On the sum of roots of a closed set

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1 Introduction

Root systems and their Weyl groups of automorphisms appear naturally in the description of semi-simple Lie algebras and their linear representations and in the classification of semi-simple algebraic groups. In terms of root systems one can answer many questions concerning the structure of Lie algebras and Chevalley groups.

For example, connected overgroups of a split maximal torus in a Chevalley group and regular subalgebras of a Lie algebra are parametrized by closed subsets of a root system.

In case of the root system A_l, closed subsets are related to topologies on a finite set. By reduction first to T_0-topologies and then to V-topologies, Z.I.Borevich [**?**] counted the number of closed sets in the systems A_l for $l \leq 11$.

Other classical root systems — B_l, C_l, and D_l — are more complicated, so that questions about classifying their closed subsets and even about the reduction to special closed sets is unsolved. Partial results in this direction were obtained by F.M.Malyshev, see for instance [**?**].

The question of classifying closed subsets of exceptional root systems of type E_6, E_7, and E_8 is also interesting.

The goal of the present paper is to describe the structure of closed sets. To do this we introduce a new invariant of a closed set, the **sum vector**. It is a generalization of the construction used in [**?**] for classifying closed sets such that their supplements are closed. The sum vector indicates important properties of the structure of a closed set, such as the disposition of its symmetric part, the disposition and size of its special part, and its stabilizer in the Weyl group. In terms of the notion "sum vector" we define the concept of the **associated base**. A closed set has the simplest form in such bases. Using these notions we obtain a new elementary proof of Chevalley's theorem on parabolic subsets. The main theorem of the current paper is the determination of the set of all possible values of a sum vector. The last section of the paper is devoted to "assembling" a closed set from a closed subsystem and a special set.

2 Preliminaries

Let Φ denote a root system, l its rank, and $W(\Phi)$ its Weyl group. If a base Π of fundamental roots of Φ is fixed, then Φ^+ will denote the set of all positive roots of Φ with respect to Π. In this situation $C(\Phi^+)$ denotes the fundamental Weyl cell.

We denote the inner product of vectors x and y by $\langle x, y\rangle$. If the angle between roots α and β is more than $\pi/2$ (in the other words $\langle \alpha, \beta\rangle \leq 0$), then $\alpha+\beta$ is a root.

The Weyl group $W(\Phi)$ is by definition the group generated by the reflections w_α for all roots $\alpha \in \Phi$. It is known that the Weyl group acts simply transitively on the set of all bases of fundamental roots (or on the set of all Weyl cells). Moreover, the closure $\overline{C(\Phi^+)}$ of the Weyl cell is a fundamental domain for the action of the Weyl group on the span $\mathbb{R}^l$ of Φ. In particular, this implies that the stabilizer in the Weyl group $W(\Phi)$ of a vector is the Weyl group of the root system of all roots in Φ which are orthogonal to the given vector.

Definition. A subset S of a root system Φ is called **closed**, if for any two roots $\alpha, \beta \in S$ such that $\alpha+\beta \in \Phi$, $\alpha+\beta \in S$.

Examples. 1. The set Φ^+ of all positive roots is closed.

2. The intersection of Φ and any linear subspace of $\mathbb{R}^l$ is closed.

Any closed set S is a disjoint union of its *symmetric* part $S^r = \{\alpha \in S | -\alpha \in S\}$ and its *special* part $S^u = \{\alpha \in S | -\alpha \notin S\}$.

The following lemma is well-known.

Lemma 1. *1. S^r is a closed subsystem of Φ.*

2. For any two roots $\alpha \in S^u$ and $\beta \in S$ such that $\alpha+\beta$ is a root, $\alpha+\beta$ lies in S^u.

In particular, the second assertion of Lemma 1 implies S^u is a closed.

If $S = S^r$, equivalently $S^u = \emptyset$, the structure of S is quite simple and well explored. In particular, all closed subsystems have been classified by Borel, de Siebental and Dynkin in [?] and [?]. Namely, any closed subsystem of Φ can be obtained from Φ as a result of multiple applications of the following pair of operations on subsystems, starting with the entire system Φ :

1) for every irreducible component of a subsystem choose any base and a maximal root;

2) choose an arbitrary subset of the chosen collection of roots and form the minimal closed set containing all these roots and their opposites (the set will be a subsystem automatically).

Definition. Closed subsets S and S' of a system Φ are **conjugate**, if there exists an element $w \in W(\Phi)$ such that $w(S) = S'$.

The problem of classifying closed subsets — all or all up to conjugacy — is unsolved.

3 Sum vector: definition and basic properties

Definition. The vector $\xi(S)$, which is equal to the sum of the roots of a closed set S, is called the **sum vector** of S.

The definition implies directly the following lemma.

Lemma 2. *1.* $\xi(S) = \xi(S^u)$.

2. All coordinates of the vector $\xi(S)$ in a base of fundamental roots are integral.

Example. Consider the closed set Φ^+ consisting of all positive roots. For this set we have $\xi(\Phi^+)/2 = \rho$. Moreover ρ lies in a fundamental Weyl cell, $w_\alpha(\rho) = \rho - \alpha$, and $\langle \rho, \alpha \rangle = \langle \alpha, \alpha \rangle$ for any fundamental root α. A detailed proof can be found in [**?**].

Proposition 1. *1. $\xi(S)$ is orthogonal to any root of S^r.*

2. The inner product of $\xi(S)$ and any root of S^u is positive.

Proof. 1. Let α be a typical root of S^r. Note that S is invariant under the corresponding reflection w_α. Indeed, the reflection w_α takes any root $\beta \in S$ to a root $w_\alpha(\beta)$ that is equal to a sum of β and a few (possibly zero) copies of the root α and lies in S. Since S is invariant under w_α, its sum vector $\xi(S)$ is invariant under w_α and orthogonal to α.

2. Let α be a typical root of S^r. For each root $\beta \in S$ such that $\langle \beta, \alpha \rangle < 0$ the root $w_\alpha(\beta)$ lies in S, since $w_\alpha(\beta)$ is equal to a sum of β and a few copies of the root α. Note that $\langle \beta, \alpha \rangle + \langle w_\alpha(\beta), \alpha \rangle = 0$. Let us say that $\beta \in S$ corresponds to $w_\alpha(\beta) \in S$, if $\langle \beta, \alpha \rangle < 0$. Then the sum of any two roots that correspond to each other is orthogonal to α. The root α does not correspond to any root since $w_\alpha(\alpha) = -\alpha$ does not lie in S. So the sum $\xi(S)$ of the roots of S satisfies $\langle \xi(S), \alpha \rangle \geq \langle \alpha, \alpha \rangle > 0$.

Proposition 2. *Let S and T be closed sets such that $S^u \subset T^u$. Then $|\xi(S)| < |\xi(T)|$.*

Proof. Using the first assertion of Lemma 1 we can assume that $S = S^u$ and $T = T^u$. Consider a root $\alpha \in T \backslash S$ such that the inner product of α and $\xi(S)$ is maximal.

Note that the set $S' = S \cup \{\alpha\}$ is closed. Indeed, the set S is closed. For each root $\beta \in S$ such that $\alpha + \beta$ is a root, $\alpha + \beta$ lies in T, because T is closed. If $\alpha + \beta \in T \backslash S$ then $\langle \xi(S), \alpha + \beta \rangle = \langle \xi(S), \alpha \rangle + \langle \xi(S), \beta \rangle > \langle \xi(S), \alpha \rangle$ which contradicts the maximality of α. So S' is closed.

We have $|\xi(S')|^2 - |\xi(S)|^2 = \langle \xi(S'), \xi(S') \rangle - \langle \xi(S), \xi(S) \rangle = \langle \xi(S'), \xi(S') \rangle - \langle \xi(S') - \alpha, \xi(S') - \alpha \rangle = 2\langle \xi(S'), \alpha \rangle - \langle \alpha, \alpha \rangle \geq \langle \alpha, \alpha \rangle > 0$. So the sum vector of S' is longer then the sum vector of S.

By adding vectors α, one at a time to S, we obtain that $|\xi(T)| > |\xi(S)|$.

4 Conjugacy and associated bases

The following lemma is obvious.

Lemma 3. *1. If S and S' are closed sets such that $w(S) = S'$ for some $w \in W$ then $w(\xi(S)) = \xi(S')$.*

2. Since $S^r \subset \xi(S)^\perp \cap \Phi$ we have

$$W(S^r) \leq X(S) \leq X(\xi(S)) = W(\xi(S)^\perp \cap \Phi),$$

where $X(S)$ denotes the normalizer in $W(\Phi)$ of S, i.e. the subgroup of $W(\Phi)$ that leaves S invariant.

Definition. A base of fundamental roots is called **associated** with a closed set S, if $\xi(S) \in \overline{C(\Phi^+)}$.

It is well-known that for any closed set S there exists a base Π such that all roots of S^u are positive with respect to Π. The inductive proof of this fact which is given in [**?**], is more complicated than the following proof in terms of the sum vector.

Proposition 3. *If Π is a base associated with a closed set S then all roots of S^u are positive with respect to Π.*

Proof. Since $\xi(S)$ is in the closure of the fundamental Weyl cell, the inner product of $\xi(S)$ and each of the fundamental roots is non-negative. So the inner product of $\xi(S)$ and each positive root is non-negative and the inner product of $\xi(S)$ and each negative roots is non-positive. Using the second assertion of Proposition 1, we conclude that S^u consists of positive roots.

Corollary. *Coordinates of $\xi(S)$ in an associated base are non-negative and do not exceed the corresponding coordinates of $\xi(\Phi^+)$.*

The shape of a closed set in an associated base is "canonical" in a certain sense. For example, if a closed subset of an irreducible root system consists of a unique root α then α is the maximal root in each associated base. In the next proposition we show that any parabolic set is determined by its sum vector and contains all positive roots in any associated base. Moreover the parabolic set has the same shape in any associated base. In general, a closed set has different shapes in different associated bases.

Example. The closed set consisting of the pair of orthogonal roots $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ in a root system of type A_3 has the shape $\{\beta_2, \beta_1 + \beta_2 + \beta_3\}$ in the base $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 + \alpha_3$, $\beta_3 = -\alpha_3$.

Two closed sets S and S' are conjugate by $w \in W(\Phi)$ if and only if S has the same shape in some base Π as S' has in the base $w(\Pi)$. For any $w_1 \in W(\Phi)$ such that $w_1(\xi(S)) = \xi(S')$ the element w decomposes as $w_2 w_1$ where $w_2 \in W(\Phi \cap \xi(S')^\perp)$.

In terms of sum vectors, we can give an elementary proof of the following proposition which is known as Chevalley's theorem.

Proposition 4. *Any two closed sets containing Φ^+ coincide or are not conjugate.*

Proof. Note if S contains Φ^+ then $\xi(S)$ lies in $\overline{C(\Phi^+)}$. Indeed, each fundamental roots lies in S^u or S^r. If a fundamental root α lies in S^r, then the inner product of α and $\xi(S)$ is equal to zero by the first assertion of Proposition 1. If a fundamental root α lies in S^s then the inner product of α and $\xi(S)$ is positive by the second assertion of Proposition 1.

So we can use the first assertion of Lemma 3 and note that since $\overline{C(\Phi^+)}$ is a fundamental domain, if two closed set contain Φ^+ and are conjugate, then their sum vectors coincide. Thus their special parts consisting of the roots having a positive inner product with the sum vector coincide. And the special parts consisting the roots which are orthogonal to the sum vector coincide. So these parabolic sets coincide.

Since a parabolic set is determined by its sum vector, the set has the same shape in all associated bases.

5 Theorem on values of a sum vector

In this section we ascertain which values the sum vector $\xi(S)$ takes when S runs over the closed subsets of the system Φ.

Consider a closed set S in some associated base Π. Then the sum vector $\xi(S)$ lies in $\overline{C(\Phi^+)}$, the coefficients of $\xi(S)$ in Π are integral (by the second assertion of Lemma 1), non-negative and do not exceed the corresponding coefficients of $\xi(\Phi^+)$. We shall show that any such vector is the sum vector of some closed set S.

Theorem *Let ξ be a vector lying in the closure $\overline{C(\Phi^+)}$ of the fundamental Weyl cell. Suppose that ξ has integral non-negative coordinates which do not exceed the corresponding coordinates of $\xi(\Phi^+)$ in the base of fundamental roots. Then there exists a closed subset S in Φ^+ such that ξ is the sum vector of S.*

Proof. Consider a finite family T of roots. We define a following procedure which we call **"contraction"**:

(a) if α and β lie in T and $\alpha+\beta$ is a root which does not lie in T then we replace α and β by $\alpha+\beta$;

(b) if α and $-\alpha$ lie in T we take α and $-\alpha$ away;

repeat (a) and (b) as long as possible. The process will end since the number of roots in T decreases.

We will use the following trivial properties of contraction.

I. Contraction preserves the sum of the roots of the family T.

II. The multiplicity of a root does not increase after contraction.

III. A subset T of Φ^+ is closed if and only if contraction leaves T invariant.

Since the coordinates of ξ in the base of fundamental roots are integral and do not exceed the coordinates of $\xi(\Phi^+)$, we can decompose ξ as a sum of roots of a family T, where T consists of all positive roots and some opposites of fundamental roots. The numbers of copies, i.e. multiplicity, of each positive root in T is equal to one and the opposites to the fundamental roots can have different multiplicities.

Apply contraction to T and denote the result by S. The sum of the roots of S is the same as the sum of the roots of T and equals ξ. Note that each multiple root of S must be negative. In fact we shall show that S consists of positive roots and thus S is a closed subset of Φ^+.

Assume to the contrary that S contains negative roots. Choose a minimal root x of such roots.

Note that the inner product of x and each multiple root a of S is non-negative. Otherwise their sum $a+x$ is a root lying in S and less then x since a is the opposite of a fundamental. This contradicts the minimality of x.

Consider the set S' consisting of roots lying in S (i.e. S' consists of the same roots as S, but there is no multiple root in S'). S' is closed, since contraction leaves S fixed. By the first assertion of Proposition 1 we have $\langle x, \xi(S')\rangle > 0$.

We have

$$\langle \xi, x\rangle = \langle \xi(S'), x\rangle + (t_1 - 1)\langle a_1, x\rangle + \cdots + (t_r - 1)\langle a_r, x\rangle,$$

where t_i is the multiplicity of the root a_i. Thus $\langle \xi, x\rangle > 0$. On the other hand since x is a negative root and ξ lies in $\overline{C(\Phi^+)}$, we have $\langle \xi, x\rangle \leq 0$.

This contradiction shows that all roots of S are positive, distinct and form a closed set, whose sum vector is equal to ξ. The theorem is proved.

Remark. Two absolutely different closed sets can have one and the same sum vector.

Examples. 1. In the system of type A_l the closed sets $\{\alpha_1+\alpha_2, \alpha_2+\alpha_3\}$ and $\{\alpha_1, \alpha_1+\alpha_2+\alpha_3\}$ have one and the same sum vector. They are conjugate. (Compare with the previous example.)

2. In the root system of type A_4 the closed sets $\alpha_1, \alpha_4, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4\}$ and $\{\alpha_1+\alpha_2, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4\}$ have the same sum vector, but different numbers of roots.

6 Assembling

The following proposition concerns "assembling", i.e. the problem whether the union of a given closed subsystem and a given special closed set form a closed set. Although the result is quite natural, we could not find it in the literature.

Proposition 5. *For any special closed set S^u there is a unique maximal closed set T such that S^u is the special part of T.*

Proof. Note that a root α lies in the symmetric part T^r of some closed set T such that $T^u = S^u$ if and only if α is orthogonal to $\xi(S^u)$ and the set $S^u \cup \{\alpha\} \cup \{-\alpha\}$ is closed. In view Lemma 1 it is sufficient to prove that all such roots form a closed subsystem. In the other words it suffices to prove that if two roots α and β are orthogonal to $\xi(S^u)$, $\alpha+\beta$ is a root and

(∗) $S^u \cup \{\alpha\} \cup \{-\alpha\}$ and $S^u \cup \{\beta\} \cup \{-\beta\}$ are closed,

then $\alpha+\beta$ is orthogonal to $\xi(S^u)$ and $S^u \cup \{\alpha+\beta\} \cup \{-\alpha-\beta\}$ is closed.

It is obvious that $\alpha+\beta$ is orthogonal to $\xi(S^u)$, so we have to prove only the second assertion.

Assume to the contrary that there is a root $\gamma \in S^u$ such that $\gamma+\alpha+\beta$ is a root and does not lie in S^u. Consider the closed subsystem Δ of Φ generated by the roots α, β and γ. The rank of Δ is equal to 3, because α, β and γ are linearly independent. Since $\alpha+\beta$ and $\alpha+\beta+\gamma$ are roots the system Δ is irreducible. Thus it is of type A_3, B_3 or C_3.

Let S' be a closed set equal to $S^u \cap \Delta$. The set S' contains γ and does not contain $\gamma+\alpha+\beta$. Note that $\alpha+\gamma$ (and similarly $\beta+\gamma$) cannot be a root. Otherwise $(*)$ implies $\alpha+\gamma \in S^u$ and $(\alpha+\gamma)+\beta \in S^u$, and we get a contradiction.

We know that if the inner product of two roots is negative then their sum is a root. And if the inner product of two roots in a system of type A_3, B_3 or C_3 is positive then their sum is not a root. So we have the inequalities $\langle\alpha,\gamma\rangle \geq 0$, $\langle\beta,\gamma\rangle \geq 0$ and $\langle\alpha+\beta,\gamma\rangle \leq 0$. Thus they must be equalities.

If the sum of two orthogonal roots is a root then these two roots are short. So $\alpha+\beta$ and γ are short. If the sum of two roots is a short root then one of these two roots is short. So α or β is a short. We may assume that α is short.

Thus α and γ are short orthogonal roots in a system of type B_3 or C_3. Thus $\alpha+\beta$ is a root which contradicts the statement above. This contradiction completes the proof.

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