$

as follows. Let

$$\nu_{X_{(A,B)^{(2)}}} : |Mor(A,B)^{(2)}| \to |Mor(A,B)^{(1)}|$$

be the inclusion of sets defined by the inclusion $|B^{(2)}| \subsetneqq |B^{(1)}|$ of sets. Define the morphism

$$\nu_{\Phi_{(A,B)^{(2)}}} : \Phi_{(A,B)^{(2)}} \to \Phi_{(A,B)^{(1)}}$$

of relations by the rule

$$(\nu_{\Phi_{(A,B)^{(2)}}}\beta)(\mathrm{x}) = \begin{cases} \eta_{\Phi^{(2)}}(\beta(\mathrm{x})) & \text{if } \mathrm{x} \in |A^{(1)\setminus(2)}| \\ \beta(\mathrm{x}) & \text{if } \mathrm{x} \in |A^{(2)}|. \end{cases}$$

Define the natural transformation

$$\nu_{G_{(A,B)^{(2)}}} : G_{(A,B)^{(2)}} \to G_{(A,B)^{(1)}}$$

by the group homomorphisms

$$\begin{array}{ccc} \nu_{G_{(A,B)^{(2)}}}(\beta) : (G_{(A,B)^{(2)}})_\beta & \longrightarrow & (G_{(A,B)^{(1)}})_{\nu_{\Phi_{(A,B)^{(2)}}}(\beta)} \\ \| & & \| \\ \prod\limits_{\mathrm{x}\in|A^{(1)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})} & & \prod\limits_{\mathrm{x}\in|A^{(1)\setminus(2)}|} (G_{B^{(1)}})_{\nu_{\Phi^{(2)}}(\beta(\mathrm{x}))} \times \prod\limits_{\mathrm{x}\in|A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})} \end{array}$$

where

$$\nu_{G_{(A,B)^{(2)}}}(\beta)|_{(G_{B^{(2)}})_{\beta(\mathrm{x})}} = \begin{cases} \eta_{G^{(2)}}(\beta(\mathrm{x})) : (G_{B^{(2)}})_{\beta(\mathrm{x})} \to (G_{B^{(1)}})_{\nu_{\Phi^{(2)}}(\beta(\mathrm{x}))} & \text{if } \mathrm{x} \in |A^{(1)\setminus(2)}| \\ \mathit{identity} \quad : (G_{B^{(2)}})_{\beta(\mathrm{x})} \to (G_{B^{(2)}})_{\beta(\mathrm{x})} & \text{if } \mathrm{x} \in |A^{(2)}|. \end{cases}$$

One checks routinely that ν is a regular morphism. $\square$

PROPOSITION 4.16 As a functor taking values in pairs of global actions, $Mor(,)$ is contravariant and regular over all morphisms in the first variable, under the condition that only regular pairs are allowed in the second variable, and covariant over all normal morphisms in the second variable under, of course, the condition that only $\mathcal{P}$-defining pairs are allowed in the second variable, where $\mathcal{P}$ is the class of all pairs. More precisely the following holds.

(4.16.1) Let C be a regular pair and let $\mathrm{f} : A \to B$ be a morphism of pairs. Then f defines a regular morphism

$$\eta = Mor(\mathrm{f}, 1_C) : Mor(B, C) \to Mor(A, C)$$

of pairs as follows. Let $(\nu_\Phi, \nu_G, \nu_X) : C^{(2)} \to C^{(1)}$ be a regular structure for the inclusion map $\nu_X : |C^{(2)}| \subseteqq |C^{(1)}|$. Define the relation preserving morphism

$$\begin{aligned} \eta_{\Phi^{(1)}} : \Phi_{(B,C)^{(1)}} &\to \Phi_{(A,C)^{(1)}}. \\ \beta &\mapsto \beta \circ \mathrm{f} \end{aligned}$$

where $\beta \circ \mathrm{f}$ is defined by

$$\beta \circ \mathrm{f}(\mathrm{x}) = \begin{cases} \beta(\mathrm{f}(\mathrm{x})) & \text{if } \mathrm{x} \in |A^{(1)\setminus(2)}| \text{ and } \mathrm{f}(\mathrm{x}) \notin B^{(2)} \\ \nu_\Phi(\beta(\mathrm{f}(\mathrm{x}))) & \text{if } \mathrm{x} \in |A^{(1)\setminus(2)}| \text{ and } \mathrm{f}(\mathrm{x}) \in B^{(2)} \\ \beta(\mathrm{f}(\mathrm{x})) & \text{if } \mathrm{x} \in |A^{(2)}|. \end{cases}$$

Define the natural transformation

$$\eta_{G^{(1)}} : G_{(B,C)^{(1)}} \to G_{(A,C)^{(1)}}$$

such that for each $\beta \in \Phi_{(B,C)^{(1)}}$, the group homomorphism

$$\begin{array}{ccc} \eta_{G^{(1)}}(\beta) : (G_{(B,C)^{(1)}})_\beta & \longrightarrow & (G_{(A,C)^{(1)}})_{\eta_{\Phi^{(1)}}(\beta)} \\ \| & & \| \\ \prod_{y \in |B^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\beta(y)} \times \prod_{y \in |B^{(2)}|} (G_{C^{(2)}})_{\beta(y)} & & \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\beta\circ\mathrm{f}(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{C^{(2)}})_{\beta\circ\mathrm{f}(\mathrm{x})} \end{array}$$

is defined by the property that for $y \in |B^{(1)\setminus(2)}|$, $\eta_{G^{(1)}}(\beta)|_{(G_{C^{(1)}})_{\beta(y)}}$ is the diagonal homomorphism

$$\prod_{\mathrm{x}\in|A^{(1)\setminus(2)}|,\mathrm{f(x)}=y} (\text{identity map}) : (G_{C^{(1)}})_{\beta(y)} \longrightarrow \prod_{\mathrm{x}\in|A^{(1)\setminus(2)}|,\mathrm{f(x)}=y} (G_{C^{(1)}})_{\beta\mathrm{of(x)}},$$

under the convention that the empty product of groups, which can occur on the right hand side of the arrow above, is the trivial group; and for $y \in |B^{(2)}|, \eta_{G^{(1)}}(\beta)|_{(G_{C^{(2)}})_{\beta(y)}}$ is the homomorphism

$$\prod_{\mathrm{x}\in|A^{(1)\setminus(2)}|,\mathrm{f(x)}=y} \times \prod_{\mathrm{x}\in|A^{(2)}|,\mathrm{f(x)}=y} (\text{identity map}) : (G_{C^{(2)}})_{\beta(y)} \rightarrow \prod_{\mathrm{x}\in|A^{(1)\setminus(2)}|,\mathrm{f(x)}=y} (G_{C^{(1)}})_{\beta\mathrm{of(x)}} \times \prod_{\mathrm{x}\in|A^{(2)}|,\mathrm{f(x)}=y} (G_{C^{(2)}})_{\beta\mathrm{of(x)}}.$$

One checks straightforward that

$$\eta^{(1)} = (\eta_{\Phi^{(1)}}, \eta_{G^{(1)}}, \eta_X)$$

is a regular morphism

$$\eta^{(1)} : Mor(B, C)^{(1)} \rightarrow Mor(A, C)^{(1)}$$

of global actions and by (3.11.1), $\eta_X|_{Mor(B,C)^{(2)}}$ extends to a regular morphism

$$\eta^{(2)} : Mor(B, C)^{(2)} \rightarrow Mor(A, C)^{(2)}$$

of global actions. One checks routinely then that

$$\eta = (\eta^{(1)}, \eta^{(2)})$$

is a regular morphism

$$\eta : Mor(B, C) \rightarrow Mor(A, C)$$

of pairs.

(4.16.2) Let A be a pair and let $\mathrm{g} : B \rightarrow C$ be a morphism of pairs. Then the function

$$Mor(1_A, \mathrm{g}) : |Mor(A, B)^{(1)}| \rightarrow |Mor(A, C)^{(1)}|$$

is a morphism $Mor(A,B) \to Mor(A,C)$ of pairs $\Leftrightarrow$ g is A-normal.

PROOF (4.16.1) Nothing has been left to prove.

(4.16.2) Let (f,β) be an A-chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ be an A-frame on (f,β). Let (f',β') be an $A^{(1)}$-chart in $B^{(2)}$ and let $\mathrm{f}' = \mathrm{f}'_0, \mathrm{f}'_1, \cdots, \mathrm{f}'_\mathrm{p}$, be an $A^{(1)}$-frame on (f',β'). By definition of the term local frame, $\mathrm{f}_0, \cdots, \mathrm{f}_\mathrm{p}$ is also a local β-frame in the global action $Mor(A,B)^{(1)}$ and conversely, any local frame in $Mor(A,B)^{(1)}$ is an A-frame on some A-chart in B. Similarly $\mathrm{f}'_0, \cdots, f'_\mathrm{p}$, is a local β'-frame in the global action $Mor(A,B)^{(2)}$ and conversely, any local frame in $Mor(A,B)^{(2)}$ is an $A^{(1)}$-frame on some $A^{(1)}$-chart in $B^{(2)}$. Thus the function $Mor(1_A,\mathrm{g}) : |Mor(A,B)^{(1)}| \to |Mor(A,C)^{(1)}|$ defines a morphism $Mor(A,B) \to Mor(A,C)$ of pairs $\Leftrightarrow$ $Mor(1_A,g)$ preserves A-frames and $Mor(1_A,g)^{(2)} : |Mor(A,B)^{(2)}| \to |Mor(A,C)^{(2)}|$ preserves $A^{(1)}$-frames $\Leftrightarrow$ g is A-normal. $\square$

DEFINITION 4.17 Let $g : B \to C$ be a morphism of ∞-((pairs))-defining pairs. A sequence $A_n, \cdots, A_1$ of pairs is called a **normal chain** of length n for g if g is A_1-normal and if for each i $(1 \leqq i \leqq n-1)$, the morphism $Mor(1_{A_{i-1}}, \cdots, Mor(1_{A_1}, g)) \cdots) :$ $Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, B)) \cdots) \to Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, C)) \cdots)$ is A_{i+1}-normal. Let $\mathcal{N}$ be a class of pairs. The morphism g is called ***n*-$\boldsymbol{\mathcal{N}}$-normal** if every sequence of n objects from $\mathcal{N}$ forms a normal chain for g. The morphism g is called **$\boldsymbol{\mathcal{N}}$-normal** (resp. **$\boldsymbol{\infty}$- $\boldsymbol{\mathcal{N}}$ -normal**) if it is 1-$\mathcal{N}$-normal (resp. n-$\mathcal{N}$-normal for all $n > 0$). If $\mathcal{N} = \{A\}$ (resp. $\mathcal{N}$ = all pairs of global actions), we shall write **$\boldsymbol{\infty}$-*A*-normal** (resp. **$\boldsymbol{\infty}$-normal**) in place of ∞-$\mathcal{N}$-normal.

If the expression t-morphism denotes anyone of the notions of normality above or the notion of regularity then a t-isomorphism is a t-morphism which has a t-morphism as its inverse.

LEMMA 4.18 If g : $B \to C$ is a regular morphism of ((pairs))-defining pairs then for any pair A, the morphism $Mor(1_A,\mathrm{g}) : Mor(A,B) \to Mor(A,C)$ is regular. Thus g is ∞-normal.

PROOF By (4.13)and (4.16.2), the morphism $Mor(1_A,\mathrm{g}) : Mor(A,B) \to Mor(A,C)$ exists. Let $(\eta_\Phi, \eta_G, \eta_X = \mathrm{g})$ be the regular structure of g. We define a regular structure $(\mu_\Phi, \mu_g, \mu_X = Mor(1_A,\mathrm{g}))$ for $Mor(1_A,\mathrm{g})$ as follows.

Define the coordinate morphisms

$$\begin{aligned}\mu_{\Phi^{(1)}} : \Phi_{(A,B)^{(1)}} &\to \Phi_{(A,C)^{(1)}},\\ \beta &\mapsto \eta_{\Phi^{(1)}}\beta\\ \mu_{\Phi^{(2)}} : \Phi_{(A,B)^{(2)}} &\to \Phi_{(A,C)^{(2)}}.\\ \beta &\mapsto \eta_{\Phi^{(2)}}\beta\end{aligned}$$

Define the natural transformation

$$\mu_{G^{(1)}} : G_{(A,B)^{(1)}} \to G_{(A,C)^{(1)}}$$

such that for each $\beta \in \Phi_{(A,B)^{(1)}}$, the group homomorphism

$$\mu_{G^{(1)}}(\beta) : (G_{(A,B)^{(1)}})_\beta \to (G_{(A,C)^{(1)}})_{\mu_{\Phi^{(1)}}(\beta)}$$

is defined by the commutative diagram

$$\begin{tikzcd}
(G_{(A,B)^{(1)}})_\beta \arrow[r, "\mu_{G^{(1)}}(\beta)"] \arrow[d, equal] & (G_{(A,C)^{(1)}})_{\mu_{\Phi^{(1)}}(\beta)} \arrow[d, equal] \\
\prod\limits_{\mathrm{x}\in|A^{(1)\setminus(2)}|}(G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod\limits_{\mathrm{x}\in|A^{(2)}|}(G_{B^{(2)}})_{\beta(\mathrm{x})} \arrow[r, "\tau"'] & \prod\limits_{\mathrm{x}\in|A^{(1)\setminus(2)}|}(G_{C^{(1)}})_{\eta_{\Phi^{(1)}}(\beta(\mathrm{x}))} \times \prod\limits_{\mathrm{x}\in|A^{(2)}|}(G_{C^{(2)}})_{\eta_{\Phi^{(2)}}}(\beta(\mathrm{x}))
\end{tikzcd}$$

where

$$\tau = \prod_{\mathrm{x}\in|A^{(1)\setminus(2)}|}\eta_{G^{(1)}}(\beta(\mathrm{x})) \times \prod_{\mathrm{x}\in|A^{(2)}|}\eta_{G^{(1)}}(\beta(\mathrm{x})).$$

Define the natural transformation

$$\mu_{G^{(2)}} : G_{(A,B)^{(2)}} \to G_{(A,C)^{(2)}}$$

as in the proof of (3.13).

One checks straightforward that $(\mu_\Phi, \mu_G, Mor(1_A, \mathrm{g}))$ is a regular morphism.

That g is ∞-normal follows by a trivial induction argument from the result just proved. $\square$

DEFINITION 4.19 Let N denote the name of a kind of morphism defined in (4.17). A pair is called an ***N* pair** if it has the property that every morphism to it is an N morphism.

The following definition is needed for the exponential law.

DEFINITION 4.20 Let A and B be pairs of global actions. Define the pair

$$A \bowtie B$$

as follows.

$$(A \bowtie B)^{(1)} = A^{(1)} \times B^{(1)}$$

.

The enveloping set of $(A \bowtie B)^{(2)}$ is defined by

$$|(A \bowtie B)^{(2)}| = |A^{(1)}| \times |B^{(2)}| \cup |A^{(2)}| \times |B^{(1)}|$$

where the union is taken in $|A^{(1)}| \times |B^{(1)}|$. The coordinate system of $(A \bowtie B)^{(2)}$ is defined by

$$\Phi_{(A \bowtie B)^{(2)}} = \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \cup \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}$$

where the union is now disjoint. Relations (arrows) between coordinates are defined coordinatewise. The group function of $(A \bowtie B)^{(2)}$ is defined by

$$(G_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)} = \begin{cases} (G_{A^{(1)}})_\alpha \times (G_{B^{(2)}})_\beta \text{ if } (\alpha,\beta) \in \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \\ (G_{A^{(2)}})_\alpha \times (G_{B^{(1)}})_\beta \text{ if } (\alpha,\beta) \in \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}. \end{cases}$$

The set function for $(A \bowtie B)^{(2)}$ is defined by

$$(X_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)} = \begin{cases} (X_{A^{(1)}})_\alpha \times (X_{B^{(2)}})_\beta \text{ if } (\alpha,\beta) \in \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \\ (X_{A^{(2)}})_\alpha \times (X_{B^{(1)}})_\beta \text{ if } (\alpha,\beta) \in \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}. \end{cases}$$

The action of $(G_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)}$ on $(X_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)}$ is coordinatewise.

One checks routinely that $A \bowtie B$ satisfies the universal property of a product.

In the next definition, the notation $(S,T) = Mor_{((sets))}(S,T)$, where S and T are sets, will be used. This notation was introduced already in (3.16).

DEFINITION 4.21 Let A, B and C be pairs such that C is $2-\{A, B, A \bowtie B\}$-defining. We define a regular morphism

$$E : Mor(A, Mor(B,C)) \to Mor(A \bowtie B, C)$$

as follows. Denote the structural components of the pair $Mor(A, Mor(B,C))$ by $(\Phi_{(A,(B,C))}, G_{(A,(B,C))}, X_{(A,(B,C))})$. Define $E_{\Phi^{(1)}}$ such that the diagram

$$\begin{array}{ccc}
\Phi_{(A,(B,C))^{(1)}} & \xrightarrow{E_{\Phi^{(1)}}} & \Phi_{(A\bowtie B,C)^{(1)}} \\
\| \cap & & \| \cap \\
(|A^{(1)}|, (|B^{(1)}|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}})) & \xrightarrow[(3.16)]{} & (|A^{(1)}| \times |B^{(1)}|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}})
\end{array}$$

commutes. Clearly $E_{\Phi^{(1)}}$ is relation preserving. Define $E_{\Phi^{(2)}}$ as in (3.17), i.e. such that the diagram

$$\begin{array}{ccc}
\Phi_{(A,(B,C))^{(2)}} & \xrightarrow{E_{\Phi^{(2)}}} & \Phi_{(A\bowtie B,C)^{(2)}} \\
\| & & \| \\
(|A^{(1)}|, (|B^{(1)}|, \Phi_{C^{(2)}})) & \xrightarrow[(3.16)]{} & (|A^{(1)}| \times |B^{(1)}|, \Phi_{C^{(2)}})
\end{array}$$

commutes.

Define the natural transformation

$$E_{G^{(1)}} : G_{(A,(B,C))^{(1)}} \to (G_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\ )}$$

such that for each $\alpha \in \Phi_{(A,(B,C))^{(1)}}$, the group homomorphism

$$E_{G^{(1)}}(\alpha) : (G_{(A,(B,C))^{(1)}})_\alpha \longrightarrow (G_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\alpha)}$$

maps the factor of $(G_{(A,(B,C))^{(1)}})_\alpha = \prod_{x\in|A^{(1)\setminus(2)}|}[(\prod_{y\in|B^{(1)\setminus(2)}|}(G_{C^{(1)}})_{\alpha(x)(y)})\times(\prod_{y\in|B^{(2)}|}(G_{C^{(2)}})_{\alpha(x)(y)})]$ $\times \prod_{x\in|A^{(2)}|}[\prod_{y\in|B^{(1)}|}(G_{C^{(2)}})_{\alpha(x)(y)}]$ with the subscript $\alpha(x)(y)$ via the identity map onto the factor of $(G_{(A\bowtie B,C)^{(1)}})_\alpha = \prod_{(x,y)\in|A^{(1)\setminus(2)}|\times|B^{(1)\setminus(2)}|}(G_{C^{(1)}})_{(E_{\Phi^{(1)}}\alpha)(x,y)}\times\prod_{(x,y)\in|(A\bowtie B)^{(2)}|}(G_{C^{(2)}})_{(E_{\Phi^{(1)}}\alpha)(x,y)}$ with the subscript $(E_{\Phi^{(1)}}\alpha)(x,y)$. Define the natural transformation

$$E_{G^{(2)}} : G_{(A,(B,C))^{(2)}} \to G_{(A\bowtie B,C)^{(2)}}$$

as in (3.17), i.e. such that

$$E_{G^{(2)}}(\alpha) : (G_{(A,(B,C))^{(2)}})_\alpha \to (G_{(A\bowtie B,C)^{(2)}})_{E_{\Phi^{(2)}}(\alpha)}$$

maps the factor of $(G_{(A,(B,C))^{(2)}})_\alpha = \prod_{x\in|A^{(1)}|}(\prod_{y\in|B^{(1)}|}(G_{C^{(2)}})_{\alpha(x)(y)})$ with the subscript $\alpha(x)(y)$ via the identity map onto the factor of $(G_{(A\bowtie B,C)^{(2)}})_{E_{\Phi^{(2)}}(\alpha)} = \prod_{(x,y)\in|A^{(1)}|\times|B^{(1)}|}$ $(G_{C^{(2)}})_{(E_{\Phi^{(2)}}\alpha)(x,y)}$ with the subscript $(E_{\Phi^{(2)}}\alpha)(x,y)$. One verifies routinely that the composite mapping $|Mor(A, Mor(B,C))^{(1)}| \to (|A^{(1)}|,(|B^{(1)}|,|C^{(1)}|))$ @ > (3.16) >> $(|A^{(1)}| \times |B^{(1)}|, |C^{(1)}|)$ takes its image in $|Mor(A\bowtie B,C)^{(1)}|$ and we define

$$E_X : |Mor(A, Mor(B,C))^{(1)}| \to |Mor(A\bowtie B,C)^{(1)}|$$

to be the resulting mapping. One checks straightforward that

$$E^{(1)} = (E_{\Phi^{(1)}}, E_{G^{(1)}}, E_X)$$

is a regular morphism

$$E^{(1)} : Mor(A, Mor(B,C))^{(1)} \to Mor(A\bowtie B,C)^{(1)}$$

of global actions. (It fails in general to be an isomorphism (resp. regular isomorphism) because E_X is not necessarily surjective (resp. $E_X((X_{(A,(B,C))^{(1)}})_\alpha)$ is not necessarily all of $(X_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\alpha)}$). Let

$$E_{X^{(2)}} = E_X|_{|Mor(A,(B,C))^{(2)}|}$$

and

$$E^{(2)} = (E_{\Phi^{(2)}}, E_{G^{(2)}}, E_{X^{(2)}}).$$

Then by (3.17),

$$\begin{tikzcd}
E^{(2)} : Mor(A, Mor(B,C))^{(2)} \arrow[r] \arrow[d, equal] & Mor(A \bowtie B, C)^{(2)} \arrow[d, equal] \\
Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}) \arrow[r] & Mor(A^{(1)} \times B^{(1)}, C^{(2)})
\end{tikzcd}$$

is a regular morphism of global actions. One checks straightforward that

$$E = (E^{(1)}, E^{(2)})$$

is a regular morphism

$$E : Mor(A, Mor(B,C)) \to Mor(A \bowtie B, C)$$

of pairs.

Suppose now that C is ∞-((pair))-defining. Let $A_n, \cdots, A_1$ be an arbitrary sequence of pairs. Iterating the procedure above, one defines for any $n \geq 2$ a regular morphism

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C)) \cdots) \to Mor(A_n \bowtie \cdots \bowtie A_1, C)$$

as follows. For $n = 2$, the morphism is defined above. Suppose $n > 2$ and that the morphism has been defined for every natural number N where $2 \leqq N \leqq n-1$. Let E_{n-1} denote the morphism for the sequence $A_{n-1}, \cdots, A_1$. Define E_n for the sequence $A_n, A_{n-1}, \cdots, A_1$ as the composite of the regular morphism $Mor(1_{A_n}, E_{n-1})$ (see (4.18)) and the regular morphism $E_2 : Mor(A_n, Mor(A_{n-1} \bowtie \cdots \bowtie A_1, B)) \to Mor(A_n \bowtie \cdots \bowtie A_1, B)$.

The next definition is made to cope with the problem of finding an inverse to the morphism E_n above.

Definition 4.22 Let $\mathcal{P}$ be a class of pairs closed under finite operations by $\bowtie$. An ∞-P-defining pair C is called **∞-$\mathcal{P}$-exponential** if the morphism $E : Mor(A, Mor(B,C)) \to Mor(A \bowtie B, C)$ is an ∞- $\mathcal{P}$-normal isomorphism for all pairs $A, B \in \mathcal{P}$. C is called **regularly ∞- $\mathcal{P}$-exponential** if E is a regular isomorphism for all pairs $A, B \in$

$\mathcal{P}$. If $\mathcal{P}$ = all finite $\bowtie$ −products of A (resp. $\mathcal{P}$ = all pairs A such that $|A^{(i)}| = \cup_{\alpha\in\Phi_{A^{(i)}}}(X_{A^{(i)}})_\alpha$ $(i = 1,2)$) then C is called **∞-*A*-exponential** (resp. **∞-exponential**) if it is ∞- $\mathcal{P}$-exponential.

LEMMA 4.23 Suppose the pair C is ∞- $\mathcal{P}$-exponential (resp. regularly ∞-$\mathcal{P}$-exponential). Then for any sequence $A_n, \cdots, A_1 \in \mathcal{P}$ such that $n \geq 2$, the morphism E_n in (4.21) is an ∞-$\mathcal{P}$-normal (resp. regular) isomorphism.

PROOF The proof is exactly the same as that of (3.19).

DEFINITON 4.24 Let A be a pair of global actions. Let $U_i \subseteqq |A^{(i)}|$ $(i = 1,2)$ be local frames in $A^{(i)}$, respectively. If $U_2 \subseteqq U_1$ then (U_1, U_2) is called a **pair of local frames** in A. A pair (U_1, U_2) of local frames is called a **neat pair** if there are coordinates $a^{(i)} \in \Phi_{A^{(i)}}$ such that U_i is a local frame at $A^{(i)}$ and $(G_{A^{(2)}})_{a^{(2)}}u \subseteqq (G_{A^{(1)}})_{a^{(1)}}u$ for any $u \in U_2$. In this case, we say that (U_1, U_2) is a **neat pair at $(a^{(1)}, a^{(2)})$**. In order to define the concept of a strong neat pair of local frames, we need some additional notation. Suppose $\Delta^{(2)} \subseteqq \Phi_{A^{(2)}}$ is a finite possibly empty subset and $U_2 \subseteqq |A^{(2)}|$ is a finite subset such that for each $d \in \Delta^{(2)}$, $(X_{A^{(2)}})_d \cap U_2 \neq \varnothing$. Then by

$$(X_{A^{(2)}})_d \sqcap U_2$$

we shall mean a fixed nonempty subset of $(X_{A^{(2)}})_d \cap U_2$. At a certain point in the definition below of a strong neat pair of local frames, we shall use the set$(X_{A^{(2)}})_d \sqcap U_2$ in place of the set $(X_{A^{(2)}})_d \cap U_2$ in order to be able to carry out later (e.g. in the proofs of (4.31) and (4.32)) specialization arguments which replace functions f, α etc. by their values f(x), α(x) etc. at a fixed element x in their domains. Define

$$\begin{aligned}\Phi_{A^{(2)}}^{\Delta^{(2)}\sqcap U_2} = &\{a \in \Phi_{A^{(2)}} \mid (X_{A^{(2)}})_d \sqcap U_2 \subseteqq (X_{A^{(2)}})_a \\ &\forall\ d \in \Delta^{(2)}, (G_{B^{(2)}})_d(u) \subseteqq (G_{B^{(2)}})_a(u) \\ &\forall\ d \in \Delta^{(2)} \text{ and } \forall\ u \in (X_{A^{(2)}})_d \sqcap U_2\}.\end{aligned}$$

The set $\Phi_{A^{(2)}}^{\Delta^{(2)}\sqcap U_2}$ will replace the closely related set$\{a \in \Phi_{A^{(2)}}^{\geqq\Delta^{(2)}} \mid (X_{A^{(2)}})_d \sqcap U_2 \subseteqq (X_{A^{(2)}})_a\ \forall\ d \in \Delta^{(2)}\}$, in the specialization arguments mentioned above.

A pair (U_1, U_2) of local frames is called a **strong neat pair** if given finite possibly empty subsets $\Delta^{(i)} \subseteqq \Phi_{A^{(i)}}$ $(i = 1,2)$ such that for each $d \in \Delta^{(i)}$, $(X_{A^{(i)}})_d \cap U_i \neq \varnothing$, the following condition is satisfied. If the sets $\Psi^{(1)} = \{\alpha^{(1)} \in \Phi_{A^{(1)}}^{\geqq\Delta^{(1)}} | U_1$ is a local frame at $\alpha^{(1)}, (G_{A^{(2)}})_d$

$(u) \subseteqq (G_{A^{(1)}})_{a^{(1)}}(u) \quad \forall\, d \in \Delta^{(2)}$ and $\forall\, u \in (X_{A^{(2)}})_d \sqcap U_2\}$ and $\Psi^{(1)} = \{a^{(2)} \in \Phi_{A^{(2)}}^{\Delta^{(2)} \sqcap U_2} \mid U_2$ is a local frame at $a^{(2)}\}$ are nonempty then (U_1, U_2) is a neat pair at some $(a^{(1)}, a^{(2)}) \in \Psi^{(1)} \times \Psi^{(2)}$. In this case, we say that (U_1, U_2) is a neat pair **for** $(\mathbf{\Delta^{(1)}, \Delta^{(2)}} \sqcap \boldsymbol{U_2})$ at $(a^{(1)}, a^{(2)})$. We say that (U_1, U_2) is a **semistrong neat pair**, if it is a neat pair for $(\Delta^{(1)}, \Delta^{(2)} \sqcap U_2)$ whenever $\Delta^{(1)} = \varnothing$.

Let $U \subseteqq A^{(1)}$ be a local frame. Let $\mathbf{U}_2$ be a nonempty set of local frames U_2 in $A^{(2)}$. The pair $(U, \mathbf{U}_2)$ is called **uniformly strongly neat** or simply **strongly neat** if given any finite possibly empty subset $\Delta^{(1)} \subseteqq \Phi_{A^{(1)}}$ such that $(X_{A^{(1)}})_d \cap U \neq \varnothing$ for all $d \in \Delta^{(1)}$ and any set $\{\Delta_{U_2}^{(2)} \subseteqq \Phi_{A^{(2)}} \mid U_2 \in \mathbf{U}_2, \Delta_{U_2}^{(2)}$ finite possibly empty, $(X_{A^{(2)}})_\delta \cap U_2 \neq \varnothing$ for all $d \in \Delta_{U_2}^{(2)}$ and all $U_2 \in \mathbf{U}_2\}$, the following condition is satisfied: If the sets $\Psi^{(1)} = \{a^{(1)} \in \Phi_{A^{(1)}}^{\geqq \Delta^{(1)}} \mid U$ is a local frame at $a^{(1)}, (G_{A^{(2)}})_d u \subseteqq (G_{A^{(1)}})_{a^{(1)}} u \quad \forall\, U_2 \in \mathbf{U}_2, \forall (d, u) \in \Delta_{U_2}^{(2)} \times (U_2 \sqcap (X_{A^{(2)}})_d)\}$ and $\Psi_{U_2}^{(2)} = \{a^{(2)} \in \Phi_{A^{(2)}}^{\Delta_{U_2}^{(2)} \sqcap U_2} \mid U_2$ is a local frame at $a^{(2)}\}$ are nonempty then there is a coordinate $a^{(1)} \in \Psi^{(1)}$ and for each $U_2 \in \mathbf{U}_2$, a coordinate $a_{U_2}^{(2)} \in \Psi_{U_2}^{(2)}$ such that (U, U_2) is neat at $(a^{(1)}, a^{(2)})$. In this case, we say that $(U, \mathbf{U}_2)$ is (uniformly) **neat** for $(\Delta^{(1)}, \Delta_{\mathbf{U}_2}^{(2)} \sqcap \mathbf{U}_2)$. The pair A of global actions is called **uniformly semistrongly neat** (resp. **uniformly neat**), if it is neat for $(\Delta^{(1)}, \Delta_{\mathbf{U}_2}^{(2)} \sqcap \mathbf{U}_2)$ whenever $\Delta^{(1)} = \varnothing$ (resp. $\Delta^{(1)}, \Delta_{U_2}^{(2)} = \varnothing$ for all $U_2 \in \mathbf{U}_2$).

Definition 4.25 A pair A of global actions is called a **neat** (resp. **semistrongly neat**) (resp. **strongly neat**) **pair** if every pair (U_1, U_2) of local frames in A is neat (resp. semistrongly neat) (resp. strongly neat).

A pair A of global actions is called **uniformly neat** (resp. **uniformly semistrongly neat**) (resp. **uniformly strongly neat**), if every pair $(U, \mathbf{U}_2)$ as in (4.24) is neat (resp. semistrongly neat) (resp. strongly neat).

The following definition is given for the sake of completeness. It will not be needed technically and therefore will not be used further in this article.

Definition 4.26 Let $\mathrm{f} : A \to B$ be a morphism of pairs of global actions. There are two notions of neatness for f. One says that f preserves neat pairs of local frames and the other that f sends any pair (U_1, U_2) of local frames in A to a neat pair $(\mathrm{f}(U_1), \mathrm{f}(U_2))$ of local frames in B. Similarly we can define two notions of semistrong neatness and of strong neatness for f.

Remark Obviously if B is a neat (resp. semistrong neat) (resp. strong neat) pair of global actions then any morphism $\mathrm{f} : A \to B$ of pairs of global actions is neat (resp. semistrongly neat) (resp. strongly neat).

The next condition provides a useful criterion for guaranteeing that a pair of global actions is ∞-normal and either ∞-exponential or regularly ∞-exponential.

DEFINITION 4.27 A pair A of global actions is called a **strong infimum action** if the following conditions are satisfied.

(4.27.1) Let $\Delta^{(i)} \subseteqq \Phi_{A^{(i)}}$ $(i = 1, 2)$ be finite possibly empty subsets. Let $U_i \subseteqq |A^{(i)}|$ $(i = 1, 2)$ be finite subsets such that $U_2 \subseteqq U_1$ and such that for each $d^{(i)} \in \Delta^{(i)}, (X_{A^{(i)}})_{d^{(i)}} \cap U_i \neq \varnothing$. If the set $\Psi^{(1)} = \{a^{(1)} \in \Phi_{A^{(1)}}^{\geqq \Delta^{(1)}} \mid U_1$ is a local frame at $a^{(1)}, (G_{A^{(2)}})_{d^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{a^{(1)}}(u)\ \forall\ d^{(2)} \in \Delta^{(2)}$ and $\forall\ u \in (X_{A^{(2)}})_{d^{(2)}} \sqcap U_2\}$ is nonempty then it contains an initial element.

Obviously the special case of (4.27.1) that $\Delta^{(2)} = \varnothing$ implies that $A^{(1)}$ is a strong infimum global action.

(4.27.2) Let Δ_i $(i = 1, 2) \subseteqq \Phi_{A^{(2)}}$ be finite possibly empty subsets. Let U_i $(i = 1, 2) \subseteqq |A^{(2)}|$ be finite subsets such that $U_2 \subseteqq U_1$ and such that for each $d^{(i)} \in \Delta_i, (X_{A^{(2)}})_{d^{(i)}} \cap U_i \neq \varnothing$. If the set $\Psi^{(2)} = \{a^{(2)} \in \Phi_{A^{(2)}}^{\geqq \Delta_1} \mid U_1$ is a local frame at $a^{(2)}, (G_{A^{(2)}})_{d^{(2)}} u \subseteqq (G_{A^{(2)}})_{a^{(2)}} u\ \forall\ d^{(2)} \in \Delta_2$ and $\forall\ u \in (X_{A^{(2)}})_{d^{(2)}} \sqcap U_2\}$ is nonempty then it contains an initial element.

Obviously the special case of (4.27.2) that $\Delta_2 = \varnothing$ implies that $A^{(2)}$ is a strong infimum global action.

(4.27.3) A is a uniform strong neat pair of global actions.

This completes the definition of a strong infimum pair of global actions.

A pair A of global actions is called an **infimum pair** (resp. **semistrong infimum pair**) if (4.27.1) and (4.27.2) are satisfied whenever $\Delta^{(1)} = \Delta_1 = \Delta^{(2)} = \Delta_2 = \varnothing$ and A is a uniform neat pair of actions (resp. $\Delta^{(1)} = \Delta_1 = \varnothing$ and A is a uniform semistrong neat pair of actions).

The following lemma is easy to verify and its proof is omitted.

LEMMA 4.28

(4.28.1) Any pointed global action is a uniform strong neat pair of actions.

(4.28.2) A pointed global action A satisfies the infimum condition (resp. semistrong infimum condition) (resp. strong infimum condition) for pairs of global actions $\Leftrightarrow A^{(1)}$ satisfies the infimum condition (resp. infimum condition) (resp. strong infimum condition) for global actions.

When the hypotheses in the following lemma are satisfied, it can be used to simplify details in the proofs of (4.30) and (4.31). The proof of the lemma is straightforward and is omitted

LEMMA 4.29 There are two statements, one for global actions and the other for pairs of global actions.

(4.29.1) Let f : $A \to B$ be a morphism of global actions such that the relation on Φ_B is transitive.. Let $\beta, \gamma \in \Phi_{(A,B)}$ such that $\mathrm{f}(\mathrm{x}) \in \beta(\mathrm{x}) \cap \gamma(\mathrm{x})$ for all $\mathrm{x} \in |A|$. If $\beta \leqq \gamma$ and (f, γ) is an $A-$chart in B then so is (f, β).

(4.29.2) Let f : $A \to B$ be a morphism of pairs of global actions such that the relations on $\Phi_{B^{(1)}}$ and $\Phi_{B^{(2)}}$ are transitive. Let $\beta^{(1)}, \gamma^{(1)} \in \Phi_{(A,B)^{(1)}}$ such that $\mathrm{f}(\mathrm{x}) \in \beta^{(1)}(\mathrm{x}) \cap\ \gamma^{(1)}(\mathrm{x})$ for all $\mathrm{x} \in |A^{(1)}|$. If $\beta^{(1)} \leqq \gamma^{(1)}$ and $(\mathrm{f}, \gamma^{(1)})$ is an $A-$chart in B then so is $(\mathrm{f}, \beta^{(1)})$.

LEMMA 4.30 If B is a pair of global actions which satisfies the infimum condition (resp. semistrong infimum condition) (resp. strong infimum condition and the relations on $\Phi_{B^{(i)}}$ $(i = 1, 2)$ are transitive) then for any pair A of global actions, the same holds for $Mor(A, B)$. In fact we shall show the following.

(4.30.1) If B satisfies (4.27.1) and (4.27.2) for $\Delta^{(1)} = \Delta^{(2)} = \Delta_1 = \Delta_2 = \varnothing$ (resp. for $\Delta^{(1)} = \Delta_1 = \varnothing$) (resp. in general and the relations on $\Phi_{B^{(i)}}$ $(i = 1, 2)$ are transitive) then the same holds for $Mor(A, B)$.

(4.30.2) Assume B satisfies (4.30.1). If B is neat (resp. semistrongly neat) (resp. strongly neat) then the same holds for $Mor(A, B)$.

(4.30.3) Assume B satisfies (4.30.1). If B is uniformly neat (resp. uniformly semistrongly neat) (resp. uniformly strongly neat) then the same holds for $Mor(A, B)$.

PROOF (4.30.1) It is clear that if the relation on $\Phi_{B^{(i)}}$ $(i = 1, 2)$ is transitive then so is that on $\Phi_{(A,B)^{(i)}}$ $(i = 1, 2)$. The remainder of the proof is devoted to showing that $Mor(A, B)$ satisfies (4.27.1) and (4.27.2). We consider first (4.27.1).

Let $\Delta^{(i)} \subseteqq \Phi_{(A,B)^{(i)}}$ $(i = 1, 2)$ be finite possibly empty subsets of coordinates. Let $F_i \subseteqq |Mor(A, B)^{(i)}|$ $(i = 1, 2)$ be finite nonempty subsets such that $F_2 \subseteqq F_1$ and such that for each $\delta \in \Delta^{(i)}$, the set $(X_{(A,B)^{(i)}})_\delta \cap F_i \neq \varnothing$. Suppose that the set $\Psi^{(1)} = \{\beta^{(1)} \in \Phi^{\geqq \Delta^{(1)}}_{(A,B)^{(1)}} \mid F_1$ is a local frame at $\beta^{(1)}, (G_{(A,B)^2})_\delta \mathrm{f} \subseteqq (G_{(A,B)^{(1)}})_{\beta^{(1)}}\mathrm{f}\ \forall\ \delta \in \Delta^{(2)}$ and $\forall$ f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$ is nonempty. We shall prove that the conclusion of (4.27.1) holds, namely that $\Psi^{(1)}$ has an initial element.

For each $\mathrm{x} \in |A^{(1)}|$, let $\Delta^{(i)}(\mathrm{x}) = \{\delta(\mathrm{x}) \mid \delta \in \Delta^{(i)}\}$, let $F_i(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) \mid \mathrm{f} \in F_i\}$, and let $\Psi^{(1)}(\mathrm{x}) = \{\beta^{(1)}(\mathrm{x}) \mid \beta^{(1)} \in \Psi^{(1)}\} \subseteqq \mathbf{\Psi}^{(1)}(\mathrm{x}) := \{b^{(1)} \in \Phi^{\geqq \Delta^{(1)}(\mathrm{x})}_{B^{(1)}} \mid F_1(\mathrm{x})$ is a local frame at

$b^{(1)}, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta^{(2)}$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. Suppose $\mathrm{x} \in |A^{(1)\setminus(2)}|$. By the conclusion of (4.27.1) applied to $B, \Delta^{(i)}(\mathrm{x}), F_i(\mathrm{x})$, and $\mathbf{\Psi}^{(1)}(\mathrm{x})$, the set $\mathbf{\Psi}^{(1)}(\mathrm{x})$ has an initial element c_{x}. Suppose $\mathrm{x} \in |A^{(2)}|$. Then $\Psi^{(1)}(\mathrm{x}) \subsetneqq \mathbf{\Psi}^{(2)}(\mathrm{x}) := \{b^{(2)} \in \Phi_{B^{(2)}}^{\geqq \Delta^{(1)}(\mathrm{x})} \mid F_1(\mathrm{x})$ is a local frame at $b^{(2)}, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(2)}})_{b^{(2)}}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta^{(2)}$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. By the conclusion of (4.27.2) applied to $B^{(2)}, \Delta^{(i)}(\mathrm{x}), F_i(\mathrm{x})$, and $\mathbf{\Psi}^{(2)}(\mathrm{x})$, the set $\mathbf{\Psi}^{(2)}(\mathrm{x})$ has an initial element c_{x}. Define $\gamma : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}, \mathrm{x} \mapsto c_{\mathrm{x}}$. For any $\mathrm{f} \in F_1$, we shall show that (f, γ) is an A-chart in B. This demonstration is where the hypotheses whether or not certain Δ's are empty and whether or not the relation on $\Phi_{B^{(i)}} (i = 1, 2)$ is transitive play a role. Suppose for the moment that this has been done. By the Local-Global Lemma 4.17, F_1 is a local frame at γ. By construction, $\delta \leqq \gamma$ for all $\delta \in \Delta^{(1)}$. Moreover since for each $\mathrm{x} \in |A^{(1)}|, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_{\gamma(\mathrm{x})}\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta^{(2)}$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$, it follows that $(G_{(A,B)^{(2)}})_\delta \mathrm{f} \subsetneqq (G_{(A,B)^{(1)}})_\gamma \mathrm{f}$ for all $\delta \in \Delta^{(2)}$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. Thus $\gamma \in \Psi^{(1)}$. Obviously γ is an initial element of $\Psi^{(1)}$, by construction. This proves that $Mor(A, B)$ satisfies the conclusion of (4.27.1).

We show as promissed above that (f, γ) is an A-chart in B. It is obvious that (f, γ) satisfies (4.7.0) and (4.7.1). Let $\beta \in \Psi^{(1)}$. By construction, $\gamma(\mathrm{x}) \leqq \beta(\mathrm{x})$ for all $\mathrm{x} \in |A^{(1)}|$. Thus $\gamma \leqq \beta$. Since F_1 is a local frame at β, it follows by definition that (f, β) is an A-chart in B. Thus if the relation on $\Phi_{B^{(i)}}\ (i = 1, 2)$ is transitive then (f, γ) is an A-chart in B by (4.29.2). Hence we can assume that $\Delta^{(1)} = \varnothing$.

We shall show that (f, γ) satisfies (4.7.2). Let V be a local frame in $A^{(1)}$. Since (f, β) is an A-chart in $B, \mathrm{f}(V)$ is a local frame at a coordinate $b \in \Phi_{B^{(1)}}$ such that $\beta(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in V^{(1)\setminus(2)}$, and such that $(G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_b\mathrm{f}(\mathrm{x})$ for all $\mathrm{x} \in V^{(2)}$. Suppose $\mathrm{x} \in V^{(1)\setminus(2)}$. Since $\Delta^{(1)} = \varnothing, \gamma(\mathrm{x})$ is by definition an initial element in $\mathbf{\Psi}^{(1)}(\mathrm{x}) = \{b' \in \Phi_{B^{(1)}} \mid F_1(\mathrm{x})$ is a local frame at $b', (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_{b'}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta^{(2)}$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. We shall show that $b \in \mathbf{\Psi}^{(1)}(\mathrm{x})$. This will prove that $\gamma(\mathrm{x}) \leqq b$. Since $F_1(\mathrm{x})$ is a local frame at $\beta(\mathrm{x})$ and $\beta(\mathrm{x}) \leqq b$, it follows that $F_1(\mathrm{x})$ is a local frame at b. From the inclusion above that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_{b'}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta^{(2)}$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$ and from the fact by definition of β that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta^{(2)}$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$, it follows that $b \in \mathbf{\Psi}^{(1)}(\mathrm{x})$. Thus $\gamma(\mathrm{x}) \leqq b$. Suppose $\mathrm{x} \in V^{(2)}$. Since $\gamma(\mathrm{x}) \leqq \beta(\mathrm{x})$, it follows that $(G_{B^{(2)}})_{\gamma(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(1)}})_b\mathrm{f}(\mathrm{x})$. This completes the proof that (f, γ) satisfies (4.7.2).

We show next that (f, γ) satisfies (4.7.3). Let $\beta \in \Psi^{(1)}$. Then $\gamma \leqq \beta$ and (f, β) is an A-chart in B. Let V be a local frame in $B^{(2)}$. Then $\mathrm{f}(V)$ is a local frame at a coordinate $n \in \Phi_{B^{(2)}}$ such that $\beta(\mathrm{x}) \leqq b$ for each $\mathrm{x} \in V$. Since $\Delta^{(1)} = \varnothing, \gamma(\mathrm{x})$ is by definition an initial element in $\mathbf{\Psi}^{(2)}(\mathrm{x}) = \{b' \in \Phi_{B^{(2)}} \mid F_1(\mathrm{x})$ is a local frame at $b', (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subsetneqq (G_{B^{(2)}})_{b'}\ \forall\ \delta \in \Delta^{(2)}$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. We shall show that $b \in \mathbf{\Psi}^{(2)}(\mathrm{x})$. This will imply that $\gamma(\mathrm{x}) \leqq b$ which shows that (f,γ) satisfies (4.7.3). Since $F_1(\mathrm{x})$ is a local frame at $\beta(\mathrm{x})$, it follows that $F_1(\mathrm{x})$ is a local frame at b. From the definition of β, it follows that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta^{(2)}$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. Since $\beta(\mathrm{x}) \leqq b$, it follows that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_b\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta^{(2)}$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. Thus $b \in \mathbf{\Psi}^{(2)}(\mathrm{x})$. Thus $\gamma(\mathrm{x}) \leqq b$ for each $\mathrm{x} \in V$. Thus (f,γ) satisfies (4.7.3). This completes the proof that (f,γ) is an A-chart in B.

We show that $Mor(A,B)$ satisfies (4.27.2). Let $\Delta_i \subseteqq \Phi_{(A,B)^2}$ $(i = 1,2)$ be finite possibly empty subsets and let $F_2 \subseteqq F_1 \subseteqq Mor(A,B)^{(2)}$ be finite subsets such that for each $\delta \in \Delta_i$, the set $(X_{(A,B)^{(2)}})_\delta \cap F_i \neq \varnothing$. Suppose that the set $\Psi^{(2)} = \{\beta^{(2)} \in \Phi^{\geqq \Delta_1}_{(A,B)^{(2)}} \mid F_1$ is a local frame at $\beta^{(2)}, (G_{(A,B)^{(2)}})_\delta\mathrm{f} \subseteqq (G_{(A,B)^{(2)}})_{\beta^{(2)}}\mathrm{f}\ \forall\ \delta \in \Delta_2$ and $\forall\ \mathrm{f} \in (X_{(A,B)^{(2)}})_\delta \sqcap F_2\}$ is nonempty. We shall prove that the conclusion of (4.27.2) holds, namely that $\Psi^{(2)}$ has an initial element.

For each $\mathrm{x} \in |A^{(1)}|$, let $\Delta_i(\mathrm{x}) = \{\delta(\mathrm{x}) \mid \delta \in \Delta_i\}$, let $F_i(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) \mid \mathrm{f} \in F_i\}$, and let $\Psi^{(2)}(\mathrm{x}) = \{\beta^{(2)}(\mathrm{x}) \mid \beta^{(2)} \in \Psi^{(2)}\}$. Let $\mathbf{\Psi}^{(2)}(\mathrm{x}) = \{b \in \Phi^{\geqq \Delta_1(\mathrm{x})}_{B^{(2)}} \mid F_1(\mathrm{x})$ is a local frame at $b, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_b\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta_2$ and $\forall\ \mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. By the conclusion of (4.27.2) applied to $B, \Delta_i(\mathrm{x}), F_i(\mathrm{x}),$ and $\mathbf{\Psi}^{(2)}(\mathrm{x})$, the set $\mathbf{\Psi}^{(2)}(\mathrm{x})$ has an initial element c_x. Define $\gamma : |A^{(1)}| \to \Phi_{B^{(2)}}, \mathrm{x} \mapsto c_\mathrm{x}$. For any $\mathrm{f} \in F_1$, we shall show that (f,γ) is an $A^{(1)}$-chart in $B^{(2)}$. This demonstration is where the hypotheses whether or not certain Δ's are empty and whether or not the relation on $\Phi_{B^{(i)}}$ $(i = 1,2)$ is transitive play a role. Suppose for the moment this has been done. By the Local-Global Lemma 3.7, F_1 is a local frame at γ. By construction, $\delta \leqq \gamma$ for all $\delta \in \Delta_1$. Moreover since for each $\mathrm{x} \in |A^{(1)}|, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{\gamma(\mathrm{x})}\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta_2$ and for all $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$, it follows that $(G_{(A,B)^{(2)}})_\delta\mathrm{f} \subseteqq (G_{(A,B)^{(2)}})_\gamma\mathrm{f}$ for all $\delta \in \Delta_2$ and for all $\mathrm{f} \in (X_{(A,B)^{(2)}})_\delta \sqcap F_2$. Thus $\gamma \in \Psi^{(2)}$. Obviously γ is an initial element of $\Psi^{(2)}$, by construction. This proves that $Mor(A,B)$ satisfies (4.27.2).

We prove as promissed above that (f,γ) is an $A^{(1)}$-chart in $B^{(2)}$. Obviously (3.4.1) is satisfied. Let $\beta \in \Psi^{(2)}$. By construction, $\gamma(\mathrm{x}) \leqq \beta(\mathrm{x})$ for all $\mathrm{x} \in |A^{(1)}|$. thus $\gamma \leqq \beta$. Since F_1 is a local frame at β, it follows by definition that (f,β) is an $A^{(1)}$-chart in $B^{(2)}$. Thus if the relation on $\Phi_{B^{(2)}}$ is transitive then (f,γ) is an $A^{(1)}$-chart in $B^{(2)}$, by (4.29.1). Hence we can assume that $\Delta_1 = \varnothing$.

We shall prove that (f,γ) satisfies (3.4.2). Let V be a local frame in $A^{(1)}$. Let $\beta \in \Psi^{(1)}$. Since (f,β) is an $A^{(1)}$-chart in $B^{(2)}, \mathrm{f}(V)$ is a local frame at a coordinate $b \in \Phi_{B^{(2)}}$ such that $\beta(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in V$. To show that (f,γ) satisfies (3.4.2), it is enough to show that $\gamma(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in V$. Since $\Delta_1 = \varnothing, \gamma(\mathrm{x})$ is by definition an initial element in $\mathbf{\Psi}^{(2)}(\mathrm{x}) = \{b' \in \Phi_{B^{(2)}} \mid F_1(\mathrm{x})$ is a local frame at $b', (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{b'}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in$

Δ_2 and $\forall$ f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. We shall show that $b \in \mathbf{\Psi}^{(2)}(\mathrm{x})$. This will complete the proof. Since $\beta(\mathrm{x}) \leqq b$ and $F_1(\mathrm{x})$ is a local frame at $\beta(\mathrm{x})$ (by definition of $\beta(x)$), it follows that $F_1(\mathrm{x})$ is a local frame at b. Furthermore $\beta(\mathrm{x}) \in \mathbf{\Psi}^{(2)}(\mathrm{x})$ by definition. Thus $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_b\mathrm{f}(\mathrm{x})$ for all $\delta \in \Delta_2$ and for all f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. Thus $b \in \mathbf{\Psi}^{(2)}(\mathrm{x})$. Thus $\gamma(\mathrm{x}) \leqq b$. This completes the proof that (f,γ) satisfies (3.4.2).

(4.30.2) The proof is a specialization of that for (4.30.3), to the case $\mathbf{F}$ has one element.

(4.30.3) Let $F \subseteqq Mor(A,B)^{(1)}$ be a local-frame and let $\mathbf{F}$ be a set of local frames $F_2 \subseteqq Mor(A,B)^{(2)}$such that $F_2 \subseteqq F$. Let $\Delta^{(1)} \subseteqq \Phi_{(A,B)^{(1)}}$ be a finite possibly empty subset of coordinates such that for each $\delta \in \Delta^{(1)}, (X_{(A,B)^{(1)}})_\delta \cap F \neq \varnothing$. For each $F_2 \in \mathbf{F}$, let $\Delta^{(2)}_{F_2} \subseteqq \Phi_{(A,B)^{(2)}}$ be a finite possibly empty subset of coordinates such that for each $\delta \in \Delta^{(2)}_{F_2}, (X_{(A,B)^{(2)}})_\delta \cap F_2 \neq \varnothing$. Suppose that the set $\Psi^{(1)} = \{\beta \in \Phi^{\geqq \Delta^{(1)}}_{(A,B)^{(1)}} \mid F$ is a local frame at $\beta, (G_{(A,B)^{(2)}})_\delta \mathrm{f} \subseteqq (G_{(A,B)^{(1)}})_\beta \mathrm{f}\ \forall\, F_2 \in \mathbf{F}, \forall\, \delta \in \Delta^{(2)}_{F_2}$, and $\forall$ f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\} \neq \varnothing$ and that for each $F_2 \in \mathbf{F}$, the set $\Psi^{(2)}_{F_2} = \{\beta_{F_2} \in \Phi^{\Delta^{(2)}_{F_2} \sqcap F_2}_{(A,B)^{(2)}} \mid F_2$ is a local frame at $\beta_{F_2}\} \neq \varnothing$. We shall show that there are coordinates $\gamma \in \Phi^{\geqq \Delta^{(1)}}_{(A,B)^{(1)}}$ and $\gamma_{F_2} \in \Phi^{\Delta^{(2)}_{F_2} \sqcap F_2}_{(A,B)^{(2)}}$ for each $F_2 \in \mathbf{F}$ such that (F, F_2) is a neat pair at (γ, γ_{F_2}).

Let $\mathrm{x} \in |A^{(1)}|$. Define $F(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) \mid \mathrm{f} \in F\}$. For $F_2 \in \mathbf{F}$, define $F_2(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) \mid \mathrm{f} \in F_2\}$. Define $\Delta^{(1)}(\mathrm{x}) = \{\delta(\mathrm{x}) \mid \delta \in \Delta^{(1)}\}$ and $\Delta^{(1)}_{F_2}(\mathrm{x}) = \{\delta(\mathrm{x}) \mid \delta \in \Delta^{(2)}_{F_2}\}$. Define $\Psi^{(1)}(\mathrm{x}) = \{\beta(\mathrm{x}) \mid \beta \in \Psi^{(1)}\}$ and $\Psi^{(2)}_{F_2}(\mathrm{x}) = \{\beta_{F_2}(\mathrm{x}) \mid \beta_{F_2} \in \Psi^{(2)}_{F_2}\}$. Define $\mathbf{\Psi}^{(2)}_{F_2}(\mathrm{x}) = \{b_2 \in \Phi_{B^{(2)}} \mid F_2(\mathrm{x})$ is a local frame at $b, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{b_2}\mathrm{f}(\mathrm{x})\ \forall\ \delta \in \Delta^{(2)}_{F_2}$ and $\forall$ f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\}$. By (4.27.2) for B, the set $\mathbf{\Psi}^{(2)}_{F_2}(\mathrm{x})$ has an initial element $(c_{F_2})_\mathrm{x}$. Define $\gamma_{F_2} : |A^{(1)}| \to \Phi_{B^{(2)}}, \mathrm{x} \mapsto (c_{F_2})_\mathrm{x}$.For $\mathrm{x} \in |A^{(1)\setminus(2)}|$, define $\mathbf{\Psi}^{(1)}(\mathrm{x}) = \{b \in \Phi^{\geqq \Delta^{(1)}(\mathrm{x})}_{B^{(1)}} \mid F(\mathrm{x})$ is a local frame at $b, (G_{B^{(2)}})_{\gamma_{F_2}(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b\mathrm{f}(\mathrm{x})\ \forall\ F_2 \in \mathbf{F}$ and $\forall$ f $\in F_2\}$. Since $\Psi^{(1)} \neq \varnothing$, there is a $b \in \Phi^{\geqq \Delta^{(1)}(\mathrm{x})}_{B^{(1)}}$ such that $F(\mathrm{x})$ is a local frame at b. Since B satisfies (4.27.3), it follows that $\mathbf{\Psi}^{(1)}(\mathrm{x}) \neq \varnothing$. Since B satisfies (4.27.1), it follows that $\mathbf{\Psi}^{(1)}(\mathrm{x})$ has an initial element c_x. For $\mathrm{x} \in |A^{(2)}|$, define $\mathbf{\Psi}^{(1)}(\mathrm{x}) = \{b \in \Phi^{\geqq \Delta^{(1)}(\mathrm{x})}_{B^{(2)}} \mid F(\mathrm{x})$ is a local frame at $b, (G_{B^{(2)}})_{\gamma_{F_2}(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_b\mathrm{f}(\mathrm{x})\ \forall\ F_2 \in \mathbf{F}$ and $\forall$ f $\in F_2\}$. Since $\Psi^{(1)} \neq \varnothing$, the set $\{b \in \Phi^{\geqq \Delta^{(1)}(\mathrm{x})}_{B^{(2)}} \mid F(\mathrm{x})$ is a local frame at $b, (G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_b\mathrm{f}(\mathrm{x})\ \forall\ F_2 \in \mathbf{F},\ \forall\, \delta \in \Delta^{(2)}_{F_2}$, and $\forall$ f $\in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta\} \neq \varnothing$. By (4.27.2) for B, the set above has an initial element c_x. From the definition of $\gamma_{F_2}(\mathrm{x})$, it follows that $\gamma_{F_2}(\mathrm{x}) \leqq c_\mathrm{x}$. Thus $c_\mathrm{x} \in \mathbf{\Psi}^{(1)}(\mathrm{x})$. Define $\gamma : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}, \mathrm{x} \mapsto c_\mathrm{x}$. From the proof of (3.22), we know that $(\mathrm{f}, \gamma_{F_2})$ is an $A^{(1)}$-chart in $B^{(2)}$ for any f $\in F_2$. Thus by the Local-Global Lemma 3.7,

F_2 is a local frame in $Mor(A,B)^{(2)}$ at γ_{F_2}. We shall show that for any $\mathrm{f} \in F, (\mathrm{f}, \gamma)$ is an A-chart in B. Assume this has been done. Then it follows from the Local-Global Lemma 4.11 that F is a local frame at γ. One checks easily that $\gamma \in \Phi^{\geqq \Delta^{(1)}}_{(A,B)^{(1)}}$ and that for each $F_2 \in \mathbf{F}, \gamma_{F_2} \in \Phi^{\Delta^{(2)}_{F_2} \sqcap F_2}_{B^{(2)}}$ and (F, F_2) is a neat pair at (γ, γ_{F_2}).

We show as promissed above that (f, γ) is an A-chart in B for any $\mathrm{f} \in F$. Clearly (f, γ) satisfies (4.7.0) and (4.7.1). We prove it satisfies (4.7.2). Let U be a local frame in $A^{(1)}$. Let $\beta \in \Psi^{(1)}$. By definition, (f, β) is an A-chart in B. Thus there is a coordinate $b \in \Phi_{B^{(1)}}$ such that $\mathrm{f}(U)$ is a local frame at $b, \beta(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in U^{(1)\backslash(2)}$, and $(G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b \mathrm{f}(\mathrm{x})$ for all $\mathrm{x} \in U^{(2)}$. We shall show that $\gamma(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in U^{(1)\backslash(2)}$ and $(G_{B^{(2)}})_{\gamma(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b \mathrm{f}(\mathrm{x})$ for all $\mathrm{x} \in U^{(2)}$. This will prove that (f, γ) satisfies (4.7.2). Let $\mathrm{x} \in U^{(1)\backslash(2)}$. Since $\beta(\mathrm{x}) \leqq b$ and $F(\mathrm{x})$ is a local frame at $\beta(\mathrm{x})$, it follows that $F(\mathrm{x})$ is a local frame at b. If $\Delta^{(1)} = \varnothing$ then vacuously $\delta(\mathrm{x}) \leqq b$ for all $\delta \in \Delta^{(1)}$. Suppose $\Delta^{(1)} \neq \varnothing$ and let $\delta \in \Delta^{(1)}$. From the definition of β, it follows that $\delta(\mathrm{x}) \leqq \beta(\mathrm{x})$ and from the definition of $b, \beta(\mathrm{x}) \leqq b$. Thus by the transitivity of $\leqq$ on $\Phi_{B^{(2)}}, \delta(\mathrm{x}) \leqq b$. Let $F_2 \in \mathbf{F}$. Let $\delta \in \Delta_{F_2}$ and $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. Since $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x})$, it follows that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b \mathrm{f}(\mathrm{x})$. From the definition of γ, it follows from the above that $\gamma(\mathrm{x}) \leqq b$. Let $\mathrm{x} \in U^{(2)}$. From the definition of β and γ, it follows that $\gamma(\mathrm{x}) \leqq \beta(\mathrm{x})$. Thus for any $\mathrm{f} \in F, (G_{B^{(2)}})_{\gamma(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b \mathrm{f}(\mathrm{x})$. This completes the proof that (f, γ) satisfies (4.7.2).

We show that (f, γ) satisfies (4.7.3). Let U be a local frame in $A^{(2)}$. Let $\beta \in \Psi^{(1)}$. By definition, (f, β) is an A-chart in B. Thus there is a coordinate $b \in B^{(2)}$ such that $\mathrm{f}(U)$ is a local frame at b and $\beta(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in U$. We shall show that $\gamma(\mathrm{x}) \leqq b$ for all $\mathrm{x} \in U$. This will prove that (f, γ) satisfies (4.7.3). Let x denote an arbitrary element of U. If $\Delta^{(1)} = \varnothing$ then vacuously $\delta(\mathrm{x}) \leqq b$ for all $\delta \in \Delta^{(1)}$. Suppose $\Delta^{(1)} \neq \varnothing$ and let $\delta \in \Delta^{(1)}$. From the definition of β, we know that $\delta(\mathrm{x}) \leqq \beta(\mathrm{x})$ and from the definition of $b, \beta(\mathrm{x}) \leqq b$. Thus by the transitivity of $\leqq$ on $\Phi_{B^{(2)}}, \delta(\mathrm{x}) \leqq b$. Let $F_2 \in \mathbf{F}$. Let $\delta \in \Delta_{F_2}$ and let $\mathrm{f} \in F_2 \sqcap (X_{(A,B)^{(2)}})_\delta$. From the definition of β, it follows that $(G_{B^{(2)}})_{\delta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(2)}})_{\beta(\mathrm{x})}\mathrm{f}(\mathrm{x}) \subseteqq (G_{B^{(1)}})_b \mathrm{f}(\mathrm{x})$. From the definition of γ and the above , it follows that $\gamma(\mathrm{x}) \leqq b$. This completes the proof that (f, γ) satisfies (4.7.3). $\square$

The next theorem is a main result.

Theorem 4.31 An infimum pair of global actions is ∞-normal and ∞-exponential. A strong infimum pair of global actions such that the relations on its coordinate systems are transitive is ∞-normal and regularly ∞-exponential.

PROOF Let C be an infimum pair. We shall show that C is ∞-normal. Lemma 4.30 reduces the proof to showing that C is A-normal for any pair A. Let $g : B \to C$ be a morphism of pairs. Since $C^{(2)}$ is ∞-normal as a global action by Theorem (3.23), the morphism $g : B^{(2)} \to C^{(2)}$ is ∞-normal and thus $A^{(1)}$-normal. Let (f, β) be an A-chart in B. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at (f, β). It remains to show that $g\mathrm{f}_0, \cdots, g\mathrm{f}_p$ is an A-frame in C. We construct first a relative coordinate $(\gamma : |A^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}) \in \Phi_{(A,C)^{(1)}}$ such that $(g\mathrm{f}, \gamma)$ is an A-chart in C.

Let $F = \{\mathrm{f}_0, \cdots \mathrm{f}_p\}$. For $\mathrm{x} \in |A^{(1)}|$, let $F(\mathrm{x}) = \{\mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})\}$. By the Local-Global Lemma 4.11, $F(\mathrm{x})$ is a local frame in $B^{(1)}$ or $B^{(2)}$ depending on whether $\mathrm{x} \in |A^{(1)\setminus(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, respectively. Thus $gF(\mathrm{x})$ is a local frame in either $C^{(1)}$ or $C^{(2)}$ depending on whether $\mathrm{x} \in |A^{(1)\setminus(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, respectively. Let $\Psi^{(1)}(\mathrm{x}) = \{c \in \Phi_{C^{(i)}} |\ gF(\mathrm{x})$ local frame in $C^{(i)}, i = 1$ or 2 depending on whether $\mathrm{x} \in |A^{(1)\setminus(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, resp.$\}$. By the infimum condition for $C^{(i)}, \Psi^{(1)}(\mathrm{x})$ has an initial element c_{x}. Define $\gamma : |A^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, \mathrm{x} \mapsto c_{\mathrm{x}}$.

We show that $(g\mathrm{f}, \gamma)$ is an A-chart in C. It is clear that (gf,γ) satisfies (4.7.0) and (4.7.1). We show next that it satisfies (4.7.3). Let U be a local frame in $A^{(1)}$. By (3.24), $gF(U) = \{g\mathrm{f}_i(\mathrm{x}) \mid 0 \leqq i \leqq p, \mathrm{x} \in U\}$ is a local frame in $C^{(1)}$ and $gF(U^{(2)})$ is a local frame in $C^{(2)}$. By (4.27.3) for $C, (gF(U), gF(U^{(2)}))$ is a neat pair at some $(c^{(1)}, c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. We shall show that $c^{(1)}$ satisfies the requirements in (4.7.3). Let $\mathrm{x} \in U^{(1)\setminus(2)}$. Then $gF(x) = \{g\mathrm{f}_i(\mathrm{x}) \mid 0 \leqq i \leqq p\}$ is a local frame at $c^{(1)}$. By construction, $\gamma(\mathrm{x})$ is an initial element in $\{c \in \Phi_{C^{(1)}} \mid gF(\mathrm{x})$ is a local frame at $c\}$. Thus $\gamma(\mathrm{x}) \leqq c^{(1)}$. Let $\mathrm{x} \in U^{(2)}$. Then $gF(\mathrm{x})$ is a local frame at $c^{(2)}$ and by neatness of the pair $(gF(U), gF(U^{(2)}))$ at $(c^{(1)}, c^{(2)}), (G_{C^{(2)}})_{c^{(2)}} g\mathrm{f}_i(\mathrm{x}) \subseteqq (G_{C^{(1)}})_{c^{(1)}} g\mathrm{f}_i(\mathrm{x})$ for all $0 \leqq i \leqq p$. By construction, $\gamma(\mathrm{x})$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid gF(\mathrm{x})$ is a local frame at $c\}$. Thus $\gamma(\mathrm{x}) \leqq c^{(2)}$. Thus $(G_{C^{(2)}})_{\gamma(\mathrm{x})} g\mathrm{f}(\mathrm{x}) \subseteqq (G_{C^{(2)}})_{c^{(2)}} g\mathrm{f}(\mathrm{x}) \subseteqq (G_{C^{(1)}})_{c^{(1)}} g\mathrm{f}(\mathrm{x})$. This shows that $(g\mathrm{f}, \gamma)$ satisfies (4.7.3).

To complete the proof that $(g\mathrm{f}, \gamma)$ is an A-chart in C is enough to show that it satisfies (4.7.4). Let U be a local frame in $A^{(2)}$. By (3.24), $gF(U)$ is a local frame in $C^{(2)}$, say at $c^{(2)} \in \Phi_{C^{(2)}}$. We shall show that $c^{(2)}$ satisfies the requirements in (4.7.4). Let $\mathrm{x} \in U$. Clearly, $gF(\mathrm{x})$ is a local frame at $c^{(2)}$. By construction, $\gamma(\mathrm{x})$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid gF(\mathrm{x})$ is a local frame at $c\}$. Thus $\gamma(\mathrm{x}) \leqq c^{(2)}$. Thus $(g\mathrm{f}, \gamma)$ satisfies (4.7.4). This completes the proof that $(g\mathrm{f}, \gamma)$ is an A-chart in C.

From the definition of γ and the Local-Global Lemma 4.11, it follows immediately that $g\mathrm{f} = g\mathrm{f}_0, g\mathrm{f}_1, \cdots, g\mathrm{f}_p$ is an A-frame on $(g\mathrm{f}, \gamma)$. This completes the proof that C is A-normal.

Let C denote again a pair satisfying the infimum condition. We shall show that C is ∞-exponential. Let A and B be pairs of global actions such that $|A^{(i)}| = \cup_{\alpha \in \Phi_{A^{(i)}}} (X_{A^{(i)}})_\alpha$ and $|B^{(i)}| = \cup_{\beta \in \Phi_{B^{(i)}}} (X_{B^{(i)}})_\beta$. Let $E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$ be the morphism in (4.21). We shall prove that E has an ∞-normal inverse. By (4.30), $Mor(A, Mor(B,C))$ is an infimum action and thus by the first assertion of the current theorem, it must be ∞-normal. Thus if an inverse to E exists, it must be ∞-normal. So it suffices to show that E has an inverse. There is an obvious candidate for an inverse, namely the set theoretic map $E' : |Mor(A \bowtie B, C)^{(1)}| \to (A,(B,C)), \mathrm{f} \mapsto E'\mathrm{f}$, where $(E'\mathrm{f}(\mathrm{x}))(y) = \mathrm{f}(\mathrm{x}, y)$. We shall show that $E'\mathrm{f} \in |Mor(A, Mor(B,C))^{(1)}|$ and that the resulting map $E' : |Mor(A \bowtie B, C)^{(1)}| \to |Mor(A, Mor(B,C))^{(1)}|$ is a morphism $Mor(A \bowtie B, C) \to Mor(A, Mor(B,C))$ of pairs of global actions. From the set theoretic definition of E', it is obvious that E' will be inverse to E.

We prove that $E'\mathrm{f} : |A^{(1)}| \to (B,C)$ is a morphism $A^{(1)} \to Mor(B,C)^{(1)}$ of global actions. There are two properties to verify. First, if $\mathrm{x} \in |A^{(1)}|$ then $E'\mathrm{f}(\mathrm{x}) : |B^{(1)}| \to |C^{(1)}|,\ y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$, is a morphism $B \to C$ of pairs of actions. Second, the resulting map $E'\mathrm{f} : |A^{(1)}| \to |Mor(B,C)^{(1)}|, \mathrm{x} \mapsto E'\mathrm{f}(\mathrm{x})$, is a morphism $A^{(1)} \to Mor(B,C)^{(1)}$ of global actions.

The demonstration that $y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$ is a morphism $B^{(1)} \to C^{(1)}$ of global actions is the same as the analogous demonstration in the proof of Theorem 3.23. Furthermore it is clear that the morphism $E'\mathrm{f}(\mathrm{x}) : B^{(1)} \to C^{(1)}$ takes $|B^{(2)}|$ into $|C^{(2)}|$ and that the pattern of the demonstration above can be repeated to show that $E'\mathrm{f}(\mathrm{x})|_{|B^{(2)}|}$ is a morphism $B^{(2)} \to C^{(2)}$ of global actions. Thus $E'\mathrm{f}(\mathrm{x}) : B \to C$ is a morphism of pairs of actions.

Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame in $A^{(1)}$. We shall verify that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)^{(1)}$. For each element $y \in |B^{(1)}|, \{y\}$ is a local frame in $B^{(1)}$ because $|B^{(1)}| = \cup_{\beta \in \Phi_{B^{(1)}}} (X_{B^{(1)}})_\beta$. Thus $(\mathrm{x}_0, y), \cdots, (\mathrm{x}_p, y)$ is a local frame in $(A \bowtie B)^{(1)}$. Thus $\mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ is a local frame in $C^{(1)}$ and if $y \in B^{(2)}$ then it is also a local frame in $C^{(2)}$. By the infimum condition for $C^{(1)}$, we know that for $y \in B^{(1)\setminus(2)}$, the set $\{c \in \Phi_{C^{(1)}} \mid \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)\ c\text{-frame}\}$ has an initial element c_y. By the infimum condition for $C^{(2)}$, we know that for $y \in B^{(2)}$, the set $\{c \in \Phi_{C^{(2)}} | \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(x_p, y)\ c\text{-frame}\}$ has an initial element c_y. Define $\gamma : |B^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto c_y$. We shall show that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 4.11 that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a B-frame on $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$. But then by definition, $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)^{(1)}$, which is what we have to verify.

We show now that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C. Obviously $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ satisfies (4.7.0) and (4.7.1). We show that it satisfies (4.7.2). Let $U = \{\mathrm{x}_0, \cdots, \mathrm{x}_p\}$. Let V be a local frame in $A^{(1)}$. Then $U \times V$ is a local frame in $(A \bowtie B)^{(1)}$ and $U \times V^{(2)}$ is a local frame in $(A \bowtie B)^{(2)}$. Thus $(\mathrm{f}(U \times V), \mathrm{f}(U \times V^{(2)}))$ is a pair of local frames in C. Since C satisfies the infimum condition, it is neat. Thus $(\mathrm{f}(U \times V), \mathrm{f}(U \times V^{(2)}))$ is a neat pair at some $(c^{(1)}, c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. We shall show that $c^{(1)}$ satisfies the requirements of (4.7.2) for the local frame

V. Let $y \in V^{(1)\setminus(2)}$. Obviously $\mathrm{f}(U \times \{y\})$ is a local frame at $c^{(1)}$. By construction, $\gamma(y)$ is an initial element in $\{c \in \Phi_{C^{(1)}} \mid \mathrm{f}(U \times \{y\}) \text{ is a local frame at } c\}$. Thus $\gamma(y) \leqq c^{(1)}$. Let $y \in V^{(2)}$. Clearly $(\mathrm{f}(U \times \{y\}), \mathrm{f}(U \times \{y\}))$ is a neat pair at $(c^{(1)}, c^{(2)})$. By construction, $\gamma(y)$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid \mathrm{f}(U \times \{y\}) \text{ is a local frame at } c^{(2)}\}$. Thus $\gamma(y) \leqq c^{(2)}$. Thus $(G_{C^{(2)}})_{\gamma(y)} E'\mathrm{f}(\mathrm{x}_0)(y) \subseteqq (G_{C^{(2)}})_{c^{(2)}} E'\mathrm{f}(\mathrm{x}_0)(y) \subseteqq (G_{C^{(1)}})_{c^{(1)}} E'\mathrm{f}(\mathrm{x}_0)(y)$. This proves that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ satisfies (4.7.2).

To complete the proof that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C, it remains to show that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ satisfies (4.7.3). Let V be a local frame in $B^{(2)}$. Then $U \times V$ is a local frame in $(A \bowtie B)^{(2)}$. Thus $\mathrm{f}(U \times V)$ is a local frame in $C^{(2)}$, say at $c^{(2)} \in \Phi_{C^{(2)}}$. Let $y \in V$. Obviously $\mathrm{f}(U \times \{y\})$ is a local frame at $c^{(2)}$. By construction, $\gamma(y)$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid \mathrm{f}(U \times \{y\}) \text{ is a local frame at } c\}$. Thus $\gamma(y) \leqq c^{(2)}$. This proves that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ satisfies (4.7.3). This completes the proof that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C.

It is clear that the morphism $E'\mathrm{f} : A^{(1)} \to Mor(B, C)^{(1)}$ takes $|A^{(2)}|$ into $|Mor(B, C)^{(2)}|$. We show that the function $(E'\mathrm{f})|_{|A^{(2)}|} : |A^{(2)}| \to |Mor(B, C)^{(2)}|$ is a morphism $A^{(2)} \to Mor(B, C)^{(2)}$ of global actions. This will complete the proof that $E'\mathrm{f} : A \to Mor(B, C)$ is a morphism of pairs of actions. Observe first that $Mor(B, C)^{(2)} = Mor(B^{(1)}, C^{(2)})$. Then observe that the function $(E'\mathrm{f})|_{|A^{(2)}|}$ is identical with the function $E'(\mathrm{f}|_{A^{(2)} \times B^{(1)}})$ where the latter E' is the morphism $E' : Mor(A^{(2)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))$ of global actions, which is constructed in the proof of (3.23). By the conclusion of Theorem 3.23, $E'(\mathrm{f}|_{A^{(2)} \times B^{(1)}})$ is a morphism of global actions.

Next we show that the function $E' : |Mor(A \bowtie B, C)^{(1)}| \to |Mor(A, Mor(B, C))^{(1)}|$ is a morphism $Mor(A \bowtie B, C) \to Mor(A, Mor(B, C))$ of pairs of actions.

To begin we show that E' is a morphism $Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B, C))^{(1)}$ of global actions. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be a local frame in $Mor(A \bowtie B, C)^{(1)}$. We must show that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B, C))^{(1)}$. For each element (x, y) in $A \times B$, $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is a local frame in $C^{(1)}$, by either the Local-Global Lemma 4.11 or (3.24). Furthermore if $(\mathrm{x}, y) \in |(A \bowtie B)^{(2)}|$ then $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is also a local frame in $C^{(2)}$, by the same reasons. By the infimum condition for $C^{(1)}$, it follows that for $(\mathrm{x}, y) \in |(A \bowtie B)^{(1)\setminus(2)}|$, the set $\{c \in \Phi_{C^{(1)}} | \ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y) \ c\text{-frame}\}$ has an initial element $c_{(\mathrm{x}, y)}$. By the infimum condition for $C^{(2)}$, it follows that for $(\mathrm{x}, y) \in |(A \bowtie B)^{(2)}|$, the set $\{c \in \Phi_{C^{(2)}} | \ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y) \ c\text{-frame}\}$ has an initial element $c_{(\mathrm{x}, y)}$. Define $\gamma : |A| \to (|B|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}), \mathrm{x} \mapsto c_{(\mathrm{x}, -)}$. We claim that $(E'\mathrm{f}_0, \gamma)$ is an A-chart in $Mor(B, C)$. It will follow then from the definition of γ and the Local-Global Lemma 4.11 that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is A- an frame at $(E'\mathrm{f}, \gamma)$. But this says by definition that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B, C))^{(1)}$ and we are finished.

We prove now that $(E'\mathrm{f},\gamma)$ is an A-chart in $Mor(B,C)$. Obviously $(E'\mathrm{f},\gamma)$ satisfies (4.7.0). We show next that $(E'\mathrm{f},\gamma)$ satisfies (4.7.1). Our task is to prove that if $\mathrm{x} \in |A^{(1)\setminus(2)}|$ then $E'\mathrm{f}(\mathrm{x}) \in (X_{(B,C)^{(1)}})_{\gamma(\mathrm{x})}$, i.e. $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ is a B-chart in C, and if $\mathrm{x} \in |A^{(2)}|$ then $E'\mathrm{f}(\mathrm{x}) \in (X_{(B,C)^{(2)}})_{\gamma(\mathrm{x})}$, i.e. $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ is a $B^{(1)}$-chart in $C^{(2)}$.

Let $\mathrm{x} \in |A^{(1)\setminus(2)}|$. From the definition of γ, it follows that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ satisfies (4.7.0) and (4.7.1). We show that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ satiesfies (4.7.2). Let V be a local frame in $B^{(1)}$. Let $F = \{\mathrm{f}_0,\cdots,\mathrm{f}_p\}$. Recall that $\mathrm{f} = \mathrm{f}_0$. Clearly $(\{\mathrm{x}\} \times V, \{\mathrm{x}\} \times V^{(2)})$ is a pair of local frames in $A \bowtie B$. By (3.24), $(F(\{\mathrm{x}\} \times V), F(\{\mathrm{x}\} \times V^{(2)}))$ is a pair of local frames. Since C satisfies the infimum condition, C is neat. Thus $(F(\{\mathrm{x}\} \times V), F(\{\mathrm{x}\} \times V^{(2)}))$ is a neat pair of local frames at some $(c^{(1)},c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. We shall show that $c^{(1)}$ satisfies the requirements of (4.7.2) for V. Let $y \in |V^{(1)\setminus(2)}|$. Obviously $F(\mathrm{x},y)$ is a local frame at $c^{(1)}$. By construction, $\gamma(\mathrm{x})(y)$ is an initial element in $\{c \in \Phi_{C^{(1)}} \mid F(\mathrm{x},y) \text{ is a local frame at } c\}$. Thus $\gamma(\mathrm{x})(y) \leqq c^{(1)}$. Let $\mathrm{x} \in |V^{(2)}|$. Obviously $(F(\mathrm{x},y),F(\mathrm{x},y))$ is a neat pair at $(c^{(1)},c^{(2)})$. By construction, $\gamma(\mathrm{x})(y)$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid F(\mathrm{x},y) \text{ is a local frame at } c\}$. Thus $\gamma(\mathrm{x})(y) \leqq c^{(2)}$. Thus $(G_{C^{(2)}})_{\gamma(\mathrm{x})(y)}E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(2)}})_{c^{(2)}}E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(1)}})_{c^{(1)}}E'\mathrm{f}(\mathrm{x})(y)$. This completes the proof that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ satisfies (4.7.2). To complete the proof that$(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ is a B-chart in C, it is enough to show that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ satisfies (4.7.3). Let V be a local frame in $B^{(2)}$. Clearly $\{\mathrm{x}\} \times V$ is a local frame in $(A \bowtie B)^{(2)}$. By (3.24), $F(\{\mathrm{x}\} \times V)$ is a local frame in $C^{(2)}$, say at the coordinate $c^{(2)} \in \Phi_{C^{(2)}}$. Clearly $F(\mathrm{x},y)$ is a local frame at $c^{(2)}$. By construction, $\gamma(\mathrm{x})(y)$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid F(\mathrm{x},y) \text{ is a local frame at } c\}$. Thus $\gamma(\mathrm{x})(y) \leqq c^{(2)}$. This shows that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ satisfies (4.7.3) and completes the proof that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ is a B-chart in C when $\mathrm{x} \in |A^{(1)\setminus(2)}|$.

Let $\mathrm{x} \in |A^{(2)}|$. It follows from the proof of Theorem 3.23 applied to $F|_{A^{(2)} \times B^{(1)}} \subseteqq Mor(A^{(2)} \times B^{(1)}, C^{(2)})$ that $(E'\mathrm{f}(\mathrm{x}),\gamma(\mathrm{x}))$ is a $B^{(1)}$-chart in $C^{(2)}$. This completes the proof that $(E'\mathrm{f},\gamma)$ satisfies (4.7.1).

We show that $(E'\mathrm{f},\gamma)$ satisfies (4.7.2). Let $F = \{\mathrm{f}_0,\cdots,\mathrm{f}_p\}$ as above and recall that $\mathrm{f} = \mathrm{f}_0$. Let U be a local frame in $A^{(1)}$. We shall show that $(E'F(U),E'F(U^{(2)}))$ is a neat pair at some $(\delta^{(1)},\delta^{(2)}) \in \Phi_{(B,C)^{(1)}} \times \Phi_{(B,C)^{(2)}}$. Assume this has been done. Obviously for each $\mathrm{x} \in U$, $E'F(\mathrm{x})$ is a local frame at $\delta^{(1)}$. Let $\mathrm{x} \in U^{(1)\setminus(2)}$. By construction and the Local-Global Lemma 4.11, $\gamma(\mathrm{x})$ is an initial element in $\{\delta' \in \Phi_{(B,C)^{(1)}} \mid E'F(\mathrm{x}) \text{ is a local frame at } \delta'\}$. Thus $\gamma(\mathrm{x}) \leqq \delta^{(1)}$. Let $\mathrm{x} \in U^{(2)}$. By construction and the Local-Global Lemma 3.7, $\gamma(\mathrm{x})$ is an initial element in $\{\delta' \in \Phi_{(B,C)^{(2)}} \mid E'F(\mathrm{x}) \text{ is a local frame at } \delta'\}$. Thus $\gamma(\mathrm{x}) \leqq \delta^{(2)}$. Thus $(G_{(B,C)^{(2)}})_{\gamma(\mathrm{x})}E'\mathrm{f}(\mathrm{x}) \subseteqq (G_{(B,C)^{(2)}})_{\delta^{(2)}}E'\mathrm{f}(\mathrm{x}) \subseteqq (G_{(B,C)^{(1)}})_{\delta^{(1)}}E'\mathrm{f}(\mathrm{x})$. This proves that $(E'\mathrm{f},\gamma)$ satisfies (4.7.2).

We show now, as promissed above, that $(E'F(U), E'F(U^{(2)}))$ is a neat pair of local frames in $Mor(B,C)$. Since C satisfies the infimum condition, it follows from (4.30) that $Mor(B,C)$ also satisfies the infimum condition. Thus $Mor(B,C)$ is a neat pair of global actions. Thus it is enough to show that $(E'F(U), E'F(U^{(2)}))$ is a pair of local frames.

We show first that $E'F(U)$ is a local frame in $Mor(B,C)^{(1)}$. Let U be a local frame in $A^{(1)}$. Let $y \in |B^{(1)\setminus(2)}|$. Then $U \times \{y\}$ is a local frame in $(A \bowtie B)^{(1)}$. By (3.24), $F(U \times \{y\})$ is a local frame in $C^{(1)}$. By the infimum condition for $C^{(1)}$, the set $\{d \in \Phi_{C^{(1)}} \mid F(U\times\{y\})$ is a local frame at $d\}$ has an initial element $d_{U,y}$. Let $y \in |B^{(2)}|$. Then $U \times \{y\}$ is a local frame in $(A \bowtie B)^{(2)}$. By (3.24), $F(U \times \{y\})$ is a local frame in $C^{(2)}$. By the infimum condition for $C^{(2)}$, the set $\{d \in \Phi_{C^{(2)}} \mid F(U \times \{y\})$ is a local frame at $d\}$ has an initial element $d_{U,y}$. Define $\delta : |B^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto d_{U,y}$. Let $\mathrm{x} \in U$. We shall show that $(E'\mathrm{f}(\mathrm{x}), \delta)$ is a B-chart in C. Once this has been done, it will follow from the Local-Global Lemma 4.11 that $E'F(U)$ is a local frame at δ in $Mor(B,C)^{(1)}$.

It is obvious that $(E'\mathrm{f}(\mathrm{x}), \delta)$ satisfies (4.7.0) and (4.7.1). We show that it satisfies (4.7.2). Let V be a local frame in $B^{(1)}$. Then $U \times V$ is a local frame in $(A \bowtie B)^{(1)}$ and $U \times V^{(2)}$ is a local frame in $(A \bowtie B)^{(2)}$. By (3.24), $(F(U \times V), F(U \times V^{(2)}))$ is a pair of local frames in C. Since C satisfies the infimum condition, C is neat. Thus $(F(U \times V), F(U \times V^{(2)}))$ is a neat pair at some $(c^{(1)}, c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. Let $y \in V^{(1)\setminus(2)}$. Obviously $F(U\times\{y\})$ is a local frame at $c^{(1)}$. By construction, $\delta(y)$ is an initial element in $\{c \in \Phi_{C^{(1)}} \mid F(U \times \{y\})$ is a local frame at $c\}$. Thus $\delta(y) \leqq c^{(1)}$. Let $y \in V^{(2)}$. Obviously $F(U \times \{y\})$ is a local frame at $c^{(2)}$. By construction, $\delta(y)$ is an initial element in $\{c \in \Phi_{C^{(2)}} \mid F(U \times \{y\})$ is a local frame at $c\}$. Thus $\delta(y) \leqq c^{(2)}$. Thus $(G_{C^{(2)}})_{\delta(y)} E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(2)}})_{c^{(2)}} E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(1)}})_{c^{(1)}} E'\mathrm{f}(\mathrm{x})(y)$. This proves that $(E'\mathrm{f}(\mathrm{x}), \delta)$ satisfies (4.7.2). The proof that $(E'\mathrm{f}(\mathrm{x}), \delta)$ satisfies (4.7.3) follows from the proof of Theorem 3.23 applied to $F|_{A^{(1)}\times B^{(2)}} \subseteqq Mor(A^{(1)} \times B^{(2)}, C^{(2)})$. This shows $(E'\mathrm{f}(\mathrm{x}), \delta)$ is an B-chart in C and completes the proof that $(E'\mathrm{f}, \gamma)$ satisfies (4.7.2).

The proof that $(E'\mathrm{f}, \gamma)$ satisfies (4.7.3) follows from the proof of Theorem 3.23 applied to $F|_{A^{(2)}\times B^{(1)}} \subseteqq Mor(A^{(2)} \times B^{(1)}, C^{(2)})$. This completes the proof that $(E'\mathrm{f}, \gamma)$ is an A-chart in $Mor(B,C)$ and the proof that $E' : Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B,C))^{(1)}$ is a morphism of global actions.

Clearly $E' : Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B,C))^{(1)}$ takes $|Mor(A \bowtie B, C)^{(2)}|$ into $(A^{(2)}, (B^{(1)}, C^{(2)}))$. In fact, $Mor(A \bowtie B, C)^{(2)} = Mor(A^{(1)} \times B^{(1)}, C^{(2)})$ and $E' |_{|Mor(A^{(1)}\times B^{(1)}, C^{(2)})|}$ is identical with the function $|Mor(A^{(1)}\times B^{(1)}, C^{(2)})| \to |Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))|$ defined by the morphism $E' : Mor(A^{(1)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))$ of global actions in Theorem 3.23. Thus $E' : Mor(A \bowtie B, C)^{(2)} \to Mor$

$(A, Mor(B,C))^{(2)}$ is a morphism of global actions. Thus $E' : Mor(A \bowtie B, C) \to Mor$ $(A, Mor(B,C))$ is a morphism of pairs of actions. This completes the proof that C is ∞-exponential.

Suppose finally that C is a strong infimum pair. We shall show that C is regularly ∞-exponential. Our task is to show that the morphism $E : Mor(A, Mor(B,C)) \to Mor(A \bowtie B, C)$ has a regular inverse E'. There are obvious candidates for the structural components (E'_Φ, E'_G, E'_X) of E'. Define

$$\text{(4.32)} \qquad E'_X : Mor(A \bowtie B, C)| \longrightarrow |Mor(A, Mor(B,C))| \\ \mathrm{f} \longmapsto E'\mathrm{f}$$

where $\mathrm{f} \mapsto E'\mathrm{f}$ is the map constructed above. Define

$$E'_{\Phi^{(i)}} : \Phi_{(A \bowtie B, C)^{(i)}} \longrightarrow \Phi_{(A,(B,C))^{(i)}} \quad (i = 1,2)$$

as the set theoretic inverse (see (3.16)) of $E_{\Phi^{(i)}}$. Define the natural transformations

$$E'_{G^{(i)}} : G_{(A \bowtie B, C)^{(i)}} \longrightarrow (G_{(A,(B,C))^{(i)}})_{E'_{\Phi^{(i)}}(\ )} \quad (i = 1,2)$$

as follows. For $\alpha \in \Phi_{(A \bowtie B, C)^{(i)}}$, define the group homomorphism

$$E'_{G^{(1)}}(\alpha) : (G_{(A \bowtie B, C)^{(1)}})_\alpha \longrightarrow (G_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)}$$

such that the factor of $(G_{(A \bowtie B, C)^{(1)}})_\alpha = \prod_{(\mathrm{x},y) \in |A^{(1)\setminus(2)}| \times |B^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\alpha(\mathrm{x},y)} \times \prod_{(\mathrm{x},y) \in |(A \bowtie B)^{(2)}|}$ $(G_{C^{(2)}})_{\alpha(\mathrm{x},y)}$ with the subscript $\alpha(\mathrm{x}, y)$ is mapped via the identity map onto the factor of $(G_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)} = \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} [\prod_{y \in |B^{(1)\setminus(2)}|} (G_{C^{(1)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)} \times \prod_{y \in |B^{(2)}|} (G_{C^{(2)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)}] \times$ $\prod_{\mathrm{x} \in |A^{(2)}|} [\prod_{y \in |B^{(1)}|} (G_{C^{(2)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)}$ with the subscript $(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)$. For $\alpha \in \Phi_{(A \bowtie B, C)^{(2)}}$, define the group homomorphism

$$E'_{G^{(2)}}(\alpha) : (G_{(A \bowtie B, C)^{(2)}})_\alpha \to (G_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)}$$

such that the factor of $(G_{(A\bowtie B,C)^{(2)}})_\alpha = \prod_{(\mathrm{x},y)\in|A^{(1)}|\times|B^{(1)}|} (G_{C^{(2)}})_{\alpha(\mathrm{x},y)}$ with the subscript $\alpha(\mathrm{x},y)$ is mapped via the identity map onto the factor of $(G_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)} = \prod_{\mathrm{x}\in|A^{(1)}|} (\prod_{y\in|B^{(1)}|} (G_{C^{(2)}})_{(E'_{\Phi^{(2)}}\alpha)(\mathrm{x})(y)})$ with the subscript $(E'_{\Phi^{(2)}}\alpha)(\mathrm{x})(y)$.

All the properties for E' to be a regular morphism are obvious, except the one that $E'_X(X_{(A\bowtie B,C)^{(i)}})_\alpha \subseteqq (X_{(A,(B,C))^{(i)}})_{E'_{\Phi^{(i)}}(\alpha)}$ for any $\alpha \in \Phi_{(A\bowtie B,C)^{(i)}}$ $(i = 1, 2)$.

We prove first the case $i = 1$. To establish this, it is enough to show that if (f, α) is an $(A \bowtie B)$-chart in C then $(E'_X(\mathrm{f}), E'_{\Phi^{(1)}}(\alpha))$ is an A-chart in $Mor(B, C)$. To simplify notation, we shall write $(E'\mathrm{f}, E'\alpha)$ in place of $(E'_X(\mathrm{f}), E'_{\Phi^{(1)}}(\alpha))$.

Clearly $(E\mathrm{f}, E'\alpha)$ satisfies (4.7.0). We show that it satisfies (4.7.1). Let $\mathrm{x} \in |A^{(1)\setminus(2)}|$. We must show that $E\mathrm{f}(\mathrm{x}) \in (X_{(A,B)^{(1)}})_{E'\alpha(\mathrm{x})}$, i.e. that $(E'\mathrm{f}(\mathrm{x}), E'\alpha(\mathrm{x}))$ is an B-chart in C. Clearly $(E'\mathrm{f}(\mathrm{x}), E'\alpha(\mathrm{x}))$ satisfies (4.7.0) and (4.7.1). We show that it satisfies (4.7.2). Let V be a local frame in $B^{(1)}$. Then $\{\mathrm{x}\} \times V$ is a local frame in $(A \bowtie B)^{(1)}$. Since (f, α) is an $A \bowtie B$-chart in C, there is a coordinate $c^{(1)} \in \Phi_{C^{(1)}}$, such that $\mathrm{f}(\{\mathrm{x}\} \times V)$ is a local frame at $c^{(1)}, \alpha(\mathrm{x}, y) \leqq c^{(1)}$ for all $y \in V^{(1)\setminus(2)}$, and $(G_{C^{(2)}})_{\alpha(\mathrm{x},y)}\mathrm{f}(\mathrm{x}, y) \subseteqq (G_{C^{(1)}})_{c^{(1)}}\mathrm{f}(\mathrm{x}, y)$ for all $y \in V^{(2)}$. But then $E'\mathrm{f}(\mathrm{x})(V)$ $(= \mathrm{f}(\{\mathrm{x}\} \times V))$ is a local frame at $c^{(1)}, E'\alpha(\mathrm{x})(y)$ $(= \alpha(\mathrm{x}, y)) \leqq c^{(1)}$ for all $y \in V^{(1)\setminus(2)}$, and $(G_{C^{(2)}})_{E'\alpha(\mathrm{x})(y)}E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(1)}})_{c^{(1)}}E'\mathrm{f}(\mathrm{x})(y)$ for all $y \in V^{(2)}$. This proves that $(E'\mathrm{f}(\mathrm{x}), E'\alpha(\mathrm{x}))$ satisfies (4.7.2). That it satisfies (4.7.3) follows from Theorem 3.23 applied to the morphism $E' : Mor(A^{(1)} \times B^{(2)}, C^{(2)}) \to Mor(A^{(1)}, Mor(B^{(2)}, C^{(2)}))$. This completes the proof that $(E'\mathrm{f}(\mathrm{x}), E'\alpha(\mathrm{x}))$ is a B-chart in C. Let $\mathrm{x} \in |A^{(2)}|$. We must show that $E'\mathrm{f}(\mathrm{x}) \in (X_{(B,C)^{(2)}})_{E'\alpha(\mathrm{x})}$, i.e. that $(E'\mathrm{f}(\mathrm{x}), E'\alpha(\mathrm{x}))$ is a $B^{(1)}$-chart in $C^{(2)}$. But this follows from Theorem 3.23 applied to the morphism $E' : Mor(A^{(2)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))$. This completes the proof that $(E'\mathrm{f}, E'\alpha)$ satisfies (4.7.1).

We show that $(E'\mathrm{f}, E'\alpha)$ satisfies (4.7.2). Let U be a local frame in $A^{(1)}$. Let $y \in |B^{(1)\setminus(2)}|$. Then $U \times \{y\}$ is a local frame in $(A \bowtie B)^{(1)}$ and $(U \times \{y\})^{(1)\setminus(2)} = U^{(1)\setminus(2)} \times \{y\}$. Since (f, α) is an $A \bowtie B$-chart in C, there is a coordinate $c \in \Phi_{C^{(1)}}$ such that $\mathrm{f}(U \times \{y\})$ is a local frame at $c, \alpha(\mathrm{x}, y) \leqq c$ for all $\mathrm{x} \in U^{(1)\setminus(2)}$, and $(G_{C^{(2)}})_{\alpha(\mathrm{x},y)}\mathrm{f}(\mathrm{x}, y) \subseteqq (G_{C^{(1)}})_c\mathrm{f}(\mathrm{x}, y)$ for all $\mathrm{x} \in U^{(2)}$. By the strong infimum condition for C, the set of all c's as above has an initial element c_y. Let $y \in |B^{(2)}|$. Then $U \times \{y\}$ is a local frame in $(A \bowtie B)^{(2)}$. Since (f, α) is an $A \bowtie B$-chart in C, there is a coordinate $c \in \Phi_{C^{(2)}}$ such that $\mathrm{f}(U \times \{y\})$ is a local frame at c and $\alpha(\mathrm{x}, y) \leqq c$ for all $\mathrm{x} \in U$. Since $C^{(2)}$ satisfies the strong infimum condition, the set of all c's above has an initial element c_y. Define $\gamma : |B^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto c_y$. Let $\mathrm{x} \in U$. We shall show that $(E'\mathrm{f}(\mathrm{x}), \gamma)$ is a B-chart in C. Assume this has been done. By the Local-Global Lemma

4.11, $E\mathrm{f}(U)$ is a local frame at γ. By the construction of $\gamma, E'\alpha(\mathrm{x}) \leqq \gamma$ for all $\mathrm{x} \in U^{(1)\backslash(2)}, (G_{C^{(2)}})_{E'\alpha(\mathrm{x})(y)}E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(1)}})_{\gamma(y)}E'\mathrm{f}(\mathrm{x})(y)$ for all $\mathrm{x} \in U^{(2)}$ and all $y \in |B^{(1)}|$, and $E'\alpha(\mathrm{x})(y) \leqq \gamma(y)$ for all $\mathrm{x} \in U^{(2)}$ and all $y \in |B^{(2)}|$. From the last two assertions in the sentence above, it follows that $(G_{(B,C)^{(2)}})_{E'\alpha(\mathrm{x})}E'\mathrm{f}(\mathrm{x}) \subseteqq (G_{(B,C)^{(1)}})_{\gamma}E'\mathrm{f}(x)$ for all $\mathrm{x} \in U^{(2)}$. Thus $(E'\mathrm{f}, E'\alpha)$ satisfies (4.7.2).

We show as promised above that $(E'\mathrm{f}(\mathrm{x}), \gamma)$ is a B-chart in C. Clearly $(E'\mathrm{f}(\mathrm{x}), \gamma)$ satisfies (4.7.0) and (4.7.1). From the proof of Theorem 3.23 applied to $E' : Mor(A^{(1)} \times B^{(2)}, C^{(2)}) \to Mor(A^{(1)}, Mor(B^{(2)}, C^{(2)}))$, it follows that $(E'\mathrm{f}(\mathrm{x}), \gamma)$ satisfies (4.7.3). To show that $(E'\mathrm{f}(\mathrm{x}), \gamma)$ satisfies (4.7.2), it suffices to show that for each $\mathrm{x} \in U^{(1)\backslash(2)}, E'\alpha(\mathrm{x}) \leqq \gamma$ and that for each $\mathrm{x} \in U^{(2)}, (G_{(B,C)^{(2)}})_{E'\alpha(\mathrm{x})}E'\mathrm{f}(\mathrm{x}) \subseteqq (G_{(B,C)^{(1)}})_{\gamma}E'\mathrm{f}(\mathrm{x})$. Suppose $\mathrm{x} \in U^{(1)\backslash(2)}$. From the definition of γ, one checks straightforward that for each $y \in |B^{(1)}|, E'\alpha$ $(\mathrm{x})(y) \leqq \gamma(y)$. Thus $E'\alpha(\mathrm{x}) \leqq \gamma$. Suppose $\mathrm{x} \in U^{(2)}$. From the definition of γ, one checks straightforward that for each $y \in |B^{(1)\backslash(2)}|$ (resp. $y \in |B^{(2)}|), (G_{C^{(2)}})_{E'\alpha(\mathrm{x})(y)}E'\mathrm{f}(\mathrm{x})(y) \subseteqq (G_{C^{(1)}})_{\gamma_{(y)}}E'\mathrm{f}(\mathrm{x})(y)$ (resp. $(G_{C^{(2)}})_{\gamma(y)}E'\mathrm{f}(\mathrm{x})(y)$). Thus $(G_{(B,C)^{(2)}})_{E'\alpha(\mathrm{x})}E'\mathrm{f}(\mathrm{x}) \subseteqq (G_{(B,C)^{(1)}})_{\gamma}$ $E'\mathrm{f}(\mathrm{x})$. This completes the proof that $(E'\mathrm{f}(\mathrm{x}), \gamma)$ is a B-chart in C.

That $(E'\mathrm{f}, E'\alpha)$ satisfies (4.7.3) follows from Theorem 3.23 applied to the morphism $E' : Mor(A^{(1)} \times B^{(2)}, C^{(2)}) \to Mor(A^{(1)}, Mor(B^{(2)}, C^{(2)}))$. This completes the proof that $(E'\mathrm{f}, E'\alpha)$ is an A-chart in $Mor(B, C)$.

This completes the proof that $E'_X(X_{(A \bowtie B, C)^{(1)}})_\alpha \subseteqq (X_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)}$ for any $\alpha \in \Phi_{(A \bowtie B, C)^{(1)}}$.

To complete the proof of the theorem, it remains now to show that $E'_X(X_{(A \bowtie B, C)^{(2)}})_\alpha \subseteqq (X_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)}$ for any $\alpha \in \Phi_{(A \bowtie B, C)^{(2)}}$. Observe that $Mor(A \bowtie B, C)^{(2)} = Mor(A^{(1)} \times B^{(1)}, C^{(2)})$ and $Mor(A, Mor(B, C))^{(2)} = Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}))$. But by Theorem 3.23, the morphism $E'_X : Mor(A^{(1)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}))$ of global actions is regular and therefore has the desired property above. □

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