Normality for Elementary Subgroup Functors

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Abstract We define a notion of group functor G, on categories of graded modules, which unifies all previous concepts of a group functor G possessing a notion of elementary subfunctor E. We show under a general condition, which is easily checked in practice, that the elementary subgroup $E(M)$ of $G(M)$ is normal for all weak and quasi-weak Noetherian objects M in the source category of G. This result includes all previous ones on Chevalley and classical groups G of rank $\geqq 2$ over a commutative or module finite ring M and settles positively the still unanswered cases of normality for these group functors.

1. Introduction.

Beginning with H. Bass' pioneering paper [Bs1], much attention has been given to determining whether or not the elementary subgroup $E(M)$ of a Chevalley or classical group $G(M)$ over a module finite ring M is normal in $G(M)$. Bass' result [Bs1] was valid for $G = GL_n$ (general linear group of rank n) under the condition that n was large with respect to the dimension of the ring M. It was immediately asked whether or not similar looking subgroups of other classical groups were also normal. This question was settled by H. Bass [Bs2] in the symplectic case and in general by A. Bak [B1], [B2], again under the assumption that the rank of the functor was large with respect to the dimension of the ring. The approach taken in [B1], [B2] introduced a generalization GQ_{2n} (referred to by the notation U_{2n} in [B1], [B2], [Bs2] and by GQ_{2n} in [B3]) of the notion of classical group, which depended on the concept of a form parameter and allowed treating in a uniform way all the classical and classical-like groups other than those in the GL_n-family. The groups corresponding to the minimum form parameter are treated also in L. Vaserstein [V]. After a gap of roughly ten years, A. Suslin showed in [S], [Tu] that the elementary subgroup $E_n(M)$ of $GL_n(M)$ is normal whenever $n \geq 3$ and M is commutative or module finite. Following his results, several papers Kopeiko [K], Kopeiko-Suslin [KS], Taddei [T1], [T2], Golubchik-Mikhalev [GM] and Vavilov [Vv1], [Vv2] were devoted to analogous questions for Chevalley groups and general quadratic groups GQ_{2n}. A detailed survey of developments to-date is given in [Vv3, §9]. However, the articles above left open the case of twisted Chevalley groups and the general case (in classical language, nonsplit case) of the general quadratic group $GQ_{2n}(M, \Lambda)$ $(n \geq 2)$ for an arbitrary module finite ring M with involution and form parameter Λ. They are settled in (3.12) and (3.10) respectively, as consequences of our Main Theorem (2.9). The result (3.10) is included in those announced in [B4, §1].

The approach we take to prove our results is inspired by the technique used in [B4], particularly in [B4, §4], for establishing results concerning GL_n. This technique hinges for the most part only on a simple minded, flexible, and general concept of group functor, which is divorced from the confines of linear algebra and at the same time contains as special cases the group functors mentioned in the previous paragraph. Specifically, we consider group functors G on a category $\mathbf{\Lambda}$ of graded modules M (for example, a category of rings or form rings) such that $G(M)$ contains homogeneous components of M in a functorial way. The **elementary subgroup** $E(M)$ of $G(M)$ is defined as the subgroup of $G(M)$ generated by the embedded homogeneous components of M. This construction obviously defines a subfunctor $E \rightarrowtail G$, which we christen the **elementary subfunctor** of G. In the case of Chevalley or general quadratic groups G, it is clear that the elementary matrices define embeddings into G of the ring and form parameter over which G is defined and that the elementary subgroup of G is indeed the subgroup generated by these embedded entities. On the other hand, any Chevalley or general quadratic group is itself embedded set theoretically, but still functorially, into a direct sum of copies of the ring over which it is defined. So, we assume in the general setting that our groups $G(M)$ embed set theoretically and functorically into a direct sum $\oplus M$ of copies of M. This extra structure coupled with the notion of weak Noetherian module (2.6) is enough to proof our main result (2.9) concerning the normality of $E(M)$ in $G(M)$, in just a few words.

The rest of the article is organized as follows. In §2, we define in detail the concepts we use and prove the Main Theorem (2.9) concerning the normality of $E(M)$. In §3, we explain how Chevalley and general quadratic groups fit into our setting and then apply our main theorem to these groups.

2. Main theorem.

We fix the following notation for the remainder of the article. We let $\boldsymbol{G}$ denote the category of all groups and group homomorphisms. The letter $\mathfrak{I}$ will denote an index set and we let $\boldsymbol{M}(\mathfrak{I})$ be equal to the category whose objects are all pairs (k_M, M) where k_M is a commutative, associative ring with identity and $M = \bigoplus_{i \in \mathfrak{I}} M_i$ is an $\mathfrak{I}$-graded, right k_M-module. A morphism $(k_M, M) \longrightarrow (k_{M'}, M')$ in $\boldsymbol{M}(\mathfrak{I})$ is any pair $g : k_M \longrightarrow k_{M'}, f : M \longrightarrow M'$ where g is a ring homomorphism preserving the identity, f a graded homomorphism of abelian groups of degree zero, and $f(ma) = f(m)g(a)$ for all $m \in M$ and $a \in k_M$. We let $\boldsymbol{M} = \boldsymbol{M}(\mathfrak{I}_0)$ where $\mathfrak{I}_0$ is a set containing exactly one element. There is a canonical inclusion $\boldsymbol{M} \subset \boldsymbol{G}$ and for each $i \in \mathfrak{I}$, there is a canonical coordinate functor $\lambda_i : \boldsymbol{M}(\mathfrak{I}) \longrightarrow \boldsymbol{M}, M \mapsto M_i$, and $((g, f) : (k_M, M) \longrightarrow (k_{M'}, M')) \mapsto ((g, f|_{M_i}) : (k_m, M_i) \longrightarrow (k_{M'}, M_i'))$.

The letter $\boldsymbol{\Lambda}$ will always denote a subcategory of $\boldsymbol{M}(\mathfrak{I})$, for some $\mathfrak{I}$, such that if $(k_M, M) \in \mathrm{Obj}(\boldsymbol{\Lambda})$ and $S \subset k_M$ is a multiplicative set then $(k_{S^{-1}M}, S^{-1}M) \in \mathrm{Obj}(\boldsymbol{\Lambda})$ where $k_{S^{-1}M} := S^{-1}k_M$ and the canonical morphism $(k_M, M) \longrightarrow (k_{S^{-1}M}, S^{-1}M)$ is in Mor $(\boldsymbol{\Lambda})$.

Definition 2.1 A **$\boldsymbol{\Lambda}$-group functor** (on $\boldsymbol{\Lambda}$) consists of a functor $G : \boldsymbol{\Lambda} \longrightarrow \boldsymbol{G}$ and for each $i \in \mathfrak{I}$, a natural inclusion $\epsilon_i : \lambda_i \longrightarrow G$ of functors. If $M \in \boldsymbol{\Lambda}$ and $U \subseteq M$ is a homogeneous subgroup of M, let $E(M, U)$ denote the subgroup of $G(M)$ generated by all subgroups $\epsilon_i^U(U_i) \subset G(M)(i \in \mathfrak{I})$ where $U_i :=$ i'th homogeneous component of U. Let $E(M) = E(M, M)$. Clearly, the construction $M \mapsto E(M)$ defines a subfunctor $E \rightarrowtail G$. If $M \in \boldsymbol{\Lambda}$ and $s \in k_M$, define $A(M, s) = \{\sigma \in G(M) |$ given $p \in \mathbb{Z}^{\geq 0}, \exists q \in \mathbb{Z}^{\geq 0}$ such that $\sigma E(M, Ms^q)\sigma^{-1} \subseteq E(M, Ms^p)$ and $\sigma^{-1}E(M, Ms^q)\sigma^{-1} \subseteq E(M, Ms^p)\}$. Clearly, $A(M, s)$ is a subgroup of $G(M)$.

Lemma 2.2 *Let G be a $\boldsymbol{\Lambda}$-group functor and let $M \in \boldsymbol{\Lambda}$. Suppose that for each element $\sigma \in G(M)$ and each maximal ideal $\mathfrak{p} \subseteq k_M$, there is an element $s_{\sigma,\mathfrak{p}} \in k_M \setminus \mathfrak{p}$ such that $\sigma \in A(M, s_{\sigma,\mathfrak{p}})$. Then $E(M) \lhd G(M)$.*

Proof Let $i \in \mathfrak{I}, m \in M_i$, and $\sigma \in G(M)$. It suffices to show that $\sigma\epsilon_j(m)\sigma^{-1} \in E(M)$. For each maximal ideal $\mathfrak{p}$ of k_M, choose $s_\mathfrak{p} (= s_{\sigma,\mathfrak{p}})$ as above. By hypothesis, there is a $q_\mathfrak{p} \geq 0$ such that $\sigma\epsilon_i(Ms_\mathfrak{p}^{q_\mathfrak{p}})\sigma^{-1} \in E(M)$. Since $\{s_\mathfrak{p}^{q_\mathfrak{p}} | \mathfrak{p}$ maximal ideal of $k_M\}$ is not contained in any maximal of k_M, there is a finite set $\mathfrak{p}_1, ..., \mathfrak{p}_\ell$ of maximal ideals of k_M and of elements $a_1, ..., a_\ell$ of k_M such that $\sum_{j=1}^{\ell} a_j s_j{}^{q_{\mathfrak{p}_j}} = 1$. But, then $\sigma\epsilon_i(m)\sigma^{-1} = \prod_{j=1}^{\ell} \sigma\epsilon_i(ma_j s_j{}^{q_{\mathfrak{p}_j}})\sigma^{-1} \in E(M)$. Q.E.D.

DEFINITION 2.3 Let G be a $\mathbf{\Lambda}$-group functor. If $M \in \mathbf{\Lambda}$ and $s \in k_M$, let $< s >$ denote the multiplicative set generated by s and let "Ms" denote the image of Ms in $< s >^{-1} M$ under the canonical homomorphism $M \longrightarrow < s >^{-1} M$. Define $B(M,s) = \{\sigma \in G(< s >^{-1} M) |$ for each $p \in \mathbb{Z}^{\geq 0}$, there is a $q \in \mathbb{Z}^{\geq 0}$ such that $\sigma E(< s >^{-1} M, \text{"}Ms^q\text{"})\sigma^{-1} \subseteq E(< s >^{-1} M, \text{"}Ms^p\text{"})$ and $\sigma^{-1} E(< s >^{-1} M, \text{"}Ms^q\text{"})\sigma \subseteq E(< s >^{-1} M, \text{"}Ms^p\text{"})$. Clearly, $B(M,s)$ is a subgroup of $G(< s >^{-1} M)$. Define $\overline{E}(M, Ms) =$ normal closure of $E(M, Ms)$ in $G(M)$.

LEMMA 2.4 *Let G be a $\mathbf{\Lambda}$-group functor. Let $M \in \mathbf{\Lambda}$ and $s \in k_M$. Then the canonical homomorphism $\Phi : G(M) \longrightarrow G(< s >^{-1} M)$ takes $A(M,s)$ into $B(M,s)$. Moreover, if for some $p \in \mathbb{Z}^{\geq 0}$, $\Phi |_{\overline{E}(M,Ms^p)}$ is injective then $\Phi^{-1}(B(M,s)) = A(M,s)$.*

PROOF Straightforward.

LEMMA 2.5 *Let G be a $\mathbf{\Lambda}$-functor. Let $M \in \mathbf{\Lambda}$. Suppose that the following conditions hold:*
(2.5.1) For each element $\sigma \in G(M)$ and each maximal ideal $\mathfrak{p}$ of k_M, there is an element $s_{\sigma,\mathfrak{p}} \in k_M \setminus \mathfrak{p}$ such that the image of σ in $G(< s_{\sigma,\mathfrak{p}} >^{-1} M)$ lies in $B(M,s)$.

(2.5.2) For each element $s \in k_M \setminus \bigcap_{\mathfrak{p}\ maximal} \mathfrak{p}$, there is a $p \in \mathbb{Z}^{\geq 0}$ such that $\overline{E}(M, Ms^p)$ injects into $G(< s >^{-1} M)$ under the canonical homomorphism $G(M) \longrightarrow G(< s >^{-1} M)$.
Then $E(M) \lhd G(M)$.

PROOF This follows immedeately from (2.2) and (2.4).

DEFINITION 2.6 An object $(k_M, M) \in \boldsymbol{M}(\mathfrak{I})$ will be called **weak Noetherian** if given $s \in k_M \setminus \bigcap_{\mathfrak{p}\ \text{maximal}} \mathfrak{p}$, there is an element $p \in \mathbb{Z}^{\geq 0}$ such that $\text{Ann}_M(s^p) = \text{Ann}_M(s^{p+1})$ (where $\text{Ann}_M(s^p) = \{m \in M \mid ms^p = 0\}$). **Note** that the index set $\mathfrak{I}$ need not be finite; for example, if each M_i is Noetherian over k_M and the number of isomorphism classes $[M_i]$ of modules M_i is finite then (k_M, M) is weak Noetherian. An object $(k_M, M) \in \mathbf{\Lambda}$ will called **quasi-weak**

Noetherian (in $\mathbf{\Lambda}$) if it is a direct limit of weak Noetherian objects in $\mathbf{\Lambda}$. An object $(k_M, M) \in \boldsymbol{M}(\mathfrak{I})$ will be called **Noetherian** if M is Noetherian over k_M. An object $(k_M, M) \in \mathbf{\Lambda}$ will be called **quasi-Noetherian** (in $\mathbf{\Lambda}$) (or **quasi-finite** as in [B4, §3]) if it is a direct limit of Noetherian objects in $\mathbf{\Lambda}$. **Clearly**, the property of being Noetherian (resp. quasi-Noetherian) implies the property of being weak Noetherian (resp. quasi-weak Noetherian).

The next lemma generlizes [B 4, (4.10)].

KEY LEMMA 2.7 *Let G be a $\mathbf{\Lambda}$-group functor. Let $M \in \mathbf{\Lambda}$. Suppose the following:*

(2.7.1) For each element $s \in k_M \setminus \bigcap_{\substack{\mathfrak{p} \\ maximal}} \mathfrak{p}$ and $p \in \mathbb{Z}^{\geqq 0}$, there is a commutative diagram of sets

$$\begin{array}{ccccc} \overline{E}(M, Ms^p) & \rightarrowtail & G(M) & \xrightarrow{\Phi} & G(< s >^{-1} M) \\ \downarrow & & \downarrow & & \downarrow \\ (\oplus M)s^p & \rightarrowtail & \oplus M & \underset{\Psi}{\longrightarrow} & < s >^{-1} (\oplus M) \end{array}$$

where $\overline{E}(M, Ms^p)$ is as in (2.3), $\oplus M$ is a finite direct sum of M's, the horizontal maps are the canonical ones, and the left vertical map is injective. Then if M is weak Noetherian over k_M, the group $\overline{E}(M, Ms^p)$ maps injectively for some p into $G(< s >^{-1} M)$, under Φ.

PROOF It suffices to show that for some $p \in \mathbb{Z}^{\geq 0}, \Psi$ maps Ms^p injectively into $< s >^{-1} M$. But, this is an easy exercise which is done, for example, in the proof of [B4, (4.10)]. Q.E.D.

THEOREM 2.8 *Let G be a $\mathbf{\Lambda}$-group functor. Let $M \in \boldsymbol{M}$ and suppose that (2.6.1) and the following are satisfied:*
(2.8.1) For each element $\sigma \in G(M)$ and each maximal ideal $\mathfrak{p}$ of k_M, there is an element $s_{\sigma,\mathfrak{p}} \in k_M \setminus \mathfrak{p}$ such that the image of σ in $G(< s_{\sigma,\mathfrak{p}} >^{-1} M)$, under the canonical homomorphism, lies in $B(M, s_{\sigma,\mathfrak{p}})$.
Then if M is weak Noetherian, $E(M) \lhd G(M)$.

PROOF This follows immediately from (2.5) and (2.7).

MAIN THEOREM 2.9. *Let G be a $\mathbf{\Lambda}$-group functor. Suppose that conditions (2.7.1) and (2.8.1) hold for each object $M' \in \mathbf{\Lambda}$ which is weak Noetherian.*

If G commutes with direct limits then $E(M) \lhd G(M)$ for every quasi-weak Noetherian object M of $\mathbf{\Lambda}$.

PROOF This follows immediately from (2.8) and the definition of direct limit.

3. Applications to Chevalley and general quadratic groups.

In this section, we show that Chevalley, twisted Chevalley, and general quadratic groups are $\mathbf{\Lambda}$-groups, for suitable $\mathbf{\Lambda}$, satisfying (2.7.1) and that they satisfy (2.8.1) under the usual conditions excluding small ranks. It will follow then from the Main Theorem (2.9) that their elementary subgroups are normal.

The demonstration that (2.7.1) is fulfilled will be essentially trivial. The proof that (2.8.1) is satisfied will be executed by showing that the following condition fulfilled.

CONDITION 3.1 Let $\mathfrak{I}, \boldsymbol{M}(\mathfrak{I})$, and $\mathbf{\Lambda} \subset \boldsymbol{M}(\mathfrak{I})$ be as in section 2. Let G be a $\mathbf{\Lambda}$-group functor. Let $M \in \mathbf{\Lambda}$ and if $\mathfrak{p}$ is a maximal ideal of k_M, let $S_{\mathfrak{p}}$ denote the multiplicative set $k_M \setminus \mathfrak{p}$. Suppose the following holds for G, M, and each maximal ideal $\mathfrak{p}$ of k_M.

(3.1.1) For each $j \in \mathfrak{I}$, there is a subset $\mathfrak{I}(j)$ of $\mathfrak{I}$ such that the following holds:

Given $s \in S_{\mathfrak{p}}$ and $p \in \mathbb{Z}^{\geq 0}$, there is a $q \in \mathbb{Z}^{\geq 0}$ such that for each $m_j \in M_j$ $(= \lambda_j(M))$ and $i \in \mathfrak{I}(j), \epsilon_j(\frac{m}{s})\epsilon_i(M_i s^q)\epsilon_j(\frac{m}{s})^{-1} \subseteq E(< s >^{-1} M, \text{“}M s^p\text{”})$.

(3.1.2) Given $p \in \mathbb{Z}^{\geq 0}$, there is a $q \in \mathbb{Z}^{\geq 0}$ such that for all $i, j \in \mathfrak{I}$, $\epsilon_i(M_i s^q) \subseteq < \epsilon_{i'}(M_{i'} s^p) | i' \in \mathfrak{I}(j) >$ where $\mathfrak{I}(j)$ is as in (3.1.1).

(3.1.3) If $\sigma \in G(M)$, there is an element $s \in S_{\mathfrak{p}}$ such that the image “σ” of σ in $G(< s >^{-1} M)$ has a product decomposition “σ ”$= \delta\epsilon$ where $\epsilon \in E(< s >^{-1} M)$ and δ has the property that given $i \in \mathfrak{I}$ and $p \in \mathbb{Z}^{\geq 0}$, there is a $q \in \mathbb{Z}^{\geq 0}$ such that $\delta\epsilon_i(\text{“}M_i s^q\text{”})\delta^{-1} \subseteq E(< s >^{-1} M, \text{“}M s^p\text{”})$ and $\delta^{-1}\epsilon_i(\text{“}M_i s^q\text{”})\delta \subseteq E(< s >^{-1} M, \text{“}M s^p\text{”})$.

LEMMA 3.2 *Let G a $\mathbf{\Lambda}$-group and $M \in \mathbf{\Lambda}$ such that (2.7.1) and (3.1) hold. Then if M is weak Noetherian over $k_M, E(M) \lhd G(M)$. Furthermore, if for all weak Noetherian objects in $\mathbf{\Lambda}$, (2.7.1) and (3.1) hold and G commutes with direct limits then for all quasi-weak Noetherian objects $M \in \mathbf{\Lambda}, E(M) \lhd G(M)$.*

PROOF Conditions (3.1.1) and (3.1.2) show that for each maximal ideal $\mathfrak{p}$ of k_M and element $s \in k_M \setminus \mathfrak{p}, E(< s >^{-1} M) \subseteq B(M, s)$. Since the element δ in

(3.1.3) clearly lies in $B(M,s)$, it follows from (3.1.1) - (3.1.3) that condition (2.8.1) is satisfied. Thus, for M weak Noetherian over $k_M, E(M) \lhd G(M)$ by (2.8). The second assertion of the lemma follows from (2.9). Q.E.D.

Let $\mathbf{alg}(\mathfrak{I})$ denote the subcategory of $\boldsymbol{M}(\mathfrak{I})$ of all objects (k_A, A) such that A_i $(= \lambda_i(A))$ is an associative k_A-algebra with identity , $A_i = A_j$ for all $i,j \in \mathfrak{I}$, and each morphism $f : (k_A, A) \longrightarrow (k_{A'}, A')$ is a direct sum $f = \underset{i\in I}{\oplus} f_i$ of algebra homomorphism f_i preserving the identity such that $f_i = f_j$ for all $i,j \in \mathfrak{I}$. Let $\mathbf{alg} = \mathbf{alg}\ (\mathfrak{I}_0)$ where $\mathfrak{I}_0$ is some index set with exactly one element. There are mutually inverse canonical functors $\Psi(\mathfrak{I})$: $\mathbf{alg} \longrightarrow \mathbf{alg}\ (\mathfrak{I}), (k_A, A) \longmapsto (k_A, \underset{i\in I}{\oplus} A)$ and $(f : (k_A, A) \longrightarrow (k_{A'}, A')) \longmapsto (\underset{i\in\mathfrak{I}}{\oplus} f : (k_A, \underset{i\in\mathfrak{I}}{\oplus} A) \longrightarrow (k_{A'}, \underset{i\in\mathfrak{I}}{\oplus} A'))$, and $\Phi(\mathfrak{I}) : \mathbf{alg}\ (\mathfrak{I}) \longrightarrow \mathbf{alg}$, $(k_A, \underset{i\in\mathfrak{I}}{\oplus} A_i) \longmapsto (k_A, A_{i_0})$ $(i_0 \in \mathfrak{I}_0, A_{i_0} = A_i)$ and $(\underset{i\in\mathfrak{I}}{\oplus} f_i : (k_A, \underset{i\in\mathfrak{I}}{\oplus} A_i) \longrightarrow (k_{A'}, \underset{i\in\mathfrak{I}}{\oplus} A'_i)) \longmapsto (f_{i_0} : (k_A, A_{i_0}) \longrightarrow (k_{A'}, A'_{i_0}))$ $(f_{i_0} = f_i)$. For $n \in \mathbb{Z}^{\geq 2}$, let $\mathfrak{I}(n) = \{(i,j)|\ 1 \leq i,j \leq n, i \neq j\}$. Let $\mathbf{alg}\ (n) = \mathbf{alg}\ (\mathfrak{I}(n)), \Psi(n) = \Psi(\mathfrak{I}(n))$ and $\Phi(n) = \Phi(\mathfrak{I}(n))$. Let $GL_n : \mathbf{alg} \longrightarrow \boldsymbol{G}$ denote the usual general linear functor. Composing GL_n with $\Phi(n)$, we get a functor $\mathbf{alg}\ (n) \longrightarrow \boldsymbol{G}$, which we also denote by GL_n. Thus, if $A \in \mathbf{alg}$ then

$$GL_n(A) = GL_n(\Psi(n)(A)).$$

If $A \in \mathbf{alg}$, $a \in A$, and $(i,j) \in \mathfrak{I}(n)$, let $\epsilon_{ij}^{\Psi(n)(A)}(a)$ denote the $n \times n$-elementary matrix whose (i,j)'th coefficient is a. Clearly, the assignment $a \longmapsto \epsilon_{ij}^{\Psi(n)(A)}(a)$ defines a group embedding $\epsilon_{ij}^{\Psi(n)(A)} : A \longrightarrow GL_n(A)$ and these embeddings taken over all objects $\Psi(n)(A)$ of $\mathbf{alg}\ (n)$ define a natural transformation $\epsilon_{ij} : \lambda_{ij} \longrightarrow GL_n$. Thus, the functor GL_n :$\mathbf{alg}\ (n) \longrightarrow \boldsymbol{G}$ is an $\mathbf{alg}(n)$-group functor in the sense of (2.1). Clearly, the usual elementary subgroup $E_n(A) \subseteq GL_n(A)$ is just the subgroup generated by the subgroups image $(\epsilon_{ij}^{\Psi(n)(A)})$ where (i,j) ranges over the elements of $\mathfrak{I}(n)$.

If $A \in \mathbf{alg}$, let $\mathbb{M}_n(A)$ denote the set of all $n\times n$-matrices with coefficients in A. Let $\tau^A : GL_n(A) \longrightarrow \mathbb{M}_n(A), \sigma \mapsto \sigma - I$, where I denotes the identity $n \times n$-matrix. The functions τ^A define a natural transformation $\tau : GL_n \longrightarrow \mathbb{M}_n$ of functors defined on $\mathbf{alg}$. Using τ, one shows easily that the functor GL_n on $\mathbf{alg}\ (n)$ satisfies (2.7.1). Moreover, if $n \geq 3$, the standard commutator relations (cf. [B4, (2.2)]) for elementary matrices show that GL_n satisfies (3.1.1) and (3.1.2). Suppose now that A is module finite over k_A. Let $\mathfrak{p}$ denote a maximal ideal of k_A and let $S_\mathfrak{p} = k_A \setminus \mathfrak{p}$. Let $\sigma \in GL_n(A)$ and 'σ' its image in $GL_n(S_\mathfrak{p}^{-1}A)$. Since $S_\mathfrak{p}^{-1}A$ is semilocal, it is well known (cf. [B4, (4.9) (b)] with $s = 1$) that 'σ'$= \delta_1\epsilon_1$ where δ_1 is a diagonal matrix and ϵ_1 a product of elementary matrices. It follows that for some $s \in S_{\mathfrak{p}}$, the image "σ" of σ in $GL_n(< s >^{-1} A)$ has a product decomposition "σ "$= \delta\epsilon$ where δ is a diagonal matrix and ϵ a product of elementary matrices. Since δ is diagonal, one deduces easily that it has the property ascribed to it in (3.1.3). Thus, we have shown the following:

LEMMA 3.3 *The functor GL_n on* **alg** *(n) is an* **alg** *(n)-group functor satisfying (2.7.1). Furthermore, if $n \geq 3$ then GL_n satisfies (3.1) on objects A which are module finite over k_A.*

THEOREM 3.4 *Let GL_n denote the general linear group functor on associative algebras A with identity. If $n \geq 3$ and A is quasi-Noetherian then $E_n(A) \lhd GL_n(A)$.*

PROOF This follows immediately from (3.2) and (3.3). Q.E.D.

COROLLARY 3.5 (Suslin [S], [Tu]) *Let GL_n denote the general linear group functor on associative algebras A with identity. If $n \geq 3$ and A is module finite over k_A then $E_n(A) \lhd GL_n(A)$.*

PROOF One shows easily that A is quasi-Noetherian. The corollary follows now from (3.4). Q.E.D.

Following procedures similar to those used in the proof above of Suslin's theorem, one can prove the analog of his theorem for the general quadratic functor GQ_{2n}. We do this next.

Let A denote an associative ring with identity and involution $a \longmapsto \overline{a}$. Thus, $\overline{ab} = \overline{b}\,\overline{a}$ and $\overline{\overline{a}} = a$ for all $a, b \in A$. Let $\lambda \in$ center (A) such that $\lambda\overline{\lambda} = 1$. Let $\min^{\lambda}(A) = \{a - \lambda\overline{a}|\ a \in A\}$ and $\max^{\lambda}(A) = \{a \in A \mid a = -\lambda\overline{a}\}$. Let Λ denote a **form parameter** on A. By definition, Λ is an additive subgroup of A such that $\min^{\lambda}(A) \subseteq \Lambda \subseteq \max^{\lambda}(A)$ and $a\Lambda\overline{a} \subseteq \Lambda$ for all $a \in A$. The triple (A, λ, Λ) is called a **form ring** and is frequently abbreviated by (A, Λ). A **form algebra** is a 4-tuple $(k_A, A, \lambda, \Lambda)$ where $(k_A, A) \in$ **alg** and (A, λ, Λ) is a form ring such that the trivial involution on k_A is compatible with that on A and $k_A\Lambda \subseteq \Lambda$. We shall frequently abbreviate the form algebra $(k_A, A, \lambda, \Lambda)$ by (A, Λ). A **morphism** $(k_A, A, \lambda, \Lambda) \longrightarrow (k_{A'}, A', \lambda', A')$ of form algebras is a morphism $(g, f) : (k_A, A) \longrightarrow (k_{A'}, A')$ of algebras such that f preserves the involution, $f(\lambda) = \lambda'$ and $f(\Lambda) \subseteq \Lambda'$. Let **form alg** denote the category of all form algebras and morphisms of form algebras.

If $n \in \mathbb{Z}^{\geq 2}$, let $\mathcal{J}(n, n) = \{(i, j) \mid 1 \leq i, j \leq n, i \neq j\}$. Define the subsets $\mathcal{J}^{\ell,+}(n, n) = \{(i, n + i) \mid 1 \leq i \leq n\}, \mathcal{J}^{\ell,-}(n, n) = \{(n + i, i) \mid 1 \leq$

$i \leq n\}, \mathfrak{I}^{\ell}(n,n) = \mathfrak{I}^{\ell,+}(n,n) \cup \mathfrak{I}^{\ell,-}(n,n)$ and $\mathfrak{I}^{s}(n,n) = \mathfrak{I}(n,n) \setminus \mathfrak{I}^{\ell}(n,n)$. The superscript ℓ (resp. s) is derived from the terminology long (resp. short) root in the theory of Chevalley and classical groups. Let **form alg** (n) denote the subcategory of $\boldsymbol{M}(\mathfrak{I}(n,n))$ whose objects are all (k_M, M) such that for some form algebra $(k_A, A, \lambda, \Lambda)$,

(3.6.1)

$$k_M = k_A$$

$$M_{ij} = \begin{cases} A \text{ if } (i,j) \in \mathfrak{I}^{s}(n,n) \\ \Lambda \text{ if } (i,j) \in \mathfrak{I}^{\ell}(n,n). \end{cases}$$

A morphism $f = \underset{(i,j)\in\mathfrak{I}(n,n)}{\oplus} f_{i,j} : (k_M, M) \longrightarrow (k_{M'}, M')$ is any morphism f in $\boldsymbol{M}(\mathfrak{I}(n,n))$ such that the following holds:

(3.6.2) For each (i,j) and $(i',j') \in \mathfrak{I}^{s}(n,n), f_{i,j} = f_{i',j'}$ and the map $f_{i,j} : A \longrightarrow A'$ defines a morphism $(k_A, A, \lambda, \Lambda) \longrightarrow (k_{A'}, A', \lambda', \Lambda')$ of form algebras.

(3.6.3) For each $(i,j) \in \mathfrak{I}^{\ell}(n,n), f_{i,j} = f_{i',j'}|_{\Lambda}$ for any $(i',j') \in \mathfrak{I}^{s}(n,n)$.

There are mutually inverse functors $\Psi(n)$: **form alg** $\longrightarrow$ **form alg** (n) and $\phi(n)$: **form alg** $(n) \longrightarrow$ **form alg** which are analogous to those between **alg** and **alg** (n).

Let $(A,\Lambda) \in$ **form alg**. If σ is an $n \times n$-matrix $(a_{k\ell})$ with coefficients $a_{k\ell} \in A$, let $\overline{\sigma} =$ transpose $(\overline{a}_{k\ell})$. Let $GQ_{2n}(A,\Lambda)$ denote the **general quadratic group** [B3, §3]of rank $2n$ over (A,Λ). By definition, $GQ_{2n}(A,\Lambda) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_n(A) \mid \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \overline{\gamma} & \lambda\overline{\beta} \\ \overline{\lambda\gamma} & \overline{\alpha} \end{pmatrix}$, diagonal coefficients of $\overline{\gamma}\alpha$ and $\overline{\delta}\beta$ lie in $\Lambda\}$. The elementary matrices of $GQ_{2n}(A,\Lambda)$ are described in detail in [B3, §3]. If $M = \Psi(n)(A,\Lambda), (i,j) \in \mathfrak{I}(n,n)$, and $a \in M_{ij}$ then the description of elementary matrices in [B3, p. 27 - 29] makes it clear that there is precisely one elementary matrix $\epsilon_{ij}^{\Psi(n)(A,\Lambda)}(a)$ corresponding to the data above. The assignment $a \longmapsto \epsilon_{ij}^{\Psi(n)(A,\Lambda)}(a)$ clearly defines an embedding $M_{ij} \longrightarrow GQ_{2n}(A,\Lambda)$ of groups. As usual, we define $EQ_{2n}(A,\Lambda)$ to be the subgroup of $GQ_{2n}(A,\Lambda)$ generated by the images of the embeddings above where (i,j) ranges over $\mathfrak{I}(n,n)$. Composing the functor GQ_{2n} : **form alg** $\longrightarrow \boldsymbol{G}$ with the functor $\phi(n)$: **form alg** $(n) \longrightarrow$ **form alg**, we get a functor from **form alg** (n) to $\boldsymbol{G}$, which we also denote by GQ_{2n}. Evidently,

$$GQ_{2n}(A,\Lambda) = GQ_{2n}(\Psi(n)(A,\Lambda))$$

and the embeddings $\epsilon_{ij}^{\Psi(n)(A,\Lambda)} : \lambda_{ij}(\Psi(n)(A,\Lambda)) \longrightarrow GQ_{2n}(A,\Lambda)$ where $(i,j) \in \mathcal{I}(n,n)$ is fixed and $\Psi(n)(A,\Lambda)$ ranges over the objects of **form alg** (n) define a natural transformation $\epsilon_{ij} : \lambda_{ij} \longrightarrow GQ_{2n}$. This shows that the functor GQ_{2n} on **form alg** (n) is a **form alg** (n)-group functor.

If $(A,\Lambda) \in$ **form alg**, let $\mathbb{M}_{2n}(A)$ denote the set of all $n \times n$-matrices with coefficients in A. Let $\tau^{A,\Lambda} : GQ_{2n}(A,\Lambda) \longrightarrow \mathbb{M}_{2n}(A), \sigma \longmapsto \sigma - I$, where I denotes the $2n \times 2n$-identity matrix. The functions $\tau^{A,\Lambda}$ where (A,Λ) ranges over **form alg** define a natural transformation $\tau : GQ_{2n} \longrightarrow \mathbb{M}_{2n}$ of functors defined on **form alg**. Using τ, one shows easily that the functor GQ_{2n} on **form alg** (n) satisfies (2.7.1). Moreover, if $n \geq 3$, the standard commutator relations [B3, (3.16)] for the elementary matrices on $GQ_{2n}(A,\Lambda)$ show that GQ_{2n} satisfies (3.1.1) and (3.1.2) over all objects of **form alg** (n). And, if $n = 2$ and the form algebra (A,Λ) has the property that the subgroup $A\Lambda + \Lambda A$ of A, generated additively by all products ax and xa where $a \in A$ and $x \in \Lambda$, equals A then again by [B3, (3.16)], $GQ_{2n}(\Psi(n)(A,\Lambda))$ satisfies (3.1.1) and (3.1.2). Furthermore, for any $n \geq 2$ and any $(A,\Lambda) \in$ **form alg** such that A is module finite over $k_A, GQ_{2n}(\Psi(n)(A,\Lambda))$ satisfies (3.1.3). One checks this as follows. Let $\mathfrak{p}$ be a maximal ideal of k_A and let $S_{\mathfrak{p}} = (k_A \setminus \mathfrak{p})$. Let $\sigma \in GQ_{2n}(A,\Lambda)$. If '$\sigma$' denotes the image of σ in $GQ_{2n}(S_{\mathfrak{p}}^{-1}A, S_{\mathfrak{p}}^{-1}\Lambda)$ then by [B1, III (4.9)] (cf. also [Bs2, III (3.11)]), one can write 'σ'$= \rho\tau$ where $\rho \in GQ_2(S_{\mathfrak{p}}^{-1}A, S_{\mathfrak{p}}^{-1}\Lambda)$ and $\tau \in EQ_{2n}(S_{\mathfrak{p}}^{-1}A, S_{\mathfrak{p}}^{-1}\Lambda)$. Thus, for some $s \in S_{\mathfrak{p}}$, one can write the image "σ" of σ in $GQ_{2n}(< s >^{-1} A, < s >^{-1} \Lambda)$ as a product "σ"$= \delta\epsilon$ where $\delta \epsilon GQ_2(< s >^{-1} A, < s >^{-1} \Lambda)$ and $\epsilon \in EQ_{2n}(< s >^{-1} A, < s >^{-1} \Lambda)$. It is straightforward to check that δ satisfies (3.1.3). Thus, we have shown the following:

LEMMA 3.7 *The functor GQ_{2n} on* **form alg** (n) *is a* **form alg** (n)*-group functor satisfying (2.7.1). Furthermore, if* $(A,\Lambda) \in$ **form alg** *is such that A is module finite over k_A (resp. A is module finite over k_A and $A = A\Lambda + \Lambda A$) and $n \geq 3$ (resp. $n \geq 2$) then GQ_{2n} satisfies (3.1) on $\Psi(n)(A,\Lambda)$.*

DEFINITION 3.8 A form algebra (A,Λ) is called **quasi-Noetherian** if it satisfies one of the following equivalent conditions:

(i) (A,Λ) is a direct limit of form algebras (A',Λ') such that A' is Noetherian over $k_{A'}$.

(ii) $\Psi(n)(A,\Lambda)$ is quasi-Noetherian in **form alg** (n) for any $n > 1$.

THEOREM 3.9 *Let GQ_{2n} denote the general quadratic functor on form algebras (A,Λ). Suppose that (A,Λ) is quasi-Noetherian. If either $n \geq 3$ or $n = 2$ and $A = A\Lambda + \Lambda A$ then $EQ_{2n}(A,\Lambda) \lhd GQ_{2n}(A,\Lambda)$.*

PROOF By definition, $\Psi(n)(A,\Lambda)$ is a direct limit of objects $\Psi(n)(A',\Lambda')$ such that A' is Noetherian over $k_{A'}$. Moreover, if the condition $A = A\Lambda + \Lambda A$ is satisfied then we can arrange that each (A',Λ') above has the property that $A' = A'\Lambda' + \Lambda' A'$. The assertion of the theorem follows now from (3.2), (3.7), and the fact that GQ_{2n} commutes with direct limits. Q.E.D.

COROLLARY 3.10 *Let GQ_{2n} denote the general quadratic functor on form algebras. Suppose that (A,Λ) is a form algebra such that A is module finite over k_A. If either $n \geq 3$ or $n = 2$ and $A = A\Lambda + \Lambda A$ then $EQ_{2n}(A,\Lambda) \lhd GQ_{2n}(A,\Lambda)$.*

PROOF One shows easily that (A,Λ) is quasi-Noetherian. The corollary follows now from (3.9). Q.E.D.

REMARK 3.10.1 There is a slight generalization of the general quadratic group, given in [B3, §13], for which the conclusions of (3.9) and (3.10) are also valid. The generalization is based on replacing the notion of form ring above by a pair (A,Λ) where A is a ring with antiautomorphism $a \longmapsto \overline{a}$ (by definition, $\overline{ab} = \overline{b}\,\overline{a}$ for all $a \in A$) such that for some fixed element $\lambda \in A, \overline{\overline{a}} = \lambda a \overline{\lambda}$ for all $a \in A$ and where Λ is a form parameter in the sense defined above. One can then define a generalized form algebra to be a triple (k_A, A, Λ) where (A,Λ) is as above, (k_A, A) is an algebra over k_A, the antiautomorphism $a \longmapsto \overline{a}$ on A is compatible with the trivial involution on k_A, and $k_A\Lambda \subset \Lambda$. The notion quasi-Noetherian for (k_A, A, Λ) is defined as in (3.8). The general quadratic group $GQ_{2n}(A,\Lambda)$ and its elementary subgroup $EQ_{2n}(A,\Lambda)$ are defined as for ordinary form rings above. This done, the conclusions in (3.9) and (3.10) remain valid. Checking details is straightforward.

The next result for Chevalley groups overlaps with (3.5) and (3.10).

THEOREM 3.11 *(Taddei [T2]) Let Φ be a reduced, irreducible root system of rank $\geqq 2$. Let R denote a commutative ring (which we view as an algebra over itself). Let $G(\Phi,R)$ denote the (untwisted) Chevalley group of Φ with coefficients in R and let $E(\Phi,R)$ denote its elementary subgroup. Then $E(\Phi,R) \lhd G(\Phi,R)$.*

PROOF Let the index set $\mathfrak{I} = \Phi$. Let as agree to denote the elements of $\mathfrak{I}$ by small Greek letters $\alpha, \beta, \ldots$, as is customary when dealing with root systems. Let $\boldsymbol{\Lambda}$ denote the subcategory of $\boldsymbol{M}(\mathfrak{I})$ whose objects are all pairs $(k, \bigoplus_{\alpha\in\mathfrak{I}} k_\alpha)$ where k is a commutative ring and $k_\alpha = k$ for all $\alpha \in \mathfrak{I}$. A morphism $(g,f) : (k, \oplus k_\alpha) \longrightarrow (k', \oplus k'_\alpha)$ is a pair (g,f) where $g : k \longrightarrow k'$

is a ring homomorphism and $f : \oplus k_\alpha \longrightarrow \oplus k'_\alpha$ a degree 0 map such that $f|_{k_\alpha} = g$ for all $\alpha \in \mathfrak{I}$. The assignment $k \longmapsto (k, \oplus k_\alpha)$ defines an isomorphism ((commutative rings)) $\longrightarrow \mathbf{\Lambda}$ of categories. The Chevalley functor $G(\Phi, -)$: ((commutative rings)) $\longrightarrow \boldsymbol{G}$ defines a $\mathbf{\Lambda}$-group $G : \mathbf{\Lambda} \longrightarrow \boldsymbol{G}$ such that $G(k, \oplus k_\alpha) = G(\Phi, k)$ and for each $\alpha \in \mathfrak{I}, \epsilon_\alpha : \lambda_\alpha \longrightarrow G(\Phi, \_)$ is the natural transformation with the property that if $a \in k$ then $\epsilon^k_\alpha(a) \in G(\Phi, k)$ is the usual elementary transformation defined by the root $\alpha \in \mathfrak{I}(= \Phi)$ and the element $a \in k$.

We want to apply now (3.2) to deduce that $E(\Phi, R) \lhd G(\Phi, R)$. So, we must check that all the hypotheses of (3.2) are satisfied. It is well known that there is a natural number n and an embedding $G(\Phi, R) \rightarrowtail GL_n(R)$ which is functorial in R. Using the set theoretic map $GL_n(R) \longrightarrow \mathbb{M}_n(R), \sigma \longmapsto \sigma - I$ (where I is the $n \times n$-identity matrix), one deduces easily that condition (2.7.1) is satisfied. Clearly, any commutative ring R is a direct limit of Noetherian subrings of itself and the functor $G(\Phi, \_)$ commutes with direct limits. Thus, to complete the proof of the theorem, it suffices to show that condition (3.1) is satisfied.

Since $\mathfrak{I}$ is a root system Φ, the negation $-\alpha$ of an element $\alpha \in \mathfrak{I}$ is defined. Define $\mathfrak{I}(\alpha) = \mathfrak{I} \setminus \{-\alpha\}$. Condition (3.1.1) follows directly from the standard commutator formula [C,(5.2)]

$$[\epsilon_\alpha(a), \epsilon_\beta(b)] = \prod_{\substack{i\alpha + j\beta \in \mathfrak{I} \\ i,j \in \mathbb{N}}} \epsilon_{i\alpha + j\beta}(N_{\alpha,\beta,i,j} a^i b^j)$$

where $\beta \in \mathfrak{I}(\alpha)$. It is an easy exercise to check that the hypotheses on Φ guarentee that given $\gamma \in \mathfrak{I}$, there are roots $\alpha, \beta \in \mathfrak{I}, \beta \in \mathfrak{I}(\alpha), \alpha, \beta \in \mathfrak{I}(\gamma)$ such that $\alpha + \beta = \gamma$ and $N_{\alpha,\beta,1,1} = 1$. Rewriting the commutator formula above as

$$\epsilon_\gamma(ab) = [\epsilon_\alpha(a), \epsilon_\beta(b)] (\prod_{\substack{i\alpha + j\beta \in \mathfrak{I} \\ i,j \in \mathbb{N} \\ (i,j) \neq (1,1)}} \epsilon_{i\alpha + j\beta}(-N_{\alpha,\beta,i,j} a^i b^j)),$$

one deduces easily that (3.1.2) holds. Let $\mathfrak{p}$ denote a maximal ideal of R and let $S_\mathfrak{p} = R \setminus \mathfrak{p}$. Let $\sigma \in G(\Phi, R)$ and 'σ ' its image in $G(\Phi, S_\mathfrak{p}^{-1} R)$. Since $S_\mathfrak{p}^{-1} R$ is a local ring, it follows from a result of Matsumoto [M, (4.4)] in the simply connected case and Abe-Suzuki [AS] in general that 'σ '$= \delta_1 \epsilon_1$ where δ_1 is a diagonal matrix, i.e. an element of the split maximal torus of $G(\Phi, S_\mathfrak{p}^{-1} R)$, and ϵ_1 a product of elementary transformations. It follows that for some element $s \in S_{\mathfrak{p}}$, the image "σ" of σ in $G(\Phi, < s >^{-1} R)$ has a product decomposition "σ "$= \delta\epsilon$ where δ is a diagonal matrix and ϵ a product of elementary transformations. Since δ is diagonal, one deduces easily that it has the property ascribed to it in (3.1.3). Q.E.D.

Let Φ be a reduced, irreducible root system of type A_ℓ, D_ℓ, or E_6 and rank $\geqq 2$. After fixing a system of fundamental roots for Φ, we have the notion [St,], [A,], [P,] of a canonical automorphsim ϱ of Φ. Except for D_4, there is exactly one such automorphism and it has order 2. In the case of D_4, there are two such automorphisms, one of order 3 and the other of order 3. Given a pair (Φ, ϱ) and a commutative ring R with a Ring action of $< \varrho >$ (: = the group generated by ϱ), one has the notion ${}^{\varrho}G(\Phi, R)$ of a twisted Chevalley group with coefficients in R and the notion ${}^{\varrho}E(\Phi, R)$ of its elementary subgroup.

The next theorem is the analog of the one above, for twisted Chevalley groups of type $\neq A_{2\ell}$, and overlaps also with (3.10).

THEOREM 3.12 *Let Φ be a completeley reduced, irreducible root system of type $A_{2\ell+1}, D_\ell$ or E_6 and rank $\geqq 2$. Let ϱ be a canonical automorphism of Φ. Let R denote a commutative ring with a ring action of ϱ. (We view R as an algebra over the fixed ring R^{ϱ}). Let ${}^{\varrho}G(\Phi, R)$ denote the twisted Chevalley group of Φ with coefficients in R and let ${}^{\varrho}(\Phi, R)$ denote its elementary subgroup. Then $E(\Phi, R) \triangleleft G(\Phi, R)$.*

PROOF Our proof is the same as the one above, except $\mathfrak{I}$ is set equal to the root system defined by the orbits of the action of ϱ on Φ, the standard commutator formula [C, (5.2)]is replaced by its twisted analogs [A, §3], [P,] and the reference to Matsumoto [M, (4.)] and Abe-Suzuki [AS] are replaced by reference Suzuki [Sz]. We fill in now the details.

The action of ϱ on the enveloping Euclidean space $\mathbb{E}$ of Φ preserves the usual inner product. If V denotes the subspace of $\mathbb{E}$ of all elements fixed by the action of ϱ then by [St,], [A], the orthogonal projection of $\mathbb{E}$ on V maps the orbit space of the action of ϱ on Φ onto a root system in V. Let $\mathfrak{I}$ denote this root system. If $(\Phi, \varrho) = (A_{2\ell+1,\varrho}), (D_\ell, \varrho), (E_6, \varrho)$ and throughout ϱ has order 2, or (D_4, ϱ) and ϱ has order 3 then $\mathfrak{I} = C_{\ell+1}, B_{\ell-1}, F_4, G_2$, respectively. (The twisted Chevalley group $G(A_{2\ell+1}, R)$ is the general quadratic group $GQ_{2(\ell+1)}(R, R^{\varrho})$ in (3.10).)

Let k denote a commutative ring with a ring action of ϱ. Let $k^{\varrho} = \{a \in k | \varrho(a) = a\}$. Let $\mathbf{\Lambda}$ denote the subcategory of $\boldsymbol{M}(\mathfrak{I})$ whose objects are all pairs $(k, \underset{\alpha\in\mathfrak{I}}{\oplus} k_\alpha)$ such that

$$k_\alpha = \begin{cases} k, & \text{if } \alpha \text{ is a short root} \\ k^\varrho, & \text{if } \alpha \text{ is a long root.} \end{cases}$$

A morphism $(g,f) : (k, \underset{\alpha\in\mathfrak{I}}{\oplus} k_\alpha) \longrightarrow (k', \underset{\alpha\in\mathfrak{I}}{\oplus} k'_\alpha)$ is a pair (g,f) where $g : k \longrightarrow k'$ is a ϱ-equivariant, ring homomorphism and $f : \oplus k_\alpha \longrightarrow \oplus k'_\alpha$ a degree 0 map such that $f|_{k_\alpha} = g|_{k_\alpha}$ for all $\alpha \in \mathfrak{I}$. (Note that each $k_\alpha \subseteq k$.) Let ((comm ϱ-rings)) denote the category of all commutative rings with a ring action of ϱ and all ϱ-equivariant ring homomorphisms. The assignment $k \longmapsto (k, \oplus k_\alpha)$ defines an isomorphism ((comm ϱ-rings)) $\longrightarrow \mathbf{\Lambda}$ of categories. The twisted Chevalley functor ${}^\varrho G(\Phi, -)$: ((comm ϱ-rings)) $\longrightarrow \boldsymbol{G}$ defines a $\mathbf{\Lambda}$-group $G : \mathbf{\Lambda} \longrightarrow \boldsymbol{G}$ such that $G(k, \oplus k_\alpha) = {}^\varrho G(\Phi, k)$ and for each $\alpha \in \mathfrak{I}, \epsilon_\alpha : \lambda_\alpha \longrightarrow {}^\varrho G(\Phi, -)$ is a natural transformation with the property that if $a \in k_\alpha (= \lambda_\alpha(\oplus k_\beta))$ then $\epsilon^k_\alpha(a) \in {}^\varrho G(\Phi, k)$ is the elementary transformation (which is also referred to in the literature by the expression elementary root unipotent element) (cf. [C,(13.6)]) defined by the root $\alpha \in \mathfrak{I}$ and the element $a \in k_\alpha$.

We want to apply now (3.2) to deduce that ${}^\varrho E(\Phi, R) \triangleleft {}^\varrho G(\Phi, R)$. So, we must check that all the hypotheses of (3.2) are satisfied. It is well known that there is a natural number n and an embedding ${}^\varrho G(\Phi, R) \rightarrowtail GL_n(R)$ which is functorial in R. Using the set theoretic map $GL_n(R) \longrightarrow \mathbb{M}_n(R), \sigma \longmapsto \sigma - I$ (where I is the $n \times n$-identity matrix), one deduces easily that condition (2.7.1) is satisfied. Clearly, any commutative ϱ-ring R is a direct limit in ((comm ϱ-rings)) of Noetherian ϱ-subrings of itself and the functor ${}^\varrho G(\Phi, \_)$ commutes with direct limits. Thus, to complete the proof of the theorem, it suffices to show that condition (3.1) is satisfied.

Since $\mathfrak{I}$ is a root system, the negation $-\alpha$ of an element $\alpha \in \mathfrak{I}$ is defined. Define $\mathfrak{I}(\alpha) = \mathfrak{I} \setminus \{-\alpha\}$. Condition (3.1.1) follows directly form the twisted analogs [A, §3], [P,] of the standard commutator formula [C, (5.2)] used in the proof of (3.11). It is an easy exercise to check that the hypotheses on Φ guarente that given $\gamma \in \mathfrak{I}$, there are roots $\alpha, \beta \in \mathfrak{I}, \alpha, \beta \in \mathfrak{I}(\gamma)$ such that $\alpha + \beta = \gamma$ and $N_{\alpha,\beta,1,1} = 1$. Using the analogs above of the standard commutator formula, one can write

$$\epsilon_\gamma(ab) = [\epsilon_\alpha(a), \epsilon_\beta(b)] \prod_{\substack{i\alpha+j\beta\in\mathfrak{I} \\ i,j\in\mathbb{N} \\ (i,j)\neq(1,1)}} \epsilon_{i\alpha+j\beta}(\star(\alpha,\beta,i,j,a,b)),$$

for suitable coefficients $\star(\alpha, \beta, i, j, a, b)$ and deduces that (3.1.2) holds. We leave the details here to the reader. Let $\mathfrak{p}$ denote a maximal ideal of R^{ϱ} and let $S_{\mathfrak{p}} = R^{\varrho} \setminus \mathfrak{p}$. Let $\sigma \in {}^{\varrho}G(\Phi, R)$ and 'σ' its image in ${}^{\varrho}G(\Phi, S_{\mathfrak{p}}^{-1}R)$. Since $S_{\mathfrak{p}}^{-1}R^{\varrho}$ is a local ring, it follows from a result of Suzuki [Sz] that 'σ' $= \delta_{1\epsilon 1}$ where δ_1 is a diagonal matrix, i.e. an element of the Cartan subgroup of ${}^{\varrho}G(\Phi, S_{\mathfrak{p}}^{-1}R)$, and ϵ_1 a product of elementary transformations. It follows that for some elements $s \in S_{\mathfrak{p}}$, the image "σ" of σ in ${}^{\varrho}G(\Phi, < s >^{-1} R)$ has a product decomposition "σ" $= \delta\epsilon$ where δ is a diagonal matrix and ϵ a product of elementary transformations. Since δ is diagonal, one deduces easily that it has the property ascribed to it in (3.1.3). Q.E.D.