Attractors of reaction diffusion systems on infinite lattices

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Abstract

In this paper, we study global attractors for implicit discretizations of a semilinear parabolic system on the line.

It is shown that under usual dissipativity conditions there exists a global (Z_u, Z_ρ)-attractor $\mathcal{A}$ in the sense of Babin, Vishik and Mielke, Schneider. Here Z_ρ is a weighted Sobolev space of infinite sequences with a weight that decays at infinity, while the space Z_u carries a locally uniform norm obtained by taking the supremum over all Z_ρ norms of translates. We show that the absorbing set containing $\mathcal{A}$ can be taken uniformly bounded (in the norm of Z_u) for small time and space steps of the discretization.

We establish the following upper semicontinuity property of the attractor $\mathcal{A}$ for a scalar equation: if $\mathcal{A}_N$ is the global attractor for a discretization of the same parabolic equation on the finite segment $[-N, N]$ with Dirichlet boundary conditions, then the attractors $\mathcal{A}_N$ (properly embedded into the space Z_u) tend to $\mathcal{A}$ as $N \to \infty$ with respect to the Hausdorff semidistance generated by the norm in Z_ρ.

We describe a possibility of "embedding" certain invariant sets of some planar dynamical systems into the global attractor $\mathcal{A}$.

Finally, we give an example in which the global attractor $\mathcal{A}$ is infinite-dimensional.

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1 Introduction

The theory of global attractors for partial differential equations and systems is now a well-developed part of the modern theory of infinite dimensional dynamical systems (see, for example, [H88], [L91a], [BV92], [T88], [SY02]). For numerical approximations it is of interest to understand whether global attractors persist in dynamical systems generated by discretizations of the PDE's and how their shape depends on various types of discretization both in time and space (see [H94] for a general reference).

For a wide class of parabolic equations and discretization schemes, attractors were studied by Ladyzhenskaya [L91b] (cf. also the approximation results in [HLR88],[HR89]). There are several papers (see [EP96], for example) which treat global attractors for the discretized semilinear parabolic equation

$$u_t = u_{xx} + f(u). \tag{1.1}$$

In all of these references the domain for the spatial variable x is assumed to be bounded.

Investigation of global attractors for PDEs on unbounded domains began in the last decade. The basic problem is that the semiflow lacks compactness properties. If the underlying domain has a translational invariance then attractors should have this invariance also (for example contain all translates of a traveling wave) and hence cannot be compact in a norm that is translationally invariant. In a pioneering paper [BV90] Babin and Vishik made two very important contributions to the above-mentioned problem. First, the corresponding semigroups are studied in weighted spaces with weights that decay at infinity. Second, they work with pairs of spaces H, H_z and define so-called (H, H_z)-attractors. Here H is usually the Hilbert space on which the semigroup $S(t)$ acts and H_z is the same space endowed with the topology of weak convergence.

The set $\mathcal{A}$ is called a global (H, H_z)-attractor if

(1) $\mathcal{A}$ is compact in H_z;

(2) $\mathcal{A}$ is positively invariant with respect to $S(t)$;

(3) $\mathcal{A}$ attracts bounded subsets of H with respect to the topology of H_z.

This concept was further developed and applied to important examples, such as the Ginzburg-Landau equation, by Feireisl [F94], Collet [C94], Mielke and Schneider [MS95] and Mielke [M97], [M99]. There, the space H_z is taken to be a weighted Sobolev space whereas H carries a locally uniform norm (obtained by taking the supremum over all H_z norms of translates, cf. (1.6) below).

In the present paper, we realize the approach of [BV90] with the norms taken from [MS95] for a dynamical system $\mathcal{S}$ that is generated by an implicit discretization of the system

$$u_t = \mathcal{D}u_{xx} + f(u), \tag{1.2}$$

where $u(x,t) \in \mathbb{R}^k, \mathcal{D} \in \mathbb{R}^{k\times k}$ and $x \in \mathbb{R}$. Taking h as time step and d as space step the system reads

$$\frac{1}{h}(u_m^{n+1} - u_m^n) = \frac{\mathcal{D}}{d^2}(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) + f(u_m^{n+1}), \quad m \in \mathbb{Z}, n \in \mathbb{N}. \tag{1.3}$$

It is assumed that the nonlinearity $f(u)$ is globally Lipschitz continuous and satisfies for some $\alpha, \beta > 0$ a dissipativity estimate

$$< u, f(u) > \leq -\alpha < u, u > +\beta \quad \text{for all } u \in \mathbb{R}^k, \tag{1.4}$$

where $<,>$ is the usual inner product in $\mathbb{R}^k$. This is a standard assumption for proving upper semicontinuity of attractors in the context of ODE's, see [SH96, Ch.7].

We introduce the weight $\rho(x) = (1 + \varepsilon^2 x^2)^{-\gamma}$, where $\gamma > 1/2$ and ε will be chosen appropriately. Consider the space H_ρ of sequences $\{v_m : m \in \mathbb{Z}\}$ with norm defined by

$$\|v\|_{0,\rho}^2 = d\sum \rho_m |v_m|^2, \quad \text{where} \quad \rho_m = \rho(md), \tag{1.5}$$

and let

$$||v||_{0,u}^2 = \sup_y d\sum \rho_m |v_{m+y}|^2, \tag{1.6}$$

where $|v|$ is the Euclidean norm of v.

For the theory we choose as underlying spaces the subspaces Z_ρ and Z_u of H_ρ defined by finiteness of the norms

$$\|v\|_{1,\rho} = \|v\|_{0,\rho} + \|\partial_- v\|_{0,\rho}$$

and

$$\|v\|_{1,u} = \|v\|_{0,u} + \|\partial_- v\|_{0,u},$$

respectively, where $\partial_- v$ is the discrete analog of the derivative v_x.

The main result of the paper (Theorem 5.1) states that the system $\mathcal{S}$ has a global (Z_u, Z_ρ)-attractor $\mathcal{A}$. It is shown that the size of the absorbing set containing $\mathcal{A}$, measured in the norm of the space Z_u, is uniformly bounded for small time and space steps. Morover, given a bounded set in Z_u, the time nh needed to attract this set (in the norm of Z_u) into the absorbing set is uniform as well.

We also establish the following upper semicontinuity property of the attractor $\mathcal{A}$ in the case of a scalar equation (1.1). Here it is assumed that the discretization steps are fixed. Let $\mathcal{A}_N$ be the global attractor for equation (1.1) restricted to the finite segment $[-N, N]$ and subject to Dirichlet boundary conditions. In Theorem 6.3 it is shown that, if $\mathcal{A}_N^*$ is a proper embedding of $\mathcal{A}_N$ into the space Z_u, then $\text{dist}(\mathcal{A}_N^*, \mathcal{A}) \to 0$ as $N \to \infty$, where dist is the Hausdorff semidistance generated by the norm in Z_ρ.

One of the first results in this direction was obtained in the recent paper [BLW01] where a space discretization of the equation

$$u_t = \nu u_{xx} - \lambda u + f(u) + g(x)$$

is studied under the assumptions $\nu, \lambda > 0$ and $uf(u) \leq 0$ (see [Z02] for an extension to a damped wave equation). In this case, the authors establish results close to our Theorems 5.1 and 6.3 for the standard phase space ℓ^2. Notice that their assumption on f excludes traveling waves (in contrast to (1.4)) and hence allows for a compactness proof in ℓ^2.

It is well known that the dynamics of a discretization of a scalar equation (1.1) considered on a bounded x-interval is rather simple – there exists a global Lyapunov function (see [OKM93]), the global attractor consists of fixed points and their unstable manifolds, and any trajectory tends to the union of these fixed points.

We show that the dynamics of the system $\mathcal{S}$ is more complicated even in the scalar case. Considering the discretization scheme as a lattice system, it is possible to "embed" into the global attractor $\mathcal{A}$ invariant sets of some planar dynamical system of quite a general form. These invariant sets are realized by families of traveling waves or stationary solutions (compare the "spatial chaos" analyzed in [AP93]).

Finally, we give an example in which the global attractor $\mathcal{A}$ is infinite-dimensional. Our construction is guided by a result of [BV90] in the continuous case. There it was shown that infinite dimensional attractors arise because they contain the unstable manifold of the origin and the latter one may be infinite dimensional in case of a bimodal nonlinearity.

2 The setting and the basic energy estimate

Consider the discretization (1.3) of system (1.2) such that $h, d < 1$.

It is assumed that the matrix $\mathcal{D}$ is positive definite (and not necessarily symmetric). We fix a number $\sigma > 0$ such that

$$< \mathcal{D}v, v > \geq \sigma < v, v > . \tag{2.1}$$

We write scheme (1.3) in the form

$$\frac{u^{n+1} - u^n}{h} = Au^{n+1} + \overline{f}(u^{n+1}), \tag{2.2}$$

where

$$u^n = \{u_m^n \in \mathbb{R}^k : m \in \mathbb{Z}\},$$

the operator A is defined by

$$(Au)_m = \frac{1}{d^2}\mathcal{D}(u_{m+1} - 2u_m + u_{m-1}),$$

and

$$(\overline{f}(u))_m = f(u_m).$$

Note that we may write

$$(Av)_m = \mathcal{D}(\partial_+\partial_- v)_m,$$

where

$$(\partial_+ v)_m = \frac{v_{m+1} - v_m}{d} \quad \text{and} \quad (\partial_- v)_m = \frac{v_m - v_{m-1}}{d}.$$

It is assumed that the nonlinearity f in (1.2) is globally Lipschitz continuous with constant $\mathcal{L}$ and satisfies condition (1.4). We assume everywhere below that

$$h\mathcal{L} \leq \frac{1}{4}. \tag{2.3}$$

Now let us describe the spaces where we study our system. We fix a number $\gamma > 1/2$ and introduce a weight function

$$\rho(x) = \left(1 + \varepsilon^2 x^2\right)^{-\gamma},$$

where $\varepsilon > 0$ will be chosen later. We set

$$\rho_m = \rho(md) = \left(1 + \varepsilon^2 m^2 d^2\right)^{-\gamma}.$$

It is easy to see that

$$|\rho'(x)|, |\rho''(x)| \leq c_1(\varepsilon)\rho(x), \tag{2.4}$$

where $c_1(\varepsilon) = \max(\gamma\varepsilon, \gamma(\gamma+1)\varepsilon^2)$. It follows from (2.4) that

$$c_2\rho_m \leq \rho(md + \theta) \leq c_3\rho_m \tag{2.5}$$

for any $m \in \mathbb{Z}$ and $|\theta| \leq 1$, where $c_2 = c_3^{-1} = \exp(-\gamma)$.

It follows from relations (2.4) and (2.5) that there exists a function $a_1(\varepsilon)$ (independent of d) such that $a_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$|\partial_-\rho_m|, |\partial_-\partial_-\rho_m|, |\partial_+\partial_-\rho_m| \leq a_1(\varepsilon)\rho_m. \tag{2.6}$$

We specify our conditions on ε as follows. We fix ε so small that

$$c_3 a_1(\varepsilon) < \min\left(\frac{1}{2}, \frac{\sigma}{||\mathcal{D}||^2}, \alpha, 2\mathcal{L}\right). \tag{2.7}$$

Below we work with this fixed ε. Note that there exists $a_2(\varepsilon) > 0$ (independent of d) such that

$$\kappa(\varepsilon, d) := d \sum \rho_m^2 \leq a_2(\varepsilon) \tag{2.8}$$

for any $d \in (0,1)$. In (2.8) and below,

$$\sum := \sum_{m=-\infty}^{\infty} .$$

For two sequences $v = \{v_m : m \in \mathbb{Z}\}$ and $w = \{w_m : m \in \mathbb{Z}\}$, let

$$\langle v, w \rangle = d \sum < v_m, w_m > .$$

Let H_ρ be the Hilbert space of sequences $v = \{v_m : m \in \mathbb{Z}\}$ with the scalar product

$$\langle v, w \rangle_\rho = \langle v, \rho w \rangle = d \sum \rho_m < v_m, w_m >$$

and the corresponding norm defined by

$$\|v\|_{0,\rho}^2 = \langle v, \rho v \rangle = d \sum \rho_m |v_m|^2.$$

For $y \in \mathbb{Z}$, we define the shift T_y by $(T_y v)_m = v_{y+m}$. Let H_u be the space with the norm

$$\|v\|_{0,u} = \sup_{y \in \mathbb{Z}} \|T_y v\|_{0,\rho}.$$

Let Z_ρ and Z_u be the same spaces H_ρ and H_u but equipped with the norms

$$\|v\|_{1,\rho} = \|v\|_{0,\rho} + \|\partial_- v\|_{0,\rho}$$

and

$$\|v\|_{1,u} = \|v\|_{0,u} + \|\partial_- v\|_{0,u},$$

respectively.

Let us write Eq. (2.2) in the following equivalent form:

$$(I - hA)u^{n+1} = u^n + h\overline{f}(u^{n+1}), \tag{2.9}$$

where I is the identity operator.

Lemma 2.1 *Consider $y \in \mathbb{Z}$ and let $\hat{\rho} = T_{-y}\rho$. For any $v \in H_\rho$, the inequality*

$$\langle Av, \hat{\rho} v \rangle \leq -\left(\sigma - \frac{a_1(\varepsilon)||\mathcal{D}||^2}{2}\right) \|T_y \partial_- v\|_{0,\rho}^2 + \frac{c_3 a_1(\varepsilon)}{2} \|T_y v\|_{0,\rho}^2 \tag{2.10}$$

holds.

Proof. Let us denote

$$R := \langle Av, \hat{\rho}v\rangle = \langle \mathcal{D}\partial_+\partial_- v, \hat{\rho}v\rangle.$$

Applying the usual formula

$$\langle \partial_+ v', v''\rangle = -\langle v', \partial_- v''\rangle,$$

let us transform R as follows:

$$R = -\langle \mathcal{D}\partial_- v, \partial_-(\hat{\rho}v)\rangle = -R_1 - R_2,$$

where

$$R_1 = \langle \mathcal{D}\partial_- v, \hat{\rho}\partial_- v\rangle \quad \text{and} \quad R_2 = \langle \mathcal{D}\partial_- v, (\partial_-\hat{\rho})(T_{-1}v)\rangle.$$

It follows from inequality (2.1) that

$$R_1 \geq \sigma||T_y\partial_- v||_\rho^2. \tag{2.11}$$

The Cauchy inequality and estimates (2.5) and (2.6) imply that

$$|R_2| \leq \frac{da_1(\varepsilon)}{2}\sum\left(|\mathcal{D}(\partial_- v)_m|^2 + |(T_{-1}v)_m|^2\right)(\hat{\rho})_m \leq$$

$$\leq \frac{1}{2}\left(a_1(\varepsilon)||\mathcal{D}||^2||T_y\partial_- v||_\rho^2 + c_3a_1(\varepsilon)||T_y v||_\rho^2\right). \tag{2.12}$$

The statement of our lemma follows from inequalities (2.11) and (2.12). ■

Remark 2.2 *Since inequality (2.7) holds, it follows from Lemma 2.1 that*

$$\langle Av, \rho v\rangle \leq \frac{c_3a_1(\varepsilon)}{2}\|v\|_{0,\rho}^2. \tag{2.13}$$

Lemma 2.3 *Under the assumptions (2.3) and (2.7), equation (2.9) defines a dynamical system $u^{n+1} = \mathcal{S}(u_n), n \in \mathbb{Z}$, on each of the spaces H_ρ, H_u, Z_ρ, and Z_u. Both mappings $\mathcal{S}$ and $\mathcal{S}^{-1}$ are continuous in the respective topologies and commute with shifts, i.e.,*

$$\mathcal{S}\circ T_y = T_y \circ \mathcal{S} \quad \text{and} \quad \mathcal{S}^{-1}\circ T_y = T_y\circ\mathcal{S}^{-1} \quad \text{for} \quad y\in\mathbb{Z}. \tag{2.14}$$

Proof. To study conditions under which the operator $B = I - hA$ is invertible and to estimate the norm of the inverse operator, we apply the theorem stated in the end of Sec. 104 in [RN72]. Denote by A^* and B^* the operators adjoint to A and B with respect to $\langle\cdot,\cdot\rangle_\rho$.

It follows from the above-mentioned theorem that if there exists a positive number μ such that

$$\langle B^*Bv, v\rangle_\rho \geq \mu\langle v, v\rangle_\rho \quad \text{and} \quad \langle BB^*v, v\rangle_\rho \geq \mu\langle v, v\rangle_\rho,$$

then the operator B is invertible.

Let us estimate

$$\langle B^*Bv, v\rangle_\rho = \langle Bv, Bv\rangle_\rho = \langle (I - hA)v, (I - hA)v\rangle_\rho =$$

$$= \langle v, v\rangle_\rho - h\langle v, Av\rangle_\rho - h\langle Av, v\rangle_\rho + h^2\langle Av, Av\rangle_\rho \geq$$

$$\geq \langle v, v\rangle_\rho - 2h\langle v, Av\rangle_\rho.$$

In this estimate, we take into account that $\langle v, Av\rangle_\rho = \langle Av, v\rangle_\rho$ and $h^2\langle Av, Av\rangle_\rho \geq 0$.

It follows from (2.7) that $1 - ha_1(\varepsilon) > 1/2$. Applying estimate (2.13), we obtain the inequality

$$\langle B^*Bv, v\rangle_\rho \geq \frac{1}{2}\langle v, v\rangle_\rho. \tag{2.15}$$

Since

$$\langle BB^*v, v\rangle_\rho = \langle B^*v, B^*v\rangle_\rho = \langle (I - hA^*)v, (I - hA^*)v\rangle_\rho =$$

$$= \langle v, v\rangle_\rho - h\langle v, A^*v\rangle_\rho - h\langle A^*v, v\rangle_\rho + h^2\langle A^*v, A^*v\rangle_\rho$$

and

$$\langle v, Av\rangle_\rho = \langle v, A^*v\rangle_\rho = \langle A^*v, v\rangle_\rho,$$

we see that

$$\langle BB^*v, v\rangle_\rho \geq \frac{1}{2}\langle v, v\rangle_\rho. \tag{2.16}$$

Inequalities (2.15) and (2.16) and the theorem mentioned above imply that the operator B is invertible. If $w = Bv$, then inequality (2.13) implies that

$$\langle w, v\rangle_\rho = \langle Bv, v\rangle_\rho \geq \frac{1}{2}\langle v, v\rangle_\rho = \frac{1}{2}\|v\|^2_{0,\rho}.$$

It follows from the Cauchy inequality that

$$\frac{1}{2}\|v\|^2_{0,\rho} \leq \langle w, v\rangle_\rho \leq \|v\|_{0,\rho}\|w\|_{0,\rho},$$

and we see that

$$\left\|B^{-1}w\right\|_{0,\rho} = \|v\|_{0,\rho} \leq 2\|w\|_{0,\rho}.$$

This means that $\left\|B^{-1}\right\|_{0,\rho} \leq 2$.

Thus, system (2.9) is equivalent to the relations

$$u^{n+1} = Qu^n + hQ\overline{f}(u^{n+1}), \tag{2.17}$$

where $Q = B^{-1}$.

For a fixed $w \in H_\rho$, consider the operator

$$F(v) = Qw + hQ\overline{f}(v). \tag{2.18}$$

Since $\|Q\|_{0,\rho} = \left\|B^{-1}\right\|_{0,\rho} \leq 2$, we have the estimates

$$\|F(v)\|_{0,\rho} \leq 2\|w\|_{0,\rho} + 2h\left(\left\|\overline{f}(0)\right\|_{0,\rho} + \mathcal{L}\|v\|_{0,\rho}\right) \leq$$

$$\leq 2\|w\|_{0,\rho} + 2h(\kappa(\varepsilon, d)|f(0)| + \mathcal{L}\|v\|_{0,\rho}),$$

hence the operator F maps the space H_ρ into itself. Condition (2.3) implies that if $v, v' \in H_\rho$, then

$$\|F(v) - F(v')\|_{0,\rho} \leq 2h\mathcal{L}\|v - v'\|_{0,\rho} \leq \frac{1}{2}\|v - v'\|_{0,\rho},$$

i.e., for any given w, F has a unique fixed point in H_ρ.

Thus, for any given $u^0 \in H_\rho$, relations (2.17) determine a unique sequence $u^n \in H_\rho, n \geq 0$. It is obvious that any given $u^0 \in H_\rho$ also determines a unique sequence $u^n \in H_\rho, n < 0$, satisfying relations (2.9). This defines an invertible mapping $\mathcal{S}$ on the space H_ρ such that $\mathcal{S}(u^n) = u^{n+1}$ for all $n \in \mathbb{Z}$. We are going to show that both $\mathcal{S}$ and $\mathcal{S}^{-1}$ are continuous.

Let us first notice that both A and $\overline{f}$ commute with shifts T_y:

$$A \circ T_y = T_y \circ A \quad \text{and} \quad \overline{f} \circ T_y = T_y \circ \overline{f} \quad \text{for all} \quad y \in \mathbb{Z}, \tag{2.19}$$

from which (2.14) follows. Moreover, together with (2.17) we obtain that

$$\partial_- u^{n+1} = Q\partial_- u^n + \frac{h}{d}Q(\overline{f}(u^{n+1}) - \overline{f}(T_{-1}u^{n+1})).$$

Since

$$\left\|\overline{f}(u^{n+1}) - \overline{f}(T_{-1}u^{n+1})\right\|_{0,\rho}^2 = d\sum \rho_m |f(u_m^{n+1}) - f(u_{m-1}^{n+1})|^2 \leq$$

$$\leq \mathcal{L}^2\left\|u^{n+1} - T_{-1}u^{n+1}\right\|_{0,\rho}^2,$$

we see that

$$\left\|\partial_- u^{n+1}\right\|_{0,\rho} \leq 2\|\partial_- u^n\|_{0,\rho} + \frac{1}{2}\left\|\partial_- u^{n+1}\right\|_{0,\rho}.$$

Hence,

$$\left\|\partial_- u^{n+1}\right\|_{0,\rho} \leq 4\|\partial_- u^n\|_{0,\rho}.$$

This means that the same relations (2.17) define a dynamical system on the space Z_ρ; we denote this system by the same symbol.

Now let us pass to the spaces H_u and Z_u. From the equivariance (2.19) we obtain

$$\left\|T_y(B^{-1}w)\right\|_{0,\rho} = \left\|B^{-1}T_y w\right\|_{0,\rho} \le 2\|T_y w\|_{0,\rho} \le 2\|w\|_{0,u} \quad \text{for} \quad y \in \mathbb{Z},$$

and hence $\left\|B^{-1}\right\|_{0,u} \le 2$.

The same reasoning as above shows that relations (2.2) generate dynamical systems on the spaces H_u and Z_u. We denote these systems by the same symbol $\mathcal{S}$ which will not lead to confusion. We claim that the operator $\mathcal{S}$ is continuous in all our spaces. Take $w, w' \in H_\rho$ and let $v = \mathcal{S}(w)$ and $v' = \mathcal{S}(w')$. Since

$$v = Qw + hQ\overline{f}(v) \quad \text{and} \quad v' = Qw' + hQ\overline{f}(v'), \tag{2.20}$$

we see that

$$\|v - v'\|_{0,\rho} \le 2\|w - w'\|_{0,\rho} + \frac{1}{2}\|v - v'\|_{0,\rho},$$

hence

$$\|\mathcal{S}(w) - \mathcal{S}(w')\|_{0,\rho} = \|v - v'\|_{0,\rho} \le 4\|w - w'\|_{0,\rho}. \tag{2.21}$$

A similar estimate holds in the space H_u.

The obvious estimate

$$\|v\|_{1,\rho} \le \left(1 + \frac{2}{d}\right)\|v\|_{0,\rho}$$

implies the continuity of $\mathcal{S}$ in the spaces Z_ρ and Z_u.

To establish the continuity of $\mathcal{S}^{-1}$ generated by the mapping

$$v \mapsto (I - hA)v - h\overline{f}(v),$$

we apply the same reasoning and the estimate

$$|f(v_m)| \le |f(0)| + \mathcal{L}|v_m|.$$

■

3 The absorbing set

Now we show that the system $\mathcal{S}$ has a bounded absorbing set in the space Z_u.

First we prove a variant of the discrete Gronwall lemma.

Lemma 3.1 *Let $a_n, b_n, n \geq 0$, be two sequences of nonnegative numbers. Assume that there exist positive numbers H and Λ such that the inequalities*

$$\frac{a_{n+1} - a_n}{H} + a_{n+1} + b_{n+1} \leq \Lambda \tag{3.1}$$

and

$$\frac{b_{n+1} - b_n}{H} \leq \Lambda(1 + b_{n+1}) \tag{3.2}$$

hold for $n \geq 0$. Then the following estimates are satisfied for $n \geq 0$:

$$a_n \leq \Lambda + a_0, \quad b_n \leq \Gamma(\Lambda, H) + a_0(\Lambda + 1) + b_0, \tag{3.3}$$

where $\Gamma(\Lambda, H) = \Lambda[(1+\Lambda)(1+H)+1]$. Moreover, for any $\delta > 0$ and $M > 0$ there exists some $T_0 = T_0(\delta, M, \Lambda) > 0$ such that

$$a_n, b_n \leq \Gamma(\Lambda, H) + \delta \quad \text{for} \quad nH \geq T_0, \tag{3.4}$$

if $a_0, b_0 \leq M$.

Proof. Consider the number $\lambda = 1 + H$ and note that

$$\left(\frac{a_{n+1} - a_n}{H} + a_{n+1}\right) \lambda^n = \frac{\lambda^{n+1} a_{n+1} - \lambda^n a_n}{H}.$$

Let us multiply inequality (3.1) by λ^n and sum the obtained inequalities for $n = 0, \ldots, k$. We get the following inequality:

$$\frac{\lambda^{k+1} a_{k+1} - a_0}{H} + \sum_{n=0}^{k} \lambda^n b_{n+1} \leq \Lambda \frac{\lambda^{k+1} - 1}{H}. \tag{3.5}$$

Since $b_n \geq 0$, we have the estimate

$$\lambda^{k+1} a_{k+1} \leq \Lambda(\lambda^{k+1} - 1) + a_0,$$

and hence the estimate

$$a_{k+1} \leq \Lambda + \frac{a_0}{\lambda^{k+1}}. \tag{3.6}$$

Since $a_{k+1} \geq 0$, it follows from (3.5) that

$$\sum_{n=0}^{k} \lambda^n b_{n+1} \leq \Lambda \frac{\lambda^{k+1} - 1}{H} + \frac{a_0}{H}. \tag{3.7}$$

Now we substitute (3.1) into (3.2):

$$\frac{b_{n+1} - b_n}{H} \leq \Lambda \left(1 + \Lambda - \frac{a_{n+1} - a_n}{H} - a_{n+1}\right).$$

Let us write the latter inequality as follows:

$$\Lambda\left(\frac{a_{n+1}-a_n}{H}+a_{n+1}\right)+\frac{b_{n+1}-b_n}{H}\leq\Delta(\Lambda), \tag{3.8}$$

where $\Delta(\Lambda)=\Lambda(1+\Lambda)$.

Let us multiply inequality (3.8) by λ^n and sum the obtained inequalities for $n=0,\dots,k$. We get the following inequality:

$$\Lambda\frac{\lambda^{k+1}a_{k+1}-a_0}{H}+\frac{\lambda^k b_{k+1}}{H}-\sum_{n=0}^{k-1}\lambda^n b_{n+1}-\frac{b_0}{H}\leq$$

$$\leq\Delta(\Lambda)\frac{\lambda^{k+1}-1}{H}.$$

Now we substitute estimate (3.7) with k replaced by $k-1$ into the latter inequality and get the following estimate (taking into account that $a_n\geq 0$):

$$\lambda^k b_{k+1}\leq\Delta(\Lambda)(\lambda^{k+1}-1)+\Lambda(\lambda^k-1)+a_0(\Lambda+1)+b_0.$$

Thus, we get the estimate

$$b_{k+1}\leq\Delta(\Lambda)\lambda+\Lambda+\frac{a_0(\Lambda+1)+b_0}{\lambda^k}. \tag{3.9}$$

Estimates (3.6) and (3.9) prove (3.3) and since $\lambda^k\geq 1+kH$ we obtain (3.4) by taking kH sufficiently large. ■

Denote by $B^R_{1,u}$ the closed ball of radius R in the space Z_u.

Lemma 3.2 *There exists a number $K>0$ with the following property: for any $M>0$ there exists $T_0=T_0(M)$ such that if $u^0\in B^M_{1,u}$, then $u^n\in B^K_{1,u}$ for $nh\geq T_0$.*

Proof. The idea is to apply Lemma 3.1 to the sequences $a_n=\|T_y u^n\|^2_{0,\rho}$ and $b_n=\|T_y\partial_- u^n\|^2_{0,\rho}$ for any given $y\in\mathbb{Z}$. First, fix any $u\in H_\rho$ and let $v=\mathcal{S}(u)$. Take the scalar products of both sides of the equality

$$\frac{1}{h}(v-u)=Av+\overline{f}(v) \tag{3.10}$$

with $\hat\rho v$, where $\hat\rho=T_{-y}\rho$. Note that $\|T_y v\|^2_{0,\rho}=\langle v,\hat\rho v\rangle$. For the left-hand side product

$$L:=\frac{1}{h}\langle v-u,\hat\rho v\rangle=\frac{1}{2h}\langle v-u,\hat\rho(v-u+v+u)\rangle,$$

we have the following expression:

$$L=\frac{\|T_y v\|^2_{0,\rho}-\|T_y u\|^2_{0,\rho}}{2h}+\frac{\|T_y v-T_y u\|^2_{0,\rho}}{2h}. \tag{3.11}$$

We estimate the first term of the right-hand side of the scalar product, $R := \langle Av, \hat{\rho}v\rangle$, applying Lemma 2.1.

Let us estimate the second term of the right-hand side of our product. Due to our growth condition (1.4),

$$\langle \overline{f}(v), \hat{\rho}v\rangle = d\sum \hat{\rho}_m < f(v_m), v_m > \leq$$

$$\leq \beta d \sum \hat{\rho}_m - d\alpha \sum \hat{\rho}_m |v_m|^2 = \beta\kappa(\varepsilon, d) - \alpha\|T_y v\|^2_{0,\rho}.$$

Combining the latter estimate with (3.11) and (2.10), we get the following estimate:

$$\frac{\|T_y v\|^2_{0,\rho} - \|T_y u\|^2_{0,\rho}}{2h} \leq$$

$$\leq -\left(\sigma - \frac{a_1(\varepsilon)\|\mathcal{D}\|^2}{2}\right)\|T_y\partial_- v\|^2_{0,\rho} + \left(\frac{c_3 a_1(\varepsilon)}{2} - \alpha\right)\|T_y v\|^2_{0,\rho} + \beta\kappa(\varepsilon, d).$$

By our condition (2.7), the inequalities

$$\frac{\|T_y v\|^2_{0,\rho} - \|T_y u\|^2_{0,\rho}}{2h} \leq -\frac{\alpha}{2}\|T_y v\|^2_{0,\rho} - \frac{\sigma}{2}\|T_y \partial_- v\|^2_{0,\rho} + \beta\kappa(\varepsilon, d) \qquad (3.12)$$

hold.

Now let us consider any positive trajectory $\{u^n : n \geq 0\}$ of our system $\mathcal{S}$. Taking $v = u^{n+1}$ and $u = u^n$ in our considerations above, we see that the inequalities

$$\frac{\left\|T_y u^{n+1}\right\|^2_{0,\rho} - \|T_y u^n\|^2_{0,\rho}}{2h} + \frac{\alpha}{2}\left\|T_y u^{n+1}\right\|^2_{0,\rho} + \frac{\sigma}{2}\left\|T_y \partial_- u^{n+1}\right\|^2_{0,\rho} \leq \beta\kappa(\varepsilon, d) \qquad (3.13)$$

are valid for any $y \in \mathbb{Z}$ and $n \geq 0$.

We again fix $u \in H_\rho$, $y \in \mathbb{Z}$, and $v = \mathcal{S}(u)$. As previously, we denote $\hat{\rho} = T_{-y}\rho$. Now we take the scalar products of both sides of equality (3.10) with $-\hat{\rho}\partial_+\partial_- v$.

Let us first consider the left-hand side L of the obtained equality:

$$L = -\frac{1}{h}\langle \hat{\rho}(v - u), \partial_+\partial_- v\rangle =$$

$$= \frac{1}{h}\langle \partial_-\left(\hat{\rho}(v-u)\right), \partial_- v\rangle = \frac{1}{h}\langle \hat{\rho}\partial_-(v-u) + (\partial_-\hat{\rho})T_{-1}(v-u), \partial_- v\rangle.$$

We see that $L = L_1 + L_2$, where

$$L_1 = \frac{1}{h}\langle \hat{\rho}\partial_-(v-u), \partial_- v\rangle =$$

$$= \frac{1}{2h}\langle \hat{\rho}(\partial_- v - \partial_- u), \partial_- v - \partial_- u + \partial_- v + \partial_- u\rangle =$$

$$= \frac{\|T_y\partial_- v\|^2_{0,\rho} - \|T_y\partial_- u\|^2_{0,\rho}}{2h} + \frac{\|T_y(\partial_- v - \partial_- u)\|^2_{0,\rho}}{2h} \tag{3.14}$$

and

$$L_2 = \frac{1}{h}\langle(\partial_-\hat{\rho})T_{-1}(v-u), \partial_- v\rangle. \tag{3.15}$$

The first term on the right in the obtained equality is

$$R_1 := -\langle\mathcal{D}\partial_+\partial_- v, \hat{\rho}\partial_+\partial_- v\rangle \leq -\sigma\|T_y\partial_+\partial_- v\|^2_{0,\rho}. \tag{3.16}$$

Let us consider the second term on the right,

$$R_2 := -\langle\overline{f}(v), \hat{\rho}\partial_+\partial_- v\rangle = \langle\partial_-(\hat{\rho}\overline{f}(v)), \partial_- v\rangle.$$

We represent this term in the form $R_2 = R_{2,1} + R_{2,2}$, where

$$R_{2,1} = \langle\hat{\rho}\partial_-\overline{f}(v), \partial_- v\rangle = d\sum\hat{\rho}_m\frac{1}{d^2} < f(v_m) - f(v_{m-1}), v_m - v_{m-1} >$$

and

$$R_{2,2} = \langle\partial_-(\hat{\rho})(T_1\overline{f}(v)), \partial_- v\rangle.$$

It was assumed that f is Lipschitz continuous with constant $\mathcal{L}$, hence

$$R_{2,1} \leq \mathcal{L}\|T_y\partial_- v\|^2_{0,\rho}. \tag{3.17}$$

Taking equality (3.15) into account, we conclude that

$$R' := R_{2,2} - L_2 = d\sum(\partial_-\hat{\rho})_m\left[f(v_{m-1}) - \frac{v_{m-1} - u_{m-1}}{h}\right](\partial_- v)_m.$$

Since

$$f(v_{m-1}) - \frac{1}{h}(v_{m-1} - u_{m-1}) = -AT_{-1}v,$$

we see that

$$R' = -\langle\partial_-\hat{\rho}(AT_{-1}v), \partial_- v\rangle = \langle\mathcal{D}\partial_+\partial_-(T_{-1}v), \partial_-\hat{\rho}\partial_- v\rangle.$$

Similarly to (2.12), we get the estimate

$$|R'| \leq \frac{1}{2}\left(c_3 a_1(\varepsilon)||\mathcal{D}||^2\|T_y\partial_+\partial_- v\|^2_{0,\rho} + a_1(\varepsilon)\|T_y\partial_- v\|^2_{0,\rho}\right). \tag{3.18}$$

Combining estimates (3.16)–(3.18), we obtain the following inequality:

$$\frac{\|T_y\partial_- v\|^2_{0,\rho} - \|T_y\partial_- u\|^2_{0,\rho}}{2h} \leq \left(-\sigma + \frac{1}{2}c_3 a_1(\varepsilon)||\mathcal{D}||^2\right)\|T_y\partial_+\partial_- v\|^2_{0,\rho} +$$

$$+\left(\mathcal{L} + \frac{1}{2}a_1(\varepsilon)\right)\|T_y\partial_- v\|^2_{0,\rho}.$$

It follows from our condition (2.7) that

$$\frac{\|T_y\partial_- v\|_{0,\rho}^2 - \|T_y\partial_- u\|_{0,\rho}^2}{2h} \le 2\mathcal{L}(1 + \|T_y\partial_- v\|_{0,\rho}^2).$$

We again consider a positive trajectory $\{u^n : n \ge 0\}$ of our system $\mathcal{S}$ and take $v = u^{n+1}$ and $u = u^n$. We obtain the inequalities

$$\frac{\left\|T_y\partial_- u^{n+1}\right\|_{0,\rho}^2 - \|T_y\partial_- u^n\|_{0,\rho}^2}{2h} \le 2\mathcal{L}(1 + \left\|T_y\partial_- u^{n+1}\right\|_{0,\rho}^2). \qquad (3.19)$$

Now let us complete the proof of our lemma. Set

$$\eta = \frac{1}{2}\min(\sigma, \alpha) \quad \text{and} \quad H = 2\eta h.$$

Let

$$\Lambda = \frac{1}{\eta}\max(\mathcal{L}, \beta\kappa(\varepsilon, d))$$

and note that

$$\Lambda \le \frac{1}{\eta}\max(\mathcal{L}, \beta a_2(\varepsilon))$$

by (2.8).

Fix $M > 0$, take a point $u^0 \in B_{1,u}^M$, and consider the corresponding positive trajectory $\{u^n\}$ of our system $\mathcal{S}$. For a fixed $y \in \mathbb{Z}$, let

$$a_n = \|T_y u^n\|_{0,\rho}^2 \quad \text{and} \quad b_n = \|T_y\partial_- u^n\|_{0,\rho}^2 \qquad (3.20)$$

and note that $a_0 + b_0 \le M^2$.

Inequalities (3.13) and (3.19) imply that the sequences a_n and b_n satisfy the conditions of Lemma 3.1, hence there exists $T_0 = T_0(M)$ such that

$$a_n, b_n \le K_1^2 := \Lambda[(\Lambda + 1)(1 + 2\eta) + 1] + 1$$

for $nh \ge T_0$. Since y is arbitrary, this means that

$$\|u^n\|_{0,u}, \|\partial_- u^n\|_{0,u} \le K_1$$

for $nh \ge T_0$. Our lemma is proved with $K = 2K_1$. ∎

Remark 3.3 *If we fix the number γ in the definition of the weight ρ, then relations (2.7) show that our choice of ε depends only on $\sigma, ||\mathcal{D}||, \mathcal{L}$, and α. Analyzing the proof of Lemma 3.2, we see that the number K (the radius of the absorbing ball in the space Z_u) depends only on γ, σ, $||\mathcal{D}||$, $\mathcal{L}$, α, and β (below we call these numbers characteristics of the system $\mathcal{S}$). We show in our main theorem that the size of the global attractor of $\mathcal{S}$ in Z_u is determined by these characteristics.*

4 Compactness

The following, almost obvious, statement holds.

Lemma 4.1 *If B is a bounded subset of the space Z_u, then B is precompact in Z_ρ.*

Proof. Let $B^* = \text{clos}_{1,\rho} B$. We claim that the set B^* is compact in Z_ρ.

First we claim that the set B^* is bounded in Z_u. Indeed, there exists a number $N > 0$ such that $\|w\|_{1,u} \le N$ for any $w \in B$.

Fix any $v \in B^*$ and find a sequence $\{v^n\} \subset B$ such that

$$\|v^n - v\|_{0,\rho} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Take $y \in \mathbb{Z}$ such that

$$\|v\|_{0,u} < \|T_y v\|_{0,\rho} + 1.$$

It follows from (4.1) that

$$\|T_y v^n - T_y v\|_{0,\rho} \to 0 \quad \text{as} \quad n \to \infty,$$

hence there exists n_0 such that

$$\|T_y v^n - T_y v\|_{0,\rho} < 1$$

for $n \ge n_0$. Recall that $\|v^n\|_{1,u} \le N$, hence if $n \ge n_0$, then

$$\|v\|_{0,u} < \|T_y v^n\|_{0,\rho} + 2 \le \|v^n\|_{0,u} + 2 \le N + 2. \tag{4.2}$$

The same arguments applied to $\|\partial_- v\|_{0,u}$ prove that the set B^* is bounded in Z_u.

Now let us consider a sequence $\{v^n\} \subset B^*$, where $v^n = \{v^n_m : m \in \mathbb{Z}\}$. We can find a subsequence (denoted again $\{v^n\}$) and an element $v = \{v_m : m \in \mathbb{Z}\}$ such that $v^n_m \to v_m$ as $n \to \infty$ for any $m \in \mathbb{Z}$. Estimates (4.2) imply that there exists a number $M > 0$ such that

$$|v^n_m| \le M \quad \text{for} \quad n \ge 0, m \in \mathbb{Z},$$

and hence

$$|v_m| \le M \quad \text{for} \quad m \in \mathbb{Z}.$$

It follows that for any $\delta > 0$ there exists $R = R(\delta)$ such that

$$d \sum_{|m|>R} \rho_m \left[|v^n_m - v_m|^2 + |\partial_- v^n - \partial_- v|^2_m \right] < \delta.$$

Since the set $\{m : |m| \le R\}$ is finite, there exists $n_1 = n_1(\delta)$ such that

$$d \sum_{|m|\le R} \rho_m \left[|v^n_m - v_m|^2 + |\partial_- v^n - \partial_- v|^2_m \right] < \delta$$

for $n \geq n_1$. The relations above mean that

$$\|v^n - v\|_{1,\rho} \to 0$$

as $n \to \infty$. Our lemma is proved. ∎

Corollary 4.2 *Let B be a bounded subset of the space Z_u. For any $t \geq 0$, the set $B' = \mathcal{S}^t(B)$ is precompact in Z_ρ.*

Proof. Since our system $\mathcal{S}$ has an absorbing set in Z_u, it follows from the continuity of $\mathcal{S}$ established above that, for any $t \geq 0$, the set $B' = \mathcal{S}^t(B)$ is bounded in Z_u. ∎

5 The global attractor

Now we establish the main result.

Theorem 5.1 *I. Under our assumptions, the system $\mathcal{S}$ has a global (Z_u, Z_ρ)-attractor $\mathcal{A}$, i.e., the following statements hold:*

(1) $\mathcal{A}$ is nonempty, closed, bounded in Z_u, and compact in Z_ρ;

(2) $\mathcal{A}$ is invariant under $\mathcal{S}$, i.e., $\mathcal{S}^t(\mathcal{A}) = \mathcal{A}$ for any $t \in \mathbb{Z}$;

(3) $\mathcal{A}$ attracts any bounded subset B of Z_u with respect to the distance induced by the norm of Z_ρ, i.e.,

$$\mathrm{dist}_{1,\rho}(\mathcal{S}^t(B), \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty,$$

where

$$\mathrm{dist}_{1,\rho}(A, A') = \sup_{v \in A} \inf_{w \in A'} \|v - w\|_{1,\rho}.$$

II. For any h and d, the attractor $\mathcal{A}$ belongs to the closed ball $B^K_{1,u}$ of radius N in the space Z_u, where the number K (depending only on $\gamma, \sigma, ||\mathcal{D}||, \mathcal{L}, \alpha$, and β) was introduced in Lemma 3.2.

III. The attractor $\mathcal{A}$ is translationally invariant, and an analog of statement I(3) holds for the metric

$$dist^*_{1,\rho}(A, A') = \sup_{y \in \mathbb{Z}} \sup_{v \in A} \inf_{w \in A'} \|T_y v - T_y w\|_{1,\rho}.$$

Proof. The proof mostly repeats the proof of Theorem 2.6 in [MS95]. It is shown in [MS95] that statements I (in a slightly different form, see below) and III of our theorem are implied by the following conditions:

(c1) the system $\mathcal{S}$ is translationally invariant (i.e., $T_y\mathcal{S} = \mathcal{S}T_y$ for any $y \in \mathbb{Z}$) and continuous in the spaces Z_ρ and Z_u;

(c2) $\mathcal{S}$ has a nonempty, bounded, and positively invariant absorbing set $\mathcal{B}$ in Z_u;

(c3) for any $B \subset \mathcal{B}$, there exists $\tau > 0$ such that the set $\mathcal{S}^\tau(B)$ is precompact in Z_ρ.

Condition (c1) is satisfied due to Lemma 2.3. Condition (c3) follows from Corollary 4.2.

The only difference between the assumptions of our theorem and Theorem 2.6 in [MS95] is the following one: in (c2), it is assumed that there exists a positively invariant absorbing set, while the absorbing set constructed in Lemma 3.2 above is not necessarily positively invariant. The only difference in our conclusions is $t \in \mathbb{Z}$ instead of $t > 0$ in statement I(2) (it is easy to understand that this difference is due to the following reason: our system $\mathcal{S}$ is invertible, while the system in [MS95] is not necessarily invertible).

It is well known that having an absorbing set, it is easy to construct a positively invariant absorbing set. Indeed, let $\mathcal{B} = B^K_{1,u}$ be the absorbing set for the system $\mathcal{S}$ given by Lemma 3.2. There exists $T_0 > 0$ (depending only on K, i.e., on the characteristics of our system) such that

$$\mathcal{S}^\tau(\mathcal{B}) \subset \mathcal{B} \quad \text{for} \quad \tau h \geq T_0.$$

Consider the set

$$\mathcal{B}' = \bigcup_{0 \leq \tau h \leq T_0} \mathcal{S}^\tau(\mathcal{B}).$$

It follows from (3.3) (when applied to the sequences from (3.20)) that there exists a number Q (depending only on the characteristics of $\mathcal{S}$) such that $\mathcal{B}' \subset B^Q_{1,u}$. Since $\mathcal{B} \subset \mathcal{B}'$, the set $\mathcal{B}'$ is absorbing and by construction it is also positively invariant. Following [MS95], we define the global attractor for $\mathcal{S}$ by the formula

$$\mathcal{A} = \bigcap_{t \geq 0} \sigma_t,$$

where

$$\sigma_t = \operatorname{clos}_{1,\rho} \mathcal{S}^t(\mathcal{B}').$$

Since $B^K_{1,u}$ is an absorbing set, $\sigma_t \subset B^K_{1,u}$ for large t, this implies statement II of our theorem.

It remains to show that $\mathcal{S}^t(\mathcal{A}) = \mathcal{A}$ for any $t \in \mathbb{Z}$. For this, it is enough to show that

$$\mathcal{S}(v) \in \mathcal{A} \quad \text{and} \quad \mathcal{S}^{-1}(v) \in \mathcal{A} \tag{5.1}$$

for $v \in \mathcal{A}$.

Take $v \in \mathcal{A}$ and consider an arbitrary $t \geq 0$. We claim that

$$\mathcal{S}(v) \in \sigma_t \quad \text{and} \quad \mathcal{S}^{-1}(v) \in \sigma_t. \tag{5.2}$$

Since t is arbitrary, the inclusions above and the definition of $\mathcal{A}$ prove inclusions (5.1). The set $\mathcal{B}'$ is positively invariant, hence $\sigma_{t+1} \subset \sigma_t$, and we may assume that $t \geq 1$.

Since $v \in \mathcal{A}$, we have the inclusion $v \in \sigma_{t-1}$. There exists a sequence $v_k \in \mathcal{B}'$ such that $\mathcal{S}^{t-1} v_k \to v$ in the space Z_ρ. Since $\mathcal{S}$ is continuous in Z_ρ (see Lemma 2.3), $\mathcal{S}^t(v_k) \to \mathcal{S}(v)$, hence the first inclusion in (5.2) holds.

Similar arguments applying the continuity of $\mathcal{S}^{-1}$ prove the second inclusion in (5.2). ∎

Remark 5.2 *Notice that according to Lemma 3.2 the absorbing ball $B^K_{1,u}$ attracts bounded sets in Z_u after times that are uniform in h and d but we are unable to prove that, given a neighborhood of the attractor and a bounded set, there is a uniform bound for the corresponding attraction time.*

6 Upper semicontinuity of attractors on finite segments

For the sake of presentation, here and below we consider the scalar equation (1.1) (and note that similar results hold also for systems).

Fix $h, d \in (0,1)$ and a natural number N. Consider the tridiagonal $(2N+1) \times (2N+1)$ matrix

$$A_N = \frac{1}{d^2} \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix}.$$

Let

$$v = (v_{-N}, \dots, v_N) \in \mathbb{R}^{2N+1} \tag{6.1}$$

and let f be a function satisfying the basic conditions formulated in Sec. 2.

Consider the implicit scheme

$$\frac{v^{n+1} - v^n}{h} = A_N v^{n+1} + \overline{f}(v^{n+1}), \tag{6.2}$$

where $v^n = (v^n_{-N}, \dots, v^n_N) \in \mathbb{R}^{2N+1}$ and $\overline{f}(v) = (f(v_{-N}), \dots, f(v_N))$ for $v = (v_{-N}, \dots, v_N) \in \mathbb{R}^{2N+1}$.

Scheme (6.2) corresponds to a discretization of the parabolic equation (1.1) on the segment $[-(N+1)d, (N+1)d]$ with the Dirichlet boundary conditions.

For a vector v of the form (6.1), we set

$$v^e = (\dots, 0, v_{-N}, \dots, v_N, 0, \dots),$$

i.e., we set $v^e_m = v_m$ for $|m| \le N$ and $v^e_m = 0$ for $|m| > N$. Let $\mathcal{H}_N$ be the set of sequences v^e.

We denote

$$\sum{}' = \sum_{|m|\le N}.$$

Let $\rho = \{\rho_m\}$ be the weight introduced above. For a vector $v \in \mathbb{R}^{2N+1}$, we introduce the values

$$\|v\|_{0,\rho} = \|v^e\|_{0,\rho}, \quad \|v\|_{0,u} = \|v^e\|_{0,u},$$

and so on. For $u, v \in \mathbb{R}^{2N+1}$, the scalar product is defined by the formula

$$\langle u, v\rangle_N = d\sum{}' u_m v_m.$$

Further, we set

$$\langle \partial_- u, \partial_- v\rangle_N = d \sum_{m=-N+1}^{N} (\partial_- u)_m (\partial_- v)_m$$

and define the norms $\|v\|_{1,\rho}$ and $\|v\|_{1,u}$ analogously.

Lemma 6.1 *Let $y \in \mathbb{Z}$ and let $\hat\rho = T_y\rho$. For any $v \in \mathcal{H}_N$, the inequality*

$$\langle A_N v, \hat\rho v\rangle_N \le \frac{c_3 a_1(\varepsilon)}{2}\langle v, \hat\rho v\rangle_N \tag{6.3}$$

holds.

Proof. First we note that if $v^e \in \mathcal{H}_N$ corresponds to $v \in \mathbb{R}^{2N+1}$, then $(Av^e)_m = (A_N v)_m$ for $|m| \le N$. Since $v^e_m = 0$ for $|m| > N$, the following relations hold:

$$\langle Av^e, \hat\rho v^e\rangle = d\sum \hat\rho_m (Av^e)_m v^e_m =$$

$$= d\sum{}' \hat\rho_m (A_N v)_m v_m = \langle A_N v, \hat\rho v\rangle_N.$$

On the other hand, it follows from Remark 2.2 that

$$\langle Av^e, \hat\rho v^e\rangle \le \frac{c_3 a_1(\varepsilon)}{2}\langle v^e, \hat\rho v^e\rangle = \frac{c_3 a_1(\varepsilon)}{2}\langle v, \hat\rho v\rangle_N.$$

■

Now the same reasoning as in Sec. 2 and estimate (6.3) show that both mappings S_N and S_N^{-1} are continuous, hence scheme (6.2) defines a dynamical system S_N on $\mathbb{R}^{2N+1}$.

Lemma 6.2 *There exists a number $\Delta > 0$ (independent of N) having the following property: for any system S_N, its global attractor $\mathcal{A}_N$ belongs to the ball*

$$\|v\|_{1,u} \le \Delta.$$

Proof. One can repeat the proof of Lemma 3.2 and Theorem 5.1; we refer to shorter arguments since h and d are fixed (and we do not have to show that Δ is independent of h and d) and the system S_N is finite-dimensional.

Consider $w \in \mathbb{R}^{2N+1}$, let $v = S_N(w)$, fix $y \in \mathbb{Z}$, and denote $\hat{\rho} = T_y \rho$. Take the scalar product of both parts of the equality

$$\frac{1}{h}(v - w) = A_N v + \overline{f}(v)$$

with $\hat{\rho} v$.

The left-hand side of the scalar product equals

$$\frac{1}{2h}\langle v - w, \hat{\rho}(v - w + v + w)\rangle_N =$$

$$= \frac{1}{2h}[\langle v, \hat{\rho} v\rangle)_N - \langle w, \hat{\rho} w\rangle)_N + d{\sum}' \hat{\rho}_m (v_m - w_m)^2].$$

We estimate the term $\langle A_N v, \hat{\rho} v\rangle_N$ using inequality (6.3). For the second term of the right-hand side of the scalar product, the following estimates hold:

$$\langle \overline{f}(v), \hat{\rho} v\rangle_N \leq d{\sum}' \hat{\rho}_m \beta - d\alpha {\sum}' \hat{\rho}_m v_m^2 \leq$$

$$\leq \kappa(\varepsilon, d)\beta - \alpha\langle v, \hat{\rho} v\rangle_N.$$

Hence, if we take $v \in \mathbb{R}^{2N+1}$ and $n \geq 0$ and set $v^n = S_N^n(v)$ and $a_n = \langle v^n, \hat{\rho} v^n\rangle_N$, then the inequality

$$\frac{a_{n+1} - a_n}{2h} \leq -\frac{\alpha}{2} a_{n+1} + \kappa(\varepsilon, d)\beta$$

holds. It follows that there exists a number Δ' (depending only on h and f) such that

$$\limsup a_n \leq \Delta'.$$

Since there exists a constant C' such that

$$\langle \partial_- v^n, \hat{\rho} \partial_- v^n\rangle_N \leq \frac{C'}{d^2}\langle v^n, \hat{\rho} v^n\rangle_N,$$

and y is arbitrary, our statement is proved. ■

Let $\mathcal{A}_N$ be the global attractor of the system S_N, denote

$$\mathcal{A}_N^* = \{v^e \in \mathcal{H}_N : v \in \mathcal{A}_N\}.$$

Theorem 6.3 *(Upper semicontinuity in Z_ρ). The relation*

$$\operatorname{dist}_{1,\rho}(\mathcal{A}_N^*, \mathcal{A}) \to 0 \tag{6.4}$$

holds as $N \to \infty$.

Remark 6.4 *Note that in the statement of Theorem 6.3, the step size d is assumed to be fixed while we let N tend to* ∞.

Proof. Since the attractor $\mathcal{A}$ is compact in Z_ρ, it is enough to show that for any sequence $v_N \in \mathcal{A}_N$ there exists a subsequence v_{N_k} such that

$$\mathrm{dist}_{1,\rho}(v^e_{N_k}, \mathcal{A}) \to 0. \tag{6.5}$$

Lemma 6.2 implies that the sequence v^e_N belongs to a bounded subset of the space Z_u, hence it follows from Lemma 4.1 that v^e_N has a convergent subsequence in Z_ρ. To simplify notation, we assume that

$$v^e_N \to v \quad \text{in} \quad Z_\rho. \tag{6.6}$$

To prove relation (6.5), it is enough to show that

$$v \in \mathcal{A}. \tag{6.7}$$

It follows from (6.6) that $v \in Z_\rho$. Let us show (compare with [BV90]) that if there exists a bounded subset B of the space Z_u such that $\mathcal{S}^n(v) \in B$ for $n \leq 0$, then inclusion (6.7) holds. Indeed, if $v \notin \mathcal{A}$, then the compactness of $\mathcal{A}$ in Z_ρ implies that

$$\mathrm{dist}_{1,\rho}(v, \mathcal{A}) = a > 0.$$

On the other hand, there exists n such that

$$\mathrm{dist}_{1,\rho}(\mathcal{S}^n(B), \mathcal{A}) < \frac{a}{2}.$$

Since

$$v = \mathcal{S}^n(\mathcal{S}^{-n}(v)) \in \mathcal{S}^n(B),$$

we have the inequality

$$\mathrm{dist}_{1,\rho}(v, \mathcal{A}) < \frac{a}{2}$$

leading to a contradiction.

Now let us assume that the sequence $\|\mathcal{S}^n(v)\|_{1,u}$ is unbounded for $n \leq 0$. Fix a number $n < 0$ such that

$$\|\mathcal{S}^n(v)\|_{1,u} \geq 3\Delta \tag{6.8}$$

(where the number Δ is given by Lemma 6.2). In this case, there exists $y \in \mathbb{Z}$ such that

$$\|T_y \mathcal{S}^n(v)\|_{1,\rho} \geq 2\Delta. \tag{6.9}$$

The rest of the proof is based on the following statement.

Lemma 6.5 *If $u_N \in \mathbb{R}^{2N+1}$ is a sequence such that $u_N^e \to u$ in Z_ρ as $N \to \infty$, then the inverse images $w_N = S_N^{-1}(u_N)$ satisfy the relation $w_N^e \to \mathcal{S}^{-1}(u)$ in Z_ρ.*

Proof. Since the mapping $\mathcal{S}^{-1}$ maps the space Z_ρ into itself (see Lemma 2.3), there exists $w \in Z_\rho$ such that $\mathcal{S}(w) = u$. In this case,

$$w_m = -\frac{h}{d^2}u_{m+1} + \left(1 + \frac{2h}{d^2}\right) u_m - \frac{h}{d^2}u_{m-1} - hf(u_m).$$

The vectors u_N and w_N satisfy the equality

$$w_N = (I - hA_N)u_N - h\overline{f}(u_N),$$

hence

$$w_{N,m} = -\frac{h}{d^2}u^e_{N,m+1} + \left(1 + \frac{2h}{d^2}\right) u^e_{N,m} - \frac{h}{d^2}u^e_{N,m-1} - hf(u_{N,m})$$

for $|m| \le N$. Introducing the restriction $w^r \in \mathbb{R}^{2N+1}$ by $w^r_m = w_m$ for $|m| \le N$ and subtracting the equation for w_m, we see that

$$\|w_N - w^r\|^2_{0,\rho} = d{\sum}' \rho_m[-\frac{h}{d^2}(u^e_{N,m+1} - u_{m+1})+$$

$$+\left(1 + \frac{2h}{d^2}\right)(u^e_{N,m} - u_m) - \frac{h}{d^2}(u^e_{N,m-1} - u_{m-1}) - hf(u_{N,m}) + hf(u_m)]^2.$$

It follows that

$$\|w^e_N - w\|^2_{0,\rho} \le C\|u^e_N - u\|^2_{0,\rho} + \sum_{|m|>N} \rho_m |w_m|^2, \tag{6.10}$$

where the constant C depends only on h, d, and f but not on N.

Since

$$\|\partial_-(w^e_N - w)\|_{0,\rho} \le \frac{1 + c_3}{d}\|w^e_N - w\|_{0,\rho}, \tag{6.11}$$

the statement of our lemma follows from estimates (6.10) and (6.11). ■

Let $w^n_N = S^n_N(v_N)$ and $w = \mathcal{S}^n(v)$, where the v_N and v satisfy (6.6) and the number n has property (6.8). Applying Lemma 6.5 inductively ($|n|$ times), we see that

$$w_N^{n,e} \to w$$

in Z_ρ as $N \to \infty$. Let $y \in \mathbb{Z}$ be such that inequality (6.9) holds. Obviously,

$$T_y w_N^{n,e} \to T_y w \tag{6.12}$$

in Z_ρ as $N \to \infty$.

The vectors w^n_N belong to the attractors $\mathcal{A}_N$, hence it follows from Lemma 6.2 that

$$\left\|T_y w_N^{n,e}\right\|_{1,\rho} \le \Delta.$$

The latter inequality, relation (6.12), and inequality (6.9) are contradictory if N is large enough. Theorem 6.3 is proved. ■

7 Embedded dynamics

Let us describe some dynamical properties of the global attractor $\mathcal{A}$ corresponding to equation (1.1). Consider scheme (1.3) as a lattice dynamical system [AP93] on the two-dimensional lattice $\mathbb{Z} \times \mathbb{Z}$ with coordinates (m, n). Three types of solutions for lattice systems are usually studied: traveling waves, steady-state solutions, and spatially-homogeneous solutions.

Traveling waves. Let us introduce a "traveling-wave" coordinate $q = n + m$ in our discretization scheme, i.e., let us set $u_m^n = z(q) = z(n+m)$. Of course this is only one type of coordinate that belongs to waves of velocity equal to one. We obtain the following equations for $z(q)$:

$$\frac{z(q+1) - z(q)}{h} = \frac{z(q+2) - 2z(q+1) + z(q)}{d^2} + f(z(q+1)). \tag{7.1}$$

If we introduce the vector $(x_q, y_q) = (z(q), z(q+1))$, then the equations above are reduced to the system

$$x_{q+1} = y_q, \quad y_{q+1} = -\left(1 + \frac{d^2}{h}\right) x_q + \left(2 + \frac{d^2}{h}\right) y_q - d^2 f(y_q). \tag{7.2}$$

Let Λ be an invariant set for the two-dimensional dynamical system

$$(x_{q+1}, y_{q+1}) = \Phi(x_q, y_q), \quad q \in \mathbb{Z}, \tag{7.3}$$

where

$$\Phi(x, y) = (y, -Kx + (1+K)y + F(y)),$$

$K > 1$, and the function F is globally Lipschitz continuous with a constant l. We assume that

$$4l < K - 1. \tag{7.4}$$

Remark 7.1 *Equality (7.3) generates a dynamical system since the mapping Φ is invertible. Indeed, the equality $\Phi(x, y) = (x', y')$ is equivalent to the equalities $y = x'$ and $x = -(y' - (1+K)x' - F(x'))/K$.*

Let us show that if Λ is a bounded invariant set of system (7.3), then the dynamics of Φ on Λ is realizable by a family of traveling waves for scheme (1.3) under a proper choice of the steps h and d and the nonlinearity f.

Since Λ is bounded, there exists a positive N such that $|y_q| \leq N$ for any trajectory $(x_q, y_q) \in \Lambda$ and for any $q \in \mathbb{Z}$. Obviously, there exists a globally Lipschitz continuous (with constant l) function F^* with the following properties:

- $F(y) = F^*(y)$ for $|y| \leq N$;
- there exist positive constants α' and β' such that

$$yF^*(y) \geq \alpha' y^2 - \beta'. \tag{7.5}$$

Now we take $h, d \in (0,1)$ such that $1+d^2/h = K$, set $f(u) = -F^*(u)/d^2$, and consider the corresponding scheme (1.3). It is easy to see that our conditions (7.4) and (7.5) imply conditions (2.3) and (1.4), respectively. Now equalities (7.2) imply that any trajectory $(x_q, y_q) \in \Lambda$ of system (7.2) generates a traveling wave $u_m^n = x_{m+n}$ of (1.3). Since $|u_m^n| \le N$, the traveling wave belongs to the global attractor $\mathcal{A}$.

Steady-state solutions. Steady-state (or stationary) solutions do not depend on "time" n, i.e., $u_m^n = z(m)$ for these solutions. In this case, scheme (1.3) is reduced to the equalities

$$\frac{z(m+1) - 2z(m) + z(m-1)}{d^2} + f(z(m)) = 0.$$

The same reasoning as in the case of traveling waves shows that any bounded invariant set of system (7.3), where $\Phi(x,y) = (y, -x+2y+F(y))$, generates a family of stationary solutions of scheme (1.3) belonging to its global attractor (we leave the details to the reader).

Spatially-homogeneous solutions. Spatially-homogeneous solutions do not depend on the "space" coordinate m, i.e., $u_m^n = z(n)$ for these solutions. In this case, scheme (1.3) is reduced to the equalities

$$\frac{z(n+1) - z(n)}{h} = f(z(n)).$$

The corresponding dynamical system is one-dimensional and has the form

$$x_{q+1} = \Phi(x_q).$$

Similarly to the first two cases, bounded invariant sets of this system generate families of solutions of scheme (1.3) belonging to its global attractor (in this case, $f(u) = (\Phi(u) - u)/h$).

8 An example of an infinite dimensional attractor

Let us give an example of a system $\mathcal{S}$ corresponding to equation (1.1) such that its global attractor $\mathcal{A}$ is infinite-dimensional.

Fix a space step d such that the ratio d/π is irrational. Let l be a natural number. Define $\alpha(l)$ by the equality

$$\alpha^{-1}(l) = 1 - h\left(\frac{2}{d^2}(\cos ld - 1) + 1\right).$$

Since

$$\sin(m+1)ld + \sin(m-1)ld = 2\sin mld \cos ld,$$

the sequence $u^n = \{u_m^n\}$, where

$$u_m^n = \alpha^n(l) b \sin mld, \quad n, m \in \mathbb{Z},$$

and $b \in \mathbb{R}$, satisfies the equalities

$$\frac{u^{n+1} - u^n}{h} = Au^{n+1} + u^{n+1}, \quad n \in \mathbb{Z}.$$

Since the ratio d/π is irrational, there exists an infinite set L of natural numbers l such that the inequalities

$$|\cos ld - 1| < \frac{d^2}{4} \tag{8.1}$$

hold. It is easy to see that inequality (8.1) implies the inequality

$$|\alpha(l)| > 1. \tag{8.2}$$

Consider an arbitrary natural number N and let $l_1, \dots, l_N$ be the first N elements of the set L. Take numbers $b_1, \dots, b_N$ such that $|b_i| < 1/N$. Inequalities (8.2) imply that the sequence U^n, where

$$U_m^n = b_1 \alpha^n(l_1) \sin ml_1 d + \dots + b_N \alpha^n(l_N) \sin ml_N d,$$

satisfies the inqualities $|U_m^n| < 1$ for $n \le 0$, hence U^n is a negative trajectory of scheme (1.3), where the nonlinearity f is such that $f(u) = u$ for $|u| \le 1$.

Since the values

$$(\partial_- U^n)_m = b_1 \alpha^n(l_1) \frac{1}{d} (\sin ml_1 d - \sin(m-1)l_1 d) + \dots$$

satisfy the inequalities $|(\partial_- U^n)_m| \le 2/d$, our negative trajectory is bounded in the space Z_u (and hence it belongs to the global attractor $\mathcal{A}$, see the proof of Theorem 6.3).

The numbers $(l_i - l_j)d/\pi$ are irrational, hence the $N \times N$ matrix

$$G = \begin{pmatrix} \sin l_1 d & \dots & \sin l_N d \\ \dots & \dots & \dots \\ \sin Nl_1 d & \dots & \sin Nl_N d \end{pmatrix}$$

is nonsingular.

Consider the initial value U^0 of our trajectory U^n. The N-dimensional vector $V = (u_1^0, \dots, u_N^0)$ satisfies the relations

$$V_m = b_1 \sin ml_1 d + \dots b_N \sin ml_N d$$

for $m = 1, \dots, N$, hence $V = Gb$, where $b = (b_1, \dots, b_N)$. Since the matrix G is nonsingular, there exists $\varepsilon > 0$ such that the image of the ball $|b_i| < 1/N, i = 1, \dots, N$, contains the ball $|u_i^0| < \varepsilon, i = 1, \dots, N$. Since the initial values U^0 belong to the global attractor $\mathcal{A}$, we see that the dimension of $\mathcal{A}$ is not less that N. Since N is arbitrary, the attractor $\mathcal{A}$ is infinite-dimensional.

References

[AP93] V. Afraimovich and Ya. Pesin. Travelling waves in lattice models of multi-dimensional and multi-component media: I. General hyperbolic properties. Nonlinearity 6, 429–455, 1993.

[BV90] A.V. Babin and M.I. Vishik. Attractors of partial differential evolution equations in an unbounded domain. Proc. R. Soc. Edinburgh 116A, 221–243, 1990.

[BV92] A.V. Babin and M.I. Vishik. Attractors of Evolutionary Equations. North-Holland, 1992.

[BLW01] P.W. Bates, K. Lu, and B. Wang. Attractors for lattice dynamical systems. Int. J. Bifurcation and Chaos 11, 143–153, 2001.

[C94] P. Collet. Thermodynamic limit of the Ginzburg-Landau equations. Nonlinearity 7, 1175-1190, 1994.

[EP96] T. Eirola and S.Yu. Pilyugin. Pseudotrajectories generated by a discretization of a parabolic equation. J. Dynam. Diff. Equat. 8, 281–297, 1996.

[F94] E. Feireisl et al. Compact attractors for reaction-diffusion equations in $\mathbb{R}^n$. C. R. Acad. Sci. Paris, Ser. I 319, 147–151, 1994.

[H88] J.K. Hale. Asymptotic Behavior of Dissipative Systems. Math. Surv. Monogr. 25, Am. Math. Soc., 1988.

[HLR88] J.K.Hale, X.B.Lin, and G.Raugel. Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, Math. of Comp., vol. 50, 89-123, 1988.

[HR89] J.K.Hale and G.Raugel. Lower semicontinuity of attractors of gradient systems and applications, Annali di Matematica pura ed applicata, vol. CLIV, 281-326, 1989.

[H94] J.K. Hale. Numerical dynamics. In: Chaotic Numerics (P. Kloeden, K. Palmer Eds.) Am. Math. Soc., 1994, pp. 1–30.

[L91a] O.A. Ladyzhenskaya. Attractors for Semi-Groups and Evolution Equations. Cambridge Univ. Press, 1991.

[L91b] O.A. Ladyzhenskaya. Globally stable difference schemes and their attractors. Preprint POMI P-5-91, St. Petersburg, 1991.

[MS95] A. Mielke and G. Schneider. Attractors for modulation equations on unbounded domains – existence and comparison. Nonlinearity 8, 743–768, 1995.

[M97] A. Mielke. The complex Ginzburg-Landau equation on large and unbounded domains: sharper bounds and attractors. Nonlinearity 10, 199-222, 1997.

[M99] A. Mielke. The Ginzburg-Landau equation in its role as a modulation equation. Preprint 20/99, DFG-SPP 'Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme' 1999. To appear in Handbook of Dynamical Systems II (B. Fiedler Ed.).

[OKM93] W.M. Oliva, N.M. Kuhl, and L.T. Magalhães. Diffeomorhisms of $\mathbb{R}^n$ with oscillatory Jacobians. Publ. Mat. 37, 255–269, 1993.

[RN72] F. Riesz and B. Sz.-Nagy. Leçons d'Analyse Fonctionelle. Budapest, 1972.

[SY02] G.R. Sell and Y. You. Dynamics of Evolutionary Systems. Appl. Math. Sci. 143, Springer, 2002.

[SH96] A. M. Stuart and A. R. Humphries. Dynamical Systems and Numerical Analysis. Cambridge Univ. Press, 1996.

[T88] R. Temam. Infinite Dimensional Systems in Mechanics and Physics. Springer, 1988.

[Z02] S. Zhou. Attractors for second order lattice dynamical systems. J. Diff. Equ. 179, 605–624, 2002.