K-Theory **20** (2000), 299 - 310

PRESENTING POWERS OF AUGMENTATION IDEALS AND PFISTER FORMS

Anthony Bak, Nikolai Vavilov

Fakultät für Mathematik
Universität Bielefeld
33615 Bielefeld, Deutschland
e-mail: bak@mathematik.uni-bielefeld.de

and

Department of Mathematics and Mechanics
Sanct-Petersburg State University
Petrodvorets, 198904, Russia
e-mail: vavilov@mathematik.uni-bielefeld.de

Dedicated to Professor Daniel Quillen on his sixtieth birthday

Abstract. *The article proposes a solution to the fundamental question of finding all additive relations among n-fold Pfister classes in the Witt ring and provides evidence for the solution by presenting the n'th power of the augmentation ideal of an integral group ring of a group of exponent 2. The*

The first named author gratefully acknowledges the hospitality of Prof. E. Tan and the support of the National Science Council of Taiwan during the preparation of the initial stages of the current article.

The second named author gratefully acknowledges the support of the *Alexander von Humboldt-Stiftung* during the preparation of the present article.

proposed solution generalizes Milnor's conjecture for quadratic forms, whose proof has been announced by V. Voevodsky.

Mathematics Subject Classifications (2000):
16S34, 11E81, 11E04, 19D45

Key Words. *Milnor conjecture, Witt ring, Pfister form, quadratic form, integral group ring, augmentation ideal, 2-cocycle*

§0. Introduction

We describe first our results presenting powers of the augmentation ideal of an integral group ring and then relate these to the fundamental question of finding all relations among n-fold Pfister classes.

Let G be a group, $\mathbb{Z}[G]$ its integral group ring and $\eta\colon\ \mathbb{Z}[G] \to \mathbb{Z}$ the augmentation homomorphism. The kernel $J(G)$ of the homomorphism η is called the *augmentation ideal* of $\mathbb{Z}[G]$. It is clear that $J(G)$ is freely generated as an abelian group by the elements

$$[g] := 1 - g, \quad g \in G \setminus \{1\}$$

(cf. [3], [6]). This implies that the n-th power $J^n(G) := J(G)^n$ of the augmentation ideal $J(G)$ is generated as an abelian group by the products

$$[g_1, \dots , g_n] := (1 - g_1) \dots (1 - g_n), \quad g_1, \dots , g_n \in G.$$

It is a classical unsolved problem in the theory of group rings to find the relations among the generators $[g_1, \dots , g_n]$ for $J^n(G)$. There are two obvious relations which hold for any group G. The first is the normalizing relation

$$\text{(N)} \qquad [g_1, \dots, g_n] = 0 \ \ (n \geq 1), \text{ whenever some } g_i = 1.$$

The second is that the symbol $[g_1, g_2]$ is a 2-cocycle:

$$[g_2, g_3] - [g_1 g_2, g_3] + [g_1, g_2 g_3] - [g_1, g_2] = 0.$$

This implies the relation

(R) $[g_1, \dots, g_n]$ $(n \geq 2)$ is a 2-cocycle in g_{i-1}, g_i

when the other variables are fixed. Other relations seem to depend essentially on the subgroup structure of the group G. Even in the case of a finite abelian group, these relations may be quite complicated and are not completely understood.

In the present article, we solve completely this problem in the important special case when G is a group of exponent 2. It turns out that the only relations one has to impose on the symbols $[g_1, \dots, g_n]$, apart from (N) and (R), are the relation

(S) $$[g_1, \dots, g_i, \dots, g_j, \dots, g_n] = [g_1, \dots, g_j, \dots, g_i, \dots, g_n] \qquad (n \geq 2)$$

which follows from the commutativity of G (every group G of exponent 2 is commutative), and the relation

(T) $$[f, g, fg, g_4, \dots, g_n] = 0 \qquad (n \geq 3)$$

which is verified easily by direct computation.

Our main result is the following.

Theorem A. *Let G be a group of exponent 2. Then the group $J^n(G)$ is the free abelian group on the symbols $[g_1, \dots, g_n]$ $(g_i \in G,\ i = 1, \dots, n)$ modulo the relations*

(N),	if $n = 1$;
(N), (R), and (S),	if $n = 2$;
(N), (R), (S), and (T),	if $n \geq 3$.

In the process of proving Theorem A, we establish the following mod 2 result. Let $\mathbb{F}_2$ denote the field of 2 elements. Let $\mathbb{F}_2[G]$ denote the group ring over $\mathbb{F}_2$, $J_2(G)$ the kernel of the augmentation homomorphism $\eta_2 : \mathbb{F}_2[G] \longrightarrow \mathbb{F}_2$, and $J_2^n(G) := J_2(G)^n$. In the mod 2 situation, the relations $(N), (R)$, and (S) remain unchanged, but the relation (T) is replaced by

(T′) $$[g, g, g_3, \dots, g_n] = 0 \qquad (n \geq 2).$$

Theorem B. *Let G be a group of exponent 2. Then the group $J_2^n(G)$ is the free abelian group on the symbols $[g_1, \dots, g_n]$ $(g_i \in G,\ i = 1, \dots, n)$ modulo the relations*

$$\begin{array}{ll} \textit{(N) and } 2[g] = 0, & \textit{if } n = 1; \\ \textit{(N), (R), (S), and (T'),} & \textit{if } n \geq 2. \end{array}$$

For general groups G, the consecutive factors $J^m(G)/J^{m+1}(G)$ of the filtration $\dots \supseteq J^m(G) \supseteq J^{m+1}(G) \supseteq \dots$ have been studied much more intensively than the filtration itself. In [7], the consecutive factors have been calculated when the ground ring is a field. In [9], a general homological procedure for calculating these factors with integral coefficients has been proposed. Finally, in the case of an elementary abelian p-group $G = G(m)$ of rank m, these factors are completely calculated in [5] (see also [3], [4]). Namely, it is shown there that $J^m(G)/J^{m+1}(G) \cong \mathbb{Z}[x_1, \dots, x_m]/(2x_i, x_i^p x_j - x_j^p x_i)$. Of course, this result follows from ours when $p = 2$. However, our presentation of $J^m(G)$ provides more detailed information concerning $J^m(G)$ than the result above, and it is not at all clear, how to recover this information from a presentation of $J^m(G)/J^{m+1}(G)$.

The relation of Theorem A above to n-fold Pfister classes and Milnor's conjecture is as follows. Let F be a field of characteristic $\neq 2$. In [2], J.Milnor conjectured that

$$\mathrm{K}_n(F)/2\mathrm{K}_n(F) \cong I^n(F)/I^{n+1}(F),$$

where $I(F)$ is the fundamental ideal of the Witt ring $W(F)$ of (anisotropic) quadratic spaces over F and $K_n(F)$ the n'th Milnor K-group of F. The proof of this conjecture has been announced by V. Voevodsky [11]. In terms of group rings, the conjecture takes the following form. For an element $a \in F^* :=$ units(F), let q_a denote the quadratic form $q_a : F \longrightarrow F$, $x \mapsto ax^2$. It is well known that $W(F)$ is generated as an $\mathbb{F}_2$-algebra by the isomorphism classes $\langle a \rangle$ of the one dimensional quadratic spaces (F, q_a). (We refer the reader to [1], [8] for the fundamentals of the algebraic theory of quadratic forms.) The map $F^* \longrightarrow W(F)$, $a \mapsto \langle a \rangle$, is a homomorphism of F^* into the group of multiplicative units of $W(F)$ and induces a surjective ring homomorphism $\mathbb{Z}[F^*/F^{*2}] \to W(F)$ which takes $J^n(F^*/F^{*2})$ onto $I^n(F)$ and maps a typical generator $[a_1F^{*2}, \dots, a_nF^{*2}]$ of $J^n(F^*/F^{*2})$ onto the n-fold Pfister class

$$\langle\langle a_1, \dots, a_n \rangle\rangle := \ (\langle 1 \rangle - \langle a_1 \rangle) \dots (\langle 1 \rangle - \langle a_n \rangle)$$

in $I^n(F)$.

Lemma C. *Suppose $n = 1$ or 2. Then a complete set of additive relations for the generators $\langle\langle a_1, \dots, a_n \rangle\rangle$ of $I^n(F)$ is given by the relations in Theorem A for the symbols*

$$\{a_1, \dots, a_n\} := [a_1F^{*2}, \dots, a_nF^{*2}]$$

and the additional relations

$$\{a\} + \{-a\} \ - \{-1\} = 0 = \{a\} + \{b\} - \{a+b\} - \{(a+b)ab\},$$

if $n = 1$ *and* $a, b, a+b \in F^*$*;*

$$\{a, b\} = \{a, -ab\} = \{a, (1-a)b\},$$

if $n = 2$ *and* $a, 1-a, b \in F^*$*.*

The lemma above follows routinely from [1, II § 4] and [10, (6.3)]. The proof will be omitted. The following question has fundamental significance for the algebraic theory of quadratic forms. It was posed independently two decades ago by R. Elman and the first author and generalized Milnor's conjecture.

Fundamental Question. *Is every additive relation among the* n*-fold Pfister classes* $\langle\langle a_1, \ldots , a_n\rangle\rangle$ *in* $I^n(F)$ *for* $n \geq 3$ *a consequence of the relations* (N), (R), (S)*, and* (T) *for the symbols* $\{a_1, \ldots , a_n\}$ *and the additional relation*

$$\text{(Q)} \qquad \{a, b, a_3, \ldots , a_n\} = \{a, -ab, a_3, \ldots , a_n\} = \{a, (1-a)b, a_3, \ldots , a_n\},$$

if $a, 1-a, b, a_3, \ldots , a_n \in F^*$*?*

A positive answer to this question allows one to show easily that the assignment

$$\langle\langle a_1, \ldots a_n\rangle\rangle \mapsto a_1 \otimes \ldots \otimes a_n \in K_n(F)/2K_n(K)$$

defines a surjective homomorphism $I^n(F) \longrightarrow K_n(F)/2K_n(K)$ whose kernel is $I^{n+1}(F)$, which proves Milnor's conjecture.

Whereas Milnor's conjecture proposed how to account for all additive relations among the images of n-fold Pfister classes in the consecutive factor $I^n(F)/I^{n+1}(F)$, the question above proposes an account in $I^n(F)$ itself. Thanks to Witt cancellation, the latter would provide an answer to the question when orthogonal sums of n-fold Pfister forms are isomorphic or more generally differ by an orthogonal sum of hyperbolic planes.

The remainder of the article is organized as follows. In § 1, we review well known relations in $J^n(G)$ for general groups G. For groups G of exponent 2, $2J^{n-1}(G) \subseteq J^n(G)$. It follows that the relations in $J^n(G)$ are essentially visible already modulo 2. Thus, we prove first Theorem B for $J_2^n(G)$ and then lift the result to $J^n(G)$ to obtain Theorem A. The case $J_2^n(G)$ is handled in § 2 and the case $J^n(G)$ in § 3.

§1. General relations

The purpose of this section is to review in a convenient form well known results concerning powers of the augmentation ideal in any group ring.

Let G be a group and R a commutative, associative ring with identity. Let RG denote the group ring of G with coefficients in R. Thus, the elements of R are all formal, finite sums $\sum\limits_{g\in G} r_g g$ such that $r_g \in R$ and $r_g = 0$ for almost all g. Addition is defined by the rule $\sum r_g g + \sum r'_g g = \sum (r_g + r'_g) g$ and multiplication by $(\sum r_g g)(\sum r'_g g) = \sum\limits_{g\in G} (\sum\limits_{fh=g} r_f r'_h) g$. The ring homomorphism $\eta_R : RG \longrightarrow R,\ \sum r_g g \longmapsto \sum r_g$, is called the *augmentation homomorphism.* The $\mathrm{Ker}(\eta_R)$ will be denoted by $J_R(G)$ and is called the *augmentation ideal* of RG. We let $J^n_R(G)$ denote the n'th power ideal $J_R(G)^n$ of $J_R(G)$. By definition, $J^n_R(G)$ is additively generated by all products $x_1, \cdots, x_n$ of elements x_i such that $x_i \in J_R(G)$. For $g_1, \dots, g_n \in G$, let

$$[g_1, \dots, g_n] = (1 - g_1)\ \dots (1 - g_n)\ \in RG.$$

Lemma 1.1. *The R-module $J_R(G)$ is the free R-module generated by the elements $[g]$ $(g \in G)$ modulo the relation $[1] = 0$.*

Proof. Let $\sum r_g g \in J_R(G)$. By definition, $\sum r_g = 0$. Thus, $\sum r_g g = -(\sum r_g - \sum r_g g) = -\sum r_g (1-g) = -\sum r_g [g]$. Thus, the elements $[g]$ $(g \in G)$ generate $J_R(G)$ as an R-module. Clearly, $[1] = 0$. Suppose $\sum r_g g\ [g] = 0$. Since $[1] = 0$, we conclude that $\sum\limits_{g\neq 1} r_g [g] = 0$. Thus, $\sum\limits_{g\neq 1} r_g - \sum\limits_{g\neq 1} r_g g = 0$. Since, $R[G]$ is a free R-module on the elements of G, it follows that $r_g = 0$ for all $g \neq 1$. Thus, $J_R(G)$ is a free R-module on $\{[g] |\ g \in G,\ g \neq 1\}$. Q.E.D.

Corollary 1.2. *The R-module $J^n_R(G)$ is generated as an R-module by the elements $[g_1, \dots, g_n]$ $(g_i \in G, 1 \leq i \leq n)$ and these elements satisfy the relation (N); namely, $[g_1, \dots, g_n] = 0$ whenever some $g_i = 1$.*

Proof. By definition, $[g_1, \dots, g_n] = [g_1] \dots [g_n]$. Thus, the corollary is a trivial consequence of (1.1).

Lemma 1.3. *For $n \geq 2$, the generators $[g_1, \dots, g_n]$ $(g_i \in G,\ 1 \leq i \leq n)$ satisfy relation (R); namely, the symbol $[g_1, \dots, g_n]$ is a cocycle in g_i, g_{i+1} when the other variables are fixed.*

Proof. Since $[g_1, \dots, g_n] = [g_1] \dots [g_i][g_{i+1}] \dots [g_n]$, it suffices to prove the result when $n = 2$. But, this is a straightforward computation, which anyone can check for himself.

Lemma 1.4. *The R-module $J_R^{n+1}(G)$ measures the failure of the generators $[g_1, \dots, g_n]$ of $J_R^n(G)$ to be n-multiplicative. Specifically, $[g_1, \dots, g_i^{(1)} g_i^{(2)}, \dots, g_n]$ $-[g_1, \dots, g_i^{(1)}, \dots, g_n] - [g_1, \dots, g_i^{(2)}, \dots, g_n] = -[g_1, \dots, g_i^{(1)}, g_i^{(2)}, \dots, g_n]$.*

Proof. Since $[g_1, \dots, g_n] = [g_1] \dots [g_n]$, it suffices to prove the result for the case $n = 1$. But, this is a trivial computation.

Lemma 1.5. *If G is abelian then the generators $[g_1, \dots g_n]$ $(g_i \in G,\ 1 \leq i \leq n)$ satisfy relation (S); namely, $[g_1, \dots, g_i, \dots, g_j, \dots, g_n] = [g_1, \dots, g_j, \dots, g_i, \dots, g_n]$.*

Proof. This follows from the fact that $[g_1, \dots, g_n] = [g_1] \dots [g_n]$ and the group ring RG is commutative when G is abelian.

§2. Relations modulo 2

Throughout this section, G denotes a group of exponent 2. Thus, G must be abelian. Let $\mathbb{F}_2$ denote the field of 2 elements, $\mathbb{F}_2 G$ the group ring of G with coefficients in $\mathbb{F}_2$, $J_2(G)$ the kernel of the augmentation homomorphism $\eta_2 : \mathbb{F}_2 G \longrightarrow \mathbb{F}_2$, $\sum_{g \in G} n_g g \longmapsto \sum_{g \in G} n_g$, and $J_2^n(G)$ the n'th power ideal $J_2(G)^n$ of $J_2(G)$. Let

$$[g_1, \dots, g_n] = (1 + g_1) \dots (1 + g_n),\ g_i \in G.$$

By (1.2), the elements $[g_1, \dots, g_n]$ generate additively the ideal $J_2^n(G)$. We are interested in the relations among these generators.

Lemma 2.1. *The generators $[g_1, \dots, g_n]$ satisfy the relations (N), (R), (S), and (T') in the introduction.*

Proof. By easy, direct computation.

Eventually, we shall show that (N), (R), (S), and (T') form a complete set of relations for $J_2^n(G)$.

Lemma 2.2. *Under the relations (N), (R), and (S), the relation (T') is equivalent to the relation*

$$[f, g, g_2, \dots, g_n] = [f, fg, g_3, \dots, g_n] \ (n \geq 2). \tag{T$''$}$$

Moreover, under (N), (R), (S), and (T') (or (T"), one has

$$2[f, g, g_2, \dots, g_n] = 0. \tag{2.2.1}$$

Proof. In our notation, we shall suppress $g_3, \dots, g_n$. The assertion (T") $\Longrightarrow$ (T') follows from the equations $[f, f] =$ (by (T")) $[f, 1] =$ (by (N)) 0. Conversely,

$$[f, g] = -[f, fg] \tag{2.2.1}$$

because $[f, g] =$ (by (R)) $= -[f, fg] + [f^2, g] + [f, f] = -[f, fg] + [1, g] + [f, f] =$ (by (N), (T')) $= -[f, fg]$. Thus, (T") will follow from (T'), once we show that for any symbol $[f, g]$, $2[f, g] = 0$. For any f and g, $[f, fg] =$ (by (S)) $[fg, f] =$ (by (2.2.1)) $= -[fg, (fg)f] = -[fg, g]$. Coupling this result with (2.2.1), one obtains $[f, g] = [fg, g]$. But, $[f, g] =$ (by (S)) $[g, f] =$ (by (2.2.1)) $= -[g, fg] =$ (by (S)) $= -[fg, g]$. Thus, $[f, g] = -[f, g]$. Q.E.D.

For $g_1, \dots, g_n \in G$, set
$H_{g_1, \dots, g_n} =$ subgroup of G generated by $g_1, \dots, g_n$.

Lemma 2.3. *Let $g \in H_{g_1, \dots, g_n}$ and write $g = \prod_{i=1}^{n} g_i^{e_i}$ where $e_i = 0$ or 1. Then relations (S) and (T") imply that $[g_1^{e_1}, g_2, \dots, g_n] = [g, g_2, \dots, g_n]$.*

Proof. The result follows from (2.2) by an easy computation.

Corollary 2.4. *The relations (N), (S), and (T") imply that the symbol $[g_1, \dots, g_n]$ $(n \geq 1)$ depends on the group $H_{g_1, \dots, g_n}$ as follows:*

$$\text{If } |H_{g_1, \dots, g_n}| < 2^n \text{ then } [g_1, \dots, g_n] = 0. \tag{2.4.1}$$

$$\text{If } H_{g_1, \dots, g_n} = H_{g'_1, \dots, g'_n} \text{ then } [g_1, \dots, g_n] = [g'_1, \dots, g'_n]. \tag{2.4.2}$$

Proof. One can prove the assertion easily, using (2.3). Details are left to the reader.

On the basis of the lemma above, we make the following definition.

Definition 2.5. If $H \subseteq G$ is a subgroup of order $|H| \leqq 2^n$, let

$$[H] = [g_1, \dots, g_n] \in J_2^n(G)$$

where $g_1, \dots, g_n \in G$ such that $H = H_{g_1}, \dots, g_n$. (The fact that $[H]$ is well defined depends only on $(N), (R), (S)$, and (T').)

Theorem 2.6. *The abelian group $J_2^n(G)$ is generated by the symbols $[g_1, \dots, g_n]$ where $g_1, \dots, g_n \in G$, subject to the defining relations (N), (R), (S), (T').*

Proof. Let $\mathcal{F}(G)$ denote the set of all finite subgroups of G. Since

$$J_2^n(G) = \varinjlim_{F \in \mathcal{F}(G)} J_2^n(F),$$

one can reduce to the case $G = G(m)$ is an elementary abelian 2-group of order 2^m.

Let $J^n(m)$ denote the abelian group whose generators and relations are as in the theorem with $G = G(m)$. By (2.2.1), $J^n(m)$ is an $\mathbb{F}_2$-vector space of finite dimension over $\mathbb{F}_2$. We shall construct by induction in n and m a basis $\mathcal{B}(n,m)$ of $J_2^n(G(m))$ consisting of symbols $[H]$ as in (2.5), $|H| = 2^n$ such that the $\mathbb{F}_2$-linear map $\varphi_{n,m} : J_2^n(G(m)) \longrightarrow J_2^n(m)$ obtained by sending each basis element $[H]$ of $\mathcal{B}(n,m)$ to its class in $J_2^n(m)$ is surjective. Since the canonical linear map $J_2^n(m) \longrightarrow J_2^n(G(m))$ is also surjective and the groups are finite, it will follow that $\varphi_{n,m}$ is an isomorphism and the proof will be complete.

If $n > m$ then by (2.3.1), $J_2^n(G(m)) = J_2^n(m) = 0$. So we are done in this case. Suppose $n \leq m$. We take as a base for induction in n and m the cases $(1,m)$ and (m,m). Set $\mathcal{B}(1,m) = \{[H] \mid H \subseteq G(m),\ |H| = 2\}$ and $\mathcal{B}(m,m) = \{G(m)\}$. By (1.1) and (2.3), respectively, the sets $\mathcal{B}(1,m)$ and $\mathcal{B}(m,m)$ are bases for $J_2^n(G(m))$ and $J_2^m(G(m))$, respectively. In these cases, it is clear that the map $\varphi_{n,m} : J_2^n(G(m)) \longrightarrow J_2^n(m)$ is surjective, because $\mathcal{B}(1,m)$ and $\mathcal{B}(m,m)$ exhaust the nonzero generators of $J_2^n(m)$. We assume now that $n < m$ and that $\mathcal{B}(i,j)$ have been constructed when $i + j < n + m$. We construct $\mathcal{B}(n,m)$ from $\mathcal{B}(n-1,m-1)$ and $\mathcal{B}(n,m-1)$. Fix an embedding $G(m-1) \subseteq G(m)$. This establishes embeddings $\mathcal{B}(n-1,m-1) \subseteq G(m)$ and $\mathcal{B}(n,m-1) \subseteq G(m)$. Let $g \in G(m) \setminus G(m-1)$. Set $\mathcal{B}(n,m) := \{[\langle g,H\rangle] \mid [H] \in \mathcal{B}(n-1,m-1)\} \cup \mathcal{B}(n,m-1)$. We shall show that $\mathcal{B}(n,m)$ has the desired properties. The fact that the elements of $\mathcal{B}(n,m)$ are linearly independent follows trivially from the observations that $\mathcal{B}(n,m-1) \subseteq \mathbb{F}_2G(m-1)$, $\{[\langle g,H\rangle] \mid [H] \in \mathcal{B}(n-1,m-1)\} \subseteq (1-g)\mathbb{F}_2G(m-1)$, $\mathbb{F}_2G(m-1) \cap (1-g)\mathbb{F}_2G(m-1) = 0$, and the induction assumption that the elements of $\mathcal{B}(n,m-1)$ and $\mathcal{B}(n-1,m-1)$ are linearly independent. Identify now each element of $\mathcal{B}(n,m)$ with its image in $J_2^n(m)$. Since the canonical homomorphism $J_2^n(m) \longrightarrow J_2^n(G(m))$ is surjective, to complete the proof it suffices to show that $\mathcal{B}(n,m)$ generates $J_2^n(m)$. Since by (2.3) the symbols $[H]$ such that $H \subseteqq G(m)$ and $|H| = 2^n$ generate $J_2^n(m)$, it suffices to show that each $[H]$ is a linear combination of elements of $\mathcal{B}(n,m)$. Suppose first that $H \subseteq G(m-1)$. Then, by the induction assumption, $[H]$ is a linear combination of elements of $\mathcal{B}(n,m-1)$. Suppose now that $H \nsubseteqq G(m-1)$. Thus, $|H \cap G(m-1)| = 2^{n-1}$. Assume $g \in H$. By the induction assumption, $[H \cap G(m-1)]$ is a linear combination of elements $[K] \in \mathcal{B}(n-1,m-1)$. Thus, $[H] = [\langle g, H \cap G(m-1)\rangle]$ is a linear combination of elements $[\langle g,K\rangle]$ in $\mathcal{B}(n,m)$. Assume $g \notin H$. Then $H = \langle gf, H \cap G(m-1)\rangle$ for some $f \in G(m-1)$. Choose $L \subseteqq H \cap G(m-1)$ such that $|L| = 2^{n-2}$. Let $h \in H \cap G(m-1) \setminus L$. Then $[H] = [\langle gf,h,L\rangle] =$ (by (R)) $[\langle g,f,L\rangle] + [\langle g,fh,L\rangle] + [\langle f,h,L\rangle]$. By the induction assumption, $[\langle f,L\rangle]$ and $[\langle fh,L\rangle]$ are linear combinations of elements in $\mathcal{B}(n-1,m-1)$ and $[\langle f,g,L\rangle]$ a linear combination of elements in $\mathcal{B}(n,m-1)$. Q.E.D.

It is tempting to give a presentation of $J_2^n(G)$, which uses only symbols $[H]$ such that $H \subseteq G$ is a subgroup of order $|H| = 2^n$. We do this next.

Lemma 2.7. *Under the relations (N), (R), and (S), the relation (T') is equivalent to the relation*

$$\text{(T}'''\text{)} \qquad [f,g,g_3,\dots,g_n] + [g,h,g_3,\dots,g_n] + [h,f,g_3,\dots,g_n] =$$

$$[fg,fh,g_3,\dots,g_n] \ (n \geq 2).$$

Proof. In our notation, we shall suppress $g_3,\dots,g_n$. Clearly, $[fg,fh] =$ (by (R) and (2.2.1)) $[g,f] + [g,f^2h] + [f,fh] =$ (by (R) and (2.2.1)) $[g,f] + [g,h] + [f,h] + [f^2,h] + [f,f] =$ (by $(S),(N),(T')$) $[f,g] + [g,h] + [h,f]$. Conversely, setting $g = h = 1$, we get $[f,f] = [f,1] + [1,1] + [1,f] =$ (by (N)) 0.

Lemma 2.8. *Let $H_i \subseteqq G$ $(i = 1,2,3,4)$ be subgroups of G of order 2^n. Then the following are equivalent:*

(2.8.1) $n \geq 2$, $|H_i \cap H_j| = 2^{n-1}$ and $|H_i \cap H_j \cap H_k| = |H_1 \cap H_2 \cap H_3 \cap H_4| = 2^{n-2}$ for all pairwise distinct indices i,j,k.

(2.8.2) There are elements $f, g, h, g_3, \dots, g_n \in G$ *with* f, g, h *linearly independent such that* $H_1 = H_{f,g,g_3,\dots,g_n}, H_2 = H_{g,h,g_3,\dots,g_n}, H_3 = H_{h,f,g_3,\dots,g_n}$, *and* $H_4 = H_{fg,fh,g_3,\dots,g_n}$.

Proof. It is trivial to check that (2.8.2) $\implies$ (2.8.1). To prove the converse, one can reduce to the case $n = 2$. But here the conclusion is a routine exercise finding bases for the vector spaces H_1, H_2, H_3, and H_4. Details are left to the reader.

Lemma 2.9. *Let* $H_i \subseteq G$ $(i = 1, 2, 3, 4)$ *be subgroups of* G *of order* 2^n. *If* n *and the* H_i $(i = 1, 2, 3, 4)$ *satisfy (2.8.1) then the relation*

$$[H_1] + [H_2] + [H_3] = [H_4]. \tag{H}$$

is satisfied in $J_2^n(G)$.

Proof. The conclusion follows immediately from (2.8) and (2.7).

Theorem 2.10. *The abelian group* $J_2^n(G)$ *is generated by the symbols* $[H]$ *such that* $H \subseteq G$ *is a subgroup of order* $|H| = 2^n$, *subject to the relations* $2[H] = 0$ *and (H) above.*

Proof. The proof is the same as that of Theorem (2.6), except one replaces the equation $[\langle gf, h, L\rangle] =$ (by (R)) $[\langle g, f, L\rangle] + [\langle g, fh, L\rangle] + [\langle f, h, L\rangle]$ towards the end of the proof by the equation $[\langle gf, h, L\rangle] =$ (setting $h = f'f$ with $f' \in G(m-1)$) $[\langle gf, f'f, L\rangle] =$ (by (H)) $= [\langle g, f', L\rangle] + [\langle g, f, L\rangle] = [\langle f', f, L\rangle]$.

One interesting aspect of the result above is that the cocycle relation (R) does appear. It turns out, one can also eliminate the cocycle relation from the presentation of $J_2^n(G)$ in Theorem (2.6), as follows.

Lemma 2.11. *Under relations (N) and (S), the relations (R) and (T') are equivalent to the relations (T”) and (T”').*

Proof. In our notation, we shall suppress as usual $g_3, \dots g_n$. By (2.2) and (2.7), (T”) and (T”') follow from (R) and (T'). Conversely, the proof of (2.7) shows that (T') follows from (N) and (T”'). We establish now (R). Let $h' = fh$. Then $[gf, h'] = [fg, h'] = [fg, fh] =$ (by (T”')) $= [f, g] + [g, h] + [h, f] =$ (by (S)) $= [g, f] + [g, h] + [f, h] = [g, f] + [g, fh'] + [f, fh'] =$ (by (T”)) $= [g, f] + [g, fh'] + [f, h']$, which proves (R).

§3. Relations over $\mathbb{Z}$

Throughout this section, G denotes a group of exponent 2. Let $\mathbb{Z}$ denote the natural integers, $\mathbb{Z}[G]$ the group ring of G with coefficients in $\mathbb{Z}$, $J(G)$ the kernel of the augmentation homomorphism $\eta : \mathbb{Z}[G] \longrightarrow \mathbb{Z},\ \sum n_g g \longmapsto \sum n_g$, and $J^n(G)$ the n'the power ideal $J(G)^n$ of $J(G)$. Let

$$[g_1, \dots, g_n] = (1 - g_1) \dots (1 - g_n) \ \ (g_i \in G,\ 1 \le i \le n).$$

By (1.2), the elements $[g_1, \dots, g_n]$ generate additively the ideal $J^n(G)$. We shall determine the relations among these generators.

Lemma 3.1. *Under the relations (N), (R), and (S), the relation*

$$\text{(T)} \qquad [f, g, fg, g_4, \dots g_n] = 0 \qquad (n \ge 3)$$

is equivalent to the relation

$$\text{(t)} \qquad [f, f, g, g_4, \dots, g_n] = [f, g, g, g_4, \dots, g_n] \ \ (n \ge 3).$$

Proof. In our notation, we suppress $g_4, \dots, g_n$. Trivially, the equivalence of (T) and (t) follows from the equations $[f, g, fg] =$ (by (R))$-[f, g, g] + [1, g, g] + [f, g, f] =$ (by (N), (S)) $-[f, g, g] + [f, f, g]$.

Lemma 3.2. *In $J^n(G)$ $(n \ge 1)$,* $2[g, g_2, \dots, g_n] = [g, g, g_2, \dots, g_n]$.

Proof. Clearly $2[g] = 2(1 - g) = (1 - g)^2 = [g]^2$. Thus, $2[g, g_2, \dots g_n] = 2[g][g_2] \dots [g_n] = [g]^2 [g_2] \dots [g_n] = [g, g, g_2, \dots, g_n]$.

Theorem 3.3. *The abelian group $J^n(G)$ is generated by the symbols $[g_1, \dots, g_n]$ $(g_i \in G,\ 1 \le i \le n)$, subject to the defining relations (N), (R), (S), and (T).*

Proof. Let J^n denote the abelian group whose generators and relations are as in the theorem. Consider the canonical homomorphism $\psi : J^n \longrightarrow J^n(G)$. By (1.1), this map is an isomorphism for $n = 1$. We procede now by induction on n and suppose that the canonical homomorphism $J^{n-1} \longrightarrow J^{n-1}(G)$ is an isomorphism. Let J^n_{double} denote the subgroup of J^n generated by all symbols $[g, g, g_3, \dots, g_n]$. We shall show that $\psi|_{J^n_{\text{double}}}$ is injective. Assume this has been done. We want to show that ψ is an isomorphism. It is clearly surjective. But, it is also injective, because by (3.1) and (2.6), Ker $\psi \subseteq J^n_{\text{double}}$.

Let $\psi' = \psi|_{J^n_{\text{double}}}$. By (3.2), image $\psi' = 2J^{n-1}(G)$. Since $\mathbb{Z}[G]$ is $\mathbb{Z}$-torsion free, so is $J^{n-1}(G)$. Thus, from our induction assumption that $J^{n-1} = J^{n-1}(G)$, is follows that $2J^{n-1}(G)$ has as generators and relations the set $\{[[g_1, \ldots, g_{n-1}]] \mid [[g_1, \ldots, g_{n-1}]] = 2[g_1, \ldots, g_{n-1}]\}$ and the relations (N), (R), (S), and (T) for the symbol $[[g_1, \ldots, g_{n-1}]]$. We shall show that the assignment $[[g_1, \ldots, g_{n-1}]] \longmapsto [g_1, g_1, g_2, \ldots, g_{n-1}] \in J^n_{\text{double}}$ defines a homomorphism $\varphi : 2J^{n-1}(G) \longrightarrow J^n_{\text{double}}$. It is clear that it φ must be inverse to ψ', which finishes the proof of the theorem. It is trivial to check that the assignment preserves relations (N) and (T). By (3.1), the assignment is the same as the assignment $[[g_1, \ldots, g_{n-1}]] \longmapsto [[g_i, g_1, \ldots, g_{n-1}]]$ for any i. It follows that relation (S) is preserved and that relation (R) is preserved, except possibly when $(n-1) = 2$. For $(n-1) = 2$, the relation (R) $[[fg, h]] + [[f, g]] - [[f, gh]] - [[g, h]]$ is taken to the element

$$x = [fg, fg, h] + [f, f, g] - [f, f, gh] - [g, g, h] \in J^3_{\text{double}}.$$

But, by (R) for J^3,

$$[fg, fg, h] = -[f, g, h] + [g, g, h] + [fg, f, h].$$

Thus,

$$\begin{aligned} x &= -[f, g, h] + [fg, f, h] + [f, f, g] - [f, f, gh] \\ &= (\text{by (S)}) = -[f, g, h] + [f, fg, h] + [f, f, g] - [f, f, gh] \\ &= (\text{by (R)}) = \ 0. \end{aligned}$$

Bibliography

[1] Lam T.-Y., *The algebraic theory of quadratic forms*, Benjamin, N.Y., 1973.

[2] Milnor J., *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970), 318–344.

[3] Passi I.B.S., *Group rings and their augmentation ideals*, Lecture Notes in Math. **715** (1979).

[4] Passi I.B.S., *The associated graded ring of an integral group ring*, Bull. London Math. Soc. **10** (1978), 241–255.

[5] Passi I.B.S., Vermani L.R., *The associated graded ring of a group ring*, Proc. Camb. Math. Soc. **82** (1977), 25–33.

[6] Passman D.S., *The algebraic structure of group rings*, J.Wiley, N.Y., 1977.

[7] Quillen D., *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418.

[8] Scharlau W., *Quadratic and hermitian forms*, Springer, Berlin et al, 1985.

[9] Stallings J.R., *Quotients of the powers of the augmentation ideal in a group ring*, Knots, Groups and 3-manifolds, Ann. Math. Studies, vol. 84, 1975, pp. 101–118.

[10] Suslin A.A, *Torsion in K_2 of Fields*, K-Theory **1**, 1987, pp. 5–29.

[11] Voevodsky, V., *The Milnor Conjecture*, preprint, Max Planck Institut (1996).