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# Dimension Theory and Nonstable $K_1$ of Quadratic Modules

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**Abstract.** Employing Bak's dimension theory, we investigate the nonstable quadratic *K*-group $K_{1,2n}(A,\Lambda) = G_{2n}(A,\Lambda)/E_{2n}(A,\Lambda)$, $n \geqslant 3$, where $G_{2n}(A,\Lambda)$ denotes the general quadratic group of rank $n$ over a form ring $(A,\Lambda)$ and $E_{2n}(A,\Lambda)$ its elementary subgroup. Considering form rings as a category with dimension in the sense of Bak, we obtain a dimension filtration $G_{2n}(A,\Lambda) \supseteq G^0_{2n}(A,\Lambda) \supseteq G^1_{2n}(A,\Lambda) \supseteq \cdots \supseteq E_{2n}(A,\Lambda)$ of the general quadratic group $G_{2n}(A,\Lambda)$ such that $G_{2n}(A,\Lambda)/G^0_{2n}(A,\Lambda)$ is Abelian, $G^0_{2n}(A,\Lambda) \supseteq G^1_{2n}(A,\Lambda) \supseteq \cdots$ is a descending central series, and $G^{d(A)}_{2n}(A,\Lambda) = E_{2n}(A,\Lambda)$ whenever $d(A) =$ (Bass–Serre dimension of $A$) is finite. In particular $K_{1,2n}(A,\Lambda)$ is solvable when $d(A) < \infty$.

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**Key words:** non-Abelian $K_1$, general quadratic group, descending central series.

## 1. Introduction

The concepts of $\Lambda$-quadratic form, quadratic module, and general quadratic group over a form ring $(A,\Lambda)$ were introduced by A. Bak who studied their $K$-theory (see [2, 5, 10]). Although the quadratic setting is much more complicated than the linear one, it is being gradually established that most results concerning the $K$-theory of general linear groups can be carried over to the $K$-theory of general quadratic groups. In the linear situation, there have been extensive studies of normal subgroups of general linear groups and of non-stable $K_1$ of these groups. Suslin showed using his localization-patching method, that the elementary subgroup $E_n(A)$ of the general linear group $\mathrm{GL}_n(A)$ is normal providing $A$ is module finite and Bak [3] used localization-completion methods to establish that the non-stable $K$-group $K_{1,n}(A) := \mathrm{GL}_n(A)/E_n(A)$ is a nilpotent by Abelian group (and thus solvable) when the Bass–Serre dimension of $A$ is finite. In the quadratic situation, the normality of the elementary subgroup is proved in [4] by generalizing methods used in [3] and again in [5] by developing a quadratic analog of the transvection procedure used in [12]. A partial statement without proof of the normality result above is found earlier in [10]. In the current paper we prove the quadratic analog of Bak's result, namely that nonstable $K_1$ of a general quadratic group is a nilpotent by abelian group (and thus solvable) when the Bass–Serre dimension of the ground ring is finite. The presence of both short and long roots in the elementary quadratic subgroup makes the proof of the quadratic analog considerably more complicated than that of the linear result.

The rest of paper is organized as follows. In Section 2 we briefly recall the basic concepts of quadratic module and general quadratic group over form rings. The elementary subgroup of the general quadratic group is defined. We then recall the sum and product of form ideals in form rings and state the first of several conjugation results. Its proof gives an indication of the flavor of the long computations to come in Section 4 and how to deal with short and long roots in elementary quadratic groups.

In Section 3 we give a self-contained account of a portion of Bak's dimension theory, which is tailored to the needs of the current paper. Dimension theory provides for any 'good' pair $\mathcal{G}, \mathcal{E}$ of group valued functors on a category with dimension, a normal filtration $\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots \supseteq \mathcal{E}$ such that $\mathcal{G}/\mathcal{G}^0$ is Abelian and $\mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$ is a descending central series with the property that $\mathcal{G}^{\dim(A)}(A) = \mathcal{E}(A)$ whenever $\dim(A)$ is finite. We then describe the category of form rings as a category with dimension whose dimension function is Bass–Serre dimension and show that the pair of functors $G_{2n}, E_{2n}, n \geqslant 3$ satisfies all, except possibly one of the conditions for being good. Section 4 consists of several long computations whose goal is verifying that the one missing condition above is satisfied.

We fix some notation for the rest of the paper. If $a$ and $b$ are elements of some group, let ${}^a b = aba^{-1}$ and $[a, b] = aba^{-1}b^{-1}$. It is easy to see that following commutator formulas hold.

**C(1)** $[a, bc] = [a, b]{}^b[a, c]$,

**C(1)** $[ab, c] = {}^a[b, c][a, c]$.

Let $A$ be an associative ring with identity 1. For any $n \in \mathbb{N}$, let $\mathrm{GL}_n(A)$ denote the general linear group over $A$, i.e., the group of all invertible $n \times n$ matrices and $E_n(A)$ its elementary subgroup, i.e., the subgroup of $\mathrm{GL}_n(A)$ generated by all elementary matrices $e_{ij}(a)$.

## 2. General Quadratic Groups and their Elementary Subgroups

The purpose of this section is to establish notation and recall some basic results, as well as get started developing a conjugation calculus which will be required in Section 4.

We begin by recalling the basic concepts of quadratic module over a form ring and of general quadratic group.

Let $A$ be a ring with an involution denoted by $a \mapsto \overline{a}$, and let $\lambda \in \mathrm{Center}(A)$ such that $\lambda\overline{\lambda} = 1$. Let

$$\Lambda_{\min} = \{a - \lambda\overline{a} \mid a \in A\} \quad \text{and} \quad \Lambda_{\max} = \{a \in A \mid a = -\lambda\overline{a}\}.$$

Clearly $\Lambda_{\min}$ and $\Lambda_{\max}$ are additive subgroups of $A$ such that $\Lambda_{\min} \subseteq \Lambda_{\max}$ and satisfy the closure property $a\Lambda_{\min}\overline{a} \subseteq \Lambda_{\min}$ and $a\Lambda_{\max}\overline{a} \subseteq \Lambda_{\max}$ for all elements $a \in A$. Let $\Lambda$ be an additive subgroup of $A$ such that

(1) $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$,
(2) $a\Lambda\overline{a} \subseteq \Lambda$ for all $a \in A$.

$\Lambda$ is called a *form parameter* and the pair $(A, \Lambda)$ is called a *form ring*.

*Remark.* There is a generalization of the notion of form ring in [2, Section 13] for which the conclusions of the current paper are valid. Checking details is straight forward and is left to the reader. The generalization replaces the notion of involution by that of $\lambda$-involution. A $\lambda$-*involution* consists by definition of an element $\lambda \in A$ and an anti-automorphism $a \mapsto \overline{a}$ of $A$ such that $\overline{\overline{\lambda a}}\lambda = a$ for all $a \in A$. Setting $a = 1$, we obtain that $\overline{\lambda}\lambda = 1$. One defines $\Lambda_{\min} = \{a - \overline{a}\lambda | a \in A\}$ and $\Lambda_{\max} = \{a \in A | a = -\overline{a}\lambda\}$. A *form parameter* is by definition an additive subgroup $\Lambda$ of $A$ such that $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$ and $\overline{a}\Lambda a \subseteq \Lambda$ for all $a \in A$. The reason that $\lambda$ is appearing on the right instead of on the left is that $\lambda$ is not necessarily in $\mathrm{Center}(A)$ and we use right $A$-modules below in the definition of quadratic module.

Let $(A, \Lambda)$ and $(A', \Lambda')$ be form rings relative, respectively, to $\lambda$ and $\lambda'$. A ring homomorphism $\mu\colon A \longrightarrow A'$ such that for any $a \in A$, $\mu(\overline{a}) = \overline{\mu(a)}$, $\mu(\lambda) = \lambda'$ and $\mu(\Lambda) \subseteq \Lambda'$ is called a *morphism of form rings*. A morphism $\mu\colon (A, \Lambda) \longrightarrow (A', \Lambda')$ of form rings is called surjective if $\mu\colon A \longrightarrow A'$ is a surjective ring homomorphism and $\mu(\Lambda) = \Lambda'$.

In order to construct later relative groups for the general quadratic group, we introduce now the notion of form ideal in a form ring, due to Bak. Let $\mathfrak{J}$ be an ideal of $A$ which is invariant under the involution of $A$, i.e., $\overline{\mathfrak{J}} = \mathfrak{J}$. Let

$$\Gamma_{\max} = \mathfrak{J} \cap \Lambda \quad \text{and} \quad \Gamma_{\min} = \{x - \lambda\overline{x} \mid x \in \mathfrak{J}\} + \langle x\alpha\overline{x} \mid x \in \mathfrak{J}, \alpha \in \Lambda\rangle.$$

Clearly $\Gamma_{\min}$ and $\Gamma_{\max}$ depend only on the form parameter $\Lambda$ and the ideal $\mathfrak{J}$ and satisfy the closure property $a\Gamma_{\min}\overline{a} \subseteq \Gamma_{\min}$ and $a\Gamma_{\max}\overline{a} \subseteq \Gamma_{\max}$ for all $a \in A$. A *relative form parameter of* $\mathfrak{J}$ is an additive subgroup $\Gamma$ of $\mathfrak{J}$ such that

(1) $\Gamma_{\min} \subseteq \Gamma \subseteq \Gamma_{\max}$
(2) $a\Gamma\overline{a} \subseteq \Gamma$ for all $a \in A$.

The pair $(\mathfrak{J}, \Gamma)$ is called a *form ideal* in $(A, \Lambda)$.

Let $V$ be a right $A$-module and $f$ a *sesquilinear* form on $V$, i.e., a biadditive map $f\colon V \times V \longrightarrow A$ such that $f(ua, vb) = \overline{a} f(u, v) b$ for all $u, v \in V$ and $a, b \in A$. Define the maps $h\colon V \times V \longrightarrow A$ and $q\colon V \longrightarrow A/\Lambda$ by $h(u, v) = f(u, v) + \lambda \overline{f(v, u)}$ and $q(v) = f(v, v) + \Lambda$. The function $q$ is called a $\Lambda$-*quadratic form* on $V$ and $h$ its associated $\lambda$-*Hermitian form*. The triple $(V, h, q)$ is called a *quadratic module over* $(A, \Lambda)$. It is called *nonsingular*, if $V$ is finitely generated and projective over $A$ and the map $V \to \operatorname{Hom}_A(V, A)$, $v \mapsto h(v, -)$ is bijective, i.e. the Hermitian form $h$ is nonsingular. A morphism $(V, h, q) \to (V', h', q')$ of quadratic modules over $(A, \Lambda)$ is an $A$-linear map $V \to V'$ which preserves the Hermitian and $\Lambda$-quadratic forms.

Define the *general quadratic group* $G(V, h, q)$ to be the group of all automorphisms of $(V, h, q)$. Thus

$$\begin{aligned} G(V, h, q) \ = \ & \{\sigma \in \mathrm{GL}(V) \mid h(\sigma u, \sigma v) = h(u, v), q(\sigma v) = q(v) \\ & \text{for all } u, v \in V\}, \end{aligned}$$

where $\mathrm{GL}(V)$ denotes as usual the group of all $A$-linear automorphisms of $V$. Suppose $h$ and $q$ are defined by the sesquilinear form $f$. If $(\mathfrak{I}, \Gamma)$ is a form ideal in $(A, \Lambda)$, define the *relative general quadratic group*

$$\begin{aligned} G(V, h, q, (\mathfrak{I}, \Gamma)) \ = \ & \{\sigma \in G(V, h, q) \mid \sigma \equiv 1 \bmod \mathfrak{I}, f(\sigma v, \sigma v) - \\ & - f(v, v) \in \Gamma \text{ for all } v \in V\}. \end{aligned}$$

THEOREM 2.1 (Bak). *If $(V, h, q)$ is nonsingular then the group $G(V, h, q, (\mathfrak{I}, \Gamma))$ is well defined, i.e. does not depend on the choice of $f$, and is normal in $G(V, h, q)$.*

The theorem is proved in Bak's thesis (unpublished). Published proofs for the special case $G_{2n}(A, \Lambda)$ which is defined below and is all we need in the current paper, are found in Section 5.2 of the book [10] of Hahn and O'Meara or in a recent paper of Bak and Vavilov [5].

We recall now the group $G_{2n}(A, \Lambda)$. Let $V$ denote a free right $A$-module with ordered basis $e_1, e_2, \ldots, e_n, e_{-n}, \ldots, e_{-1}$. If $u \in V$, let $u_1, \ldots, u_n, u_{-n}, \ldots, u_{-1} \in A$ such that $u = \sum_{i=-n}^{n} e_i u_i$. Let $f\colon V \times V \to A$ denote the sesquilinear map defined by

$$f(u, v) = f\left( \begin{pmatrix} u_1 \\ \vdots \\ u_n \\ u_{-n} \\ \vdots \\ u_{-1} \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ v_{-n} \\ \vdots \\ v_{-1} \end{pmatrix} \right) = \overline{u}_1 v_{-1} + \cdots + \overline{u}_n v_{-n}. \tag{2.1}$$

It is easy to see that if $h$ and $q$ are the Hermitian and $\Lambda$-quadratic forms defined by $f$ then

$$h(u,v)=\overline{u}_1v_{-1}+\cdots+\overline{u}_nv_{-n}+\lambda\overline{u}_{-n}v_n+\cdots+\lambda\overline{u}_{-1}v_1$$

and

$$q(u)=\overline{u}_1u_{-1}+\cdots+\overline{u}_nu_{-n}+\Lambda.$$

Using the basis above, we can identify $G(V,h,q)$ with a subgroup of the general linear group $\mathrm{GL}_{2n}(A)$ of rank $2n$. This subgroup will be denoted by $G_{2n}(A,\Lambda)$ and is called the *general quadratic group over* $(A,\Lambda)$ *of rank* $n$. Using the basis, we can identify the relative subgroup $G(V,h,q,(\mathfrak{I},\Gamma))\subseteq G(V,h,q)$ with a subgroup denoted by $G_{2n}(\mathfrak{I},\Gamma)$ of $G_{2n}(A,\Lambda)$.

In order to describe the matrices in $G_{2n}(A,\Lambda)$, we need some notation. Let $M_n(A)$ denote the ring of $n\times n$ matrices over $A$. If $\alpha\in M_n(A)$, let $\alpha_{ij}$ denote the $(i,j)$th entry of $\alpha$. For $\alpha\in M_n(A)$ define the *conjugate transpose* $\alpha^*\in M_n(A)$ by $\alpha^*_{ij}=\overline{\alpha_{ji}}$. Let

$$p=\begin{pmatrix}0&\cdots&1\\ \vdots&\ddots&\vdots\\ 1&\cdots&0\end{pmatrix}$$

denote the matrix in $M_n(A)$, which has 1's along the second diagonal and zero elsewhere. If $\alpha\in M_n(A)$, the matrix $p\alpha p$ amounts to rotating the matrix $\alpha$ by 180 degrees. Let

$$\Lambda_n=\{\alpha\in M_n(A)\mid\alpha=-\lambda\alpha^*\text{ and }\alpha_{ii}\in\Lambda,\ \text{ for }1\leqslant i\leqslant n\}.$$

If $(\mathfrak{I},\Gamma)$ is a form ideal in $(A,\Lambda)$, let

$$\Gamma_n=\{\alpha\in M_n(A)\mid\alpha=-\lambda\alpha^*,\alpha_{ij}\in\mathfrak{I}\text{ for all }i\neq j,\alpha_{ii}\in\Gamma\text{ for }1\leqslant i\leqslant n\}.$$

If

$$g=\begin{pmatrix}a&b\\ c&d\end{pmatrix}\in\mathrm{GL}_{2n}(A),$$

then it is straightforward to check that it preserves $h$ if and only if

$$\begin{pmatrix}a&b\\ c&d\end{pmatrix}^{-1}=\begin{pmatrix}pd^*p&\overline{\lambda}pb^*p\\ \lambda pc^*p&pa^*p\end{pmatrix}$$

and it preserves $q$ if and only if $a^*pc$ and $b^*pd \in \Lambda_n$. Using the above, one establishes easily that

$$\begin{aligned} G_{2n}(A,\Lambda) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{2n}(A) \,\middle|\, d^*pa + \overline{\lambda} b^*pc \right. \\ &= \left. p \text{ and } a^*pc, b^*pd \in \Lambda_n \right\}. \end{aligned} \tag{2.2}$$

Similarly

$$\begin{aligned} G_{2n}(\mathfrak{I},\Gamma) = \Big\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{2n}(A,\Lambda) \,\Big|\, & g \in \mathrm{GL}_{2n}(\mathfrak{I}) \\ & \text{and } a^*pc, b^*pd \in \Gamma_n \Big\} \end{aligned}$$

where $\mathrm{GL}_{2n}(\mathfrak{I}) = \{\sigma \in \mathrm{GL}_{2n}(A) \mid \sigma_{ij} = 0 \bmod \mathfrak{I} \text{ for all } i \neq j \text{ and } \sigma_{ii} = 1 \bmod \mathfrak{I}\}$. Note that the description above of $G_{2n}(\mathfrak{I},\Gamma)$ proves that its definition does not depend on the choice of $f$.

Let $k \leqslant n$. Then there is a standard embedding of $G_{2k}(A,\Lambda)$ into $G_{2n}(A,\Lambda)$ as follows. If $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $G_{2k}(A,\Lambda)$ then using (2.2), it is easy to see that the rule

$$\begin{array}{c} \\ 1 \\ \vdots \\ k \\ \\ -k \\ \vdots \\ -1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1\cdots k & -k\cdots -1 \end{array} \\ \left(\begin{array}{c|c} & \\ A & B \\ & \\ \hline & \\ C & D \\ & \end{array}\right) \end{array} \mapsto \begin{array}{c} \\ 1 \\ \vdots \\ k \\ \vdots \\ n \\ -n \\ \vdots \\ -k \\ \vdots \\ -1 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1\cdots k & \cdots n & -n & -k\cdots -1 \end{array} \\ \left(\begin{array}{cc|cc} A & & & B \\ & & & \\ & 1\ddots & & \\ & & & \\ \hline & & \ddots 1 & \\ & & & \\ C & & & D \\ & & & \end{array}\right) \end{array}$$

induces an injective homomorphism $G_{2k}(A,\Lambda) \longrightarrow G_{2n}(A,\Lambda)$. We shall frequently use this standard embedding to identify $G_{2k}(A,\Lambda)$ with its image in $G_{2n}(A,\Lambda)$. Note that the above embedding depends on the choice of the basis.

With the basis which is used in the book of Bak [1], the standard embedding has the form

$$
\begin{array}{c}
\begin{array}{c|c} 1\cdots k & 1\cdots k \end{array}\\
\begin{array}{c}1\\ \vdots\\ k\\ 1\\ \vdots\\ k\end{array}
\left(\begin{array}{c|c}
A & B\\ \hline
C & D
\end{array}\right)
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{cc|cc} 1\cdots k & \cdots n & 1\cdots k & \cdots n \end{array}\\
\begin{array}{c}1\\ \vdots\\ k\\ \vdots\\ n\\ 1\\ \vdots\\ k\\ \vdots\\ n\end{array}
\left(\begin{array}{cc|cc}
A & & B & \\
 & 1\ddots & & \\ \hline
C & & D & \\
 & & & 1\ddots
\end{array}\right)
\end{array}
$$

Next we recall the definition of the elementary quadratic subgroup. For $i \in \Delta = \{1, \ldots, n, -n, \ldots, -1\}$, let $\varepsilon(i)$ denote the sign of $i$, i.e., $\varepsilon(i) = 1$ if $i \geqslant 0$ and $\varepsilon(i) = -1$ if $i < 0$. Let $i, j \in \Delta$ such that $i \neq j$. The *elementary transvection* $T_{ij}(a)$ is defined as follows:

$$
T_{ij}(a) = \begin{cases} e + ae_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{a}e_{-j,-i}, & \text{where } a \in A, \text{ if } i \neq -j\\ e + ae_{i,-i}, \quad \text{where } a \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda, & \text{if } i = -j. \end{cases}
$$

It is easy to check that $T_{ij}(a) \in G_{2n}(A, \Lambda)$. The symbol $T_{ij}$ where $i \neq -j$ is called a *short root* whereas $T_{i,-i}$ is called a *long root*.

The subgroup generated by all elementary transvections is called the *elementary quadratic group* and is denoted by $E_{2n}(A, \Lambda)$. This group is the quadratic version of the elementary group in the theory of general linear group. Note that elementary transvections corresponding to long roots are elementary matrices in $E_{2n}(A)$ and elementary transvections corresponding to short roots are a product of two elementary matrices in $E_{2n}(A)$. Let $(\mathfrak{I}, \Gamma)$ be a form ideal of $(A, \Lambda)$. The subgroup which is generated by all $(\mathfrak{I}, \Gamma)$-elementary transvections is denoted by $F_{2n}(\mathfrak{I}, \Gamma)$, i.e.,

$$
F_{2n}(\mathfrak{I}, \Gamma) = \langle T_{ij}(x), T_{i,-i}(y) \mid x \in \mathfrak{I}, y \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma\rangle.
$$

The normal closure ${}^{E_{2n}(A,\Lambda)}F_{2n}(\mathfrak{I}, \Gamma)$ of $F_{2n}(\mathfrak{I}, \Gamma)$ in $E_{2n}(A, \Lambda)$ is denoted by $E_{2n}(\mathfrak{I}, \Gamma)$ and is called the *relative* (*or principal*) *elementary quadratic subgroup* *of* $G_{2n}(A, \Lambda)$ *of level* $(\mathfrak{I}, \Gamma)$. In this note we sometimes do not distinguish between short and long roots and simply write $T_{ij}(x)$, assuming that $x \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$ whenever $i = -j$.

There are standard relations among the elementary transvections, which are analogous to those for the elementary matrices in the general linear group. In Section 4, we shall repeatedly use these relations. We list them now for future reference.

**(R1)** $T_{ij}(a) = T_{-j,-i}(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{a})$.
**(R2)** $T_{ij}(a)T_{ij}(b) = T_{ij}(a+b)$.
**(R3)** $[T_{ij}(a), T_{hk}(b)] = 1$ where $h \neq j, -i$ and $k \neq i, -j$.
**(R4)** $[T_{ij}(a), T_{jh}(b)] = T_{ih}(ab)$ where $i, h \neq \pm j$ and $i \neq \pm h$.
**(R5)** $[T_{ij}(a), T_{j,-i}(b)] = T_{i,-i}(ab - \lambda^{-\varepsilon(i)}\overline{b}\overline{a})$ where $i \neq \pm j$.
**(R6)** $[T_{i,-i}(a), T_{-i,j}(b)] = T_{ij}(ab)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{b}ab)$ where $i \neq \pm j$

We need the following theorem which determines the form of the generators of $E_{2n}(\mathfrak{I}, \Gamma)$ (See [5] for the proof).

THEOREM 2.2. *Let* $(\mathfrak{I}, \Gamma)$ *be a form ideal and suppose* $n \geqslant 3$. *Then the group* ${}^{E_{2n}(A,\Lambda)}F_{2n}(\mathfrak{I}, \Gamma)$ *is generated by all elements of the form* $T_{ji}(a)T_{ij}(x)T_{ji}(-a)$, *where* $a \in A$ *and* $x \in \mathfrak{I}$.

Again note that we didn't distinguish between the short and long roots. If in the above theorem $i = -j$ then $a$ and $x$ are in $\lambda^{-(\varepsilon(j)+1)/2}\Lambda$ and $\lambda^{-(\varepsilon(i)+1)/2}\Gamma$, respectively.

The above theorem is the quadratic version of an analogous result by A. Suslin and L. Vaserstein for the general linear group. Using the latter result, it is easy to show that ${}^{E_n(A)}E_n(\mathfrak{I}\mathfrak{J}) \subseteq E_n(\mathfrak{I} + \mathfrak{J})$, where $\mathfrak{I}$ and $\mathfrak{J}$ are two sided ideals of $A$. We need a quadratic version of this observation. For this purpose we recall the sum and product of form ideals in a form ring. Let $(\mathfrak{I}, \Gamma)$ and $(\mathfrak{J}, \Omega)$ be form ideals. We write $(\mathfrak{I}, \Gamma) \subseteq (\mathfrak{J}, \Omega)$ if $\mathfrak{I} \subseteq \mathfrak{J}$ and $\Gamma \subseteq \Omega$. It is clear if $(\mathfrak{I}, \Gamma) \subseteq (\mathfrak{J}, \Omega)$ then $G_{2n}(\mathfrak{I}, \Gamma) \subseteq G_{2n}(\mathfrak{J}, \Omega)$, $F_{2n}(\mathfrak{I}, \Gamma) \subseteq F_{2n}(\mathfrak{J}, \Omega)$ and $E_{2n}(\mathfrak{I}, \Gamma) \subseteq E_{2n}(\mathfrak{J}, \Omega)$. The *sum* and *product* of arbitrary form ideals $(\mathfrak{I}, \Gamma)$ and $(\mathfrak{J}, \Omega)$ in $(A, \Lambda)$ is defined by

$$(\mathfrak{I}, \Gamma) + (\mathfrak{J}, \Omega) = (\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega),$$

$$(\mathfrak{I}, \Gamma)(\mathfrak{J}, \Omega) = (\mathfrak{I}\mathfrak{J}, \Gamma\Omega),$$

where $\Gamma\Omega = \Gamma_{\min}(\mathfrak{I}\mathfrak{J}) + \langle y\Gamma\overline{y} \mid y \in \mathfrak{J}\rangle + \langle x\Omega\overline{x} \mid x \in \mathfrak{I}\rangle$. In the above definition, $\langle y\Gamma\overline{y} \mid y \in \mathfrak{J}\rangle$ is the subgroup generated by all elements of the form $y\gamma\overline{y}$ where $\gamma \in \Gamma$ and $y \in \mathfrak{J}$. Now we are able to give the quadratic result.

THEOREM 2.3. *Let* $(\mathfrak{I}, \Gamma)$ *and* $(\mathfrak{J}, \Omega)$ *be form ideals of* $(A, \Lambda)$. *Then*

(1) ${}^{G_2(A,\Lambda)}F_{2n}\Big((\mathfrak{I}, \Gamma)(\mathfrak{J}, \Omega)\Big) \subseteq F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$, *providing* $n \geqslant 2$.

(2) ${}^{E_{2n}(A,\Lambda)}F_{2n}\Big((\mathfrak{I}, \Gamma)(\mathfrak{J}, \Omega)\Big) \subseteq F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$, *providing* $n \geqslant 3$.

*Remark.* We can write the first statement of the above Theorem under a weaker condition. Namely if $(\mathfrak{I}, \Gamma) \subseteq (\mathfrak{J}, \Omega)$ and $\Gamma \subseteq \langle x\Omega\overline{x} \mid x \in \mathfrak{I}\rangle$, then a modification of the proof of (1) shows that ${}^{G_2(A,\Lambda)}F_{2n}(\mathfrak{I}, \Gamma) \subseteq F_{2n}(\mathfrak{J}, \Omega)$.

*Proof.* (1) First note that each element of $G_2(A, \Lambda)$ in $G_{2n}(A, \Lambda)$ has the following form:

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \\ \\ -n \\ \vdots \\ -1 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & n & & -n & \cdots & -1 \end{array} \\ \left(\begin{array}{ccc:ccc} a & & & & & b \\ & 1 & & & & \\ & & \ddots & & & \\ & & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & 1 & \\ c & & & & & d \end{array}\right) \end{array}.$$

Let $\sigma \in G_2(A, \Lambda)$ and $T_{ij}(x) \in F_{2n}((\mathfrak{I}, \Gamma)(\mathfrak{J}, \Omega))$. We shall show that ${}^{\sigma}T_{ij}(x) \in F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. Suppose $i \neq -j$, i.e. $T_{ij}$ is a short root. We shall prove a stronger statement that for any form ideal $(\mathfrak{M}, \Phi)$ and element $x \in \mathfrak{M}$, ${}^{\sigma}T_{ij}(x)$ is in $F_{2n}(\mathfrak{M}, \Phi)$. This will be required in the proof of the long root case later. If $i \neq \pm 1$ and $j \neq \pm 1$, then clearly $\sigma$ commutes with $T_{ij}(x)$ and we are done. Suppose that $j = 1$. The argument for the case $j = -1$ is the same and will be skipped. Furthermore the relation (R1) shows that the case $i = \pm 1$, follows from the case $j = \pm 1$. Thus it suffices to treat just the case $j = 1$. Since $T_{ij}$ is a short root, $i \neq \pm 1$. Furthermore, since $\sigma \in G_{2n}(A, \Lambda)$, it follows by (2.2) that $\overline{b}d \in \Lambda$. Direct matrix calculation shows that

$$\sigma T_{i1}(x)\sigma^{-1} = T_{i,-1}(\overline{\lambda}x\overline{b})T_{i1}(x\overline{d})T_{i,-i}(\overline{\lambda}x(\overline{b}d)\overline{x}). \tag{2.3.1}$$

For example if $\varepsilon(i) \geqslant 0$, the calculation above takes the form

$$\sigma T_{i1}(x)\sigma^{-1} = \left(\begin{array}{ccc:ccc} 1 & & & -b\overline{x} & & \\ & \ddots & & & & \\ & & & & & \\ x\overline{d} & & 1 & & & \overline{\lambda}x\overline{b} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & \\ & & & -d\overline{x} & & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc:ccc} 1 & & & -b\overline{x} & & \\ & \ddots & & & & \\ & & 1 & & & \overline{\lambda}x\overline{b} \\ \hdashline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)\left(\begin{array}{ccc:ccc} 1 & & & & & \\ & \ddots & & & & \\ x\overline{d} & & 1 & & & \\ \hdashline & & & 1 & & \\ & & & & \ddots & \\ & & & -d\overline{x} & & 1 \end{array}\right)\left(\begin{array}{ccc:ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & \overline{\lambda}x(\overline{b}d)\overline{x} & & \\ \hdashline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right).$$

The above decomposition can be better understood if we write elementary transvections $T_{ij}(x)$ as a special case of ESD-transvections and use the calculus of the latter which is spelled out in [5, Section 6] to make the computation above. The translation of elementary transvections into ESD-transvections is done in [5, 6.5]. For a short root $T_{ij}$ where $j = 1$ we get $T_{ij}(x) = T_{e_i,e_{-1}}(\overline{\lambda}x, 0)$. Using the conjugation property [5, 6.2] of ESD-transvections, we have

$$\sigma T_{i1}(x)\sigma^{-1} = \sigma T_{e_i,e_{-1}}(\overline{\lambda}x, 0)\sigma^{-1} = T_{\sigma e_i,\sigma e_{-1}}(\overline{\lambda}x, -\overline{b}d).$$

But $\sigma e_i = e_i$. Now a direct calculation shows that

$$T_{\sigma e_i,\sigma e_{-1}}(\overline{\lambda}x, -\overline{b}d) = T_{e_i,e_1 b}(\overline{\lambda}x, 0)T_{e_i,e_{-1}d}(\overline{\lambda}x, -\overline{b}d),$$

which leads to the above decomposition (2.3.1) thanks to [5, 6.4].

Now suppose that $i = -j$, i.e. $T_{ij}$ is a long root. If $i \neq \pm 1$ then $\sigma T_{i,-i}(x)\sigma^{-1} = T_{i,-i}(x)$. So assume that $i = 1$. The argument for $i = -1$ is the same. Let $x = \overline{\lambda}\gamma$ where $\gamma \in \Gamma\Omega$. Therefore $\gamma = \alpha + \beta + \delta$ for some $\alpha \in \Gamma_{\min}(\mathfrak{I}\mathfrak{J})$, $\beta \in \langle y\Gamma\overline{y} \mid y \in \mathfrak{J}\rangle$ and $\delta \in \langle z\Omega\overline{z} \mid z \in \mathfrak{I}\rangle$. We shall show that ${}^{\sigma}T_{i,-i}(\overline{\lambda}\alpha)$, ${}^{\sigma}T_{i,-i}(\overline{\lambda}\beta)$ and ${}^{\sigma}T_{i,-i}(\overline{\lambda}\delta)$ are all in $F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. For $T_{i,-i}(\overline{\lambda}\delta)$, it is enough by R(2) to prove this when $\delta = \overline{z}\omega z$ where $z \in \mathfrak{I}$ and $\omega \in \Omega$. The argument for $T_{i,-i}(\overline{\lambda}\beta)$ is the same. So let $\delta = \overline{z}\omega z$. Using (R6) and the fact that $n \geqslant 2$, we can write

$$T_{i,-i}(\overline{\lambda}\delta) = T_{i,-i}(\overline{\lambda}\overline{z}\omega z) = T_{k,-i}(\omega z)[T_{k,-k}(-\omega), T_{-k,-i}(z)]$$

where $k \neq \pm i$ and $k < 0$. Therefore

$${}^{\sigma}T_{i,-i}(\overline{\lambda}\delta) = {}^{\sigma}T_{k,-i}(\omega z)[{}^{\sigma}T_{k,-k}(-\omega), {}^{\sigma}T_{-k,-i}(z)].$$

Since $k \neq \pm 1$, $\sigma$ commutes with $T_{k,-k}(-\omega)$. On the other hand, by the proof of the short root case above, ${}^{\sigma}T_{k,-i}(\omega z)$ and ${}^{\sigma}T_{-k,-i}(z)$ are in $F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. Therefore

$${}^{\sigma}T_{i,-i}(\overline{\lambda}\delta) \in F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega).$$

Now let $\alpha \in \Gamma_{\min}(\mathfrak{I}\mathfrak{J})$. So $\alpha = \tau + \upsilon$ for some

$$\tau \in \{x - \lambda\overline{x} \mid x \in \mathfrak{I}\mathfrak{J}\} \quad \text{and} \quad \upsilon \in \langle x\eta\overline{x} \mid x \in \mathfrak{I}\mathfrak{J}, \eta \in \Lambda\rangle.$$

Let $\tau = x_1y_1 - \lambda\overline{y_1 x_1}$, where $x_1 \in \mathfrak{I}$, $y_1 \in \mathfrak{J}$. Using R(5), we have $T_{i,-i}(\overline{\lambda}\tau) = [T_{ij}(-\overline{y_1}), T_{j,-i}(\overline{x_1})]$, for any $j \neq \pm i$. Therefore

$${}^{\sigma}T_{i,-i}(\overline{\lambda}\tau) = [{}^{\sigma}T_{ij}(-\overline{y_1}), {}^{\sigma}T_{j,-i}(\overline{x_1})].$$

By the short root case, ${}^{\sigma}T_{ij}(-\overline{y_1})$ and ${}^{\sigma}T_{j,-i}(\overline{x_1})$ are in $F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. This shows that ${}^{\sigma}T_{i,-i}(\overline{\lambda}\tau) \subseteq F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. We are left with $T_{-i,i}(\overline{\lambda}\upsilon)$. But it is easy to see that $\upsilon$ can be written as the sum of elements from the sets

$$\{x - \lambda\overline{x} \mid x \in \mathfrak{I}\mathfrak{J}\}, \qquad \{y\Gamma\overline{y} \mid y \in \mathfrak{J}\} \quad \text{and} \quad \{x\Omega\overline{x} \mid x \in \mathfrak{I}\}.$$

Therefore the argument for $T_{-i,i}(\overline{\lambda}\upsilon)$ reduces to the cases above and the first part of the theorem is complete.

(2) By Theorem 2.2, ${}^{E_{2n}(A,\Lambda)}F_{2n}((\mathfrak{I}, \Gamma)(\mathfrak{J}, \Omega))$ is generated by elements of the form ${}^{T_{ij}(a)}T_{ji}(x)$ where $a \in A$ and $x \in \mathfrak{I}\mathfrak{J}$, if $i \neq \pm j$, and by elements of the form ${}^{T_{i,-i}(a)}T_{-i,i}(x)$ where $a \in \lambda^{-(\varepsilon(j)+1)/2}\Lambda$ and $x \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma\Omega$, if $i = -j$. Let's deal first with the short roots. We shall show that ${}^{T_{ij}(a)}T_{ji}(x)$ where $i \neq \pm j$, $a \in A$ and $x \in \mathfrak{I}\mathfrak{J}$ is in $F_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega)$. Since $x \in \mathfrak{I}\mathfrak{J}$, we can write $x = \sum_l x_l y_l$ where $x_l \in \mathfrak{I}$, $y_l \in \mathfrak{J}$. By R(2), it suffices to prove the theorem for $x = x_1y_1$ where $x_1 \in \mathfrak{I}$, $y_1 \in \mathfrak{J}$. Since $n \geqslant 3$, there is an $h \neq \pm i, \pm j$. By (R4),

$${}^{T_{ij}(a)}T_{ji}(x_1y_1) = {}^{T_{ij}(a)}[T_{jh}(x_1), T_{hi}(y_1)] = [{}^{T_{ij}(a)}T_{jh}(x_1), {}^{T_{ij}(a)}T_{hi}(y_1)].$$

Applying now (R4) to the left and right-hand entries of the last commutator, we obtain that this commutator equals

$$[T_{ih}(ax_1)T_{jh}(x_1), T_{hi}(y_1)T_{hj}(-y_1a)] \in E_{2n}(\mathfrak{I} + \mathfrak{J}, \Gamma + \Omega),$$

since $\mathfrak{I}$ and $\mathfrak{J}$ are two sided ideals in $A$. Next we turn to the case of long roots. Suppose $i = -j$. Therefore we are dealing with elements of the form ${}^{T_{i,-i}(\alpha)}T_{-i,i}(\gamma)$, where

$$\alpha \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda \quad \text{and} \quad \gamma \in \lambda^{-(\varepsilon(-i)+1)/2}\Gamma\Omega.$$

Let $\gamma = \nu + \beta + \delta$ for some

$$\nu \in \lambda^{-(\varepsilon(-i)+1)/2}\Gamma_{\min}(\mathfrak{I}\mathfrak{J}),$$

$$\beta \in \lambda^{-(\varepsilon(-i)+1)/2}\langle y\Gamma\overline{y} \mid y \in \mathfrak{J}\rangle$$

and

$$\delta \in \lambda^{-(\varepsilon(-i)+1)/2}\langle x\Omega\overline{x} \mid x \in \mathfrak{I}\rangle.$$

We shall show that

$${}^{T_{i,-i}(\alpha)}T_{-i,i}(\nu), {}^{T_{i,-i}(\alpha)}T_{-i,i}(\beta) \quad \text{and} \quad {}^{T_{i,-i}(\alpha)}T_{-i,i}(\delta)$$

are all in $F_{2n}(\mathfrak{I}+\mathfrak{J},\Gamma+\Omega)$. Let $\mu=\lambda^{-(\varepsilon(-i)+1)/2}$. For $T_{-i,i}(\delta)$, it is enough to prove it when $\delta=\mu\overline{z}\omega z$, where $z\in\mathfrak{I}$ and $\omega\in\Omega$. The argument for $T_{-i,i}(\beta)$ is the same. Let $h\neq\pm i$ such that $\varepsilon(h)=-\varepsilon(i)$. Then by (R6), we have

$$\rho={}^{T_{i,-i}(\alpha)}T_{-i,i}(\mu\overline{z}\omega z)={}^{T_{i,-i}(\alpha)}T_{hi}(\mu\omega z)[{}^{T_{i,-i}(\alpha)}T_{h,-h}(-\mu\omega),{}^{T_{i,-i}(\alpha)}T_{-h,i}(z)].$$

Since $h\neq\pm i$, $T_{i,-i}(\alpha)$ commutes with $T_{h,-h}(-\mu\omega)$. Therefore we obtain

$$\begin{aligned}\rho &= T_{i,-i}(\alpha)T_{hi}(\mu\omega z)T_{i,-i}(-\alpha)[T_{h,-h}(-\mu\omega),{}^{T_{i,-i}(\alpha)}T_{-h,i}(z)]\\ &= T_{hi}(\mu\omega z)\underbrace{T_{hi}(-\mu\omega z)T_{i,-i}(\alpha)T_{hi}(\mu\omega z)T_{i,-i}(-\alpha)}_{R(6)}\times\\ &\quad\times[T_{h,-h}(-\mu\omega),{}^{T_{i,-i}(\alpha)}T_{-h,i}(z)]\\ &= T_{hi}(\mu\omega z)T_{h,-i}(-\mu\omega z\alpha)T_{h,-h}(\lambda^{(\varepsilon(i)-\varepsilon(h))/2}\omega z\alpha\overline{\omega z})\times\\ &\quad\times[T_{h,-h}(-\mu\omega),{}^{T_{i,-i}(\alpha)}T_{-h,i}(z)].\end{aligned}$$

On the other hand,

$$\begin{aligned}&[T_{h,-h}(-\mu\omega),{}^{T_{i,-i}(\alpha)}T_{-h,i}(z)]\\ &\quad=[T_{h,-h}(-\mu\omega),T_{i,-i}(\alpha)T_{-h,i}(z)T_{i,-i}(-\alpha)]\\ &\quad=[T_{h,-h}(-\mu\omega),T_{-h,i}(z)\underbrace{T_{-h,i}(-z)T_{i,-i}(\alpha)T_{-h,i}(z)T_{i,-i}(-\alpha)}_{R(6)}]\\ &\quad=[T_{h,-h}(-\mu\omega),T_{-h,i}(z)T_{-h,-i}(-z\alpha)T_{-h,h}(z\alpha\overline{z})].\end{aligned}$$

Now a quick inspection shows that $\rho\in F_{2n}(\mathfrak{I}+\mathfrak{J},\Gamma+\Omega)$.

Next, we consider the long root case ${}^{T_{i,-i}(\alpha)}T_{-i,i}(\nu)$ where $\nu\in\mu\Gamma_{\min}(\mathfrak{I}\mathfrak{J})$ and $\mu=\lambda^{-(\varepsilon(-i)+1)/2}$. So $\nu=\tau+\upsilon$ for some

$$\tau\in\mu\{x-\lambda\overline{x}\mid x\in\mathfrak{I}\mathfrak{J}\}\quad\text{and}\quad\upsilon\in\mu\langle x\eta\overline{x}\mid x\in\mathfrak{I}\mathfrak{J},\eta\in\Lambda\rangle.$$

Let $\tau=\mu(x_1y_1-\lambda\overline{y_1x_1})$ where $x_1\in\mathfrak{I}$, $y_1\in\mathfrak{J}$. Depending on sign of $i$, two cases may occur. Suppose first $\varepsilon(i)=1$. Thus $\mu=1$. Using R(5), we have $T_{-i,i}(\tau)=[T_{-i,j}(x_1),T_{j,i}(y_1)]$, where $j\neq\pm i$. Therefore

$$\begin{aligned}{}^{T_{i,-i}(\alpha)}T_{-i,i}(\tau) &= [\underbrace{{}^{T_{i,-i}(\alpha)}T_{-i,j}(x_1)}_{R(6)},\underbrace{{}^{T_{i,-i}(\alpha)}T_{ji}(y_1)}_{R(6)}]\\ &= [T_{ij}(\alpha x_1)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{x_1}\alpha x_1)T_{-i,j}(x_1),\\ &\quad T_{ji}(y_1)T_{j,-i}(-y_1\alpha)T_{j,-j}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}y_1\alpha\overline{y_1})].\end{aligned}$$

But

$$\begin{aligned}&T_{ij}(\alpha x_1),T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{x_1}\alpha x_1)\\ &\quad=T_{-j,j}(-\lambda^{-(\varepsilon(-j)+1)/2}\overline{x_1}\alpha x_1),T_{-i,j}(x_1),T_{ji}(y_1),T_{j,-i}(-y_1\alpha)\end{aligned}$$

and

$$T_{j,-j}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}y_1\alpha\overline{y_1})=T_{j,-j}(\lambda^{-(\varepsilon(j)+1)/2}y_1\alpha\overline{y_1})$$

which appear in the above equation are in $F_{2n}(\mathfrak{I}+\mathfrak{J},\Gamma+\Omega)$.

Now consider the case $\varepsilon(i) = -1$. Therefore $\mu = \overline{\lambda}$. Thus $\tau = \overline{(-y_1)}\overline{x_1} - \overline{\lambda}(x_1)(-y_1)$. Using R(5), we have

$$T_{-i,i}(\tau) = [T_{-i,j}(-\overline{y_1}), T_{j,i}(\overline{x_1})],$$

where $j \neq \pm i$. Therefore

$$^{T_{i,-i}(\alpha)}T_{-i,i}(\tau) = [\underbrace{^{T_{i,-i}(\alpha)}T_{-i,j}(-\overline{y_1})}_{R(6)}, \underbrace{^{T_{i,-i}(\alpha)}T_{ji}(\overline{x_1})}_{R(6)}]$$

and one completes the proof as in the case $\varepsilon(i) = 1$ above.

We are left with $T_{-i,i}(\upsilon)$. But it is easy to see that elements of the form $x\eta\overline{x}$ where $\eta \in \Lambda$ can be written as a sum of elements from the sets $\{x - \lambda\overline{x} | x \in \mathfrak{I}\mathfrak{J}\}$, $\{y\Gamma\overline{y} \mid y \in \mathfrak{J}\}$ and $\{x\Omega\overline{x} \mid x \in \mathfrak{I}\}$. Therefore the argument for $T_{-i,i}(\upsilon)$ reduces to the cases above and the proof is complete. □

COROLLARY 2.4. *Let* $(A, \Lambda)$ *be a form ring and let* $s \in \mathrm{Center}(A)$ *such that* $s = \overline{s}$ *and* $s\Lambda \subseteq \Lambda$, *e.g.* $s = t\overline{t}$ *where* $t \in \mathrm{Center}(A)$. *Then*

(1) $^{G_2(A,\Lambda)}F_{2n}(s^{3k}A, s^{3k}\Lambda) \subseteq F_{2n}(s^kA, s^k\Lambda)$, *providing* $n \geqslant 2$.
(2) $^{E_{2n}(A,\Lambda)}F_{2n}(s^{3k}A, s^{3k}\Lambda) \subseteq F_{2n}(s^kA, s^k\Lambda)$, *providing* $n \geqslant 3$.

*Proof.* The corollary follows from Theorem 2.3, by letting $(\mathfrak{I}, \Gamma) = (\mathfrak{J}, \Omega) = (sA, s\Lambda)$ and recognizing that $(s^{3k}A, s^{3k}\Lambda) \subseteq (s^kA, s^k\Lambda)(s^kA, s^k\Lambda)$. □

COROLLARY 2.5. *If* $(A, \Lambda)$ *is a form ring then* $G_2(A, \Lambda)$ *normalizes* $E_{2n}(A, \Lambda)$, *providing* $n \geqslant 2$.

*Proof.* Let $s = 1$ in Theorem 2.4 (1). □

The next result is due to Bak and if $\Lambda = \Lambda_{\max}$, independently also to Vaserstein.

THEOREM 2.6. *Let* $(A, \Lambda)$ *be a form ring such that* $A$ *is semilocal. If* $n > 1$ *then*

$$G_{2n}(A, \Lambda) = G_2(A, \Lambda)E_{2n}(A, \Lambda) = E_{2n}(A, \Lambda)G_2(A, \Lambda),$$

$E_{2n}(A, \Lambda)$ *is normal in* $G_{2n}(A, \Lambda)$ *and the quotient* $G_{2n}(A, \Lambda)/E_{2n}(A, \Lambda)$ *is abelian*.

*Proof.* If $\sigma$ is a $2n \times 2n$ matrix with coefficients in $A$, let

$$^t(\sigma_1, \ldots, \sigma_n, \sigma_{-n}, \ldots, \sigma_{-1})$$

denote the $(n + 1)$st column of $\sigma$, where $\sigma_1, \ldots, \sigma_n, \sigma_{-n}, \ldots, \sigma_{-1} \in A$ and $t$ denoted the transpose operator taking row vectors to column vectors. Suppose $\sigma \in G_{2n}(A, \Lambda)$. By [9, Section IV, (3.11)], there is an $\varepsilon \in E_{2n}(A, \Lambda)$ such that

$$^t((\varepsilon\sigma)_{-n}, \ldots, (\varepsilon\sigma)_{-1})$$

is a unimodular vector, i.e. there exist $a_{-n}, \ldots, a_{-1} \in A$ such that

$$\sum_{i=-n}^{-1} a_i(\varepsilon\sigma)_i = 1.$$

It follows from [8, Section V, (3.3)(1) and (3.4)(a)] that there is a product $\tau$ of elements of the kind $T_{ij}(a)$ where $i, j \in \{-n, \dots, -1\}$ such that

$$((\tau\varepsilon\sigma)_{-n}, \dots, (\tau\varepsilon\sigma)_{-1}) = (1, 0, \dots, 0).$$

Now it is straightforward to find an element $\rho \in E_{2n}(A, \Lambda)$ such that

$$((\rho\varepsilon\sigma)_1, \dots, (\rho\varepsilon\sigma)_n, (\rho\varepsilon\sigma)_{-n}, \dots, (\rho\varepsilon\sigma)_{-1}) = (0, \dots, 0, 1, \dots, 0).$$

This says that the matrix $\rho\tau\varepsilon\sigma$ fixes the basis element $e_{-n}$. A standard argument (see the Proof of [9, Section IV, (3.4)]) shows that there is an $\delta \in E_{2n}(A, \Lambda)$ such that $\delta\rho\tau\varepsilon\sigma$ fixes not only $e_{-n}$, but also $e_n$. Thus $\delta\rho\tau\varepsilon\sigma$ leaves invariant the hyperbolic plane $H$ generated by $e_n, e_{-n}$. Since $\delta\rho\tau\varepsilon\sigma$ preserves the Hermitian form $h$, it follows that it leaves the orthogonal complement of $H$ invariant. But this is the subspace generated by $e_1, \dots, e_{n-1}, e_{-(n-1)}, \dots, e_{-1}$. Thus $\delta\rho\tau\varepsilon\sigma \in G_{2(n-1)}(A, \Lambda)$. Thus $\sigma \in E_{2n}(A, \Lambda)G_{2(n-1)}(A, \Lambda)$. Repeating the argument for each $m$ such that $2 \leqslant m \leqslant n$, we get

$$\sigma \in E_{2n}(A, \Lambda)G_2(A, \Lambda) = (\text{by (2.5)})G_2(A, \Lambda)E_{2n}(A, \Lambda).$$

This shows that

$$G_{2n}(A, \Lambda) = G_2(A, \Lambda)E_{2n}(A, \Lambda)$$

and $E_{2n}(A, \Lambda)$ is normal in $G_{2n}(A, \Lambda)$. If $\pi$ denotes the permutation matrix

$$\left(\begin{array}{cccc|cccc}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
 & & 1 & & & & & \\
 & & & \ddots & & & & \\
 & & & 1 & & & & \\
\hline
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & 1 & & \\
 & & & & & & 0 & 1 \\
 & & & & & & -1 & 0
\end{array}\right)$$

then obviously $\pi \in G_{2n}(A, \Lambda)$ (because it satisfies the defining equations in (2.2)), $\pi$ normalizes $E_{2n}(A, \Lambda)$(because conjugation by $\pi$ leaves the set of elementary transvections invariant), and

$$G_2(A, \Lambda)E_{2n}(A, \Lambda) = G_{2n}(A, \Lambda) = {}^{\pi}G_{2n}(A, \Lambda) = {}^{\pi}G_2(A, \Lambda)E_{2n}(A, \Lambda).$$

Since $G_2(A, \Lambda)$ and ${}^{\pi}G_2(A, \Lambda)$ commute, it follows that $G_{2n}(A, \Lambda)/E_{2n}(A, \Lambda)$ is Abelian. □

We close this section by recalling a lemma which will be used in Section 4.

LEMMA 2.7. *Let $A$ be module finite over a Noetherian ring $R$. Then for any $s$ in $R$, there is a nonnegative integer $k$ such that the map $s^kA \longrightarrow \langle s\rangle^{-1}A$ induced by the canonical homomorphism $A \longrightarrow \langle s\rangle^{-1}A$ is injective.*

The verification is easy and can be found in the proof of Lemma 4.10 in [3]. The above lemma shows that if $A$ is a Noetherian ring, then there is an integer $k$, such that the relative congruence subgroup $G_{2n}(s^kA, s^k\Lambda)$ embeds in $G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$. This result will be used in proving Theorem 4.6.

## 3. On Bak's Dimension Theory

In this section we give a self-contained account of a portion of Bak's dimension theory and show how to apply it to general quadratic groups.

Recall that a relation $\leqslant$ on a set is called a *quasi-ordering*, if it is reflexive and transitive. If in addition, it is anti-symmetric, then it is called a *partial ordering*. A quasi-ordering $\leqslant$ is *directed*, if given elements $a$ and $b$, there is an element $c$ such that $a \leqslant c$ and $b \leqslant c$. Following Bak [6], we define a category with structure as follows.

DEFINITION 3.1. A *category with structure* is a category $\mathcal{C}$ together with a class $\mathcal{S}(\mathcal{C})$ of commutative squares in $\mathcal{C}$ called *structure squares* and a class of $\mathcal{I}(\mathcal{C})$ of functors from directed quasi-ordered sets to $\mathcal{C}$ called *infrastructure functors*, satisfying the following conditions.

(1) $\mathcal{S}(\mathcal{C})$ is closed under isomorphism of commutative squares.

(2) For each object $A$ of $\mathcal{C}$, the *trivial* square i.e.,

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{1} & A \end{array}$$

is in $\mathcal{S}(\mathcal{C})$,

(3) $\mathcal{I}(\mathcal{C})$ is closed under isomorphism of functors.

(4) For each object $A$ of $\mathcal{C}$, the *trivial* functor $F_A$: $\{*\} \longrightarrow \mathcal{C}$, $* \mapsto A$, is in $\mathcal{I}(\mathcal{C})$, where $\{*\}$ denotes the directed quasi-ordered set with precisely one element $*$.

(5) For each $F$: $I \longrightarrow \mathcal{C}$ in $\mathcal{I}(\mathcal{C})$, the direct limit $\varinjlim_I F$ exists in $\mathcal{C}$.

Next we define a category with dimension.

DEFINITION 3.2. Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure. Let $d$: $\mathcal{O}bj(\mathcal{C}) \longrightarrow \mathbb{Z}^{\geqslant 0} \cup \infty$ be a function which is constant on isomorphism classes of objects. Let $A \in \mathcal{O}bj(\mathcal{C})$ such that $0 < d(A) < \infty$. A *d-reduction* of $A$ is a set

$$\begin{array}{ccc} A & \longrightarrow & B_i \\ \downarrow & & \downarrow {\scriptstyle (i \in I)} \\ C_i & \longrightarrow & D_i \end{array}$$

of structure squares where $I$ is a directed quasi-ordered set and $B\colon I \longrightarrow \mathcal{C}$, $i \mapsto B_i$, is an infrastructure functor such that the following holds.

(1) If $i \leqslant j \in I$ then the triangle

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow & \\ B_i & \longrightarrow & B_j \end{array}$$

commutes.

(2) $d(\varinjlim_I B) = 0$.

(3) $d(C_i) < d(A)$ for all $i \in I$.

A function $d$ is called a *dimension function* on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ if for any object $A$ of $\mathcal{C}$, such that $0 < d(A) < \infty$, $A$ has a $d$-reduction. In this case, the quadruple $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ is called a *category with dimension*.

For the rest of this section $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ will denote a category with dimension and $\mathcal{G}, \mathcal{E}\colon \mathcal{C} \longrightarrow$ Group a pair of group valued functors on $\mathcal{C}$ such that $\mathcal{E} \subseteq \mathcal{G}$.

DEFINITION-LEMMA 3.3. Let $n \geqslant 0$. Define the functor $\mathcal{G}^n : \mathcal{C} \longrightarrow$ Group, by

$$\mathcal{G}^n(A) = \bigcap_{\substack{A \longrightarrow B \\ d(B) \leqslant n}} \mathrm{Ker}(\mathcal{G}(A) \longrightarrow \mathcal{G}(B)/\mathcal{E}(B)).$$

In general $\mathcal{G}^n(A)$ is not a normal subgroup of $\mathcal{G}(A)$. Clearly $\mathcal{E}(A) \subset \mathcal{G}^n(A)$ for any object $A$ of $\mathcal{C}$ and if $d(A)$ is finite then $\mathcal{G}^n(A) = \mathcal{E}(A)$ for all $n \geqslant d(A)$, because the identity morphism $A \to A$ is now one of those occurring in the definition of $\mathcal{G}^n(A)$. The filtration

$$\mathcal{G}(A) \supseteq \mathcal{G}^0(A) \supseteq \mathcal{G}^1(A) \supseteq \cdots$$

is called the *dimension filtration on* $\mathcal{G}$ *with respect to* $\mathcal{E}$. For a fixed object $A$, a set $\mathcal{S}$ of morphisms $A \to B$ such that for any $A \to B \in \mathcal{S}$, $d(B) \leqslant n$, and such that

$$\mathcal{G}^n(A) = \bigcap_{A \longrightarrow B \in \mathcal{S}} \mathrm{Ker}(\mathcal{G}(A) \longrightarrow \mathcal{G}(B)/\mathcal{E}(B)),$$

is called a *defining set for* $\mathcal{G}^n(A)$. It is easy to check that defining sets exist, although they are not as a rule unique. However, for any defining set $\mathcal{S}$, the map

$$\mathcal{G}(A)/\mathcal{G}^n(A) \longrightarrow \prod_{A \longrightarrow B \in \mathcal{S}} \mathcal{G}(B)/\mathcal{E}(B) \qquad (3.3.1)$$

of coset spaces is injective. Clearly if $d(A) \leqslant n$, then $\mathcal{G}^n(A) = \mathcal{E}(A)$, because one can enlarge if necessary any defining set $\mathcal{S}$ for $\mathcal{G}^n(A)$ to a defining set $\mathcal{S}'$ by adding the identity morphism id: $A \to A$.

DEFINITION 3.4. A pair $\mathcal{G}, \mathcal{E}$ of group valued functors on $\mathcal{C}$ is called *good* if the following holds.

(1) $\mathcal{E}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) For any $A$ of $\mathcal{C}$, $\mathcal{E}(A)$ is a perfect group.
(3) For any zero dimensional object $A$, $K_1(A) := \mathcal{G}(A)/\mathcal{E}(A)$ is an abelian group.
(4) For any structure square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

let

$$H = \mathrm{Ker}(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B)) \quad \text{and} \quad L = \mathrm{Ker}(\mathcal{G}(A) \to \mathcal{G}(C)/\mathcal{E}(C)).$$

Then the mixed commutator $[H, L] \subseteq \mathcal{E}(A)$.

The following theorem plays a crucial role in this note and is a central result in Bak's dimension theory.

THEOREM 3.5. *Let $\mathcal{C} = (\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ be a category with dimension and $(\mathcal{G}, \mathcal{E})$ be a good pair of group valued functors on $\mathcal{C}$. Then the dimension filtration $\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$ of $\mathcal{G}$ with respect to $\mathcal{E}$ is a normal filtration of $\mathcal{G}$ such that the quotient functor $\mathcal{G}/\mathcal{G}^0$ takes its values in abelian groups and the filtration $\mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$ is a descending central series such that if $d(A)$ is finite then $\mathcal{G}^n(A) = \mathcal{E}(A)$ whenever $n \geqslant d(A)$. In particular, if $d(A)$ is finite, then $\mathcal{E}(A)$ is normal in $\mathcal{G}(A)$.*

*Proof*. If $A$ is an object of $\mathcal{C}$, let $\mathcal{S}_n(A)$ denote a set of defining morphisms for $\mathcal{G}^n(A)$.

By Lemma 3.3, the map

$$\mathcal{G}(A)/\mathcal{G}^0(A) \to \prod_{A \longrightarrow B \in \mathcal{S}_0(A)} \mathcal{G}(B)/\mathcal{E}(B)$$

is injective. Since each $\mathcal{G}(B)/\mathcal{E}(B)$ is Abelian by (3) of Definition 3.4, it follows that $\mathcal{G}^0(A)$ is normal in $\mathcal{G}(A)$ and the quotient $\mathcal{G}(A)/\mathcal{G}^0(A)$ is Abelian.

Let $n \geqslant 0$. We shall show that for any object $A$, $[\mathcal{G}^0(A), \mathcal{G}^n(A)] \subseteq \mathcal{G}^{n+1}(A)$. Since for any object $B$ such that $d(B) \leqslant n+1$, we have that $\mathcal{G}^{n+1}(B) = \mathcal{E}(B)$ and since the map

$$\mathcal{G}(A)^0/\mathcal{G}^{n+1}(A) \to \prod_{A \longrightarrow B \in \mathcal{S}_{n+1}(A)} \mathcal{G}^0(B)/\mathcal{E}(B)$$

is injective, we can reduce to the case $d(A) \leqslant n+1$. Suppose $d(A) \leqslant n+1$. Let $\sigma \in \mathcal{G}^0(A)$ and $\rho \in \mathcal{G}^n(A)$. Let

$$\begin{array}{ccc} A & \longrightarrow & B_i \\ \downarrow & & \downarrow {\scriptstyle (i\in I)} \\ C_i & \longrightarrow & D_i \end{array}$$

be a $d$-reduction of $A$. Since $d(\varinjlim_I B_i) = 0$ and since $\mathcal{G}$ and $\mathcal{E}$ commute with $\varinjlim_I$, there is an $i \in I$ such that

$$\sigma \in \operatorname{Ker}(\mathcal{G}(A) \longrightarrow \mathcal{G}(B_i)/\mathcal{E}(B_i)).$$

Since

$$d(C_i) < n+1, \mathcal{G}^n(C_i) = \mathcal{E}(C_i).$$

Thus

$$\rho \in \operatorname{Ker}(\mathcal{G}(A) \to \mathcal{G}(C_i)/\mathcal{E}(C_i)).$$

Now by property (4) of Definition 3.4, $[\sigma, \rho] \in \mathcal{E}(A) = \mathcal{G}^{n+1}(A)$.

We show finally that for any $n$, $\mathcal{G}^n$ is normal in $\mathcal{G}$. The proof is by induction on $n$. The case $n = 0$ has been done above. Suppose $n > 0$. By induction on $n$, we can assume for all $0 \leqslant m < n$ that $\mathcal{G}^m$ is normal in $\mathcal{G}$. Since the map

$$\mathcal{G}(A)/\mathcal{G}^n(A) \to \prod_{A \longrightarrow B \in \mathcal{S}_n(A)} \mathcal{G}(B)/\mathcal{E}(B)$$

is injective, it suffices to show that each $\mathcal{E}(B)$ above is normal in $\mathcal{G}(B)$. This allows us to reduce to the case that $d(A) \leqslant n$ and $\mathcal{G}^n(A) = \mathcal{E}(A)$. We have shown already that

$$[\mathcal{G}^0(A), \mathcal{G}^{n-1}(A)] \subseteq \mathcal{G}^n(A) = \mathcal{E}(A).$$

Since $\mathcal{E}(A)$ is perfect by property (3) of Definition 3.4, and $\mathcal{E}(A) \subseteq \mathcal{G}^{n-1}(A) \subseteq \mathcal{G}^0(A)$, it follows that

$$[\mathcal{G}^0(A), \mathcal{G}^{n-1}(A)] = \mathcal{G}^n(A).$$

But $\mathcal{G}^0(A)$ and $\mathcal{G}^{n-1}(A)$ are normal in $\mathcal{G}(A)$, by the induction assumption. Thus $\mathcal{G}^n(A)$ is normal in $\mathcal{G}(A)$. □

*Remark.* Bak has also an alternative version of the theorem above in which a good pair $(\mathcal{G}, \mathcal{E})$ is replaced by a natural transformation $\mathcal{S} \to \mathcal{G}$ of group valued functors such that

(1) $\mathcal{S}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) $\mathcal{S}(A)$ is perfect for any $A$.

(3) $\mathcal{G}(A)/\mathrm{image}(\mathcal{S}(A) \to \mathcal{G}(A))$ is Abelian for any zero-dimensional object $A$.
(4) $\mathrm{Ker}(\mathcal{S}(A) \to \mathcal{G}(A)) \subseteq \mathrm{Center}(\mathcal{S}(A))$ for any finite-dimensional object $A$.
(5) The extension $\mathcal{S} \to \mathcal{G}$ satisfies excision on any structure square.

The conclusion of the alternative version is the same as that above. The alternative approach is used in [11], where it is applied to general linear groups and in [7] where it is applied to net general linear groups.

There are many ways to make the category of form rings into a category with dimension such that $G_n$, $E_n$ is a good pair of group valued functors. We describe next a way based on quasi-finite localization-completion squares and the Bass–Serre dimension.

Let $A_R$ denote a pair consisting of an associative ring $A$ with identity and a commutative ring $R \subseteq \mathrm{Center}(A)$. Thus $A_R$ is an algebra over $R$. *A morphism* $A_R \to A'_{R'}$ of algebras is a ring homomorphism $f\colon A \to A'$ such that $f(R) \subseteq R'$. Next we recall the Bass–Serre dimension of $A_R$.

Let $X$ be a topological space. The *dimension* of $X$ is the length $n$ of the longest chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n$ of nonempty closed *irreducible* subsets $X_i$ of $X$, [8, Section III]. Define $\delta(X)$ to be the smallest nonnegative integer $d$ such that $X$ is a finite union of irreducible *Noetherian* subspaces of dimension $\leqslant d$. If there is no such $d$, then by definition $\delta(X) = \infty$. Let $R$ be a commutative ring. Let $\mathrm{Spec}(R)$ denote the topological space consisting of the set of all prime ideals of $R$, under the Zariski topology and let $\mathrm{Max}(R)$ denote the subspace consisting of all maximal ideals of $R$. Then the *Bass–Serre dimension* of $R$ is $\delta(\mathrm{Max}(R))$ and is denoted by $\delta(R)$. Define the *Bass–Serre dimension* $\delta(A_R)$ *of* $A_R$ *by*

$$\delta(A_R) = \begin{cases} \delta(R) & \text{if } A \text{ is quasi finite over } R, \\ \infty & \text{otherwise.} \end{cases}$$

Recall that an $R$-algebra $A$ is called quasi-finite over $R$ if there is a direct system of finite $R$-subalgebras $A_i$ of $A$ such that $\varinjlim_I A_i = A$.

A *form algebra over a commutative ring* $R$ is a form ring $(A_R, \Lambda)$ where the involution leaves $R$ invariant. A *morphism* $(A_R, \Lambda) \to (A'_{R'}, \Lambda')$ of form algebras is a morphism of form rings which defines an algebra morphism $A_R \to A'_{R'}$. A form algebra $(A_R, \Lambda)$ is called *module finite*, if $A$ is module finite over $R$ and is called *quasi-finite*, if $A_R$ is quasi-finite. If $(A_R, \Lambda)$ is a form algebra, let $R_0$ denote the subring of $R$ generated by all $a\overline{a}$ such that $a \in R$. Define the *Bass–Serre dimension of* $(A_R, \Lambda)$ by

$$\delta(A_R, \Lambda) = \begin{cases} \delta(R_0) & \text{if } (A_R, \Lambda) \text{ is quasi-finite,} \\ \infty & \text{otherwise.} \end{cases}$$

The next task is to put structure on the category $\mathcal{F}orm\ algebras$, which makes it a category with dimension under Bass–Serre dimension.

Let $\mathcal{M}od(R)$ denote the category of all modules over the commutative ring $R$ and $\mathcal{N}oeth(R) \subseteq \mathcal{M}od(R)$ the full subcategory of all Noetherian modules over $R$.

If $s \in R$ and $M \in \mathcal{M}od(R)$, let $\hat{M}_s = \varprojlim_{i \geqslant 0} M/Ms^i$ denote the *completion* of $M$ at $s$. Let $\langle s\rangle^{-1}M$ denote the module of $\langle s\rangle$-fractions of $M$ where $\langle s\rangle$ denotes the multiplicative set $\{1, s, s^2, \ldots\}$ generated by $s$. The square

$$\begin{array}{ccc} M & \longrightarrow & \langle s\rangle^{-1}M \\ \downarrow & & \downarrow \\ \hat{M}_{(s)} & \longrightarrow & \langle s\rangle^{-1}\hat{M}_{(s)} \end{array}$$

is called the *localization-completion square* of $M$ at $s$. Whereas the functor $M \mapsto \hat{M}_{(s)}$ is exact on $\mathcal{N}oeth(R)$ (in particular if $N \subseteq M$, there is a canonical embedding $\hat{N}_{(s)} \subseteq \hat{M}_{(s)}$) and whereas the localization-completion square above is a pullback square if $M \in \mathcal{N}oeth(R)$, these facts fail to hold over $\mathcal{M}od(R)$. To rectify this problem, Bak [3] has defined for any $R$-module $M$, its *finite completion* at $s$ by $\widetilde{M}_{(s)} = \varinjlim_J (\hat{M}_j)_{(s)}$ where $\{R_j | j \in J\}$ is any directed system of subrings $R_j \subseteq R$ such that each $R_j$ is finitely generated as a $\mathbb{Z}$-algebra, contains $s$, and $\varinjlim_J R_j = R$ and $\{M_j | j \in J\}$ is any directed system of Abelian subgroups $M_j \subseteq M$ such that each $M_j$ is a finitely generated $R_j$-module and $\varinjlim_J M_j = M$. It is easy to check that $\widetilde{M}_{(s)}$ does not depend on the choice of the directed system above. Clearly $\widetilde{M}_{(s)} = \hat{M}_{(s)}$ if $M \in \mathcal{N}oeth(R)$ and $R$ is finitely generated as a $\mathbb{Z}$-algebra. The square

$$\begin{array}{ccc} M & \longrightarrow & \langle s\rangle^{-1}M \\ \downarrow & & \downarrow \\ \widetilde{M}_{(s)} & \longrightarrow & \langle s\rangle^{-1}\widetilde{M}_{(s)} \end{array}$$

is called the *localization-finite-completion square* of $M$ at $s$. The exactness of finite completion on $\mathcal{M}od(R)$ and the pullback property for localization-finite-completion squares on $\mathcal{M}od(R)$ follow from the analogous properties, respectively, of ordinary completion and of ordinary localization-completion squares on $\mathcal{N}oeth(R)$.

Let $M \in \mathcal{M}od(R)$. Whereas ordinary completion $\hat{M}_{(s)}$ does *not* depend on $R$, finite completion $\widetilde{M}_{(s)}$ does. If confusion can arise, we shall write $(\widetilde{M}_{(s)})_{\widetilde{R}_{(s)}})$ in place of $\widetilde{M}_{(s)}$.

DEFINITION-LEMMA 3.6 (Bak). Let $A_R$ be an $R$-algebra. Let $s \in R$ and let $\{R_\alpha \,|\, \alpha \in J\}$ and $\{A_\alpha \,|\, \alpha \in J\}$ be directed systems in $R$ and $A$, respectively, used to construct $(\widetilde{A}_{(s)})_{\widetilde{R}_{(s)}}$. Let $x, y \in \widetilde{A}_{(s)}$. Choose $\alpha, \beta \in J$ and elements $x' \in (\hat{A}_\alpha)_{(s)}$ and $y' \in (\hat{A}_\beta)_{(s)}$ such that $x'$ and $y'$ represent $x$ and $y$, respectively. Neither $A_\alpha$ nor $A_\beta$ is necessarily closed under multiplication in $A$. However, since $A_\alpha$ is module finite over $R_\alpha$ and $A_\beta$ is module finite over $R_\beta$, there is a $\gamma \in J$ such that $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$, and $A_\alpha A_\beta \subseteq A\gamma$. Let $\prod_{i\geqslant 0} x_i \in \prod_{i\geqslant 0} A_\alpha$ represent $x'$ and $\prod_{i\geqslant 0} y_i \in \prod_{i\geqslant 0} A_\beta$ represent $y'$. Define $x \circ y$ to be the class in $\widetilde{A}_{(s)}$ of the element of $(\hat{A}_\gamma)_{(s)}$ defined by $\prod_{i\geqslant 0} x_i y_i \in \prod_{i\geqslant 0} A_\gamma$. Then the product $x \circ y$ is independent of all the choices made and makes $\widetilde{A}_{(s)}$ into an $\widetilde{R}_{(s)}$-algebra.

*Proof.* Straightforward. □

The result above paves the way for defining finite completions of form algebras. Let $(A_R, \Lambda)$ be a form algebra and let $s \in R_0$. Define the *finite completion of* $(A_R, \Lambda)$ *at* $s$ by

$$(A_R, \Lambda)\tilde{}_{(s)} = (A_R, \Lambda)\tilde{}_{(\widetilde{R}_0)_{(s)}} = \Big((\widetilde{A}_{(s)})_{\tilde{R}_{0(s)}}, (\widetilde{\Lambda}_{(s)})_{\tilde{R}_{0(s)}}\Big).$$

Define the ordinary *completion of* $(A_R, \Lambda)$ *at* $s$ *by* $(A_R, \Lambda)\hat{}_{(s)} = (\hat{A}_{(s)}, \hat{\Lambda}_{(s)})$.

LEMMA 3.7. *Let $(A_R, \Lambda)$ be a module finite form algebra such that $R$ is finitely generated as a $\mathbb{Z}$-algebra. If $s \in R_0$, then $(A_R, \Lambda)\tilde{}_{(s)} = (A_R, \Lambda)\hat{}_{(s)}$.*

*Proof.* It suffices to show that $R$ is finitely generated as an $R_0$-module and that $R_0$ is finitely generated as a $\mathbb{Z}$-algebra. Let $a_1, \ldots, a_n \in R$ such that $a_1, \ldots, a_n, \overline{a}_1, \ldots, \overline{a}_n$ generate $R$ as a $\mathbb{Z}$-algebra. Clearly each $a_i$ and $\overline{a}_i$ satisfies the monic polynomial $X^2 + (a_i + \overline{a}_i)X + a_i\overline{a}_i$ whose coefficients lie in $R_0$. Thus $R$ is finitely generated as an $R_0$-module. It is an easy exercise to show that $R_0$ is generated as a $\mathbb{Z}$-algebra by all elements $a_i\overline{a}_i$ such that $1 \leqslant i \leqslant n$ and all elements $(x_1 \ldots x_k)(\overline{y}_1 \ldots \overline{y}_l) + (y_1 \ldots y_l)(\overline{x}_1 \ldots \overline{x}_k)$ where $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_l\}$ range over all disjoint, possibly empty subsets of $\{a_1, \ldots, a_n\}$. □

The following corollary is an easy consequence of the lemma above and its proof.

COROLLARY 3.8. *Let $(A_R, \Lambda)$ be a quasi-finite form algebra. Then there is a directed system of module finite form subalgebras $((A_\alpha)_{R_\alpha}, \Lambda_\alpha) \subseteq (A_R, \Lambda)$, $(\alpha \in J)$ such that each $R_\alpha$ is finitely generated as a $\mathbb{Z}$-algebra and*

$$(A_R, \Lambda) = \varinjlim_J ((A_\alpha)_{R_\alpha}, \Lambda_\alpha).$$

*Furthermore, if $s \in R_0$, we can assume that $s \in (R_\alpha)_0$, for all $\alpha \in J$. Thus*

$$(A_R, \Lambda)\tilde{}_{(s)} = \varinjlim_J (A_\alpha, \Lambda_\alpha)\hat{}_{(s)} = \varinjlim_J ((\hat{A}_\alpha)_{(s)\hat{R}_{\alpha(s)}}, (\hat{\Lambda}_\alpha)_{(s)}).$$

*In particular $(A_R, \Lambda)\tilde{}_{(s)}$ is quasi-finite.*

REDUCTION LEMMA 3.9. *Let $(A_R, \Lambda)$ be a form algebra such that $0 < \delta(A_R, \Lambda) < \infty$. Then there is a multiplicative subset $S \subseteq R_0$ such that*

$$\delta((S^{-1}A_R)_{S^{-1}R}, S^{-1}\Lambda) = 0$$

*and for all $s \in S$, $\delta((A_R, \Lambda)\tilde{}_{(s)}) < \delta(A_R, \Lambda)$.*

*Proof.* Let $X_1 \cup \cdots \cup X_r$ be a decomposition of $\mathrm{Max}(R_0)$ into irreducible Noetherian subspaces such that $\delta(X_i) \leqslant \delta(A_R, \Lambda)$ for all $1 \leqslant i \leqslant r$ and $\delta(X_{i_0}) = \delta(A_R, \Lambda)$ for some $i_0$. For each $1 \leqslant i \leqslant r$, let $\mathfrak{M}_i \in X_i$. Let

$$S = R_0 - \mathfrak{M}_1 \cup \cdots \cup \mathfrak{M}_r.$$

Since $(S^{-1}A_{S^{-1}R}, S^{-1}\Lambda)$ is obviously quasi-finite and $S^{-1}R_0$ is semilocal, it follows that

$$\delta(S^{-1}A_{S^{-1}R}, S^{-1}\Lambda) = \delta(S^{-1}R_0) = 0.$$

By the corollary above, $(A_R, \Lambda)\tilde{}_{(s)}$ is quasi-finite and by [3,4.17], $\delta(\tilde{R}_{0(s)}) < \delta(R_0)$. Thus

$$\delta(A_R, \Lambda)\tilde{}_{(s)} = \delta(\tilde{R}_{0(s)}) < \delta(R_0) = \delta(A_R, \Lambda). \qquad \square$$

We can now make the category $\mathcal{C} = \mathcal{F}orm\ algebras$ into a category with dimension. As structure squares, we take all localization-finite-completion squares

$$\begin{array}{ccc} (A_R, \Lambda) & \longrightarrow & (\langle s\rangle^{-1}A_{\langle s\rangle^{-1}R}, \langle s\rangle^{-1}\Lambda) \\ \downarrow & & \downarrow \\ (A_R, \Lambda)\tilde{}_{(s)} & \longrightarrow & \langle s\rangle^{-1}(A_R, \Lambda)\tilde{}_{(s)} \end{array}$$

where $s \in R_0$. If $S \subseteq R_0$ is a multiplicative set, give $S$ a quasi-ordering by defining $s \leqslant t$ if and only if there is a $u \in S$ such that $su = t$. As infrastructure functors, we take all functors of the kind

$$F\colon S \to \mathcal{C}, s \mapsto (\langle s\rangle^{-1}A_{\langle s\rangle^{-1}R}, \langle s\rangle^{-1}\Lambda).$$

Clearly $\varinjlim_S F = (S^{-1}A_{S^{-1}R}, S^{-1}\Lambda)$. From the Reduction Lemma above, it follows immediately that $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \delta)$ is a category with dimension.

MAIN THEOREM 3.10 *Let $n \geqslant 3$. Let $G_{2n}$ denote the general quadratic group functor on $\mathcal{C} = \mathcal{F}orm\ algebras$ and let $E_{2n}$ denote its elementary subgroup. Let $G_{2n} \supseteq G^0_{2n} \supseteq G^1_{2n} \supseteq \cdots$ denote the dimension filtration on $G_{2n}$ with respect to $E_{2n}$. Then this filtration is normal, the quotient functor $G_{2n}/G^0_{2n}$ is Abelian, and the filtration $G^0_{2n} \supseteq G^1_{2n} \supseteq \cdots$ is a descending central series such that $G^i_{2n}(A_R, \Lambda) = E_{2n}(A_R, \Lambda)$ whenever $i \geqslant \delta(A_R, \Lambda)$.*

*Proof.* It suffices to show by Theorem 3.5 that the pair $(G_{2n}, E_{2n})$ is good on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \delta)$. Property (1) for being good is obvious. Property (2) follows from R(1)–R(6) in Section 2, because $n \geqslant 3$. We shall prove property (3) next. Property (4) is the subject of the next section.

Suppose $\delta(A_R, \Lambda) = 0$. By definition $\delta(R_0) = 0$. Thus $R_0$ is semilocal. Since $(A_R, \Lambda)$ is quasi-finite, it follows that it is a direct limit $\varinjlim_j ((A_j)_{R_j}, \Lambda_j)$ of a directed system of form subalgebras $((A_j)_{R_j}, \Lambda_j) \subseteq (A_R, \Lambda)$ such that each $A_j$ is module finite over $R_j$, $R_0 \subseteq R_j$ and $R_j$ is finitely generated as an $R_0$-module. It follows (cf. proof of Lemma 3.7) that $A_j$ is finitely generated as an $R_0$-module. Thus $A_j$ is semilocal, by [8,III(2.5), 2.11]. It follows by Theorem 2.6 that $E_{2n}(A_j, \Lambda_j)$ is normal in $G_{2n}(A_j, \Lambda_j)$ and the quotient $G_{2n}(A_j, \Lambda_j)/E_{2n}(A_j, \Lambda_j)$ is Abelian. Taking direct limits, we obtain that the same is true for $G_{2n}(A, \Lambda)$ and $E_{2n}(A, \Lambda)$. □

## 4. Computation

The goal of this section is to complete the proof of Theorem 3.10 by showing that $(G_{2n}, E_{2n})$ satisfies property (4) in Definition 3.4 of a good pair of group valued functors on a category with dimension. This is achieved in Theorem 4.6 below. Throughout the section it will be assumed that $n \geqslant 3$. We follow closely Bak's method in section 4 of [3], with an obvious complication due to the existence of long and short roots in elementary quadratic groups. In passing, we also prove that $E_{2n}(A, \Lambda)$ is a normal subgroup of $G_{2n}(A, \Lambda)$.

The following notation will be used. Suppose $(\mathfrak{I}, \Gamma) \subseteq (A, \Lambda)$ is a form ideal and $s \in R_0$. Let $(1/s)\mathfrak{I}$ (resp. $(1/s)\Gamma$) denote the additive subgroup of $\langle s\rangle^{-1}A$ (resp. $\langle s\rangle^{-1}\Gamma$) consisting of all elements $(1/s)a$ such that $a \in \mathfrak{I}$ (resp. $a \in \Gamma$). For any natural number $N$, let $E^N((1/s)\mathfrak{I}, (1/s)\Gamma)$ denote the *subset* of $G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$ consisting of all products of $N$ elementary transvections $T_{ij}(a)$ such that $a \in (1/s)\mathfrak{I}$ if $T_{ij}$ is a short root and $a \in \lambda^{-(\varepsilon(i)+1)/2}(1/s)\Gamma$ if $T_{ij}$ is a long root. If $t \in R_0$, we let $E^N(t\mathfrak{I}, t\Gamma)$ denote the subset of $E^N(1/s)\mathfrak{I}, (1/s)\Gamma)$ consisting all products of $N$ elementary transvections $T_{ij}(a)$ such that $a \in \mathrm{Im}(t\mathfrak{I} \to \langle s\rangle^{-1}\mathfrak{I})$ if $T_{ij}$ is short and such that $a \in \lambda^{-(\varepsilon(i)+1)/2}\mathrm{Im}(t\Gamma \to \langle s\rangle^{-1}\Gamma)$ if $T_{ij}$ is long. Note that if the canonical map $t\mathfrak{I} \to \langle s\rangle^{-1}A$ is injective then $E^N(t\mathfrak{I}, t\Gamma)$ is identified under the injective map $G_{2n}(t\mathfrak{I}, t\Gamma) \to G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$ with its preimage in $G_{2n}(t\mathfrak{I}, t\Gamma)$ consisting of all products of $N$ elementary transvections $T_{ij}(a)$ such that $a \in t\mathfrak{I}$ if $T_{ij}$ is short and $a \in \lambda^{-(\varepsilon(i)+1)/2}t\Gamma$ if $T_{ij}$ is long. We also use the notation $E((1/s)(\mathfrak{I}), (1/s)(\Gamma))$ for $\bigcup_N E^N((1/s)(\mathfrak{I}), (1/s)(\Gamma))$.

LEMMA 4.1. *Let $s, t \in R_0$. If $K, L$ and $\mathfrak{m}$ are given, there are $\mathfrak{k}$ and $M$, e.g. $\mathfrak{k} = (\mathfrak{m}+1)4^K + 4^{K-1} + \cdots + 4$ and $M = 14^{KL}$, such that*

$$^{E^K((t/s)A,(t/s)\Lambda)}E^L(s^{\mathfrak{k}}tA, s^{\mathfrak{k}}t\Lambda) \subseteq E^M(s^{\mathfrak{m}}tA, s^{\mathfrak{m}}t\Lambda).$$

*Proof.* Once the lemma is proved for $K = 1$, $L = 1$, then by an easy induction procedure we can establish the lemma for any pair of $K$ and $L$. Therefore we shall first show that

$$^{E^1((t/s)A,(t/s)\Lambda)}E^1(s^{(\mathrm{m}+1)4}tA, s^{(\mathrm{m}+1)4}t\Lambda) \subseteq E^{14}(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).$$

Let $\rho = {}^{T_{hk}(a)}T_{ij}(b)$. We must show that $\rho \in E^{14}(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda)$. The proof breaks into 4 cases depending on the length of the roots $T_{hk}$ and $T_{ij}$. It will be seen that the most complicated situations are when we have either two short roots such that $T_{hk} = T_{-i,-j}$ and $\rho = {}^{T_{-i,-j}(a)}T_{ij}(b)$ or two long roots such that $T_{h,-h} = T_{-i,i}$ and $\rho = {}^{T_{-i,i}(a)}T_{i,-i}(b)$.

*Case I.* $T_{hk}$ and $T_{ij}$ are short roots, namely $h \neq \pm k$ and $i \neq \pm j$. This case is handled by dividing further into four subcases:

(1) $h \neq j, k \neq i$
(2) $h = j, k \neq i$
(3) $h \neq j, k = i$
(4) $h = j, k = i$.

We shall prove (1) and leave it to the reader to reduce cases (2)–(4) to case (1). Our proof of (1) breaks further into four subcases:

(i) $h \neq -i, k \neq -j$,
(ii) $h = -i, k \neq -j$,
(iii) $h \neq -i, k = -j$,
(iv) $h = -i, k = -j$.

Thus consider $\rho = {}^{T_{hk}(a)}T_{ij}(b)$ where $h \neq \pm k$, $i \neq \pm j$, $h \neq j, k \neq i$ and $a \in (t/s)A$, $b \in s^{(\mathrm{m}+1)4}tA$.

(i) In this case $T_{hk}(a)$ commutes with $T_{ij}(b)$. Therefore $\rho = T_{ij}(b)$ and we are done.

(ii) In this case

$$\rho = T_{hk}(a)T_{-h,j}(b)T_{hk}(-a).$$

Two cases can occur. If $k \neq j$ use (R1) to write

$$T_{-h,j}(b) = T_{-j,h}(\lambda^{(\varepsilon(j)-\varepsilon(-h))/2}\overline{b}).$$

By definition, $b = s^{(\mathrm{m}+1)4}tc$ for some $c \in A$. Since $s, t \in R_0$

$$\lambda^{(\varepsilon(j)-\varepsilon(-h))/2}\overline{b} = \lambda^{(\varepsilon(j)-\varepsilon(-h))/2}s^{(\mathrm{m}+1)4}t\overline{c} \in s^{(\mathrm{m}+1)4}tA.$$

To simplify notation, we denote $\lambda^{(\varepsilon(j)-\varepsilon(-h))/2}\overline{b}$ by $b$. This done, we have

$$\begin{aligned}\rho &= T_{hk}(a)T_{-j,h}(b)T_{hk}(-a)\\ &= T_{-j,h}(b)\,\underbrace{T_{-j,h}(-b)T_{hk}(a)T_{-j,h}(b)T_{hk}(-a)}_{R(4)}\end{aligned}$$

$$
\begin{aligned}
&= T_{-j,h}(b)T_{-j,k}(-ba)(\text{but } a = ta'/s, b = s^{(\mathrm{m}+1)4}tb'))\\
&= T_{-j,h}(s^{(\mathrm{m}+1)4}tb')T_{-j,k}(s^{(\mathrm{m}+1)4-1}t^2b'a') \in E^2(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).
\end{aligned}
$$

On the other hand, if $k = j$ then

$$
\begin{aligned}
\rho &= T_{hk}(a)T_{-k,h}(b)T_{hk}(-a)\\
&= T_{-k,h}(b)\underbrace{T_{-k,h}(-b)T_{hk}(a)T_{-k,h}(b)T_{hk}(-a)}_{R(5)}\\
&= T_{-k,h}(b)T_{-k,k}(-ba + \lambda^{\varepsilon(k)}\overline{a}\overline{b})\\
&= T_{-k,h}(s^{(\mathrm{m}+1)4)}tb')T_{-k,k}(s^{(\mathrm{m}+1)4-1}t^2(-b'a' + \lambda^{\varepsilon(k)}\overline{a}'\overline{b}')) \in E^2 \times\\
&\quad \times (s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda)
\end{aligned}
$$

for some $a', b' \in A$.

(iii) The argument is similar to that in the previous case and is omitted.

(iv) In this case $\rho = T_{hk}(a)T_{-h,-k}(b)T_{hk}(-a)$. By (R1) we can rewrite $\rho$ as $T_{hk}(a)$ $T_{kh}(b)T_{hk}(-a)$, where $a = ta'/s$, $b = s^{(\mathrm{m}+1)4}tb'$. Choose $i \neq \pm h, \pm k$ and set $x = s^{(\mathrm{m}+1)2}$ and $y = s^{(\mathrm{m}+1)2}tb'$. Thus $b = xy$. Now the computation goes as follow,

$$
\begin{aligned}
\rho &= T_{hk}(a)T_{kh}(b)T_{hk}(-a)\\
&= T_{hk}(a)\,[T_{ki}(x), T_{ih}(y)]\,T_{hk}(-a)\\
&= \underbrace{T_{hk}(a)T_{ki}(x)T_{hk}(-a)T_{ki}(-x)}_{R(3)} \times\\
&\quad \times T_{ki}(x)T_{hk}(a)T_{ih}(y)T_{ki}(-x)T_{ih}(-y)T_{hk}(-a)\\
&= T_{hi}(ax)T_{ki}(x)T_{ih}(y)\underbrace{T_{ih}(-y)T_{hk}(a)T_{ih}(y)T_{hk}(-a)}_{R(3)} \times\\
&\quad \times T_{hk}(a)T_{ki}(-x)T_{ih}(-y)T_{hk}(-a)\\
&= T_{hi}(ax)T_{ki}(x)\underbrace{T_{ih}(y)T_{ik}(-ya)}_{\text{commutes}}\underbrace{T_{hk}(a)T_{ki}(-x)T_{hk}(-a)T_{ki}(x)}_{R(3)} \times\\
&\quad \times T_{ki}(-x)T_{hk}(a)T_{ih}(-y)T_{hk}(-a)\\
&= T_{hi}(ax)T_{ki}(x)T_{ik}(-ya)T_{ih}(y)T_{hi}(-ax)T_{ki}(-x)T_{ih}(-y) \times\\
&\quad \times \underbrace{T_{ih}(y)T_{hk}(a)T_{ih}(-y)T_{hk}(-a)}_{R(3)}\\
&= T_{hi}(ax)\underbrace{T_{ki}(x)T_{ik}(-ya)T_{ki}(-x)}_{T_1}\underbrace{T_{ki}(x)T_{ih}(y)T_{hi}(-ax)T_{ki}(-x)}_{T_2} \times\\
&\quad \times T_{ih}(-y)T_{ik}(ya).
\end{aligned}
$$

Clearly $-ya = -s^{(\mathrm{m}+1)2-1}t^2b'a'$. Let

$$c = -s^{\mathrm{m}}t \quad \text{and} \quad d = s^{\mathrm{m}+1}tb'a'.$$

Therefore $-ya = cd$. Thus,

$$T_1 = T_{ki}(x)\,[T_{ih}(c), T_{hk}(d)]\,T_{ki}(-x) = [T_{kh}(xc)T_{ih}(c), T_{hk}(d)T_{ki}(-dx)]\,,$$

and

$$T_2 = T_{ki}(x)T_{ih}(y)T_{hi}(-ax)T_{ki}(-x) = T_{kh}(xy)T_{ih}(y)T_{hi}(-ax).$$

A quick inspection shows that $ax, xc, c, d, dx, xy, ya \in s^{\mathrm{m}}tA$. Therefore,

$$\rho = T_{hi}(ax)\underbrace{T_1}_{8\text{terms}}\underbrace{T_2}_{3\text{terms}}\underbrace{T_{ih}(-y)T_{ik}(ya)}_{2\text{terms}} \in E^{14}(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).$$

*Case II.* $T_{hk}$ is a long root and $T_{ij}$ a short one. Thus $k = -h$ and $a \in \lambda^{-(\varepsilon(h)+1)/2}(t/s)\Lambda$ whereas $i \neq \pm j$ and $b \in s^{(\mathrm{m}+1)4}tA$. This case is handled by dividing further into three subcases:

(1) $j \neq h, i \neq -h$,
(2) $j = h, i \neq -h$,
(3) $j \neq h, i = -h$.

(1) By R(3), $T_{h,-h}(a)$ commutes with $T_{ij}(b)$. Therefore $\rho = T_{ij}(b)$ and we are done.
(2) We have

$$\begin{aligned}\rho &= T_{h,-h}(a)T_{ih}(b)T_{h,-h}(-a)\\ &= T_{ih}(b)\underbrace{T_{ih}(-b)T_{h,-h}(a)T_{ih}(b)T_{h,-h}(-a)}_{R(6)}\\ &= T_{ih}(b)T_{i,-h}(-ba)T_{i,-i}(\lambda^{(\varepsilon(h)-\varepsilon(i))/2}ba\overline{b}) \in E^3(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).\end{aligned}$$

(3) This case is similar to the above argument in (2).

*Case III.* $T_{hk}$ and $T_{ij}$ are long roots. Thus $h = -k$, $i = -j$ and

$$a \in \lambda^{\frac{-(\varepsilon(h)+1)}{2}}\frac{t}{s}\Lambda, \qquad b \in \lambda^{\frac{-(\varepsilon(i)+1)}{2}}s^{(\mathrm{m}+1)4}t\Lambda.$$

Suppose $h \neq -i$. Then $T_{i,-i}$ commutes with $T_{h,-h}$ and we are done. The only case which remains is when $h = -i$, i.e. $\rho = T_{h,-h}(a)T_{-h,h}(b)T_{h,-h}(-a)$ where

$$a \in \lambda^{\frac{-(\varepsilon(h)+1)}{2}}\frac{t}{s}\Lambda \quad \text{and} \quad b \in \lambda^{\frac{-(\varepsilon(-h)+1)}{2}}s^{(\mathrm{m}+1)4}t\Lambda.$$

Choose $p \neq \pm h$ such that $p < 0$. Write $b$ as a product $b = \mu cd\overline{c}$ where

$$\mu = \lambda^{-(\varepsilon(-h)+1)/2}, \qquad c = s^{(\mathrm{m}+1)} \quad \text{and} \quad d = s^{(\mathrm{m}+1)2}tb'$$

where $b' \in \Lambda$. By R(6), we can write

$$T_{-h,h}(\mu cd\overline{c}) = T_{ph}(-\mu cd)[T_{p,-p}(d), T_{-p,h}(c)].$$

Thus

$$\begin{aligned}
\rho &= T_{h,-h}(a)T_{ph}(-\mu cd)[T_{p,-p}(d), T_{-p,h}(c)]T_{h,-h}(-a)\\
&= \underbrace{T_{h,-h}(a)T_{ph}(-\mu cd)T_{h,-h}(-a)T_{ph}(\mu cd)}_{R(6)}\, T_{ph}(-\mu cd)T_{h,-h}(a)\times\\
&\quad\times [T_{p,-p}(d), T_{-p,h}(c)]T_{h,-h}(-a)\\
&= \underbrace{T_{h,-p}(\lambda^{(\varepsilon(h)-\varepsilon(p))/2}a\overline{\mu cd})T_{p,-p}(cda\overline{cd})T_{ph}(-\mu cd)}_{T_1}\,\underbrace{T_{h,-h}(a)T_{p,-p}(d)}_{\text{commutes}}\times\\
&\quad\times T_{-p,h}(c)\times\ T_{p,-p}(-d)T_{-p,h}(-c)T_{h,-h}(-a)\\
&= T_1T_{p,-p}(d)\,\underbrace{T_{h,-h}(a)T_{-p,h}(c)T_{h,-h}(-a)T_{-p,h}(-c)}_{R(6)}\,T_{-p,h}(c)\times\\
&\quad\times T_{h,-h}(a)T_{p,-p}(-d), T_{-p,h}(-c)T_{h,-h}(-a)\\
&= T_1\,\underbrace{T_{p,-p}(d)T_{hp}(-\lambda^{(\varepsilon(h)+1)/2}a\overline{c})T_{-p,p}(\overline{\lambda}ca\overline{c})}_{T_2}\,T_{-p,h}(c)\times\\
&\quad\times\ \underbrace{T_{h,-h}(a)T_{p,-p}(-d)}_{\text{commutes}}\,T_{-p,h}(-c)T_{h,-h}(-a)\\
&= T_1T_2T_{-p,h}(c)T_{p,-p}(-d)T_{-p,h}(-c)\times\\
&\quad\times\ \underbrace{T_{-p,h}(c)T_{h,-h}(a)T_{-p,h}(-c)T_{h,-h}(-a)}_{R(6)}\\
&= T_1T_2T_{p,-p}(-d)\,\underbrace{T_{p,-p}(d)T_{-p,h}(c)T_{p,-p}(-d)T_{-p,h}(-c)}_{R(6)}\times\\
&\quad\times\underbrace{T_{-p,-h}(ca)T_{-p,p}(\overline{\lambda}ca\overline{c})}_{T_3}\\
&= \underbrace{T_1}_{3}\,\underbrace{T_2}_{3}\,T_{p,-p}(-d)T_{ph}(dc)T_{-h,h}(-\lambda^{-(\varepsilon(-h)+1)/2}\overline{c}dc)\,\underbrace{T_3}_{2}\in\\
&\quad E^{11}(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).
\end{aligned}$$

*Case IV.* $T_{hk}$ is a short root and $T_{ij}$ is a long one. All the possibilities which may occur here reduce to one of the cases above.

Therefore we have shown that

$$^{E^1((t/s)A,(t/s)\Lambda)}E^1(s^{(\mathrm{m}+1)4}tA, s^{(\mathrm{m}+1)4}t\Lambda)\subseteq E^{14}(s^{\mathrm{m}}tA, s^{\mathrm{m}}t\Lambda).$$

Now suppose that $K>0$ and $L>0$. Since

$$^{E^K((t/s)A,(t/s)\Lambda)}E^L(s^{(\mathrm{m}+1)4}tA, s^{(\mathrm{m}+1)4}t\Lambda)$$

is the set of all products of $L$ or fewer elements of

$$^{E^K((t/s)A,(t/s)\Lambda)}E^1(s^{(\mathrm{m}+1)4}tA, s^{(\mathrm{m}+1)4}t\Lambda),$$

we will be done if we can prove the assertion of the lemma for arbitrary $K$ and $L = 1$. We proceed by induction on $K$. The case $K = 1$ is proved above. Let $K > 1$. We shall show that

$$\begin{aligned} &{}^{E^K((t/s)A,(t/s)\Lambda)}E^1(s^{(\mathfrak{m}+1)4^{K-1}+4^{K-2}+\cdots+1)4}tA, s^{(\mathfrak{m}+1)4^{K-1}+4^{K-2}+\cdots+1)4}t\Lambda) \\ &\quad\subseteq {}^{E^{K-1}((t/s)A,(t/s)\Lambda)}E^{14}(s^{(\mathfrak{m}+1)4^{K-1}+4^{K-2}+\cdots+4)}tA, s^{(\mathfrak{m}+1)4^{K-1}+4^{K-2}+\cdots+4)}t\Lambda). \end{aligned}$$

To prove this, it suffices to show that ${}^{E^1((t/s)A,(t/s)\Lambda)}E^1(s^{(\mathfrak{m}'+1)4}tA, s^{(\mathfrak{m}'+1)4}t\Lambda) \subseteq E(s^{\mathfrak{m}'}tA, s^{\mathfrak{m}'}t\Lambda)$, where $\mathfrak{m}' = (\mathfrak{m}+1)4^{K-1} + 4^{K-2} + \cdots + 4$. But this is just a special case of the first step of the induction which we have already proved. Therefore the proof is complete. $\square$

If $U$ and $V$ are subsets of a group, let $]U, V[$ denote the set of all commutators $[u, v]$ such that $u \in U$ and $v \in V$.

LEMMA 4.2. *Let $s, t \in R_0$. If $K \geqslant 1$ and $\mathfrak{l} \geqslant 0$, let $E^K(t^{\mathfrak{l}}/sA, t^{\mathfrak{l}}/s\Lambda)$ denote the subset of $G_{2n}(\langle st\rangle^{-1}A, \langle st\rangle^{-1}\Lambda)$ consisting all products of $K$ or fewer elementary transvections $T_{ij}(a)$ such that $a \in t^{\mathfrak{l}}/sA(\subseteq \langle st\rangle^{-1}A)$ if $T_{ij}$ is short and $a \in \lambda^{-(\varepsilon(i)+1)/2}t^{\mathfrak{l}}/s\Lambda(\subseteq \langle st\rangle^{-1}\Lambda)$ if $T_{ij}$ is long. If $L \geqslant 1$ and $\mathfrak{k} \geqslant 0$, define $E^L(s^{\mathfrak{k}}/tA, s^{\mathfrak{k}}/t\Lambda)$ similarly. If $M \geqslant 1$ and $\mathfrak{p}, \mathfrak{q} \geqslant 0$, let $E^M(s^{\mathfrak{p}}t^{\mathfrak{q}}A, s^{\mathfrak{p}}t^{\mathfrak{q}}\Lambda)$ denote the subset of $G_{2n}(\langle st\rangle^{-1}A, \langle st\rangle^{-1}\Lambda)$ consisting of all products of $M$ or fewer elementary transvections $T_{ij}(a)$ such that $a \in s^{\mathfrak{p}}t^{\mathfrak{q}}A(\subseteq \langle st\rangle^{-1}A)$ if $T_{ij}$ is short and $a \in \lambda^{-(\varepsilon(i)+1)/2}s^{\mathfrak{p}}t^{\mathfrak{q}}\Lambda(\subseteq \langle st\rangle^{-1}\Lambda)$ if $T_{ij}$ is long. If $K, L, \mathfrak{p}$ and $\mathfrak{q}$ are given, there are $\mathfrak{k}, \mathfrak{l}$ and $M$, e.g.*

$$\begin{aligned} &\mathfrak{k} = (\mathfrak{p}+1)4^{K+2} + 4^{K+1} + \cdots + 4, \\ &\mathfrak{l} = (\mathfrak{q}+1)4^{L+2} + 4^{L+1} + \cdots + 4, \quad \textit{and} \quad M = 14^{K+L+3}KL, \end{aligned}$$

*such that*

$$\Big]E^K\left(\frac{t^{\mathfrak{l}}}{s}A, \frac{t^{\mathfrak{l}}}{s}\Lambda\right), E^L\left(\frac{s^{\mathfrak{k}}}{tA}, \frac{s^{\mathfrak{k}}}{t\Lambda}\right)\Big[ \subseteq E^M(s^{\mathfrak{p}}t^{\mathfrak{q}}A, s^{\mathfrak{p}}t^{\mathfrak{q}}\Lambda).$$

*Proof.* If $U$ is a subset of a group and $N$ a nonnegative integer, let $\mathrm{Prod}^N(U)$ denote the set of all products of $N$ or fewer elements of $U$. Using commutator formulas, it is easy to see that

$$\big]\mathrm{Prod}^K(U_1), \mathrm{Prod}^L(U_2)\big[ \subseteq \mathrm{Prod}^{KL}\left({}^{\mathrm{Prod}^{K-1}(U_1)\mathrm{Prod}^{L-1}(U_2)}]U_1, U_2[\right). \qquad (4.2.1)$$

Let

$$U_1 = E^1\left(\frac{t^{\mathfrak{l}}}{s}A, \frac{t^{\mathfrak{l}}}{s}\Lambda\right) \quad \text{and} \quad U_2 = E^1\left(\frac{s^{\mathfrak{k}}}{t}A, \frac{s^{\mathfrak{k}}}{t\Lambda}\right).$$

Since

$$E^K\left(\frac{t^{\mathfrak{l}}}{s}A, \frac{t^{\mathfrak{l}}}{s}\Lambda\right) = \mathrm{Prod}^K(U_1) \quad \text{and} \quad E^L\left(\frac{s^{\mathfrak{k}}}{tA}, \frac{s^{\mathfrak{k}}}{t\Lambda}\right) = \mathrm{Prod}^L(U_2),$$

it follows from (4.2.1) that

$$\left]E^K\left(\frac{t^{\mathfrak{l}}}{s}A, \frac{t^{\mathfrak{l}}}{s}\Lambda\right), E^L\left(\frac{s^{\mathfrak{k}}}{tA}, \frac{s^{\mathfrak{k}}}{t\Lambda}\right)\right[ \subseteq \mathrm{Prod}^{KL}\left({}^{\mathrm{Prod}^{K-1}(U_1)\mathrm{Prod}^{L-1}(U_2)}]U_1, U_2[\right).$$

By Lemma 4.1, it suffices to show that

$$\left]E^1\left(\frac{t^{\mathfrak{l}}}{s}A, \frac{t^{\mathfrak{l}}}{s}\Lambda\right), E^1\left(\frac{s^{\mathfrak{k}}}{tA}, \frac{s^{\mathfrak{k}}}{t\Lambda}\right)\right[ \subseteq E^{14^5}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda), \tag{4.2.2}$$

where

$$\mathfrak{p}' = (\mathfrak{p}+1)4^{K-1} + 4^{K-2} + \cdots + 4 \quad \text{and}$$
$$\mathfrak{q}' = (\mathfrak{q}+1)4^{L-1} + 4^{L-2} + \cdots + 4.$$

Let $\rho = [T_{hk}((t^{\mathfrak{l}}/s)a), T_{ij}((s^{\mathfrak{k}}/t)b)]$. The proof breaks into 4 cases depending on the length of the roots $T_{hk}$ and $T_{ij}$.

*Case I.* $T_{hk}$ and $T_{ij}$ are short roots, namely $h \neq \pm k$ and $i \neq \pm j$. This case is handled by dividing further into four subcases:

(1) $h \neq j, k \neq i$,
(2) $h = j, k \neq i$,
(3) $h \neq j, k = i$,
(4) $h = j, k = i$.

We shall prove (1) and leave it to the reader to reduce cases (2)–(4) to case (1). Our proof of (1) breaks further into four subcases:

(i) $h \neq -i, k \neq -j$,
(ii) $h = -i, k \neq -j$,
(iii) $h \neq -i, k = -j$,
(iv) $h = -i, k = -j$.

(i) By R(1), $T_{hk}((t^{\mathfrak{l}}/s)a)$ commutes with $T_{ij}((s^{\mathfrak{k}}/t)b)$ and therefore $\rho = 1$. Thus we are done.

(ii) In this case $\rho = [T_{hk}((t^{\mathfrak{l}}/s)a), T_{-hj}((s^{\mathfrak{k}}/t)b)]$. Two cases can occur. If $k \neq j$ then use R(1) to write

$$T_{hk}((t^{\mathfrak{l}}/s)a) = T_{-k,-h}(\lambda^{(\varepsilon(k)-\varepsilon(h))/2}(t^{\mathfrak{l}}/s)\overline{a}).$$

Set $a' = \lambda^{(\varepsilon(k)-\varepsilon(h))/2}\overline{a}$. Then

$$\begin{aligned}\rho &= [T_{-k,-h}((t^{\mathfrak{l}}/s)a'), T_{-hj}((s^{\mathfrak{k}}/t)b)] \\ &= \big(\text{by R(4)}\big) T_{-k,j}(t^{\mathfrak{l}-1}s^{\mathfrak{k}-1}a'b).\end{aligned}$$

If $k = j$ then using R(5) we obtain

$$\begin{aligned}\rho &= [T_{-k,-h}((t^{\mathfrak{l}}/s)a'), T_{-hk}((s^{\mathfrak{k}}/t)b)]\\ &= T_{-k,k}(t^{\mathfrak{l}-1}s^{\mathfrak{k}-1}a'b - \lambda^{\varepsilon(k)}t^{\mathfrak{l}-1}s^{\mathfrak{k}-1}\overline{b}\overline{a}').\end{aligned}$$

(iii) The argument here is similar to that in the previous case.

(iv) In this case $\rho = [T_{hk}((t^{\mathfrak{l}}/s)a), T_{-h,-k}((s^{\mathfrak{k}}/t)b)]$. Choose $p \neq \pm h, \pm k$. Write $T_{-h,-k}((s^{\mathfrak{k}}/t)b) = T_{kh}((s^{\mathfrak{k}}/t)b')$ where $b' = \lambda^{(\varepsilon(-k)-\varepsilon(-h))/2}\overline{b}$. Then

$$\begin{aligned}\rho &= [T_{hk}((t^{\mathfrak{l}}/s)a), T_{kh}((s^{\mathfrak{k}}/t)b')]\\ &= [T_{hk}((t^{\mathfrak{l}}/s)a), [T_{kp}((s^{\mathfrak{k}/2}/t)b'), T_{ph}(s^{\mathfrak{k}/2})]].\end{aligned}$$

Using the commutator formula $[x,[y,z]] = [x,y]^y[x,z]^{yz}[x,y^{-1}]^{yzy^{-1}}[x,z^{-1}]$, we have

$$\begin{aligned}\rho &= \big[T_{hk}((t^{\mathfrak{l}}/s)a), [T_{kp}((s^{\mathfrak{k}/2}/t)b'), T_{ph}(s^{\mathfrak{k}/2})]\big]\\ &= [T_{hk}((t^{\mathfrak{l}}/s)a), T_{kp}((s^{\mathfrak{k}/2}/t)b')] \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')}[T_{hk}((t^{\mathfrak{l}}/s)a), T_{ph}(s^{\mathfrak{k}/2})] \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')T_{ph}(s^{\mathfrak{k}/2})}[T_{hk}((t^{\mathfrak{l}}/s)a), T_{kp}(-(s^{\mathfrak{k}/2}/t)b')] \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')T_{ph}(s^{\mathfrak{k}/2})T_{kp}(-(s^{\mathfrak{k}/2}/t)b')}[T_{hk}((t^{\mathfrak{l}}/s)a), T_{ph}(-s^{\mathfrak{k}/2})].\end{aligned}$$

Applying R(4) repeatedly, we obtain

$$\begin{aligned}\rho &= T_{hp}(t^{\mathfrak{l}-1}s^{\mathfrak{k}/2-1}ab') \times (\in E(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)) \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')}T_{pk}(t^{\mathfrak{l}}s^{\mathfrak{k}/2-1}a') \times\\ &\quad \times \text{(by Lemma 4.1 } \in E^{14}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)) \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')T_{ph}(s^{\mathfrak{k}/2})}T_{hp}(-t^{\mathfrak{l}-1}s^{\mathfrak{k}/2-1}ab') \times\\ &\quad \times \text{(by (4.1) } \in E^{14^2}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)) \times\\ &\quad \times {}^{T_{kp}((s^{\mathfrak{k}/2}/t)b')T_{ph}(s^{\mathfrak{k}/2})T_{kp}(-(s^{\mathfrak{k}/2}/t)b')}T_{pk}(-t^{\mathfrak{l}}s^{\mathfrak{k}/2-1}a') \times\\ &\quad \times \text{(by (4.1) } \in E^{14^3}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)).\end{aligned}$$

Therefore $\rho \in E^{14^4}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)$.

*Case II.* $T_{hk}$ is long and $T_{ij}$ is short. Thus $\rho = [T_{h,-h}((t^{\mathfrak{l}}/s)a), T_{ij}((s^{\mathfrak{k}}/t)b)]$ where $i \neq \pm j$, $a \in \lambda^{-(\varepsilon(h)+1)/2}\Lambda$ and $b \in A$. This case is handled by dividing further into 3 possible subcases: (1) $j \neq h, i \neq -h$ (2) $j = h, i \neq -h$ (3) $j \neq h, i = -h$.

(1) In this case $\rho = 1$ and we are done.
(2) In this case

$$\begin{aligned}\rho &= [T_{h,-h}((t^{\mathfrak{l}}/s)a), \underbrace{T_{ih}((s^{\mathfrak{k}}/t)b)}_{R(1)}] \\ &= [T_{h,-h}((t^{\mathfrak{l}}/s)a), T_{-h,-i}((s^{\mathfrak{k}}/t)b')](\text{where } b' = \lambda^{(\varepsilon(h)-\varepsilon(i))/2}\overline{b}) \\ &= (\text{by R(6) })T_{h,-i}(t^{\mathfrak{l}-1}s^{\mathfrak{k}-1}ab)T_{i,-i}(-\mu_1 s^{2\mathfrak{k}-1}t^{\mathfrak{l}-2}b'a\overline{b'}) \\ &\in E^2(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda),\end{aligned}$$

where $\mu_1 = \lambda^{(\varepsilon(-i)-\varepsilon(-h))/2}$.
(3) Here the argument is the same as in the previous case.

*Case III.* $T_{hk}$ and $T_{ij}$ are long roots. Thus $\rho = [T_{h,-h}((t^{\mathfrak{l}}/s)a), T_{i,-i}((s^{\mathfrak{k}}/t)b)]$. If $h \neq -i$ then $\rho = 1$ and we are done. The only case which remains is when $h = -i$. Then

$$\rho = [T_{h,-h}((t^{\mathfrak{l}}/s)a), T_{-h,h}((s^{\mathfrak{k}}/t)b)],$$

where $a \in \lambda^{-(\varepsilon(h)+1)/2}\Lambda$ and $b \in \lambda^{-(\varepsilon(-h)+1)/2}\Lambda$. Choose $p \neq \pm h$. By R(6), we can decompose

$$T_{-h,h}((s^{\mathfrak{k}}/t)b) = T_{ph}(-(s^{\mathfrak{k}/2}/t)b(s^{\mathfrak{k}/4}))[T_{p,-p}(\mu(s^{\mathfrak{k}/2}/t)b), T_{-p,h}(s^{\mathfrak{k}/4})],$$

where $\mu = \lambda^{(-\varepsilon(h)-\varepsilon(p))/2}$. Therefore

$$\rho = [T_{h,-h}((t^{\mathfrak{l}}/s)a), T_{ph}(-(s^{\mathfrak{k}/2}/t)b(s^{\mathfrak{k}/4}))[T_{p,-p}(\mu(s^{\mathfrak{k}/2}/t)b), T_{-p,h}(s^{\mathfrak{k}/4})]].$$

Now using the commutator formula

$$[x, \mu[y, z]] = [x, \mu]^{\mu}[x, y]^{\mu y}[x, z]^{\mu y z}[x, y^{-1}]^{\mu y z y^{-1}}[x, z^{-1}],$$

we have

$$\begin{aligned}\rho = {} & T_{h,-p}(t^{\mathfrak{l}-1}s^{3\mathfrak{k}/4-1}\mu_1 ab)T_{p,-p}(\mu_1 t^{\mathfrak{l}-2}s^{6\mathfrak{k}/4-1}ba\overline{b}) \times \\ & \times {}^{T_{ph}((s^{3\mathfrak{k}/4}/t)b)T_{p,-p}((s^{\mathfrak{k}/2}/t)b)}\Big(T_{h,p}(t^{\mathfrak{l}}s^{\mathfrak{k}/4-1}\mu_2 a)T_{-p,p}(\mu_2 s^{\mathfrak{k}/2-1}t^{l}a)\Big) \times \\ & \times {}^{T_{ph}((s^{3\mathfrak{k}/4}/t)b)T_{p,-p}((s^{\mathfrak{k}/2}/t)b)T_{-p,h}(s^{\mathfrak{k}/4})T_{p,-p}(-(s^{\mathfrak{k}/2}/t)b)}\Big(T_{h,p}(t^{\mathfrak{l}}s^{\mathfrak{k}/4-1}\mu_2 a) \times \\ & \times T_{-p,p}(-\mu_2 s^{\mathfrak{k}/2-1}t^{l}a)\Big)\end{aligned}$$

where $\mu_1 = \lambda^{(\varepsilon(h)-\varepsilon(p))/2}$ and $\mu_2 = \lambda^{(\varepsilon(h)-\varepsilon(-p))/2}$.

Now the same argument as in the case II, shows that $\rho \in E^{14^5}(s^{\mathfrak{p}'}t^{\mathfrak{q}'}A, s^{\mathfrak{p}'}t^{\mathfrak{q}'}\Lambda)$.

*Case IV.* $T_{hk}$ is short and $T_{ij}$ is long. This case is handled in the same spirit as the others above. □

LEMMA 4.3. *Let $(A_R, \Lambda)$ be a quasi-finite form algebra. Let $(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda)$ be the subgroup of $(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$. Let "$G$"$(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)$ denote the image of*

$G_{2n}(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)$ *in* $G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$. *Given* $K$ *and* $\mathfrak{m}$, *there is a* $\mathfrak{k}$, *e.g*, $\mathfrak{k} = 9((\mathfrak{m}+1)4^{K+3} + 4^{K+2} + \cdots + 4)$, *such that*

$$\left[E^K\left(\frac{1}{s}A, \frac{1}{s}\Lambda\right), ''G''(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)\right] \subseteq E(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda).$$

*Proof.* Since $(A_R, \Lambda)$ is quasi-finite, the proof reduces to the case $A$ is module finite over $R$ and $R$ is finitely generated as a $\mathbb{Z}$-algebra. This implies (cf. proof of (3.7)) that $A$ is module finite over $R_0$ and $R_0$ is also finitely generated as a $\mathbb{Z}$-algebra. In particular $A$ is a Noetherian $R_0$-module. We shall show that

$$\left[E^1\left(\frac{1}{s}A, \frac{1}{s}\Lambda\right), ''G''(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)\right] \subseteq E(s^{\mathfrak{m}'}A, s^{\mathfrak{m}'}\Lambda)$$

where $\mathfrak{m}' = (\mathfrak{m}+1)4^{K-1} + 4^{K-2} + \cdots + 4$. The conclusion of the lemma follows from this result, the commutator formulas **C(1)** and **C(2)** of the introduction, and Lemma 4.1.

Let $T_{ij}(a/s) \in E^1(\frac{1}{s}A, \frac{1}{s}\Lambda)$ and $''\sigma'' \in'' G''(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)$. We do not treat the short and long roots separately. We use the standard localization-patching method to prove our result. We shall show that for any maximal ideal $\mathfrak{M}$ of $R_0$, there is an element $t_{\mathfrak{M}} \in R_0 - \mathfrak{M}$ and a nonnegative integer $l_{\mathfrak{M}}$ such that for any $a \in A$,

$$\left[T_{ij}\left(\frac{t_{\mathfrak{M}}^{l_{\mathfrak{M}}}}{s}a\right), ''\sigma''\right] \in E(s^{(\mathfrak{m}'+1)4}A, s^{(\mathfrak{m}'+1)4}\Lambda). \tag{4.3.1}$$

Suppose this is done. Since the set $\{t_{\mathfrak{M}}^{l_{\mathfrak{M}}} \mid \mathfrak{M} \in \mathrm{Max}(R_0)\}$ is not contained in any maximal ideal of $R_0$, there is a finite set $\{t_{\mathfrak{M}_1}^{l_{\mathfrak{M}_1}}, \ldots, t_{\mathfrak{M}_r}^{l_{\mathfrak{M}_r}}\}$ such that the ideal $\langle t_{\mathfrak{M}_1}^{l_{\mathfrak{M}_1}}, \ldots, t_{\mathfrak{M}_r}^{l_{\mathfrak{M}_r}}\rangle$ is the whole ring $R_0$. Choose $x_1, \ldots, x_r \in R_0$ such that $x_1 t_{\mathfrak{M}_1}^{l_{\mathfrak{M}_1}} + \cdots + x_r t_{\mathfrak{M}_r}^{l_{\mathfrak{M}_r}} = 1$. Then

$$\begin{aligned}\left[T_{ij}\left(\frac{a}{s}\right), ''\sigma''\right] &= \left[T_{ij}\left(\frac{t_{\mathfrak{M}_1}^{l_{\mathfrak{M}_1}} x_1 a}{s}\right) \cdots T_{ij}\left(\frac{t_{\mathfrak{M}_r}^{l_{\mathfrak{M}_r}} x_r a}{s}\right), ''\sigma''\right] \\ &\in \text{(by (4.3.1) and C(2))} \\ &\quad {}^{E^1((1/s)A,(1/s)\Lambda)}E(s^{(\mathfrak{m}'+1)4}A, s^{(\mathfrak{m}'+1)4}\Lambda) \\ &\subseteq \text{(by Lemma 4.1)} \subseteq E(s^{\mathfrak{m}'}A, s^{\mathfrak{m}'}\Lambda).\end{aligned}$$

This finishes the proof.

It remains to prove (4.3.1). Let $\mathfrak{M}$ be a maximal ideal of $R_0$. Then $A_{\mathfrak{M}}$ is a semilocal ring. Recall the definition of $F_{2n}$ given prior to Theorem 2.2. By Theorem 2.6 (cf. also [10, 9.1.4]) and Corollary 2.4 we have

$$G_{2n}(s^{\mathfrak{k}}A_{\mathfrak{M}}, s^{\mathfrak{k}}\Lambda_{\mathfrak{M}}) \subseteq F_{2n}(s^{\mathfrak{k}/3}A_{\mathfrak{M}}, s^{\mathfrak{k}/3}\Lambda_{\mathfrak{M}})G_2(s^{\mathfrak{k}}A_{\mathfrak{M}}, s^{\mathfrak{k}}\Lambda_{\mathfrak{M}}). \tag{4.3.2}$$

Therefore the image of $\sigma$ over $A_{\mathfrak{M}}$ can be decomposed as a product of elements of $G_2(s^{\mathfrak{k}}A_{\mathfrak{M}}, s^{\mathfrak{k}}\Lambda_{\mathfrak{M}})$ and $F_{2n}(s^{\mathfrak{k}/3}A_{\mathfrak{M}}, s^{\mathfrak{k}/3}\Lambda_{\mathfrak{M}})$. Thus we can find an element $t \in R_0 - \mathfrak{M}$ such that over $(\langle t\rangle^{-1}A, \langle t\rangle^{-1}\Lambda)$, $\sigma$ can be factored as $\xi\delta$, where $\delta \in G_2(s^{\mathfrak{k}}\langle t\rangle^{-1}A, s^{\mathfrak{k}}\langle t\rangle^{-1}\Lambda)$ and $\xi \in F_{2n}(s^{\mathfrak{k}/3}\langle t\rangle^{-1}A, s^{\mathfrak{k}/3}\langle t\rangle^{-1}\Lambda)$. By Lemma 2.7, there is a $\mathfrak{q}$ such that the canonical homomorphism

$$G_{2n}\Big(t^{\mathfrak{q}}\langle s\rangle^{-1}A, t^{\mathfrak{q}}\langle s\rangle^{-1}\Lambda\Big) \xrightarrow{\text{inj.}} G_{2n}\Big(\langle st\rangle^{-1}A, \langle st\rangle^{-1}\Lambda\Big) \tag{4.3.3}$$

is injective. Let $\mathfrak{l} > \mathfrak{q}$. Since $T_{ij}(t^{\mathfrak{l}}a/s) \in G_{2n}\Big(t^{\mathfrak{q}}\langle s\rangle^{-1}A, t^{\mathfrak{q}}\langle s\rangle^{-1}\Lambda\Big)$, we have by Theorem 2.1 that

$$\rho = [T_{ij}(t^{\mathfrak{l}}a/s),'' \sigma''] \in G_{2n}\Big(t^{\mathfrak{q}}\langle s\rangle^{-1}A, t^{\mathfrak{q}}\langle s\rangle^{-1}\Lambda\Big).$$

Let $\underline{\rho}$ denote the image of $\rho$ in $G_{2n}\Big(\langle st\rangle^{-1}A, \langle st\rangle^{-1}\Lambda\Big)$. If we can show that

$$\underline{\rho} \in E(s^{\mathfrak{p}}t^{\mathfrak{q}}A, s^{\mathfrak{p}}t^{\mathfrak{q}}\Lambda)$$

where $\mathfrak{p} = (\mathfrak{m}'+1)4$ then because of the injectivity of the map in (4.3.3) we obtain that $\rho \in E(s^{\mathfrak{p}}A, s^{\mathfrak{p}}\Lambda)$.

Let $\underline{T_{ij}}(t^{\mathfrak{l}}a/s)$, $\underline{\sigma}$, $\underline{\delta}$ and $\underline{\xi}$ denote respectively the images of $T_{ij}(t^{\mathfrak{l}}a/s)$, $\sigma$, $\delta$ and $\xi$ in $G_{2n}(\langle st\rangle^{-1}A, \langle st\rangle^{-1}\Lambda)$. Then

$$\begin{aligned}\underline{\rho} &= \left[\underline{T_{ij}}\left(\frac{t^{\mathfrak{l}}}{s}a\right), \underline{\sigma}\right] = \left[\underline{T_{ij}}\left(\frac{t^{\mathfrak{l}}}{s}a\right), \underline{\xi}\underline{\delta}\right] = (\text{by } \mathbf{C(1)})\\ &= \left[\underline{T_{ij}}\left(\frac{t^{\mathfrak{l}}}{s}a\right), \underline{\xi}\right]^{\underline{\xi}}\left[\underline{T_{ij}}\left(\frac{t^{\mathfrak{l}}}{s}a\right), \underline{\delta}\right].\end{aligned}$$

If $\{\pm i, \pm j\} \cap \{\pm 1\} = \emptyset$ then $[\underline{T_{ij}}((t^{\mathfrak{l}}/s)a), \underline{\delta}] = 1$. If $\{\pm i, \pm j\} \cap \{\pm 1\} \neq \emptyset$ then we choose $k \notin \{\pm i, \pm j\}$ and change the embedding of $G_2$ in $G_{2n}$ to that corresponding to $\{\pm k\}$, without sacrificing the validity of Corollary 2.4, Theorem 2.6 and (4.3.2). This done, we obtain again that $[\underline{T_{ij}}((t^{\mathfrak{l}}/s)a), \underline{\delta}] = 1$. Thus, in either case, we achieve that $\underline{\rho} = [\underline{T_{ij}}((t^{\mathfrak{l}}/s)a), \underline{\xi}]$.

Since

$$\underline{\xi} \in E\left(\frac{s^{\mathfrak{k}/3}}{t}A\frac{s^{\mathfrak{k}/3}}{t}\Lambda\right), \qquad \frac{\mathfrak{k}}{3} > (\mathfrak{p}+1)4^3 + 4^2 + 4 \quad \text{and} \quad K = 1,$$

it follows from Lemma 4.2 that there is a $\mathfrak{l}$ such that $[\underline{T}_{ij}(t^{\mathfrak{l}}a/s), \underline{\xi}] \in E(s^{\mathfrak{p}}t^{\mathfrak{q}}A, s^{\mathfrak{p}}t^{\mathfrak{q}}\Lambda)$. This completes the proof. ☐

If $s = 1$ then the above lemma implies the result of Bak and Vavilov [4, 5], that $E_{2n}(A, \Lambda)$ is a normal subgroup of $G_{2n}(A, \Lambda)$ when $n \geqslant 3$.

THEOREM 4.4. *Let $(A_R, \Lambda)$ be a quasi-finite form algebra. Then $E_{2n}(A, \Lambda)$ is a normal subgroup of $G_{2n}(A, \Lambda)$.*

DEFINITION 4.5. Let $(A_R, \Lambda)$ be a quasi-finite form algebra. Let $s \in R_0$. Define

$$G(s^{-1}, A)$$
$$= \mathrm{Ker}(G_{2n}(A, \Lambda) \longrightarrow G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)/E_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)),$$

and

$$G(\hat{s}, A) = \mathrm{Ker}(G_{2n}(A, \Lambda) \longrightarrow G_{2n}(A_R, \Lambda)\tilde{}_{(s)}/E_{2n}(A_R, \Lambda)\tilde{}_{(s)}).$$

THEOREM 4.6. *Let $(A_R, \Lambda)$ be a quasi-finite form algebra. Then*

$$[G(s^{-1}, A), G(\hat{s}, A)] \subseteq E_{2n}(A, \Lambda).$$

*Proof.* As in Lemma 4.3, the proof reduces to the case $A$ is module finite over $R_0$ and $R_0$ is Noetherian. We first show that

$$E_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda) = \bigcup_{K \geqslant 0} E^K\left(\frac{1}{s}A, \frac{1}{s}\Lambda\right). \tag{4.6.1}$$

Let $m > 1$ and $T_{ij}(a/s^m) \in E_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$. Suppose first that $T_{ij}$ is a short root, namely $i \neq \pm j$. Choose $h \neq \pm i, \pm j$. By R(4), we have that

$$T_{ij}\left(\frac{a}{s^m}\right) = \left[T_{ih}\left(\frac{a}{s^{m-1}}\right), T_{hj}\left(\frac{1}{s}\right)\right].$$

By induction on $m$, we conclude that there is a $K$ such that

$$T_{ij}\left(\frac{a}{s^m}\right) \in E^K\left(\frac{1}{s}A, \frac{1}{s}\Lambda\right).$$

Suppose now that $T_{ij} = T_{i,-i}$ is a long root. If $m$ is odd, decompose

$$\frac{a}{s^m} = \frac{1}{s^{m-1/2}}\frac{a}{s}\frac{1}{s^{m-1/2}}$$

and if $m$ is even then decompose

$$\frac{a}{s^m} = \frac{1}{s^{m/2}}\frac{a}{1}\frac{1}{s^{m/2}}.$$

Suppose $m$ is odd. Then by R(6), we have

$$T_{i,-i}\left(\frac{a}{s^m}\right) = T_{ji}\left(\frac{a}{s^{m+1/2}}\right)\left[T_{j,-j}\left(\frac{-a}{s}\right), T_{-j,i}\left(\frac{1}{s^{m-1/2}}\right)\right],$$

where $j \neq \pm i$. Suppose $m$ is even. Then by R(6), we have

$$T_{i,-i}\left(\frac{a}{s^m}\right) = T_{ji}\left(\frac{a}{s^{m/2}}\right)\left[T_{j,-j}\left(\frac{-a}{1}\right), T_{-j,i}\left(\frac{1}{s^{m/2}}\right)\right].$$

Since the short roots are in

$$\bigcup_{K \geqslant 0} E^K((1/s)A, (1/s)\Lambda),$$

we conclude that there is a $K$ such that $T_{i,-i}(a/s^m) \in E^K((1/s)A, (1/s)\Lambda)$. This completes the proof of (4.6.1).

By Lemma 2.7 there is an $\mathfrak{m}$ such that the canonical homomorphism,

$$\psi : G_{2n}(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda) \longrightarrow G_{2n}(\langle s\rangle^{-1}A, \langle s\rangle^{-1}\Lambda)$$

is injective. Since $A$ is module finite over $R_0$ and $R_0$ is Noetherian, the Artin–Rees Lemma [1,10.10] tells us that given an integer $n \geqslant 0$, there is an integer $l \geqslant 0$ such that $s^{l+n}A \cap \Lambda \subseteq s^n\Lambda$.

Let $\sigma \in G(s^{-1}, A)$ and $\rho \in G(\hat{s}, A)$. We must show that $[\sigma, \rho] \in E_{2n}(A, \Lambda)$. Choose $K$ such that $''\sigma'' \in E^K((1/s)A, (1/s)\Lambda)$. Let $\mathfrak{k} = 9((\mathfrak{m}+1)4^{K+3}+4^{K+2}+\cdots+4)$ (see Lemma 4.3) and choose $\mathfrak{p} = \mathfrak{k}+l$, using the Artin–Rees Lemma. Then

$$G_{2n}(s^{\mathfrak{p}}A, s^{\mathfrak{p}}A \cap \Lambda) \subseteq G_{2n}(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda). \tag{4.6.2}$$

Let

$$\theta : G_{2n}(A, \Lambda) \longrightarrow G_{2n}\left(\frac{A}{s^{\mathfrak{p}}A}, \frac{\Lambda}{s^{\mathfrak{p}}A \cap \Lambda}\right)$$

denote the canonical map. Thus $\mathrm{Ker}\theta = G_{2n}(s^{\mathfrak{p}}A, s^{\mathfrak{p}}A \cap \Lambda)$. Since

$$\theta(\rho) \in E_{2n}\left(\frac{A}{s^{\mathfrak{p}}A}, \frac{\Lambda}{s^{\mathfrak{p}}A \cap \Lambda}\right),$$

there is an element $\xi^{-1} \in E_{2n}(A, \Lambda)$ such that $\theta(\xi^{-1}) = \theta(\rho)$. This and (4.6.2) imply that $\rho\xi \in G_{2n}(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)$. By Theorem 4.4, $E_{2n}(A, \Lambda)$ is a normal subgroup of $G_{2n}(A, \Lambda)$. Thus by **C(1)**, $[\sigma, \rho] \in E_{2n}(A, \Lambda)$ if and only if $[\sigma, \rho\xi] \in E_{2n}(A, \Lambda)$. Because $G_{2n}(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda)$ is normal in $G_{2n}(A, \Lambda)$, it follows that $[\sigma, \rho\xi] \in G_{2n}(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda)$. Since the image $''\sigma''$ of $\sigma$ is in $E^K((1/s)A, (1/s)\Lambda)$ and the image $''\rho\xi''$ of $\rho\xi$ is in $''G''_{2n}(s^{\mathfrak{k}}A, s^{\mathfrak{k}}\Lambda)$, it follows from Lemma 4.3 that

$$[''\sigma'','' \rho\xi''] \in E(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda).$$

Since $\psi$ is injective and takes $F_{2n}(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda)$ bijectively onto $E(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda)$, it follows thats

$$[\sigma, \rho\xi] \in F_{2n}(s^{\mathfrak{m}}A, s^{\mathfrak{m}}\Lambda) \subseteq E_{2n}(A, \Lambda),$$

and the proof is complete. □

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