Overgroups of Unitary Groups

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Abstract

This paper classifies under a local stable rank condition for rings with form parameter, subgroups of the general linear group GL_{2l} which contain the elementary unitary subgroup EU_{2l}.

1 Introduction

In the present paper we describe overgroups of the elementary hyperbolic unitary group over a form ring subject to a certain stability condition which holds in particular for almost commutative and semilocal rings. This paper is a continuation of previous joint papers with N. Vavilov [16, 17], where a similar description was obtained for the orthogonal and symplectic cases.

We use the setting of generalized unitary groups introduced by A. Bak [2, 6, 5] which we recall in Section 3 in slightly different terms. Let (R, Λ) be a form ring, $\mathrm{EU}_{2l}(R, \Lambda)$ be the elementary hyperbolic unitary group of degree $2l$ over (R, Λ). Consider subgroups of the general linear group $\mathrm{GL}_{2l}(R)$ which contain $\mathrm{EU}_{2l}(R, \Lambda)$. We prove that for any such subgroup H there exists a unique ideal form parameter (A, Γ) such that H normalizes the elementary subgroup $\mathrm{EEU}_{2l}(R, A, \Gamma)$ of level (A, Γ) (see Section 14 for definitions).

In the case of a skew-field R our results boil down to a special case of results by O. King [8] and Li Shangzhi [9] pertaining to hyperbolic forms. On the other hand for the case of a commutative R with trivial involution and the maximal form parameter our results spicialize to those of [16, 17].

As in [17] we use a version of local-global techniques introduced by D. Quillen, A. Suslin, L. Vaserstein and A. Bak, see, for example, [14, 15, 3, 7] and references there. We make substantial use of calculations from the theses of R. Hazrat [7]. Unlike the case of split classical groups, in our setting several new ideas are required to settle the local case. Here we prove a version of Witt's theorem in terms of a new stability condition Λsr introduced by A. Bak [4]. This theorem is an extension of a result by W. van der Kallen, B. Magurn and L. Vaserstein [12] and in particular immediately implies surjective stability for KU_1.

The paper is organized as follows. In Sections 2–6 we recall basic definitions related to unitary groups and their elementary subgroups, in Section 7 and

8 we prove several auxiliary results we need for localization procedures. In Sections 9–11 we introduce Λsr and prove Witt's theorem. In Section 12 we perform calculations with elementary matrices similar to the yoga of conjugation used by R. Hazrat [7]. Section 13 is a crucial step in the proof of the main theorem. In Section 14 we introduce the group $\mathrm{EEU}_{2l}(R, A, \Gamma)$ and calculate its normalizer. Finally the main theorem is established in Section 15.

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2 General notations

Let G by an arbitrary group. If $a, b \in G$, we write ${}^a b$ for aba^{-1} and $[a, b]$ for $aba^{-1}b^{-1}$. In the sequel we use the following commutator formulae

$$[a, bc] = [a, b] \cdot {}^b[a, c],$$
$$[ab, c] = {}^a[b, c][a, c],$$
$$[ab, cd] = {}^a[b, c] \cdot {}^{ac}[b, d] \cdot [a, c] \cdot {}^c[a, d],$$
$$[{}^c a, [b, c]] \cdot [{}^b c, [a, b]] \cdot [{}^a b, [c, a]] = 1 \text{ (the Hall identity).}$$

For a subset $X \subseteq G$ and a subgroup $H \leq G$ we denote by $\langle X \rangle$ the subgroup generated by X, and by $\langle X \rangle^H$ the smallest subgroup containing X and normalized by H. For two subgroups $F, H \leq G$ we denote by $[F, H]$ their relative commutator subgroup, generated by $[f, h]$, $f \in F$, $h \in H$. A group is called *perfect* if it coincides with its commutator subgroup.

Further, let R be an arbitrary associative ring with identity. Denote by $R^\times$ the group of all invertible elements of R. Let n be a positive integer. For a matrix $a \in \mathrm{GL}_n(R)$ we denote by a_{ij} its coefficient in the position (i, j), and we write a'_{ij} for $(a^{-1})_{ij}$. For $a_i \in \mathrm{GL}_{n_i}(R)$, $i = 1, \ldots, m$ we denote by $\mathrm{diag}(a_1, \ldots, a_m)$ the block-diagonal matrix with blocks $a_1, \ldots, a_m$.

A column $u = (u_1, \ldots, u_n)^t$ is called *unimodular* if the left ideal generated by $u_1, \ldots, u_n$ coincides with R. For example, a column of every invertible matrix is unimodular.

Let e be the identity matrix and e^{ij} be the matrix which has 1 in the position (i, j) and zeroes elsewhere. By $t_{ij}(\xi)$ ($\xi \in R$, $i \neq j$) we denote the *linear elementary transvection*: $t_{ij}(\xi) = e + \xi e^{ij}$. In the sequel we use the following relations among the elementary transvections:

$$t_{ij}(\xi) t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$$
$$[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta), \ h \neq i.$$

Let A be an ideal of R. Denote by $\mathrm{E}_n(A)$ the following subgroup of $\mathrm{GL}_n(R)$:

$$\mathrm{E}_n(A) = \langle t_{ij}(\xi), \xi \in A, i \neq j \rangle.$$

When $A = R$ the subgroup $\mathrm{E}_n(R)$ is called the *linear elementary group*. By $\mathrm{E}_n(R, A)$ we denote the normal closure of $\mathrm{E}_n(A)$ in $\mathrm{E}_n(R)$:

$$\mathrm{E}_n(R, A) = \mathrm{E}_n(A)^{\mathrm{E}_n(R)}.$$

Lemma 1. *Let $n \geq 3$. Then $\mathrm{E}_n(R, A)$ is generated by all elements of the form $z_{ij}(\xi, \zeta) = t_{ji}(\zeta)t_{ij}(\xi)t_{ji}(-\zeta)$, $\xi \in A$, $\zeta \in R$, $j \neq i$.*

Proof. See, for example, [14, Lemma 8]. □

Denote by ρ_A the reduction homomorphism modulo A:

$$\rho_A : \mathrm{GL}_{2l}(R) \to \mathrm{GL}_{2l}(R/A).$$

The kernel of ρ_A is denoted by $\mathrm{GL}_{2l}(R, A)$ and called the *general congruence subgroup of level A*.

3 Unitary group

Let R be an associative ring with identity. An additive map $\sigma : R \to R$, $\xi \mapsto \bar{\xi}$ is called a *pseudoinvolution* if $\bar{\bar{\xi}} = \xi$ and $\overline{\xi\zeta} = \bar{\zeta}\bar{1}^{-1}\bar{\xi}$. Set $\varepsilon = -\bar{1}$ and $\xi^* = \bar{\xi}\bar{1}^{-1}$. Then we have $1^* = 1$, $(\xi\zeta)^* = \zeta^*\xi^*$ and $\xi^{**} = \varepsilon\xi\varepsilon^*$. Conversely, given an additive map $\tau : R \to R$, $\xi \mapsto \xi^*$ and an element $\varepsilon \in R$ satysfying the conditions above, define the pseudoinvolution $\sigma : R \to R$ as follows: $\bar{\xi} = -\xi^*\varepsilon$. So our notion of pseudoinvolution coincides essentially with one of Magurn, van der Kallen and Vaserstein (see [12]), but we use only one parameter σ instead of τ and ε.

Set $\min^\sigma(R) = \{\xi + \bar{\xi} | \xi \in R\}$, $\max^\sigma(R) = \{\xi \in R | \bar{\xi} = \xi\}$.

An additive subgroup Λ of R is called a *σ-form parameter* if the following conditions hold:

$$\min^\sigma(R) \subseteq \Lambda \subseteq \max^\sigma(R),$$
$$\bar{\xi}\bar{1}^{-1}\Lambda\xi \subseteq \Lambda \text{ for all } \xi \in R.$$

This is Bak's notion of form parameter [2, §1B and §13], formulated in the language of pseudoinvolutions.

Let V be a right R-module. A biadditive form $B : V \times V \to R$ is *σ-sesquilinear* if

$$B(u\xi, v\zeta) = \bar{\xi}\bar{1}^{-1}B(u, v)\zeta \text{ for all } u, v \in V,\ \xi, \zeta \in R.$$

A σ-sesquilinear form H is called *σ-antihermitian* if

$$H(u, v) = -\overline{H(v, u)} \text{ for all } u, v \in V.$$

A (σ, Λ)-*quadratic form* is a pair $q = (H, Q)$ consisting of a σ-antihermitian form H and a map $Q : V \to R/\Lambda$ satisfying the following conditions for all

$u, v \in V$, $\xi \in R$:

$$Q(v\xi) = \bar{\xi}\bar{1}^{-1}Q(v)\xi,$$
$$Q(u+v) = Q(u) + Q(v) + H(u,v) + \Lambda,$$
$$H(v,v) = \alpha - \bar{\alpha} \text{ for any } \alpha \in R \text{ such that } Q(v) = \alpha + \Lambda.$$

We denote H by $(\cdot,\cdot)_q$ and Q by $|\cdot|_q$. $(\cdot,\cdot)_q$ is called the *associated antihermitian form* of q and $|\cdot|_q$ the *associated quadratic form*. The (σ,Λ)-quadratic form q is called *nonsingular*, if the map $V \to \mathrm{Hom}_R(V,R)$, $v \mapsto (v,\cdot)_q$ is bijective.

A pair (V,q) is called a (σ,Λ)*-quadratic space*.

Any σ-sesquilinear form B determines a (σ,Λ)-quadratic form by setting $(u,v)_q = B(u,v) - \overline{B(v,u)}$, $|v|_q = B(v,v) + \Lambda$.

Let B denote a σ-sesquilinear form on V. If $\lambda \in \mathrm{Cent}(R)$ then the σ-sesquilinear form λB (by definition $(\lambda B)(u,v) = \lambda B(u,v)$) is called *similar* to B. Moreover, if $\{B(u,v)|u,v \in V\}$ contains a nonzero divisor and λ is an arbitrary element of R then λB is a σ-sesquilinear if and only if $\lambda \in \mathrm{Cent}(R)$; furthermore $\lambda B = \lambda' B$ for some $\lambda' \in R$ if and only if $\lambda = \lambda'$. If in addition B and λB are σ-antihermitian then $\bar{\lambda} = \lambda\bar{1}$. Indeed, choose $u,v \in V$ such that $B(u,v)$ is a nonzero divisor in R. Then

$$B(u,v)\lambda = \lambda B(u,v) = -\overline{\lambda B(v,v)} = -\overline{B(v,u)}\bar{1}^{-1}\bar{\lambda} = B(u,v)\bar{1}^{-1}\bar{\lambda},$$

hence $\bar{\lambda} = \lambda\bar{1}$.

Let $R_\Lambda = \{\xi \in \mathrm{Cent}(R)|\bar{\xi} = \xi\bar{1}, \xi\Lambda \subseteq \Lambda\}$. Obviously R_Λ is a subring or $\mathrm{Cent}(R)$. If $q = (H,Q)$ is a (σ,Λ)-quadratic form on V and $\lambda \in R_\Lambda$ then the (σ,Λ)-quadratic form $\lambda q = (\lambda H, \lambda Q)$ is called *similar* to q. If $\{H(u,v)|u,v \in V\}$ contains a nonzero divisor then $\lambda q = \lambda' q$ if and only if $\lambda = \lambda'$, by the paragraph above.

Let (V,q) and (V',q') be two (σ,Λ)-quadratic spaces. An R-module homomorphism $f: V \to V'$ is an *isometry* if $(fu,fv)_{q'} = (u,v)_q$, $|fv|_{q'} = |v|_q$ for all $u,v \in V$, and a *similitude* if $\{(u,v)_q|u,v \in V\}$ contains a nonzero divisor and there exists (a unique) $\lambda \in R_\Lambda$ (called the *multiplicator* of f) such that $(fu,fv)_{q'} = \lambda(u,v)_q$, $|fv|_{q'} = \lambda|v|_q$ for all $u,v \in V$. So an isometry is a similitude with multiplicator 1.

If $\{e_i\}_i \in I$ is a family of generators V then it's sufficient to check identities from the definition of a similitude (or an isometry) only for the case when u and v belong to this family. Indeed, if $u = \sum_i e_i\xi_i$, $v = \sum_j e_j\zeta_j$ then

$$(fu,fv)_{q'} = \sum_{i,j} \bar{\xi}_i\bar{1}^{-1}(fe_i,fe_j)_{q'}\zeta_j = \lambda\sum_{i,j} \bar{\xi}_i\bar{1}^{-1}(e_i,e_j)_q\zeta_j = \lambda(u,v)_q,$$

$$\begin{aligned}|fu|_{q'} = \sum_i \bar{\xi}_i\bar{1}^{-1}|fe_i|_{q'}\xi_i + \sum_{i<j} \bar{\xi}_i\bar{1}^{-1}(fe_i,fe_j)_{q'}\xi_j \\ = \lambda(\sum_i \bar{\xi}_i\bar{1}^{-1}|e_i|_q\xi_i + \sum_{i<j}\bar{\xi}_i\bar{1}^{-1}(e_i,e_j)_q\xi_j) = \lambda|u|_q.\end{aligned}$$

A composition of two similitudes is clearly a similitude. So we have two categories of (σ,Λ)-quadratic spaces: in the first one, morphisms are isometries, and in the second one morphisms are similitudes.

Suppose $\{(u,v)_q | u,v \in R\}$ contains a nonzero divisor. Let $\mathrm{U}(V,q)$ denote the group of all bijective isometries of (V,q) with itself and $\mathrm{GU}(V,q)$ the group of all bijective similitudes of (V,q) with itself. $\mathrm{U}(V,q)$ is called *unitary group* and $\mathrm{GU}(V,q)$ is called *group of similitudes* of (V,q).

Under the assumption above, the multiplicator of a similitude determines a group homomorphism from $\mathrm{GU}(V,q)$ to the group $R_\Lambda^\times$, whose kernel is clearly $\mathrm{U}(V,q)$. Thus $\mathrm{U}(V,q)$ is a normal subgroup of $\mathrm{GU}(V,q)$.

4 Eichler-Siegel-Dickson transvections

Let u,v be elements of V, ξ,α be elements of R such that $|u|_q=0$, $(u,v)_q=0$, $|v|_q=\alpha+\Lambda$. Define $T_{uv}(\xi,\alpha)$ to be a transformation of V mapping $w\in V$ to $w+u\xi((v,w)_q+\alpha\bar{1}^{-1}\bar{\xi}\bar{1}^{-1}(u,w)_q)+v\bar{1}^{-1}\bar{\xi}\bar{1}^{-1}(u,w)_q$. Transformations of this form are known as *Eichler-Siegel-Dickson transvections*. The following properties are verified by a direct calculation:

$$T_{uv}(\xi\zeta,\alpha)=T_{u\xi,v}(\zeta,\alpha)=T_{u,v\bar{1}^{-1}\bar{\zeta}}(\xi,\zeta\alpha\bar{1}^{-1}\bar{\zeta}),$$

$$T_{u,v_1}(\xi,\alpha)T_{u,v_2}(\zeta,\beta)=T_{u,v_1\bar{1}^{-1}\bar{\xi}+v_2\bar{1}^{-1}\bar{\zeta}}(1,\\ \xi\alpha\bar{1}^{-1}\bar{\xi}+\zeta\beta\bar{1}^{-1}\bar{\zeta}+\xi(v_1,v_2)_q\bar{1}^{-1}\bar{\zeta}),$$

and if $g\in\mathrm{GU}(V,q)$ has multiplicator λ then

$$gT_{uv}(\xi,\alpha)g^{-1}=T_{gu,gv}(\lambda^{-1}\xi,\lambda\alpha).$$

Lemma 2. *$T_{uv}(\xi,\alpha)$ belongs to* $\mathrm{U}(V,q)$.

Proof. We may suppose that $\xi=1$. Let $w\in V$; set $a=(u,w)_q$, $b=(v,w)_q$. Then

$$|T_{uv}(1,\alpha)w|_q=|w+u(b+\alpha\bar{1}^{-1}a)+v\bar{1}^{-1}a|_q\\ =|w|_q+\bar{a}|v|_q\bar{1}^{-1}a-\bar{a}(b+\alpha\bar{1}^{-1}a)-\bar{b}\bar{1}^{-1}a.$$

But $\bar{a}b+\bar{b}\bar{1}^{-1}a\in\min^\sigma(R)\subseteq\Lambda$, and

$$\bar{a}|v|_q\bar{1}^{-1}a-\bar{a}\alpha\bar{1}^{-1}a=\overline{\bar{1}^{-1}a}\bar{1}^{-1}(|v|_q-\alpha)\bar{1}^{-1}a\in\Lambda.$$

Now let $w_1,w_2\in V$; set $a_1=(u,w_1)_q$, $a_2=(u,w_2)_q$, $b_1=(v,w_1)_q$, $b_2=(v,w_2)_q$. Then

$$(T_{uv}(1,\alpha)w_1,T_{uv}(1,\alpha)w_2)_q=(w_1+u(b_1+\alpha\bar{1}^{-1}a_1)+v\bar{1}^{-1}a_1,\\ w_2+u(b_2+\alpha\bar{1}^{-1}a_2)+v\bar{1}^{-1}a_2)_q=(w_1,w_2)_q+(\bar{b}_1+\bar{a}_1\bar{\alpha})\bar{1}^{-1}a_2\\ +\bar{a}_1b_2-\bar{a}_1(b_2+\alpha\bar{1}^{-1}a_2)-\bar{b}_1\bar{1}^{-1}a_2+\bar{a}_1(v,v)_q\bar{1}^{-1}a_2\\ =(w_1,w_2)_q+\bar{a}_1((v,v)_q-\alpha+\bar{\alpha})\bar{1}^{-1}a_2=(w_1,w_2)_q.$$

□

5 Hyperbolic unitary group

A family of vectors $\{e_i\}_{i=1,\dots,n,-n,\dots,-1}$ in (V,q) is called *hyperbolic* if $|e_i|_q = 0$ for all i, $(e_i, e_j)_q = 0$ for all $j \neq \pm i$, and $(e_i, e_{-i})_q = \bar{1}$ for $i = 1,\dots,n$. Then $(e_i, e_j) = \delta_{i,-j}\bar{1}\varepsilon_i$, where δ is the Kronecker symbol, and ε_i equals to 1 if $i = 1,\dots,n$, and to $-\bar{1}^{-1}$ if $i = -n,\dots,-1$. The greatest n such that there exists a hyperbolic family $\{e_i\}_{i=1,\dots,n,-n,\dots,-1}$ is called *Witt index* of quadratic space (V,q) and denoted by $\mathrm{ind}(V)$.

Suppose V has a hyperbolic family $\{e_i\}_{i=1,\dots,l,-l,\dots,-1}$ as a basis (that is (V,q) has dimension $2l$ and Witt index l). The group $\mathrm{U}(V,q)$ (respectively, $\mathrm{GU}(V,q)$) in this case is called *hyperbolic* and is denoted by $\mathrm{U}_{2l}(R,\sigma,\Lambda)$ (respectively, $\mathrm{GU}_{2l}(R,\sigma,\Lambda)$). We fix σ and will omit it in the notation.

Now we deduce explicit formulae determining whether an element of $\mathrm{GL}_{2l}(R)$ belongs to $\mathrm{GU}_{2l}(R,\Lambda)$ (or $\mathrm{U}_{2l}(R,\Lambda)$).

Since $\{e_i\}$ is a family of generators, $g \in \mathrm{GU}_{2l}(R,\Lambda)$ if and only if $(ge_i, ge_j)_q = \lambda\delta_{i,-j}\bar{1}\varepsilon_i = \delta_{i,-j}\bar{\lambda}\varepsilon_i$ and $|ge_i|_q = 0$ for all i,j. The second condition can be rewritten in a form

$$\sum_{j=1}^{l} \bar{g}_{ji} g_{-ji} \in \Lambda \text{ for all } i. \tag{GU2'}$$

The first condition may be written as

$$\sum_k \bar{g}_{ki}\varepsilon_k g_{-kj} = \delta_{i,-j}\bar{\lambda}\varepsilon_i \text{ for all } i,j. \tag{GU1'}$$

Multiplying by $g'_{j,-h}$ on the right and summing by j, we obtain

$$\sum_k \bar{g}_{ki}\varepsilon_k\delta_{-k,-h} = \bar{\lambda}\sum_j \delta_{i,-j}\bar{1}\varepsilon_i g'_{j,-h},$$

or, equivalently,

$$\bar{g}_{hi} = \bar{\lambda}\varepsilon_i g'_{-i,-h}\varepsilon_h^{-1} \text{ for all } i,h. \tag{GU1}$$

Conversely, (GU1') follows obviously from (GU1).

Now we can write conditions (GU1') and (GU2') for g^{-1} and apply equation (GU1). We have

$$\sum_k g_{ik}\varepsilon_k\bar{g}_{j,-k} = -\delta_{i,-j}\lambda\varepsilon_j^{-1}, \tag{GU1''}$$

and

$$\sum_{j=1}^{l} g_{i,-j}\bar{1}^{-1}\bar{g}_{ij} \in \Lambda. \tag{GU2''}$$

To obtain equation for $\mathrm{U}_{2l}(R,\Lambda)$ one has to substitute $\lambda = 1$.

Now it's easy to see that the functor mapping pair (R,Λ) to $\mathrm{U}_{2l}(R,\Lambda)$ commutes with filtered direct limits.

Denote by $\mathrm{T}_{2l}(R,\Lambda)$ subgroup of $\mathrm{GU}_{2l}(R,\Lambda)$ consisting of all diagonal matrices. Let $\lambda \in R_\Lambda^\times$. Note that a matrix $\mathrm{diag}(1,\dots,1,\lambda,\dots,\lambda)$ belongs to $\mathrm{T}_{2l}(R,\Lambda)$ and has a multiplicator λ. So we have $\mathrm{GU}_{2l}(R,\Lambda) = \mathrm{U}_{2l}(R,\Lambda)\,\mathrm{T}_{2l}(R,\Lambda)$.

We consider $\mathrm{GU}_{2l-2}(R,\Lambda)$ as a subgroup of $\mathrm{GU}_{2l}(R,\Lambda)$ consisting of elements preserving e_1 and line generated by e_{-1}. That is a matrix g maps to a matrix $\mathrm{diag}(1,g,\lambda)$ where λ is the multiplicator of g.

6 Elementary unitary group

Now we describe a subgroup of $\mathrm{U}_{2l}(R,\Lambda)$ called *elementary* unitary group.

Set

$$T_{ij}(\xi) = T_{e_i e_{-j}}(-\xi\varepsilon_j, 0),\ j \neq \pm i,$$
$$T_{i,-i}(\alpha) = T_{e_i 0}(1, \alpha\varepsilon_i^{-1}),\text{ where } \alpha \in \Lambda\varepsilon_i.$$

$T_{ij}(\xi)$ is called a *short* elementary transvection, and $T_{i,-i}(\alpha)$ a *long* elementary transvection. They satisfies the following relations:

$$T_{ij}(\xi) = T_{-j,-i}(\varepsilon_{-j}\bar{\xi}\varepsilon_i) \tag{R1}$$
$$T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi+\zeta) \tag{R2}$$
$$[T_{ij}(\xi), T_{hk}(\zeta)] = e, \quad h \neq j, -i,\ k \neq i, -j \tag{R3}$$
$$[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta), \quad i, h \neq \pm j,\ i \neq \pm h \tag{R4}$$
$$[T_{ij}(\xi), T_{j,-i}(\zeta)] = T_{i,-i}(\xi\zeta + \varepsilon_i\bar{\zeta}\bar{1}^{-1}\bar{\xi}\varepsilon_i), \quad j \neq \pm i \tag{R5}$$
$$[T_{ij}(\xi), T_{j,-j}(\alpha)] = T_{i,-j}(\xi\alpha)T_{i,-i}(-\xi\alpha\varepsilon_{-j}\bar{\xi}\varepsilon_i), \quad j \neq \pm i. \tag{R6}$$

They are related with elementary linear transvections by formulae (S1) –(S5):

$$T_{ij}(\xi) = t_{ij}(\xi)t_{-j,-i}(\varepsilon_{-j}\bar{\xi}\varepsilon_i), \quad j \neq \pm i \tag{S1}$$
$$T_{i,-i}(\alpha) = t_{i,-i}(\alpha) \tag{S2}$$
$$[T_{ij}(\xi), t_{hk}(\zeta)] = e, \quad h \neq j, -i,\ k \neq i, -j \tag{S3}$$
$$[T_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta), \quad h \neq i, -j \tag{S4}$$
$$[T_{ij}(\xi), t_{j,-j}(\zeta)] = T_{i,-j}(\xi\zeta)t_{i,-i}(-\xi\zeta\varepsilon_{-j}\bar{\xi}\varepsilon_i)t_{j,-i}(-(\zeta\varepsilon_{-j} + \varepsilon_j\bar{\zeta}\bar{1}^{-1})\bar{\xi}\varepsilon_i), j \neq \pm i \tag{S5}$$

The elementary unitary group $\mathrm{EU}_{2l}(R,\Lambda)$ is the group generated by all elementary transvections (both short and long).

Obviously, the functor mapping pair (R,Λ) to $\mathrm{EU}_{2l}(R,\Lambda)$ also commutes with filtered direct limits.

Lemma 3. *Let $l \geq 3$. Then $\mathrm{EU}_{2l}(R,\Lambda)$ is perfect.*

Proof. From (R4) it follows that commutator subgroup contains all short elementary transvections, and from (R6) it follows that it contains also all long elementary transvections. □

Lemma 4. *Let $l \geq 2$, $g \in \mathrm{GL}_{2l}(R)$ such that $g\,\mathrm{EU}_{2l}(R,\Lambda)g^{-1} \leq \mathrm{U}_{2l}(R,\Lambda)$. Then g belongs to $\mathrm{GU}_{2l}(R,\Lambda)$.*

Proof. Since $gT_{e_ie_j}(1,0)g^{-1}$ belongs to $\mathrm{U}_{2l}(R,\Lambda)$, we have

$$(ge_i, ge_{-i})_q = (gT_{e_ie_j}(1,0)e_i, gT_{e_ie_j}(1,0)e_{-i})_q = (ge_i, ge_{-i} + ge_j\varepsilon_i)_q.$$

Thus $(ge_i, ge_j)_q = 0$ for all $j \neq \pm i$.

Next, one has

$$\begin{aligned}|ge_{-i}|_q = |gT_{e_ie_j}(1,0)e_{-i}|_q = |ge_{-i} + ge_j\varepsilon_i|_q \\ = |ge_{-i}|_q + (ge_{-i}, ge_j)_q\varepsilon_i + \bar{\varepsilon}_i\bar{1}^{-1}|ge_j|_q\varepsilon_i.\end{aligned}$$

Thus $|ge_j|_q = 0$ for all j.

Since $gT_{e_ie_j}(\xi,0)g^{-1} \in \mathrm{U}_{2l}(R,\Lambda)$, we have

$$\begin{aligned}(ge_{-i}, ge_{-j})_q = (gT_{e_ie_j}(\xi,0)e_{-i}, gT_{e_ie_j}(\xi,0)e_{-j})_q = (ge_{-i} + ge_j\bar{1}^{-1}\bar{\xi}\varepsilon_i, \\ ge_{-j} + ge_i\xi\varepsilon_j)_q = (ge_{-i}, ge_{-j})_q + \bar{\varepsilon}_i\bar{1}^{-1}\xi(ge_j, ge_{-j})_q + (ge_{-i}, ge_i)_q\xi\bar{1}\varepsilon_j \\ + \bar{\varepsilon}_i\bar{1}^{-1}\xi(ge_j, ge_i)_q\xi\bar{1}\varepsilon_j.\end{aligned}$$

Thus $\bar{\varepsilon}_i^{-1}\bar{1}(ge_{-i}, ge_i)_q\xi = -\xi(ge_j, ge_{-j})_q\varepsilon_j^{-1}\bar{1}^{-1}$ for all $j \neq \pm i$. But $\bar{\varepsilon}_i^{-1}\bar{1} = \varepsilon_i$, $\varepsilon_j^{-1}\bar{1}^{-1} = -\varepsilon_{-j}$, so $\varepsilon_i(ge_{-i}, ge_i)\xi = \xi(ge_j, ge_{-j})\varepsilon_{-j}$.

Substituting $\xi = 1$, we have $\varepsilon_i(ge_{-i}, ge_i)_q = (ge_j, ge_{-j})_q\varepsilon_{-j}$. Applying this equation to the previous one, we see that $(ge_j, ge_{-j})_q\varepsilon_{-j}$ is central for all j. Now we have (since $(ge_{-i}, ge_i)_q\varepsilon_i$ is central)

$$\varepsilon_i(ge_{-i}, ge_i)_q = \varepsilon_i(ge_{-i}, ge_i)_q\varepsilon_i\varepsilon_i^{-1} = (ge_{-i}, ge_i)_q\varepsilon_i.$$

Hence, $(ge_{-i}, ge_i)_q\varepsilon_i = (ge_j, ge_{-j})_q\varepsilon_{-j}$ for all $j \neq \pm i$. Since $l \geq 2$, an element $\lambda = -(ge_j, ge_{-j})_q\varepsilon_{-j}$ does not depend on j. We show that $\lambda \in R_\Lambda$. We already proved that λ is central. Let $\alpha \in \Lambda$. Then

$$|ge_{-1}|_q = |gT_{e_10}(1,\alpha)e_{-1}|_q = |ge_{-1} + ge_1\alpha|_q = |ge_{-1}|_q - \lambda\alpha.$$

Thus $\lambda\alpha \in \Lambda$, hence $\lambda\Lambda \subseteq \Lambda$.

Now it's easy to verify that $(ge_i, ge_j)_q = \lambda(e_i, e_j)_q$ for all i, j, and therefore g is a similitude with multiplicator λ. □

7 Localization

Let R_0 be a fixed subring of R such that $R_0 \subseteq R_\Lambda$ for every form parameter Λ. The set of all maximal ideals of R_0 will be denoted by $\mathrm{Max}(R_0)$.

All multiplicative systems considered below will be in R_0.

Let S be a multiplicative system. The image of R under the localization homomorhism will be denoted by "R"$_S$.

Denote by Λ_S a set of all elements of the form $\frac{\alpha}{s}$, $\alpha \in \Lambda$, $s \in S$. It's easy to check that Λ_S is a form parameter of R_S.

A pair (R_S, Λ_S) can be represented as a filtered direct limit of pairs (R_s, Λ_s) by all $s \in S$.

Let V be a right R-module, B be a biadditive form on V, S be a multiplicative system. Define a biadditive form B_S on a module $V_S = V \otimes_R R_S$ as follows:

$$B_S(\frac{u}{s}, \frac{v}{t}) = \frac{B(u,v)}{st} \text{ where } u, v \in V,\ s, t \in S.$$

It's clear that if H is antihermitian then H_S is also antihermitian.

Similarly, if Q is a map $V \to R/\Lambda$ satisfying the condition from the definition of quadratic forms, define Q_S to be a map $V_S \to R_S/\Lambda_S$ as follows:

$$Q_S(\frac{v}{s}) = \frac{Q(v)}{s^2} \text{ where } v \in V,\ s \in S.$$

So we have a localization functor from a category of Λ-quadratic spaces to a category of Λ_S-quadratic spaces mapping (V, q) to (V_S, q_S) (we leave to the reader defining a value of this functor on morphisms). As a consequence, we have maps $\mathrm{GU}(V, q) \to \mathrm{GU}(V_S, q_S)$ and $\mathrm{U}(V, q) \to \mathrm{U}(V_S, q_S)$. They clearly coincide with restrictions of $F_S : \mathrm{GL}(V) \to \mathrm{GL}(V_S)$ defined as follows:

$$(F_S(g))(\frac{v}{s}) = \frac{g(v)}{s} \text{ where } v \in V,\ s \in S.$$

Lemma 5. *Let V be a finite generated R-module, q be a Λ-quadratic form. Assume that V contains vectors u, v such that $(u, v)_q$ is a nonzero divisor. Fix for every $\mathfrak{m} \in \mathrm{Max}(R_0)$ a multiplicative system $S_\mathfrak{m} \subseteq R_0 \setminus \mathfrak{m}$. Let $g \in \mathrm{GL}(V)$. Then $g \in \mathrm{GU}(V, q)$ if and only if $F_{S_\mathfrak{m}}(g) \in \mathrm{GU}(V_{S_\mathfrak{m}}, q_{S_\mathfrak{m}})$ for all $\mathfrak{m} \in \mathrm{Max}(R_0)$.*

Proof. Let $\{e_i\}_{i=1,\dots,n}$ be a family of generators. We can assume that $(e_1, e_2)_q$ is a nonzero divisor. Suppose for each $\mathfrak{m} \in \mathrm{Max}(R_0)$ $F_{S_\mathfrak{m}}(g) \in \mathrm{GU}(V_{S_\mathfrak{m}}, q_{S_\mathfrak{m}})$ and has multiplicator $\lambda_\mathfrak{m}$. For each $\mathfrak{m} \in \mathrm{Max}(R_0)$, there exists $s_\mathfrak{m} \in S_\mathfrak{m}$ such that

$$\begin{aligned}
&s_\mathfrak{m}\lambda_\mathfrak{m} \in \text{"}R\text{"}_{S_\mathfrak{m}}, \text{ say, } s_\mathfrak{m}\lambda_\mathfrak{m} = F_{S_\mathfrak{m}}(\mu_\mathfrak{m}),\\
&s_\mathfrak{m}(ge_i, ge_j)_q = \mu_\mathfrak{m}(e_i, e_j)_q,\\
&s_\mathfrak{m}|ge_i|_q = \mu_\mathfrak{m}|e_i|_q.
\end{aligned}$$

The ideal generated by all $s_\mathfrak{m}$ is not contained in any maximal ideal of R_0. Thus, there exist $t_\mathfrak{m} \in R_0$ such that $\sum_\mathfrak{m} t_\mathfrak{m} s_\mathfrak{m} = 1$. Set $\lambda = \sum_\mathfrak{m} t_\mathfrak{m}\mu_\mathfrak{m}$. Then we have

$$\begin{aligned}
&(ge_i, ge_j)_q = \lambda(e_i, e_j)_q,\\
&|ge_i|_q = \lambda|e_i|_q.
\end{aligned}$$

It remains only to prove that $\lambda \in R_\Lambda$.

Note that $\lambda(e_1, e_2)_q = (ge_1, ge_2)_q = \lambda_\mathfrak{m}(e_1, e_2)_q$ in $R_{S_\mathfrak{m}}$, so $\lambda = \lambda_\mathfrak{m}$ in $R_{S_\mathfrak{m}}$, since $(e_1, e_2)_q$ is a nonzero divisor also in $R_{S_\mathfrak{m}}$. Therefore, the image of λ in each $R_{S_\mathfrak{m}}$ is central, so λ is central in R by the local-global principle.

Now we have to prove that $\lambda\Lambda \subseteq \Lambda$. This is equivalent to showing that the map $m_\lambda : \Lambda \to R/\Lambda$, $\alpha \mapsto \lambda\alpha$, is trivial. By the definition, the map $m_{\lambda_\mathfrak{m}} : \Lambda_{S_\mathfrak{m}} \to R_{S_\mathfrak{m}}/\Lambda_{S_\mathfrak{m}} = (R/\Lambda)_{S_\mathfrak{m}}$, $\alpha \mapsto \lambda_\mathfrak{m}\alpha$, is trivial. Since $\lambda = \lambda_\mathfrak{m}$ in $R_{S_\mathfrak{m}}$, $m_{\lambda_\mathfrak{m}} = m_{\lambda_{S_\mathfrak{m}}}$. Thus by the local-global prinicple, m_λ is trivial. □

8 Bak-Vaserstein lemma

Let $\langle s\rangle$ be a multiplicative system generated by an element $s \in R_0$. We write simply R_s, Λ_s, F_s instead of $R_{\langle s\rangle}$, $\Lambda_{\langle s\rangle}$, $F_{\langle s\rangle}$.

For $\xi \in R_s$ we denote by $ord_s(\xi)$ the minimal non-negative integer N such that $s^N\xi \in$ "R"$_s$. Obviously, $ord_s(\bar\xi) = ord_s(\xi)$.

If $s \in S$, we denote by $\mathrm{EU}_{2l}(\text{"}s^N R\text{"}_S, \text{"}s^N\Lambda\text{"}_S)$ where N is non-negative integer the subgroup of $\mathrm{EU}_{2l}(R_S, \Lambda_S)$ generated by all elements of the form $T_{ij}(\frac{s^N\xi}{1})$, $\xi \in R$ for $j \neq \pm i$ and $\xi \in \Lambda\varepsilon_i$ for $j = -i$.

Lemma 6. *Let $e_1, \dots, e_l, e_{-l}, \dots, e_{-1}$ be a hyperbolic basis for V. Let v be a vector in V_s orthogonal to e_i and e_{-i}, $|v|_q = \alpha + \Lambda_s$. Then for each N there exists non-negative integer M (say, $M = N + L$, $L = \max\{ord_s(\xi_j), ord_s(\alpha)\}$, where ξ_j are the coefficients of v in the hyperbolic basis above) such that for all $\xi \in R$, $T_{e_iv}(s^M\xi, \alpha) \in \mathrm{EU}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$.*

Proof. Let $v = \sum_{j\neq\pm i} e_j\xi_j$. Then we have

$$\prod_j T_{e_ie_j}(s^M\xi\bar\xi_j\bar 1^{-1}, 0) = T_{e_i, v\bar 1^{-1}\bar\xi s^M}(1, s^{2M}\xi(\sum_{j=1}^{l}\bar\xi_j\xi_{-j})\bar 1^{-1}\bar\xi) =$$

$$= T_{e_i, v\bar 1^{-1}\bar\xi s^M}(1, s^{2M}\xi\alpha\bar 1^{-1}\bar\xi)T_{e_i0}(1, s^{2M}\xi(\sum_{j=1}^{l}\bar\xi_j\xi_{-j} - \alpha)\bar 1^{-1}\bar\xi)$$

$$= T_{e_iv}(s^M\xi, \alpha)T_{e_i0}(1, s^{2M}\xi(\sum_{j=1}^{l}\bar\xi_j\xi_{-j})\bar 1^{-1}\bar\xi).$$

So $T_{e_iv}(s^M\xi, \alpha) \in \mathrm{EU}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$, as claimed. □

Lemma 7. *Let $g \in \mathrm{GU}_{2l-2}(R_s, \Lambda_s)$. Then for each non-negative integer N there exists non-negative integer M (say, $M = 4(N + L)$, $L = ord_s(\lambda) + ord_s(\lambda^{-1}) + \max\{ord_s(g_{ij}), ord_s(g'_{ij})\}$, where λ is multiplicator of g) such that*

$$g\,\mathrm{EU}_{2l}(\text{"}s^M R\text{"}_s, \text{"}s^M\Lambda\text{"}_s)g^{-1} \leq \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s).$$

Proof. It's sufficient to prove that for all i, j, $\xi \in R$

$$gT_{ij}(s^M\xi)g^{-1} \in \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s).$$

First, suppose $i = \pm 1$ or $j = \pm 1$. We shall show that

$$gT_{ij}(s^{M/4}\xi)g^{-1} \in \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s).$$

By relation (R1) we can assume that $i = \pm 1$. Note that $ge_i = e_i\mu$, where $\mu = 1$ for $i = 1$ and $\mu = \lambda$ for $i = -1$.

Let $j=-i$. We have

$$gT_{ij}(s^{M/4}\xi)g^{-1}=gT_{e_i0}(1,s^{M/4}\xi\varepsilon_i^{-1})g^{-1}=T_{e_i0}(\mu\lambda^{-1},s^{M/4}\lambda\xi\varepsilon_i^{-1})$$
$$=T_{ij}(s^{M/4}\mu^2\lambda^{-1}\xi)\in \mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s,\text{"}s^N\Lambda\text{"}_s).$$

Let $j\neq\pm i$. Note that ge_j is orthogonal to e_i. We have

$$gT_{ij}(s^{M/4}\xi)g^{-1}=gT_{e_ie_{-j}}(-s^{M/4}\xi\varepsilon_j,0)g^{-1}=T_{e_i,ge_{-j}}(-s^{M/4}\mu\lambda^{-1}\xi\varepsilon_j,0)$$

lies in $\mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s,\text{"}s^N\Lambda\text{"}_s)$ by Lemma 6.

Now suppose $i,j\neq\pm1$. Let $j\neq -i$. Then we have by (R4)

$$gT_{ij}(s^M\xi)g^{-1}=[gT_{i1}(s^{M/2})g^{-1},gT_{1j}(s^{M/2}\xi)g^{-1}],$$

so $gT_{ij}(s^M\xi)g^{-1}$ lies in $\mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s,\text{"}s^N\Lambda\text{"}_s)$.

Finally, let $j=-i$. We have by (R6)

$$gT_{i,-i}(s^M\xi)g^{-1}$$
$$=gT_{i,-1}(-s^{3M/4}\xi\varepsilon_i^{-1})g^{-1}[gT_{i1}(s^{M/4})g^{-1},gT_{1,-1}(s^{M/2}\xi\varepsilon_i^{-1})g^{-1}],$$

so $gT_{i,-i}(s^M\xi)g^{-1}$ lies in $\mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s,\text{"}s^N\Lambda\text{"}_s)$ again. □

9 Λ-stable rank

We say that *stable rank* of R does not exceed n and denote this fact by $sr(R)\leq n$, if for every unimodular column $u=(u_1,\dots,u_{n+1})^t$ of length $n+1$ there exist elements $\xi_1,\dots,\xi_n\in R$ such that the column $(u_1+\xi_1u_{n+1},\dots,u_n+\xi_nu_{n+1})^t$ is also unimodular.

We say that *local stable rank* of R does not exceed n and denote this fact by $lsr(R)\leq n$, if there exists subring R_0 of R contained in $\mathrm{Cent}(R)$ such that for every $\mathfrak{m}\in\mathrm{Max}(R_0)$ there exists a multiplicative system $S_\mathfrak{m}\subseteq R_0\setminus\mathfrak{m}$ such that $sr(R_{S_\mathfrak{m}})\leq n$.

Proposition 1. *Let $n\geq lsr(R)+1,3$ and let A be an ideal of R. Then*

$$[\mathrm{GL}_n(R,A),\mathrm{E}_n(R)]=\mathrm{E}_n(R,A).$$

In particular, $\mathrm{E}_n(R)$ is normal in $\mathrm{GL}_n(R)$.

Proof. This result was proved by Vaserstein, see [14, Corollary 14]. (He supposes that R_0 coincides with $\mathrm{Cent}(R)$ but he uses only the inclusion $R_0\subseteq\mathrm{Cent}(R)$.) □

We say that Λ-*stable rank* of R does not exceed n and denote this fact by $\Lambda sr(R)\leq n$, if $sr(R)\leq n$ and for every unimodular column

$$u=(u_1,\dots,u_{n+1},u_{-n-1},\dots,u_{-1})^t$$

of length $2(n+1)$ there exists an $(n+1)\times(n+1)$-matrix a such that for all i, j

$$\bar{a}_{ij} = a_{n+2-j,n+2-i},$$
$$a_{i,n+2-i} \in \Lambda,$$

and the column $(u_1,\dots,u_{n+1})^t + a(u_{-n-1},\dots,u_{-1})^t$ is also unimodular.

The conditions imply that the matrix

$$\begin{pmatrix} e & a \\ 0 & e \end{pmatrix}$$

belongs to $\mathrm{EU}_{2(n+1)}(R,\Lambda)$; indeed, it's equal to

$$\prod_{1\le i\le j\le n+1} T_{i,-j}(a_{i,n+2-j}).$$

It's proved in [4] that $\Lambda sr(R) \le asr(R)$ where asr is absolute stable rank; for example, $\Lambda sr(R) \le 1$ for a semilocal ring R. The definition and properties of asr can be found in [12] and [10].

We say that *local* Λ*-stable rank* of R does not exceed n and denote this fact by $\Lambda lsr(R) \le n$, if there exists a subring R_0 of R such that $R_0 \subseteq R_\Gamma$ for every form parameter Γ containing Λ, and for every $\mathfrak{m} \in \mathrm{Max}(R_0)$ there exists a multiplicative system $S_\mathfrak{m} \subseteq R_0 \setminus \mathfrak{m}$ such that $\Lambda_{S_\mathfrak{m}} sr(R_{S_\mathfrak{m}}) \le n$. Obviously, $lsr(R) \le \Lambda lsr(R) \le \Lambda sr(R)$, and if $\Lambda \le \Gamma$ then $\Gamma sr(R) \le \Lambda sr(R)$ and $\Gamma lsr(R) \le \Lambda lsr(R)$.

Ring R is called *weakly finite* if every square matrix that has a one-side inverse must be invertible. For instance, commutative rings are weakly finite.

Proposition 2. *If* $sr(R) \le 1$ *then* R *is weakly finite.*

Proof. By [13, Theorem 3], $sr(\mathrm{M}_n(R)) \le 1$ for all n, where $\mathrm{M}_n(R)$ is the ring of all $n\times n$-matrices. Then by Kaplansky-Lenstra theorem (see [11]) every element of $\mathrm{M}_n(R)$ that has a one-side inverse is invertible. □

We say that the *locally finite* Λ*-stable rank* of R does not exceed n and denote this fact by $\Lambda lfsr(R) \le n$, if in the definition of local Λ-stable rank, we insist that the rings $R_{S_\mathfrak{m}}$ are weakly finite. Obviously, $\Lambda lsr(R) \le \Lambda lfsr(R)$, and $\Gamma lfsr(R) \le \Lambda lfsr(R)$ if $\Lambda \le \Gamma$.

Proposition 3. *If* $\Lambda lsr(R) \le 1$ *then* $\Lambda lfsr(R) \le 1$.

Proof. Indeed, $sr(R_{S_\mathfrak{m}}) \le 1$, so these rings are weakly finite by Proposition 2. □

Proposition 4. *If* R *is module-finite over its center then* $\Lambda lsr(R) \le 1$.

Proof. Let R_0 be a subring generated by all elements of the form $\bar{\xi}\bar{1}^{-1}\xi$ where $\xi \in \mathrm{Cent}(R)$. Set $S_\mathfrak{m} = R_0 \setminus \mathfrak{m}$. Then $R_{S_\mathfrak{m}}$ is semilocal (see [15, Lemma 1.4]), hence $\Lambda_{S_\mathfrak{m}} sr(R_{S_\mathfrak{m}}) \le 1$. Thus $\Lambda lsr(R) \le 1$, so $\Lambda lfsr(R) \le 1$. □

Proposition 5. *If* $asr(R) \le 1$ *then* $\Lambda lfsr(R) \le 1$.

Proof. Indeed, $\Lambda lsr(R) \le \Lambda sr(R) \le asr(R) \le 1$, hence $\Lambda lfsr(R) \le 1$. □

10 Witt's theorem

Proposition 6. *Let $\{v_i\}_{i=1,\dots,m}$ be a family of vectors such that there exist vectors $\{v'_i\}_{i=1,\dots,m}$ satisfying the condition $(v'_j, v_i)_q = \delta_{ij}$. Let $\{e_i\}_{i=1,\dots,n,-n,\dots,-1}$ be a hyperbolic family, where $n \geq m + \Lambda sr(R)$. Then there exists an element $a \in \mathrm{U}(V,q)$ such that $av_i = e_i + u_i$ where $u_i \in \langle e_{-i}, \dots, e_{-1}\rangle$ for all i. Moreover, if (V,q) is hyperbolic with a hyperbolic basis $\{e_i\}_{i=1,\dots,l,-l,\dots,-1}$, $l \geq n$, we can find such an element in $\mathrm{EU}_{2l}(R,\Lambda)$.*

Proof. This proposition was proved by Bak (see [1]) with $BS(R)+1$ replacing $\Lambda sr(R)$ and central $\bar{1}$, where $BS(R)$ =Bass-Serre dimension (R), and for the case $m=1$ with $asr(R)+1$ replacing $\Lambda sr(R)$ by van der Kallen, Magurn and Vaserstein (see [12, Theorem 8.1]).

We prove our proposition using induction on m. Base $m=0$ is trivial.

We shall use only elements of the form $T_{e_i w}(\xi, \alpha)$ where w is orthogonal to e_i and e_{-i}. If (V,q) is hyperbolic then these elements belong to $\mathrm{EU}_{2l}(R,\Lambda)$ by Lemma 6 (with $s=1$).

Denote $v = v_m$. By the induction assumption, there exists $a' \in \mathrm{U}(V,q)$ such that $a'v_i = e_i + u_i$ where $u_i \in \langle e_{-i}, \dots, e_{-1}\rangle$ for all $i \leq m-1$. We have $(a'v_j, a'v_i)_q = \delta_{ij}$. Therefore we may assume from the very start that $v_i = e_i + u_i$ where $u_i \in \langle e_{-i}, \dots, e_{-1}\rangle$, and there exists v' such that $(v', v_i)_q = 0$, $i \leq m-1$ and $(v', v)_q = 1$. We are looking for $a \in \mathrm{U}(V,q)$ such that $av_i = v_i$, $i \leq m-1$, $av = e_i + u$, $u \in \langle e_{-m}, \dots, e_{-1}\rangle$.

Set $\xi_i = -\varepsilon_i(e_{-i}, v)_q$. Then $v = \sum_{|i|\leq n} e_i\xi_i + w$ where $(e_i, w)_q = 0$ for all $i = 1, \dots, n, -n, \dots, -1$. Next, $(v', v)_q = 1$, hence

$$R(\sum_{|i|\leq m-1} (v', e_i)_q \xi_i + (v', w)_q) + \sum_{m\leq |j|\leq n} R\xi_j = R.$$

Since $sr(R) \leq \Lambda sr(R) \leq n-m$, there exist $\gamma_j \in R$ such that

$$\sum_{-n\leq i\leq -m} R\xi_i + \sum_{m\leq j\leq n} R(\xi_j + \gamma_j(\sum_{|i|\leq m-1} (v', e_i)_q\xi_i + (v', w)_q)) = R.$$

Let $v'' = v' + \sum_{m\leq |j|\leq n} e_j\varepsilon_j(e_{-j}, v')_q$, $|v''|_q = \alpha + \Lambda$. Note that $(v'', w)_q = (v', w)_q$, $(v'', e_i)_q = (v', e_i)_q$ and $(v'', v_i)_q = (v', v_i)_q = 0$ for all $i \leq m-1$. Set

$$b = \prod_{m\leq j\leq n} T_{e_j, v''}(\gamma_j, \alpha).$$

Then bv can be expressed in the form $\sum_{|i|\leq n} e_i\xi'_i + w'$, where $(e_i, w')_q = 0$ for all i, and

$$\xi'_j = \xi_j, \ -n \leq j \leq -m,$$
$$\xi'_j = \xi_j + \gamma_j(\sum_{|i|\leq m-1} (v', e_i)_q\xi_i + (v', w)_q) + \zeta_j\xi_{-j}, \ m \leq j \leq n$$

for some $\zeta_j \in R$. Thus $\sum_{m\leq|j|\leq n} R\xi'_j = R$.

But $bv_h = v_h$ for all $h \leq m-1$, since e_j and v'' are orthogonal to v_h for all $|m| \leq j \leq |n|$. Therefore, by multiplying by b, we can assume from the very start that $v = \sum_{|i|\leq n} e_i\xi_i + w$, where w is orthogonal to all e_i and $\sum_{m\leq|j|\leq n} R\xi_j = R$.

Let $V_1 = \langle e_m, \dots, e_n, e_{-n}, \dots, e_{-m}\rangle$ and V_2 is the orthogonal complement to V_1 in V. Since $\Lambda sr(R) \leq n-m$, there exists an elementary matrix $c \in \mathrm{U}(V_1, q|_{V_1})$ such that $c(\sum_{m\leq|j|\leq n} \varepsilon_j\xi_j)$ has a form $\sum_{m\leq|j|\leq n} \varepsilon_j\xi'_j$ where $\sum_{m\leq j\leq n} R\xi'_j = R$. Replace v by $(c \oplus id_{V_2})v$; so we can assume that already $\sum_{m\leq j\leq n} R\xi_j = R$.

Now by replacing v with $T_{e_i,e_{-j}}(-\zeta, 0)v$, $m \leq i, j \leq n$, we change ξ_i to $\xi_i + \zeta\xi_j$ without affecting the other coefficient among $\xi_m, \dots, \xi_n$. Since $sr(R) \leq n-m$, we can perform a sequence of such transvections until $\xi_m = 1$.

Set

$$d = T_{e_{-m}w}(\bar{1}^{-1}, \alpha) \cdot \prod_{m+1\leq|i|\leq n} T_{e_i,e_{-m}}(\xi_i, 0)$$

with any α such that $|w|_q = \alpha + \Lambda$. Then dv has a form $\sum_{|i|\leq m-1} e_i\xi_i + e_m + e_{-m}\xi$ and d acts trivially on v_i, $i \leq m-1$. So we can assume that $v = \sum_{|i|\leq m-1} e_i\xi_i + e_m + e_{-m}\xi$.

Let $f = \prod_{i=1}^{m-1} T_{e_ie_{-m}}(\xi_i, 0)$. Then fv has the required form $e_m + u'$ where $u' \in \langle e_{-m}, \dots, e_{-1}\rangle$. But $fv_i = v_i + e_{-m}\zeta_i$ for some $\zeta_i \in R$. Now set $g = \prod_{i=1}^{m-1} T_{e_{-m}e_{-i}}(\zeta_i, 0)$. Then $gfv_i = v_i$ for $i \leq m-1$ and $gfv = e_m + u$ where $u \in \langle e_{-m}, \dots, e_{-1}\rangle$, that finishes the proof. □

Subspace U of quadratic space (V, q) is called *non-singular relative to V* if for every R-linear map $f : U \to R$ there exists vector $v \in V$ such that $(v, u)_q = f(u)$ for all $u \in U$.

Theorem 1. *Let V_1 and V_2 be free non-singular relative to V subspaces of V, $g : V_1 \to V_2$ is isometry and $\mathrm{ind}(V) \geq \dim(V_1) + \Lambda sr(R)$. Then there exists an isometry $a : V \to V$ such that $a|_{V_1} = g$. Moreover, if V is hyperbolic, we can find elementary one.*

Proof. Let $\dim(V_1) = m$, $\{v_i\}_{i=1,\dots,m}$ be a basis of V_1. Then $\{g(v_i)\}_{i=1,\dots,m}$ is a basis of V_2. By the condition, there exist vectors $v'_1, \dots, v'_m$ such that $(v'_i, v_j)_q = \delta_{ij}$. Let $e_1, \dots, e_n$ be a hyperbolic family of V, where $n = m + \Lambda sr(R)$, By Proposition 6, there exists isometry a sich that $av_i = e_i + \sum_{j=1}^{i} e_{-j}\xi_{ij}$ where $\xi_{ij} \in R$. Similarly, there exists isometry b such that $bg(v_i) = e_i + \sum_{j=1}^{i} e_{-j}\zeta_{ij}$.

Note that $\xi_{ii} + \Lambda = \bar{1}^{-1}|av_i|_q = \bar{1}^{-1}|bg(v_i)|_q = \zeta_{ii} + \Lambda$. Therefore, multiplying a by appropriate transvections $T_{e_{-i}0}(\alpha_i)$, we can assume that $\xi_{ii} = \zeta_{ii}$. Further, if $j < i$ then $\xi_{ij} = \bar{1}^{-1}(av_i, av_j)_q = \bar{1}^{-1}(bg(v_i), bg(v_j))_q = \zeta_{ij}$. So $av_i = bg(v_i)$, and $b^{-1}a$ is required isometry. □

11 Surjective stability of KU_1

Proposition 7. *Let $l \geq \Lambda sr(R) + 1$. Then*

$$\mathrm{GU}_{2l}(R, \Lambda) = \mathrm{EU}_{2l}(R, \Lambda)\,\mathrm{GU}_{2l-2}(R, \Lambda).$$

Proof. This statement was proved by Bak (see [1]) with $BS(R)+1$ replacing $\Lambda sr(R)$ and central $\bar{1}$ and by Vaserstein and You (see [15, Proposition 2.3]) with $asr(R)+1$ replacing $\Lambda sr(R)$. See also [4, Theorem 1.1] for hermitian groups.

First note that by Lemma 7 (where we set $s=1$) $\mathrm{GU}_{2l-2}(R,\Lambda)$ normalizes $\mathrm{EU}_{2l}(R,\Lambda)$, so the right-hand side of the equality is a group.

Let $g \in \mathrm{GU}_{2l}(R,\Lambda)$. Note that $(ge_{-1}, -ge_1\lambda^{-1})_q = 1$ (λ is the multiplicator of g). By Proposition 6 there exists element $a \in \mathrm{EU}_{2l}(R,\Lambda)$ such that the $age_1 = e_1 + e_{-1}\xi$ for some $\xi \in R$. But $\xi = \bar{1}^{-1}|age_1|_q = 0$ in RLa, so, multiplying by $T_{-1,1}(-\xi)$ on the left, we can assume that $ge_1 = e_1$, that is the first column of g coincides with the first column of the identity matrix.

Now, multiplying g by $\prod_{j\neq\pm1} T_{1j}(-g_{1j})$ on the right, we obtain a matrix whose first row equals $(1,0,\ldots,0,\zeta)$. But $\zeta \in \Lambda$ (by (GU2")), so we can multiply by $T_{1,-1}(-\zeta)$ on the right in order to get a matrix, whose first column and first row coincide with the first column and the first row of the identity matrix, and, therefore, belonging to $\mathrm{GU}_{2l-2}(R,\Lambda)$. □

Corollary. *Let $l \geq \Lambda sr(R)+1$. Then* $\mathrm{U}_{2l}(R) = \mathrm{EU}_{2l}(R)\,\mathrm{U}_{2l-2}(R)$.

Remark. *This result actually says that for $l \geq \Lambda sr(R)+1$ the canonical map*

$$\mathrm{KU}_{1,2l-2}(R,\Lambda) \to \mathrm{KU}_{1,2l}(R,\Lambda)$$

is surjective, where $\mathrm{KU}_{1,2l}(R,\Lambda) = \mathrm{U}_{2l}(R,\Lambda)/\,\mathrm{EU}_{2l}(R,\Lambda)$ *is the unitary K_1-functor.*

Proposition 8. *Let $l \geq \Lambda lsr(R)+1, 3$. Then* $\mathrm{EU}_{2l}(R,\Lambda) \trianglelefteq \mathrm{GU}_{2l}(R,\Lambda)$.

Proof. Since $\mathrm{GU}_{2l}(R,\Lambda) = \mathrm{U}_{2l}(R,\Lambda)\,\mathrm{T}_{2l}(R,\Lambda)$ and $\mathrm{T}_{2l}(R,\Lambda)$ obviously normalizes $\mathrm{EU}_{2l}(R,\Lambda)$, it's sufficient to prove that $\mathrm{EU}_{2l}(R,\Lambda)$ is normal in $\mathrm{U}_{2l}(R,\Lambda)$. But this fact was proved by Vaserstein and You, [15, Theorem 1.1]. (They formulate it in the terms of *asr*, but they use only the property of stability of KU_1 in rings obtained by localization.) See also [5, Theorem 1.1] and [7, Theorem 4.4] for the case of almost commutative rings and central $\bar{1}$. □

12 Conjugation calculus

Lemma 8. *Let $l \geq 3$, $g \in \mathrm{EU}_{2l}(R_s,\Lambda_s)$. Then for every non-negative integer N there exists a non-negative integer M such that*

$$g\,\mathrm{EU}_{2l}(\text{"}s^M R\text{"}_s, \text{"}s^M\Lambda\text{"}_s)g^{-1} \leq \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s).$$

Proof. This lemma was proved by Hazrat, see [7, Lemma 4.1], but he supposes that $\bar{1}$ is central, so we reproduce his proof here. Compare also [15, Lemma 3.1].

We have to prove that for every $\xi \in R_s$ and every non-negative integer N there exists non-negative integer M such that for all $i \neq j$, $h \neq k$ and all $\zeta \in R$ ${}^{T_{ij}(\xi)}T_{hk}(s^M\zeta) \in \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$. Then induction on the length of g completes proof.

Set $M = 4(N + ord_s(\xi))$.

First, if $h \neq j, -i$, $k \neq i, -j$, we can apply (R3), and there is nothing to prove. So we can assume $h = j$ (other cases reduce to this using (R1)).

Case I. $j \neq -i$.

Subcase I.1 $k \neq \pm i, j$. We can apply (R4).

Subcase I.2 $k = -i$. We can apply (R5).

Subcase I.3 $k = -j$. We can apply (R6).

Subcase I.4 $k = j$. Pick $r \neq \pm i, \pm j$. We have

$$T_{ji}(s^M\zeta) = [T_{jr}(s^{M/2}), T_{ri}(s^{M/2}\zeta)].$$

Now ${}^{T_{ij}(\xi)}T_{jr}(s^{M/2}) = T_{ir}(s^{M/2})T_{jr}(s^{M/2}) \in \mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$, and

$$\begin{aligned}&{}^{T_{ij}(\xi)}T_{ri}(s^{M/2}\zeta) = {}^{T_{-j,-i}(\varepsilon_{-j}\bar{\xi}\varepsilon_i)}T_{-i,-r}(\varepsilon_{-i}s^{M/2}\bar{\zeta}\varepsilon_r)\\ &= T_{-j,-r}(-\varepsilon_{-j}s^{M/2}\bar{\xi}\bar{1}^{-1}\bar{\zeta}\varepsilon_r)T_{-i,-r}(\varepsilon_{-i}s^{M/2}\bar{\zeta}\varepsilon_r) \in \mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s, \text{"}s^N\Lambda\text{"}_s).\end{aligned}$$

Case II. $j = -i$.

Subcase II.1 $k \neq i$. We have

$$\begin{aligned}{}^{T_{i,-i}(\xi)}T_{-ik}(s^M\zeta) &= [T_{-ik}(s^M\zeta), T_{i,-i}(\xi)]^{-1}T_{-ik}(s^M\zeta)\\ &= [T_{-ki}(\varepsilon_{-k}s^M\bar{\zeta}\varepsilon_{-i}), T_{i,-i}(\xi)]^{-1}T_{-ik}(s^M\zeta)\\ &= (T_{-k,-i}(\varepsilon_{-k}\bar{\zeta}\varepsilon_{-i}s^M\xi)T_{-k,k}(-\varepsilon_{-k}\bar{\zeta}\varepsilon_{-i}s^{2M}\xi\zeta))^{-1}T_{-ik}(s^M\zeta)\end{aligned}$$

lies in $\mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$.

Subcase II.2 $k = i$. Pick $r \neq \pm i$. We have

$$T_{-ii}(s^M\zeta) = T_{-i,-r}(-s^{3M/4}\zeta\varepsilon_i\varepsilon_{-r}^{-1})[T_{-ir}(s^{M/4}), T_{r,-r}(s^{M/2}\zeta\varepsilon_i\varepsilon_{-r}^{-1})].$$

Now, as in the Subcase II.1, one can verify that ${}^{T_{i,-i}(\xi)}T_{-i,-r}(-s^{3M/4}\zeta\varepsilon_i\varepsilon_{-r}^{-1})$ and ${}^{T_{i,-i}(\xi)}T_{-i,r}(s^{M/4})$ belong to $\mathrm{EU}_{2l}(\text{"}s^NR\text{"}_s, \text{"}s^N\Lambda\text{"}_s)$. □

Lemma 9. *Let S be a multiplicative system of R_0, $l \geq \Lambda sr(R_S) + 1$. Let $g \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$, $\xi \in R_S$, $j \neq i$. Then there exists $s \in S$ satisfying the following condition: for every non-negative integer N there exists M_0 not depending on the pair (i, j) such that for all $M \geq M_0$*

$$gT_{ij}(s^M\xi)g^{-1} \in \mathrm{EU}_{2l}(\text{"}s^NR\text{"}_S, \text{"}s^N\Lambda\text{"}_S),$$

Proof. Since $l \geq \Lambda sr(R_S) + 1$, we can find elements $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$, $b \in \mathrm{U}_{2l-2}(R_S, \Lambda_S)$, $d \in \mathrm{T}_{2l}(R_S, \Lambda_S)$ such that $g = abd$. Denote the multiplicator of d by λ. Then we have:

$$d_{hk} = 0,\ h \neq k, \tag{T1}$$

$$\bar{d}_{hh} = \bar{\lambda}\varepsilon_h d'_{-h,-h}\varepsilon_h^{-1}, \text{ and, if } j = -i, \tag{T2}$$

$$\lambda^{-1}\xi\varepsilon_i^{-1} \in \Lambda_S. \tag{T3}$$

Since R_S is a filtered direct limit of R_s, $s \in S$, and EU and U commutes with limits, we can find $s \in S$ such that $s\xi \in "R"_s$, and there exist elements $\tilde{a} \in \mathrm{EU}_{2l}(R_s, \Lambda_s)$, $\tilde{b} \in \mathrm{U}_{2l-2}(R_s, \Lambda_s)$ and $\tilde{d} \in \mathrm{GL}_{2l}(R_s, \Lambda_s)$ satisfying condition $F_S(\tilde{a}) = a$, $F_S(\tilde{b}) = b$, $F_S(\tilde{d}) = d$, and equations (T1)–(T3) hold (with $\tilde{d}$ instead of d and Λ_s instead of Λ_S). Set $\tilde{g} = \tilde{a}\tilde{b}\tilde{d}$; so $F_S(\tilde{g}) = g$.

Let K denote a non-negative integer. Let $j \neq -i$. Then $\tilde{d}T_{ij}(s^M\xi)\tilde{d}^{-1} = T_{ij}(\tilde{d}_{ii}s^M\xi\tilde{d}'_{jj})$ obviously belongs to $\mathrm{EU}_{2l}("s^KR"_s, "s^K\Lambda"_s)$ for a sufficient large M.

Let $j = -i$. Then $\tilde{d}T_{i,-i}(s^M\xi)\tilde{d}^{-1} = T_{i,-i}(d_{ii}s^M\lambda^{-1}\xi\varepsilon_i^{-1}\bar{1}^{-1}\bar{d}_{ii}\varepsilon_i)$ also belongs to $\mathrm{EU}_{2l}("s^KR"_s, "s^K\Lambda"_s)$ for a sufficiently large M.

Now we can apply Lemma 7 and Lemma 8 and conclude that $\tilde{g}T_{ij}(s^M\xi)\tilde{g}^{-1}$ lies in $\mathrm{EU}_{2l}("s^NR"_s, "s^N\Lambda"_s)$. Hence $gT_{ij}(s^M\xi)g^{-1} = F_S(\tilde{g}T_{ij}(s^M\xi)\tilde{g}^{-1})$ belongs to $\mathrm{EU}_{2l}("s^NR"_S, "s^N\Lambda"_S)$. □

Lemma 10. *Let $l \geq 3$, s an element of R, and H a subgroup of $\mathrm{GL}_{2l}(R_s)$ containing $\mathrm{EU}_{2l}("s^NR"_s, "s^N\Lambda"_s)$. Set $A_{ij} = \{\xi \in R_s | t_{ij}(\xi) \in H\}$. Then A_{ij} are additive subgroups of R_s and the following relations hold:*

1) $s^{4N}RA_{ij}R \subseteq A_{hk}$, $i \neq \pm j$, $h \neq \pm k$,

2) $s^{4N}R\bar{A}_{ij}R \subseteq A_{hk}$, $i \neq \pm j$, $h \neq \pm k$,

3) $s^{5N}RA_{ij}R \subseteq A_{h,-h}$, $i \neq \pm j$,

4) *if* $\alpha \in A_{i,-i}$ *and* $s^K\alpha \in "R"_s$, *then* $s^{6N+K}(\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})R \in A_{jh}$, $j \neq \pm h$,

5) *if* $\alpha \in A_{i,-i}$ *and* $s^K\alpha \in "R"_s$, *then* $s^{12N+2K}\xi\alpha\varepsilon_{-i}\bar{\xi}\varepsilon_j \in A_{j,-j}$ *for all* $\xi \in R$.

Proof. 1) Let $h \neq \pm i, \pm j$, $\xi \in R_s, \zeta \in R$. Then we have

$$[T_{hi}(s^N\zeta), t_{ij}(\xi)] = t_{hj}(s^N\zeta\xi),$$

so $s^NRA_{ij} \subseteq A_{hj}$. If $h = i$ or $h = -i$, we can pick $k \neq \pm i, \pm j$. Then $s^NRA_{ij} \subseteq A_{kj}$ and $s^NRA_{kj} \subseteq A_{hi}$, so $s^{2N}RA_{ij} \subseteq A_{hj}$.

Similarly,

$$[t_{ij}(\xi), T_{jk}(s^N\zeta)] = t_{ik}(s^N\xi\zeta),$$

hence $s^{2N}A_{ij}R \subseteq A_{ik}$ for all k. Now $s^{4N}RA_{ij}R \subseteq s^{2N}A_{hj}R \subseteq A_{hk}$.

2) Let $\xi \in A_{ij}$. Then $t_{-j,-i}(\varepsilon_{-j}\bar{\xi}\varepsilon_i) = T_{ij}(\xi)t_{ij}(-\xi)$. So $\bar{\xi} \in RA_{-j,-i}R$, and $s^{4N}R\bar{\xi}R \in A_{hk}$.

3) Let $\xi \in A_{k,-h}$. Pick $k \neq \pm h$. Then

$$[T_{hk}(s^N), t_{k,-h}(\xi)] = t_{h,-h}(s^N\xi).$$

Now $s^{5N}RA_{ij}R \subseteq s^NA_{k,-h} \subseteq A_{h,-h}$.

4) Pick $k \neq \pm i$, $r \neq \pm i, \pm k$. We have

$$\begin{aligned}&[T_{ki}(s^{N+K}), t_{i,-i}(\alpha)]\\ &\qquad = T_{k,-i}(s^{N+K}\alpha)t_{k,-k}(s^{2N+2K}\alpha\varepsilon_i^{-1}\varepsilon_k)t_{k,-i}(s^{N+K}(\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})\varepsilon_k).\end{aligned}$$

So we have $t_{k,-k}(s^{2N+2K}\alpha\varepsilon_i^{-1}\varepsilon_k)t_{k,-i}(s^{N+K}(\alpha\varepsilon_i^{-1}-\varepsilon_i\bar{\alpha})\varepsilon_k) \in H$. By commuting with $T_{rk}(s^N)$ on the left, we obtain $t_{r,-i}(s^{2N+K}(\alpha\varepsilon_i^{-1}-\varepsilon_i\bar{\alpha})\varepsilon_k)) \in H$. Hence $s^{6N+K}(\alpha\varepsilon_i^{-1}-\varepsilon_i\bar{\alpha})R \in A_{jh}$.

5) We have

$$[T_{ji}(s^{6N+K}\xi), t_{i,-i}(\alpha)] =$$
$$T_{j,-i}(s^{6N+K}\xi\alpha)t_{j,-j}(-s^{12N+2K}\xi\alpha\varepsilon_{-i}\bar{\xi}\varepsilon_j)t_{j,-i}(s^{6N+K}(\alpha\varepsilon_i^{-1}-\varepsilon_i\bar{\alpha})\bar{1}^{-1}\bar{\xi}\varepsilon_j).$$

But by 4) $t_{j,-i}(s^{6N+K}(\alpha\varepsilon_i^{-1}-\varepsilon_i\bar{\alpha})\bar{1}^{-1}\bar{\xi}\varepsilon_j)$ belongs to H, so $s^{12N+2K}\xi\alpha\varepsilon_{-i}\bar{\xi}\varepsilon_j$ lies in $A_{j,-j}$. □

Lemma 11. *Let $l \geq 3$, $g \in \mathrm{EU}_{2l}(R_s, \Lambda_s)$, $\xi \in R_s$, $j \neq i$. Then for every non-negative integer N there exists a non-negative integer M_0 not depending on the pair (i,j) such that for all $M \geq M_0$*

$$gt_{ij}(s^M\xi)g^{-1} \in \langle t_{ij}(\xi),\ \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)\rangle.$$

Proof. Set $H = \langle t_{ij}(\xi),\ \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)\rangle$.

We have to prove that given $\xi, \zeta \in R_s$ and a non-negative integer N there exists non-negative integer M_0 such that for all $i \neq j$, $h \neq k$ and all $M \geq M_0$, ${}^{T_{hk}(\zeta)}t_{ij}(s^M\xi) \in H$. Then induction on the length of g (using Lemma 8) completes proof.

Set $M_0 = 12N + 2ord_s(\xi) + 2ord_s(\zeta)$.

First, if $h \neq j, -i$, $k \neq i, -j$, we can apply (S3), and there is nothing to prove. So we can assume $k = i$ (other cases reduce to this using (R1), (S1), and Lemma 8).

Case I. $j \neq -i$.

Subcase I.1 $h \neq j$. We can apply (S4) and Lemma 10.

Subcase I.2 $h = j$. Pick $r \neq \pm i, \pm j$. We have

$$t_{ij}(s^M\xi) = [T_{ir}(s^{M/2}), t_{rj}(s^{M/2}\xi)].$$

Now ${}^{T_{ji}(\zeta)}T_{ir}(s^{M/2}) = T_{jr}(s^{M/2}\zeta)T_{ir}(s^{M/2}) \in H$, and

$${}^{T_{ji}(\zeta)}t_{rj}(s^{M/2}\xi) = t_{ri}(-s^{M/2}\xi\zeta)t_{rj}(s^{M/2}\xi)$$

belongs to H by Lemma 10.

Case II. $j = -i$.

Subcase II.1 $h \neq -i$. We can apply (S5) and Lemma 10.

Subcase II.2 $h = -i$. Pick $r \neq \pm i$. We have

$$t_{i,-i}(s^{M/2}\xi) = [T_{ir}(s^{M/2}), t_{r,-i}(s^{M/2}\xi)].$$

Now

$${}^{T_{-ii}(\zeta)}T_{ir}(s^{M/2}) = (T_{-ri}(-\varepsilon_r^{-1}\varepsilon_i s^{M/2}\zeta)T_{-rr}(\varepsilon_r^{-1}\varepsilon_i s^M\zeta))^{-1}T_{ir}(s^{M/2}) \in H,$$

and ${}^{T_{-ii}(\zeta)}t_{r,-i}(s^{M/2}\xi) = t_{ri}(-s^{M/2}\xi\zeta) \in H$ by Lemma 10. □

Lemma 12. *Let $l \geq 2$, A be an ideal of R. Then*

$$^{\mathrm{EU}_{2l}(R,\min^{\sigma}(R))}\,\mathrm{E}_{2l}(A) = \mathrm{E}_{2l}(R, A).$$

Proof. Denote the left-hand side by H. Clearly, $H \leq \mathrm{E}_{2l}(R, A)$. By Lemma 1 it's sufficient to prove that $z_{ij}(\xi, \zeta) = {}^{t_{ji}(\zeta)}t_{ij}(\xi) \in H$ for all $j \neq i$, $\zeta \in R$, $\xi \in A$. If $j \neq -i$ one has $z_{ij}(\xi, \zeta) = {}^{T_{ji}(\zeta)}t_{ij}(\xi)$ belongs to H. It remains to verify that $z_{i,-i}(\xi, \zeta) \in H$.

Pick $j \neq \pm i$. Then $t_{i,-i}(\xi) = [t_{ij}(\xi), t_{j,-i}(1)]$. We have

$$z_{i,-i}(\xi, \zeta) = {}^{t_{-ii}(\zeta)}[t_{ij}(\xi), t_{j,-i}(1)] = [t_{ij}(\xi)t_{-ij}(\zeta\xi), t_{ji}(-\zeta)t_{j,-i}(1)].$$

Set $a = t_{ij}(\xi)$, $b = t_{-ij}(\zeta\xi)$, $c = t_{ji}(-\zeta)$, $d = t_{j,-i}(1)$. We have to prove that $[ab, cd] = {}^{a}[b, c] \cdot {}^{ac}[b, d] \cdot [a, c] \cdot {}^{c}[a, d] \in H$. We show that all factors belong to H.

$$[a, c] = [t_{ij}(\xi), t_{ji}(-\zeta)] = [t_{ij}(\xi), T_{ji}(-\zeta)] \in H.$$

$$[b, c] = [t_{-ij}(\zeta\xi), t_{ji}(-\zeta)] = t_{-ii}(-\zeta\xi\zeta) \in \mathrm{E}_{2l}(A),$$

so ${}^{a}[b, c] \in H$, since $a \in \mathrm{E}_{2l}(A)$.

$${}^{c}[a, d] = {}^{t_{ji}(-\zeta)}[t_{ij}(\xi), t_{j,-i}(1)] = {}^{t_{ji}(-\zeta)}t_{i,-i}(\xi) = t_{j,-i}(-\zeta\xi)t_{i,-i}(\xi) \in \mathrm{E}_{2l}(A).$$

$$\begin{aligned}
{}^{c}[b, d] &= {}^{t_{ji}(-\zeta)}[t_{-ij}(\zeta\xi), t_{j,-i}(1)] = {}^{T_{ji}(-\zeta)t_{-i,-j}(-\varepsilon_{-i}\bar{\zeta}\varepsilon_j)}[t_{-ij}(\zeta\xi), t_{j,-i}(1)] \\
&= {}^{T_{ji}(-\zeta)}([t_{-ij}(\zeta\xi), t_{j,-i}(1)t_{j,-j}(-\varepsilon_{-i}\bar{\zeta}\varepsilon_j)]) \\
&= {}^{T_{ji}(-\zeta)}([t_{-ij}(\zeta\xi), t_{j,-i}(1)]t_{j,-j}(-\zeta\xi\varepsilon_{-i}\bar{\zeta}\varepsilon_j)t_{-i,-j}(-\zeta\xi\varepsilon_{-i}\bar{\zeta}\varepsilon_j)) \\
&= {}^{T_{ji}(-\zeta)}([t_{-ij}(\zeta\xi), T_{j,-i}(1)]t_{j,-j}(-\zeta\xi\varepsilon_{-i}\bar{\zeta}\varepsilon_j)t_{-i,-j}(-\zeta\xi\varepsilon_{-i}\bar{\zeta}\varepsilon_j)) \in H,
\end{aligned}$$

hence ${}^{ac}[b, d] \in H$. □

13 Extraction of transvections

Throughout this section H is a subgroup of $\mathrm{GL}_{2l}(R)$ containing $\mathrm{EU}_{2l}(R, \Lambda)$, $l \geq 3$.

We say that an elementary linear transvection $t_{ij}(\xi)$ in $\mathrm{GL}_{2l}(R)$ is *non-trivial*, if $j \neq -i$ and $\xi \neq 0$, or $j = -i$ and $\xi \notin \Lambda\varepsilon_i$.

Lemma 13. *Assume that there exist elements $g \in H$, $s \in R_0$, $\xi \in R$ such that $F_s(g) = t_{ij}(\xi)$ is a non-trivial transvection. Then H contains a non-trivial transvection.*

Proof. We have $F_s(t_{ij}(-\xi)g) = e$, so $g = t_{ij}(\xi)a$ where $a \in \mathrm{GL}_{2l}(R, \mathrm{Ann}(s^K))$ for some K. So the element a commutes with every element of the form $T_{hk}(s^K)$.

First, let $j \neq \pm i$. Pick $k \neq \pm i, \pm j$. We have

$$[g, T_{jk}(s^K)] = [t_{ij}(\xi)a, T_{jk}(s^K)] = [t_{ij}(\xi), T_{jk}(s^K)] = t_{ik}(s^K\xi).$$

It's a non-trivial transvection, since $\xi \neq 0$ in R_s.

Let now $j=-i$. Pick $k \neq \pm i$, $h \neq \pm i, \pm k$. We have

$$[g, T_{-ik}(s^K)] = [t_{i,-i}(\xi)a, T_{-ik}(s^K)] = [t_{i,-i}(\xi), T_{-ik}(s^K)]$$
$$= T_{-k,-i}(s^K\varepsilon_k^{-1}\varepsilon_{-i}\xi)t_{-kk}(-s^{2K}\varepsilon_k^{-1}\varepsilon_{-i}\xi))t_{ik}(s^K(\xi-\varepsilon_i\bar{\xi}\varepsilon_i)).$$

Hence H contains the factor $x = t_{-kk}(-s^{2K}\varepsilon_k^{-1}\varepsilon_{-i}\xi)t_{ik}(s^K(\xi-\varepsilon_i\bar{\xi}\varepsilon_i))$. Consider $[T_{hi}(1), x] = t_{hk}(s^K(\xi-\varepsilon_i\bar{\xi}\varepsilon_i)) \in H$. If it's trivial, $s^K(\xi-\varepsilon_i\bar{\xi}\varepsilon_i)=0$, so x itself is a non-trivial transvection. □

Fix a multiplicative system S.

Lemma 14. *Assume that there exist elements $g \in H$, $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$ such that $aF_S(g)a^{-1} = t_{ij}(\xi)$ is a non-trivial transvection. Then H contains a non-trivial transvection.*

Proof. Since R_S is a filtered direct limit of R_s, $s \in S$, and EU commutes with limits, we can assume that $S = \langle s \rangle$. We know that $F_s(H)$ contains $a^{-1}t_{ij}(\xi)a$. On the other hand, by Lemma 8 there exists N such that

$$a^{-1}\,\mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)a \leq \mathrm{EU}_{2l}(\text{"}R\text{"}_s, \text{"}\Lambda\text{"}_s) \leq F_s(H).$$

So $F_s(H)$ contains the group

$$a^{-1}\langle t_{ij}(\xi),\ \mathrm{EU}_{2l}(\text{"}s^N R\text{"}_s, \text{"}s^N\Lambda\text{"}_s)\rangle a.$$

By Lemma 11, $F_s(H)$ contains $t_{ij}(s^M\xi)$ for a sufficient large M. Now we can apply Lemma 13. □

In the sequel we assume that $l \geq \Lambda sr(R_S)+1$.

Lemma 15. *Assume that there exist elements $g \in H$, $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$, $b \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$ such that $aF_S(g)b = t_{1,-1}(\alpha)t_{2,-2}(\beta)t_{1,-2}(\xi)t_{2,-1}(\zeta)$ does not belong to $\mathrm{U}_{2l}(R_S, \Lambda_S)$. Then H contains a non-trivial transvection.*

Proof. Multiplying by $T_{2,-1}(-\zeta)$ on the right, we can assume that $\zeta = 0$. By Lemmas 9 and 8 there exist $s \in S$ and M such that $bT_{-23}(s^M)b^{-1} \in \mathrm{EU}_{2l}(\text{"}R\text{"}_S, \text{"}\Lambda\text{"}_S)$ and

$$a^{-1}\,\mathrm{EU}_{2l}(\text{"}s^{M-ord_s(\beta)}R\text{"}_S, \text{"}s^{M-ord_s(\beta)}\Lambda\text{"}_S)a \leq \mathrm{EU}_{2l}(\text{"}R\text{"}_S, \text{"}\Lambda\text{"}_S).$$

So $x = aF_S(g)bT_{-23}(s^M)b^{-1}F_S(g)^{-1}a^{-1}$ is a product $aF_S(g_1)a^{-1}$ where $g_1 \in H$. On the other hand, we have

$$x = T_{23}(s^M\beta)t_{-3,-2}(-s^M\bar{1}^{-1}(\beta-\bar{\beta}))t_{-33}(s^{2M}\bar{1}^{-1}\beta)t_{13}(s^M\xi)T_{-23}(s^M).$$

Hence $aF_S(H)a^{-1}$ contains the factor

$$y = t_{-3,-2}(-s^M\bar{1}^{-1}(\beta-\bar{\beta}))t_{-33}(s^{2M}\bar{1}^{-1}\beta)t_{13}(s^M\xi).$$

Consider an element $[T_{-21}(s^M), y] = t_{-23}(s^{2M}\xi)$. We see that if $\xi \neq 0$ then $aF_S(H)a^{-1}$ contains a non-trivial transvection, and we can apply Lemma 14. So we can assume $\xi = 0$. Similarly, by commuting with $T_{-21}(s^M)$ on the right, we see that $\bar\beta = \beta$. But then $y = t_{-33}(s^{2M}\bar{1}^{-1}\beta)$, so $\beta \in \Lambda$.

Similarly we can prove that $\alpha \in \Lambda$. But this contradicts our assumption that $aF_S(g)b \notin \mathrm{U}_{2l}(R_S, \Lambda_S)$. □

Let Y be a subgroup of $\mathrm{GL}_{2l}(R_S)$ generated by all transvections of the following forms: $t_{1j}(\xi)$, $t_{2j}(\xi)$, $t_{i,-1}(\xi)$ and $t_{i,-2}(\xi)$, $j \neq 1, 2$, $i \neq -2, -1$, $\xi \in R_S$. Clearly, $x \in Y$ if and only if $x_{ij} = \delta_{ij}$ for all i, j such that $j = 1, 2$ or $i = -2, -1$ or $i, j \neq \pm1, \pm2$.

Lemma 16. *Assume that there exist elements $g \in H$, $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$, $b \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$ such that $aF_S(g)b = y \in Y \setminus \mathrm{U}_{2l}(R_S, \Lambda_S)$. Then H contains a non-trivial transvection.*

Proof. Multiplying y by $\prod_{i\neq\pm1,\pm2} T_{1i}(-y_{1i}) \prod_{j\neq\pm1,\pm2} T_{2j}(-y_{2j})$ on the right, we can assume that all columns of y except the two last one coincide with the corresponding columns of the identity matrix.

Fix an index $i \neq \pm1, \pm2$. By Lemma 9 there exist $s \in S$ and M such that $bT_{1j}(s^M)b^{-1}$ and $aT_{1j}(s^N)a^{-1}$ belong to $\mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$. So

$$x = [y, T_{1j}(s^M)] = aF_S(g)bT_{ij}(s^M)b^{-1}F_S(g)^{-1}a^{-1}T_{1j}(-s^M)aa^{-1}$$

has a form $aF_S(g_1)a^{-1}$ for some $g_1 \in H$. On the other hand, by direct calculation we obtain that $x = t_{1,-2}(s^M y'_{i,-2})t_{1,-1}(s^M y'_{i,-1})$. Therefore, if H does not contain any non-trivial transvection then by Lemma 15 $y'_{i,-1} = 0$. Similarly, we can prove that $y'_{i,-2} = 0$. But then y itself has the form required to apply Lemma 15. □

In the sequel we assume additionally that R_S is weakly finite.

Lemma 17. *Assume that there exist elements $g \in H$, $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$, $b \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$ such that $x = aF_S(g)b \notin \mathrm{GU}_{2l}(R_S, \Lambda_S)$ satisfies the following conditions: the first and the second columns of x coincides with the corresponding columns of the identity matrix, and, moreover, $x_{-2j} = x_{-1j} = 0$ for all $j \neq -2, -1$. Then H contains a non-trivial transvection.*

Proof. Note that the matrix

$$\begin{pmatrix} x_{-2,-2} & x_{-2,-1} \\ x_{-1,-2} & x_{-1,-1} \end{pmatrix}$$

is invertible on the right. But then it's invertible, since R_S is weakly finite. Thus $x'_{-2j} = x'_{-1j} = 0$ for all $j \neq -2, -1$.

Fix an index $j \neq \pm1, \pm2$. By Lemma 9 there exist $s \in S$ and non-negative integer M such that $bT_{1j}(s^M)b^{-1} \in \mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$. So $y_j = xT_{1j}(s^M)x^{-1}$

has the form $aF_S(g_1)a^{-1}$ for some $g_1 \in H$. On the other hand, direct calculation shows that

$$y_j = e + s^M(\sum_k x'_{jk}e^{1k} - \sum_k x_{k,-j}\varepsilon_j^{-1}x'_{-1,-2}e^{k,-2} - \sum_k x_{k,-j}\varepsilon_j^{-1}x'_{-1,-1}e^{k,-1}).$$

Note that this matrix belongs to Y. Hence, if H does not contain a non-trivial transvection, by Lemma 16 $y_j \in \mathrm{U}_{2l}(R_S, \Lambda_S)$. So we have the following equations:

$$\begin{aligned} &x_{k,-j}x'_{-1,-2} = 0,\ \ k \neq \pm 1, \pm 2, \\ &\bar{x}'_{jk} = \bar{1}\varepsilon_k x_{-k,-j}\varepsilon_j^{-1}x'_{-1,-1},\ \ k \neq \pm 1. \end{aligned} \tag{X1}$$

Similarly, by considering matrix $z_j = xT_{2j}(s^M)x^{-1}$, we obtain the following equations:

$$\begin{aligned} &x_{k,-j}x'_{-2,-1} = 0,\ \ k \neq \pm 1, \pm 2, \\ &\bar{x}'_{jk} = \bar{1}\varepsilon_k x_{-k,-j}\varepsilon_j^{-1}x'_{-2,-2},\ \ k \neq \pm 2. \end{aligned} \tag{X2}$$

Now note that our assumption on x implies that matrix $\tilde{x}$ obtained from x by deleting the 1-st, the 2-nd, the -2-nd and the -1-st rows and columns is invertible. Hence the left ideal generated by the elements $x_{k,-j}$, $k \neq \pm 1, \pm 2$ coincides with R. Therefore, $x'_{-1,-2} = x'_{-2,-1} = 0$ and $x'_{-1,-1} = x'_{-2,-2}$. Denote $x'_{-1,-1}$ by λ.

We show that $\lambda \in R_{S_{\Lambda_S}}$. By Lemma 9, for every $\xi \in R_S$ there exist $s \in S$ and M such that $bT_{1,-2}(s^M\xi)b^{-1} \in \mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$; so $c = xT_{1,-2}(s^M\xi)x^{-1}$ has the form $aF_S(g_2)a^{-1}$ for some $g_2 \in H$. But $c = t_{1,-2}(s^M\xi\lambda)t_{2,-1}(s^M\bar{\xi}\lambda)$. Hence, by Lemma 15, if H does not contain any non-trivial transvection, $c \in \mathrm{U}_{2l}(R_S, \Lambda_S)$. So $\bar{\xi}\lambda = \bar{\lambda}\bar{1}^{-1}\bar{\xi}$ for all $\xi \in R_S$, hence $\bar{\lambda} = \lambda\bar{1}$ and $\lambda \in \mathrm{Cent}(R_S)$.

Let $\alpha \in \Lambda_S$. By Lemma 9 there exist $s \in S$ and M such that $bT_{1,-1}(s^M\alpha)b^{-1}$ lies in $\mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$; so $d = xT_{1,-1}(s^M\alpha)x^{-1}$ has the form $aF_S(g_3)a^{-1}$ for some $g_3 \in H$. But $d = t_{1,-1}(s^M\lambda\alpha)$. Hence, by Lemma 15, if H does not contain any non-trivial transvection, $d \in \mathrm{U}_{2l}(R_S, \Lambda_S)$, so $\lambda\alpha \in \Lambda_S$.

Now it follows from (X1) that $\tilde{x}$ satisfies (GU1) (with multiplicator λ^{-1}). We show that $\tilde{x}$ satisfies also (GU2'). By (GU2') for y_j we have

$$s^M(\bar{x}'_{j,-1} - \overline{\lambda x_{1,-j}\varepsilon_j^{-1}}) + s^{2M}\lambda^2\varepsilon_j(\sum_{k=2}^{l}\bar{x}_{k,-j}x_{-k,-j})\varepsilon_j^{-1} \in \Lambda_S.$$

But from (X2) for $k = -1$ we obtain that the first summand lies in $\min^\sigma(R_S)$. So $\tilde{x}$ belongs to $\mathrm{GU}_{2l-4}(R_S, \Lambda_S)$ with multiplicator λ^{-1}.

Now consider matrix $x \cdot \mathrm{diag}(1, 1, \tilde{x}^{-1}, \lambda, \lambda)$. It has the form $aF_S(g)b_1$ with $b_1 \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$ and lies in Y; so we can apply Lemma 16. □

Lemma 18. *Assume that there exist elements $g \in H$, $a \in \mathrm{EU}_{2l}(R_S, \Lambda_S)$, $b \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$ such that the first and the second columns of the matrix $x = aF_S(g)b \notin \mathrm{GU}_{2l}(R_S, \Lambda_S)$ coincide with the corresponding columns of the identity matrix. Then H contains a non-trivial transvection.*

Proof. By Lemma 9 there exist $s \in S$ and M such that the matrix $bT_{1,-2}(s^M)b^{-1}$ belongs to $\mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$. So $y = xT_{1,-2}(s^M)x^{-1}$ has the form $aF_S(g_1)a^{-1}$ for some $g_1 \in H$. On the other hand, direct calculation shows that

$$y = e + s^M(\sum_i x'_{-2i}e^{1i} + \bar{1}x'_{-1i}e^{2i}).$$

Note that this matrix belongs to Y. Hence, if H does not contain a non-trivial transvection, by Lemma 16 $y \in \mathrm{U}_{2l}(R_S, \Lambda_S)$. But then we have $x'_{-2i} = x'_{-1i} = 0$ for all $i \neq \pm 1, \pm 2$, and we can apply Lemma 17. $\square$

Lemma 19. *Assume that there exist elements $g \in H$, $a \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$, such that $x = aF_S(g) \notin \mathrm{GU}_{2l}(R_S, \Lambda_S)$ satisfies the following condition: $x_{i1} = x_{i2} = 0$ for all $i \neq \pm 1, \pm 2$, and $l \geq 4$. Then H contains a non-trivial transvection.*

Proof. Fix indices $i, j \neq \pm 1, \pm 2$ and an element $\xi \in R_S$. By Lemma 9 there exist $s \in S$ and M such that $a^{-1}T_{ij}(s^M\xi)a \in \mathrm{EU}_{2l}("R"_S, "\Lambda"_S)$. So $y = x^{-1}T_{ij}(\xi)x$ has the form $F_S(g_1)$ for some $g_1 \in H$. But it's easy to see that the first and the second columns of y coincide with the corresponding columns of the identity matrix. Hence, by Lemma 18, if H does not contain any non-trivial transvections then $y \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$. Denote the multiplicator of y by λ.

Since $y \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$, the -2-nd and the -1-st rows of y coincides with the corresponding rows of the matrix λe. So we have

$$s^M(x'_{-1i}\xi x_{jk} + x'_{-1,-j}\varepsilon_{-j}\bar{\xi}\varepsilon_i x_{-ik}) = (\lambda - 1)\delta_{-1k} \text{ for all } k, \text{ if } j \neq -i,$$
$$s^M x'_{-1i}\xi x_{-ik} = (\lambda - 1)\delta_{-1k} \text{ for all } k, \text{ if } j = -i.$$

Suppose that $\xi = 1$, $j \neq -i$. Multiplying by x'_{kh} on the right and summing over k, we obtain

$$s^M(x'_{-1i}\delta_{jh} - x'_{-1,-j}\varepsilon_j^{-1}\varepsilon_i\delta_{-ih}) = (\lambda - 1)x'_{-1h} \text{ for all } h,$$

that is,

$$x'_{-1i} = s^{-M}(\lambda - 1)x'_{-1j},$$
$$x'_{-1,-j} = -s^{-M}(\lambda - 1)x'_{-1,-i}\varepsilon_i^{-1}\varepsilon_j,$$
$$(\lambda - 1)x'_{-1h} = 0, \ h \neq j, -i.$$

Hence $(\lambda - 1)^2 x'_{-1h} = 0$ for all h, so $(\lambda - 1)^2 = 0$.

Consider the ideal I generated by all elements of the form $\lambda - 1$, where λ is the multiplicator of $x^{-1}T_{ij}(1)x$, $j \neq -i$. Since λ is central and $(\lambda - 1)^2 = 0$, I is contained in Jacobson radical of R_S. Let A be the left ideal generated by the elements x'_{-1i}, $i \neq \pm 1, \pm 2$. From our equations we obtain that $IA = A$. By Nakayama lemma, $A = 0$, so all $x'_{-1i} = 0$. Similarly, $x'_{-2i} = 0$.

Now we see that the multiplicator of $x^{-1}T_{ij}(\xi)x$ is equal to 1, that is $x^{-1}T_{ij}(\xi)x \in \mathrm{U}_{2l}(R_S, \Lambda_S)$.

Denote by $\tilde{x}$ (respectively, $\tilde{x}'$) the matrix obtained from x (respectively, x^{-1}) by deleting the 1-st, the 2-nd, the -2-nd and the -1-st rows and columns.

We see that $\tilde{x}\tilde{x}' = e$; so $\tilde{x}' = \tilde{x}^{-1}$. Further, $\tilde{x}^{-1}T_{ij}(\xi)\tilde{x} \in \mathrm{U}_{2l-4}(R_S, \Lambda_S)$. Therefore, by Lemma 4, $\tilde{x} \in \mathrm{GU}_{2l-4}(R_S, \Lambda_S)$; denote its multiplicator by μ. Now consider matrix $z = x^{-1}\,\mathrm{diag}(1, 1, \tilde{x}, \mu, \mu)$. It has the form $F_S(g)b$, $b \in \mathrm{GU}_{2l}(R_S, \Lambda_S)$, and $z_{ij} = \delta_{ij}$ for all $i \neq 1, 2$, $j \neq \pm 1, \pm 2$. Multiplying z by some elementary unitary transvections on the left, we get a matrix whose 3-rd and 4-th columns coincide with the corresponding columns of the identity matrix; further, multiplying on the right by some elementary unitary permutation, we obtain a matrix z_1 satisfying the properties of the element x in Lemma 18. This finishes the proof. □

Lemma 20. *Assume that* $l \geq \Lambda sr(R) + 2, 4$, *and there exists an element* $g \in H$ *such that* $F_S(g)$ *does not belong to* $\mathrm{GU}_{2l}(R_S, \Lambda_S)$. *Then* H *contains a non-trivial transvection.*

Proof. If u is any row of length $2l$, denote by $\tilde{u}$ the column

$$(\bar{u}_{-1}, \ldots, \bar{u}_{-l}, -\bar{1}^{-1}\bar{u}_l, \ldots, -\bar{1}^{-1}\bar{u}_1)^t.$$

It's clear that $(\tilde{u}, v)_q = uv$ for every column $v \in R^{2l}$.

Denote by v_1 and v_2 the first and the second columns of $x = F_S(g)$ and by u_1 and u_2 the first and the second rows of x^{-1}. Set $v_1' = \tilde{u}_1$, $v_2' = \tilde{u}_2$. We have $(v_i', v_j)_q = \delta_{ij}$, $i, j = 1, 2$. By Proposition 6 there exists element $a \in \mathrm{EU}_{2l}(R, \Lambda)$ such that $av_1 \in \langle e_1, e_{-1}\rangle$ and $av_2 \in \langle e_2, e_{-2}, e_{-1}\rangle$. But then matrix ax satisfies the conditions of matrix x in Lemma 19. □

Lemma 21. *Let* $l \geq \Lambda lfsr(R) + 2, 4$. *Then either* $H \leq \mathrm{GU}_{2l}(R, \Lambda)$, *or* H *contains a non-trivial transvection.*

Proof. Fix for every $\mathfrak{m} \in \mathrm{Max}(R_0)$ a multiplicative system $S_\mathfrak{m}$ such that $l \geq \Lambda_{S_\mathfrak{m}} sr(R_{S_\mathfrak{m}}) + 2$ and $R_{S_\mathfrak{m}}$ is weakly finite. Suppose that there exists $g \in H \setminus \mathrm{GU}_{2l}(R, \Lambda)$. By Lemma 5, there exists $\mathfrak{m} \in \mathrm{Max}(R_0)$ such that $F_{S_\mathfrak{m}}(g) \notin \mathrm{GU}_{2l}(R_{S_\mathfrak{m}}, \Lambda_{S_\mathfrak{m}})$. Hence by Lemma 20 H contains a non-trivial transvection. □

14 Ideal form parameters and corresponding subgroups

An *ideal form parameter* is a pair (A, Γ) consisting of an ideal A of R and an additive subgroup Γ of R such that the following conditions hold:

$$\bar{A} = A,$$
$$\min^\sigma(R) + A \leq \Gamma,$$
$$\Gamma/A \text{ is a form parameter of } R/A.$$

We say that (A, Γ) *contains* (B, Δ) if $A \geq B$ and $\Gamma \geq \Delta$.

The *elementary group of level* (A, Γ) is by definition the group generated by $\mathrm{EU}_{2l}(R, \min^\sigma(R))$, $\mathrm{E}_{2l}(A)$ and all elements of the form $t_{i,-i}(\alpha)$ where $\alpha \in \Gamma\varepsilon_i$. We denote it by $\mathrm{EEU}_{2l}(R, A, \Gamma)$. It's clear that $\mathrm{EEU}_{2l}(R, 0, \Lambda) = \mathrm{EU}_{2l}(R, \Lambda)$.

Lemma 22. *Let $l \geq 3$. Then* $\mathrm{EEU}_{2l}(R, A, \Gamma)$ *is perfect.*

Proof. Denote by H the commutator subgroup of $\mathrm{EEU}_{2l}(R, A, \Gamma)$. By Lemma 3, $\mathrm{EU}_{2l}(R, \min^\sigma(R)) \leq H$. Consider an element $t_{ij}(\xi)$, $j \neq \pm i$, $\xi \in A$. Pick an index $k \neq \pm i, \pm j$. We have $t_{ij}(\xi) = [T_{ik}(1), t_{kj}(\xi)] \in H$. Finally, let $\alpha \in \Gamma\varepsilon_i$. Pick $k \neq \pm i$. Then

$$[T_{ik}(1), t_{k,-k}(\alpha\varepsilon_i^{-1}\varepsilon_k)] = T_{i,-k}(\alpha\varepsilon_i^{-1}\varepsilon_k)t_{k,-i}(\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})t_{i,-i}(\alpha).$$

But we already proved that H contains $\mathrm{EU}_{2l}(R, \min^\sigma(R))$ and $\mathrm{E}_{2l}(A)$. So $t_{i,-i}(\alpha) \in H$. □

Let ρ_A denote the reduction homomorphism $\mathrm{GL}_{2l}(R) \to \mathrm{GL}_{2l}(R/A)$. We obviously have $\rho_A(\mathrm{EEU}_{2l}(R, A, \Gamma)) = \mathrm{EU}_{2l}(R/A, \Gamma/A)$. Define the *congruence general unitary group of level* (A, Γ) to be the group $\rho_A^{-1}(\mathrm{GU}_{2l}(R/A, \Gamma/A))$. We denote it by $\mathrm{CGU}_{2l}(R, A, \Gamma)$. For example $\mathrm{CGU}_{2l}(R, 0, \Lambda)$ is simply $\mathrm{GU}_{2l}(R, \Lambda)$.

An element $g \in \mathrm{GL}_{2l}(R, \Lambda)$ belongs to $\mathrm{CGU}_{2l}(R, A, \Gamma)$ if and only if there exists $\lambda \in R_\Gamma$ such that the following conditions hold:

$$\bar{g}_{hi} = \bar{\lambda}\varepsilon_i g'_{-i,-h}\varepsilon_h^{-1} \bmod A \text{ for all } i, h, \qquad \text{(CGU1)}$$

$$\sum_{j=1}^{l} \bar{g}_{ji} g_{-ji} \in \Gamma \text{ for all } i. \qquad \text{(CGU2')}$$

The condition (CGU2') may be replaced by

$$\sum_{j=1}^{l} g_{i,-j}\bar{1}^{-1}\bar{g}_{ij} \in \Gamma \text{ for all } i. \qquad \text{(CGU2'')}$$

Theorem 2. *Let $l \geq lsr(R) + 1, (\Gamma/A)lsr(R/A) + 1, 3$. Then* $\mathrm{CGU}_{2l}(R, A, \Gamma)$ *is the normalizer of* $\mathrm{EEU}_{2l}(R, A, \Gamma)$.

Proof. Denote $\mathrm{CGU}_{2l}(R, A, \Gamma)$ by C and $\mathrm{EEU}_{2l}(R, A, \Gamma)$ by E.

By Proposition 8, $\mathrm{EU}_{2l}(R/A, \Gamma/A)$ is normal in $\mathrm{GU}_{2l}(R/A, \Gamma/A)$. Hence $\rho_A([C, E]) \leq \mathrm{EU}_{2l}(R/A, \Gamma/A)$, so

$$[C, E] \leq E \cdot \mathrm{GL}_{2l}(R, A). \qquad \text{(C1)}$$

Note that

$$[\mathrm{GL}_{2l}(R, A), \mathrm{E}_{2l}(R)] \leq \mathrm{E}_{2l}(R, A) \leq E$$

by Proposition 1 and Lemma 12. So we have

$$[[C, E], E] \leq E. \qquad \text{(C2)}$$

By Lemma 22, it's sufficient to prove that for all $a \in C$, $b, c \in E$ $[a, [b, c]] \in E$. The Hall identity implies that

$$[a, [b, c]] = [[a, c^{-1}], [c^{-1}ac, b]b][[c^{-1}ac, b], {}^{b}c].$$

The second factor lies in E by (C2). By (C1) we can express $[a, c^{-1}]$ in the form xy where $x \in E$, $y \in \mathrm{GL}_{2l}(R, A)$. Denote $[c^{-1}ac, b]b$ by z. We have to prove that $[xy, z] \in E$. Now $[xy, z] = {}^x[y, z][x, z]$. But

$$[y, z] \leq [\mathrm{GL}_{2l}(R, A), \mathrm{E}_{2l}(R)] \leq E.$$

So it remains to prove that $[x, z] \in E$. But

$$[x, z] = [x, [c^{-1}ac, b]][[c^{-1}ac, b], [x, b]][x, b],$$

and by (C2) all factors belong to E.

Conversely, if $g \in \mathrm{GL}_{2l}(R)$ normalizes $\mathrm{EEU}_{2l}(R, A, \Gamma)$, then $\rho_A(g)$ normalizes $\mathrm{EU}_{2l}(R/A, \Gamma/A)$, hence by Lemma 4 $\rho_A(g) \in \mathrm{GU}_{2l}(R/A, \Gamma/A)$, so $g \in \mathrm{CGU}_{2l}(R, A, \Gamma)$. □

15 Main theorem

Theorem 3. *Suppose that $l \geq 4$ and for every ideal A in R such that $A = \bar{A}$ $l \geq (\Lambda + A/A)lfsr(R/A) + 2$. Then for every subgroup H in $\mathrm{GL}_{2l}(R)$ containing $\mathrm{EU}_{2l}(R, \Lambda)$ there exists a unique ideal form parameter (A, Γ) containing $(0, \Lambda)$ such that*

$$\mathrm{EEU}_{2l}(R, A, \Gamma) \leq H \leq \mathrm{CGU}_{2l}(R, A, \Gamma).$$

Proof. It follows from the conditions (CGU1), (CGU2') that if

$$\mathrm{EEU}_{2l}(R, B, \Delta) \leq \mathrm{CGU}_{2l}(R, A, \Gamma)$$

then $(B, \Delta) \leq (A, \Gamma)$. So the uniqueness of (A, Γ) is obvious.

Set $A = \{\xi | t_{ij}(\xi) \in H\}$ $(j \neq \pm i)$ and $\Gamma = \{\alpha | t_{i,-i}(\alpha\varepsilon_i) \in H\}$. It follows from Lemma 10 (with $s = 1$) that A and Γ do not depend on choice of i, j and (A, Γ) is an ideal form parameter. It's clear that H contains $\mathrm{EEU}_{2l}(R, A, \Gamma)$.

Consider the group $\rho_A(H)$. It contains $\mathrm{EU}_{2l}(R/A, \Gamma/A)$. We show that $\rho_A(H)$ does not contain any non-trivial transvection.

Let $t_{ij}(\xi + A) \in \rho_A(H)$, $j \neq i$, $\xi \notin A$. Then there exist $a \in \mathrm{GL}_{2l}(R, A)$ such that $t_{ij}(\xi)a \in H$. Pick $k \neq \pm i, \pm j$. We have

$$[T_{ki}(1), t_{ij}(\xi)a] = t_{kj}(\xi) \cdot {}^{t_{ij}(\xi)}[T_{ki}(1), a] \in H.$$

But by Proposition 1 $[T_{ki}(1), a] \in \mathrm{E}_{2l}(R, A)$ and, since $\mathrm{E}_{2l}(R, A)$ is normal in $\mathrm{E}_{2l}(R)$,

$${}^{t_{ij}(\xi)}[T_{ki}(1), a] \in \mathrm{E}_{2l}(R, A) \leq \mathrm{EEU}_{2l}(R, A, \Gamma) \leq H$$

by Lemma 12. So we have $t_{kj}(\xi) \in H$, which contradicts the definition of A.

Now let $t_{i,-i}(\alpha + A) \in \rho_A(H)$, $\alpha \notin \Gamma\varepsilon_i$. Pick $k \neq \pm i$. By the same argument as above we see that

$$[T_{ki}(1), t_{i,-i}(\alpha)] = T_{k,-i}(\alpha)t_{k,-k}(\alpha\varepsilon_i^{-1}\varepsilon_k)t_{i,-k}((\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})\varepsilon_k) \in H.$$

So $t_{k,-k}(\alpha\varepsilon_i^{-1}\varepsilon_k)t_{i,-k}((\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})\varepsilon_k) \in H$.

Pick $h \neq \pm i, \pm k$. By commuting with $T_{hi}(1)$ on the left, we see that $t_{h,-k}((\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})\varepsilon_k) \in H$. So $\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha} \in A$, and $t_{i,-k}((\alpha\varepsilon_i^{-1} - \varepsilon_i\bar{\alpha})\varepsilon_k) \in \mathrm{E}_{2l}(A) \leq H$. But then $t_{k,-k}(\alpha\varepsilon_i^{-1}\varepsilon_k) \in H$, which contradicts the definition of Γ.

Note that $l - 2 \geq (\Lambda + A/A)lfsr(R/A) \geq (\Gamma/A)lfsr(R/A)$. Now by Lemma 21 we have $\rho_A(H) \leq \mathrm{GU}_{2l}(R/A, \Gamma/A)$, that is $H \leq \mathrm{CGU}_{2l}(R, A, \Gamma)$. □

Corollary 1. *Suppose that $l \geq 4$ and R is module finite over its center. Then the conclusion of Theorem 3 holds.*

Proof. It follows from main theorem and Proposition 4. □

Corollary 2. *Suppose that $l \geq 4$ and $asr(R) \leq 1$. Then the conclusion of Theorem 3 holds.*

Proof. Note that $asr(R/A) \leq asr(R)$ for every ideal A. Now the assertion follows from main theorem and Proposition 5. □

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