A Model Function for Polynomial Rates in Discrete Dynamical Systems

Thorsten Hüls*

Fakultät für Mathematik, Universität Bielefeld
Postfach 100131, 33501 Bielefeld, Germany
huels@mathematik.uni-bielefeld.de

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Abstract

In this paper we construct a one-dimensional map with a non hyperbolic fixed point at zero for which the orbits converging to zero and the solution of the associated variational equation can be determined explicitly. We extend the construction to parameterized systems where the fixed point undergoes bifurcations. Applications are indicated to heteroclinic orbits that connect a hyperbolic to a non hyperbolic fixed point with one-dimensional center manifold.

Keywords: polynomial rate of convergence, bifurcation, discrete dynamical systems.

1 Introduction

Consider a time discrete dynamical system depending on a parameter

$$x_{n+1} = f(x_n, \lambda), \quad x \in \mathbb{R}^k,\ \lambda \in \mathbb{R}, \tag{1}$$

where $k \geq 1$ and the map f is smooth with respect to both x and λ. Further assume that at $\lambda = \bar{\lambda}$, the map $f(\cdot, \lambda)$ is a diffeomorphism with two fixed points $\xi_\pm$, such that ξ_- is hyperbolic and ξ_+ has a one-dimensional center direction. This situation arises for example, when ξ_+ undergoes a fold, flip or pitchfork bifurcation at $\bar{\lambda}$ (cf. [4], [6]).

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In order to analyze the bifurcation of such heteroclinic orbits as well as the numerical approximation, it is essential to understand the polynomial behavior of the restriction of (1) to the one-dimensional center manifold at ξ_+ (cf. [1], [3], [2]).

In this paper we discuss a very useful model function that allows us to study the orbit and the solution of the associated variational equation explicitly.

In section 3 this function is used to give an explicit example for a discrete time dynamical systems with a non-unique center manifold.

To analyze the pitchfork bifurcation the model function is extended by a bifurcation parameter in section 4. In particular we will see how the exponential rate of convergence for $\lambda < \bar{\lambda}$ turns into a polynomial rate for $\lambda = \bar{\lambda}$.

The general bifurcation analysis and an approximation theorem for non-hyperbolic orbits can be found in my PhD thesis [2] and will be published in forthcoming papers.

2 The model function for a non-hyperbolic orbit

Consider for $q \in \mathbb{N}$ and $b > 0$ the following map

$$g(x, q, b) = \frac{x}{(1 + bqx^q)^{1/q}}. \tag{2}$$

Taylor expansion at 0 gives

$$g(x, q, b) = x - bx^{q+1} + \mathcal{O}(x^{q+2}).$$

This map has the nice property that an orbit $x_{n+1} = g(x_n, q, b)$ with starting point $x_0 = \frac{1}{\gamma(bq)^{1/q}}$, $\gamma \geq 1$ has for all $n \in \mathbb{Z}^+$ the explicit representation

$$x_n = \frac{1}{(bq)^{1/q}(\gamma^q + n)^{1/q}} = \frac{x_0}{(1 + bqx_0^q n)^{1/q}}.$$

If q is even, the starting point $y_0 = -x_0$ leads to the orbit $y_n = -x_n$ for $n \geq 0$.

The solution of the associated variational equation

$$u_{n+1} = g_x(x_n, b, q)u_n, \quad n \in \mathbb{Z}^+ \tag{3}$$

with starting point $u_0 = \frac{1}{\gamma^q} x_0$ has the explicit form

$$u_n = \frac{x_n}{\gamma^q + n} = \frac{1}{(bq)^{1/q}(\gamma^q + n)^{1+1/q}}$$

and all other solutions $(v_n)_{n \geq 0}$ are multiples.

By comparing Taylor's expansion of $g(x, q, b)$ with an arbitrary function of the form $f(x) = x - bx^{q+1} + \mathcal{O}(x^{q+2})$ one can prove the polynomial rate of convergence for an f-orbit x_n towards 0. It holds $x_n \approx \frac{1}{(bq)^{1/q}} \frac{1}{n^{1/q}}$, see [3]. Furthermore we get

$u_n \approx C\frac{1}{n^{1+1/q}}$ with a constant $C > 0$ (cf. [2]). For a different approach to these asymptotics see [5].

Notice that the solution operator $\Phi(n,m)$ of (3), defined for $n \geq m \geq 0$ by $\Phi(n,m) = \prod_{i=m}^{n-1} g_x(x_i, q, b)$, has the following representation

$$\Phi(n,m) = \left(\frac{n+\gamma^q}{m+\gamma^q}\right)^{-1-1/q}.$$

For an arbitrary $f(x) = x - bx^{q+1} + \mathcal{O}(x^{q+2})$ one can show the estimate

$$|\Phi(n,m)| \leq C\left(\frac{n+1}{m+1}\right)^{-1-1/q} \quad \text{for some constant } C > 0.$$

This leads to the definition of a polynomial dichotomy (cf. [2]).

3 Non-uniqueness of center manifold; an example

It is well known that the center manifold is in general not unique. The standard example for continuous time systems is

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y. \end{aligned}$$

This system has the equilibrium $(x,y) = (0,0)$ and possesses a family of one-dimensional center manifolds (see [4]) given by

$$W_\alpha^c(0) = \big\{(x,y) : y = h_\alpha(x)\big\}, \quad h_\alpha(x) = \begin{cases} \alpha e^{\frac{1}{x}} & \text{for } x < 0 \\ 0 & \text{for } x \geq 0. \end{cases}$$

With the map above we can construct an analogous explicit example for discrete time systems. Consider the map

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{1+x} \\ \frac{1}{2}y \end{pmatrix}. \tag{4}$$

Orbits, with positive x-component, converge to the fixed point $(0,0)$, in particular the starting point $(x_0, y_0) = (1,1)$ leads to the orbit $(x_n, y_n) = \left(\frac{1}{n+1}, \frac{1}{2^n}\right)$, $n \geq 0$.

A family of center manifolds (see Figure 1) is given by

$$W_\alpha^c(0) := \big\{\big(x, h_\alpha(x)\big), x \in (-1,1)\big\}, \quad h_\alpha(x) := \begin{cases} \alpha e^{-\frac{1}{x}\log 2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

To verify this, the invariance condition must be fulfilled:

$$\begin{aligned} h\left(\tfrac{x}{1+x}\right) &= \alpha e^{-\frac{1+x}{x}\log 2} \\ &= \alpha e^{-\frac{1}{x}\log 2} e^{-\log 2} = \tfrac{1}{2}\alpha e^{-\frac{1}{x}\log 2} = \tfrac{1}{2}h(x) \text{ for } x > 0. \end{aligned}$$

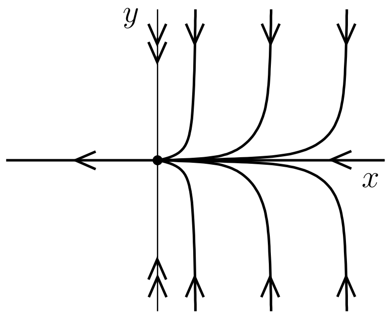

Figure 1: *A family of center manifolds for the discrete time system (4)*

4 The parameterized model function

In this section, we extend the model function with a parameter λ such that bifurcations of the fixed point $(0,0)$ can be studied explicitly. Taylor's expansion of the extended model function

$$g(x,\lambda,q,b) := \frac{\lambda x}{\left(1+\frac{bq}{\lambda}x^q\right)^{1/q}}$$

at 0 has the form

$$g(x,\lambda,q,b) = \lambda x - bx^{q+1} + \mathcal{O}(x^{q+2}).$$

At $\lambda = 1$ this function coincides with (2).

First we show that every $g(\cdot,\lambda,q,b)$-orbit has an explicit representation.

Proposition 1 *For the orbit $x_n = g^n(x_0,\lambda,q,b)$, $n \in \mathbb{Z}^+$ we get*

$$x_0 = \frac{1}{\gamma\left(\frac{bq}{\lambda}\right)^{1/q}}, \quad \gamma \geq 1 \quad \Longrightarrow \quad x_n = \frac{\lambda^n}{\left(\frac{bq}{\lambda}\right)^{1/q}\left(\gamma^q + \sum_{i=0}^{n-1}\lambda^{qi}\right)^{1/q}}. \tag{5}$$

Proof: The proof follows by induction:

$$\begin{aligned}
x_{n+1} &= g(x_n,\lambda,q,b) \\
&= \frac{\lambda \dfrac{\lambda^n}{\left(\frac{bq}{\lambda}\right)^{1/q}\left(\gamma^q+\sum_{i=0}^{n-1}\lambda^{qi}\right)^{1/q}}}{\left(1+\dfrac{bq}{\lambda}\dfrac{\lambda^{qn}}{\frac{bq}{\lambda}\left(\gamma^q+\sum_{i=0}^{n-1}\lambda^{qi}\right)}\right)^{1/q}} = \frac{\dfrac{\lambda^{n+1}}{\left(\frac{bq}{\lambda}\right)^{1/q}\left(\gamma^q+\sum_{i=0}^{n-1}\lambda^{qi}\right)^{1/q}}}{\dfrac{\left(\gamma^q+\sum_{i=0}^{n-1}\lambda^{qi}+\lambda^{qn}\right)^{1/q}}{\left(\gamma^q+\sum_{i=0}^{n-1}\lambda^{qi}\right)^{1/q}}} \\
&= \frac{\lambda^{n+1}}{\left(\frac{bq}{\lambda}\right)^{1/q}\left(\gamma^q+\sum_{i=0}^{n}\lambda^{qi}\right)^{1/q}}.
\end{aligned}$$

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The following lemma gives us the solution of the associated variational equation.

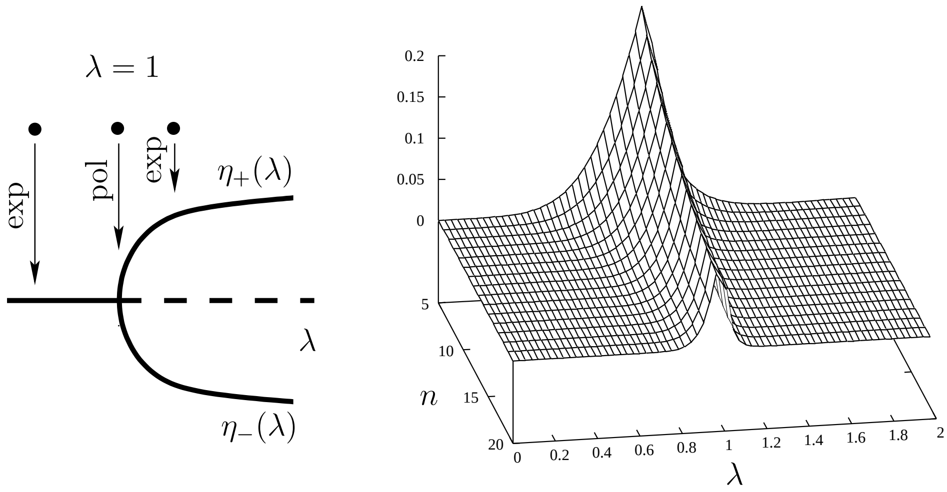

Figure 2: *The left picture shows the bifurcation diagram of the fixed point in case of a pitchfork bifurcation ($q = 2$). The arrows symbolize the exponential/polynomial rate of convergence for an orbit with positive starting point towards the fixed point. The right picture shows the distance of an orbit $x_n(\lambda)$ from the fixed point, as a function of n and λ.*

Proposition 2 *For a $g(\cdot, \lambda, q, b)$-orbit $(x_n)_{n\geq 0}$ the variational equation (3) has the solution*

$$u_n = \frac{x_n}{\gamma^q + \sum_{i=0}^{n-1} \lambda^{qi}} = \frac{\lambda^n}{\left(\frac{bq}{\lambda}\right)^{1/q} \left(\gamma^q + \sum_{i=0}^{n-1} \lambda^{qi}\right)^{1+1/q}}. \tag{6}$$

Furthermore the solution operator has for all $n \geq m \geq 0$ the form

$$\Phi(n, m) = \lambda^{n-m} \left(\frac{\gamma^q + \sum_{i=0}^{n-1} \lambda^{qi}}{\gamma^q + \sum_{i=0}^{m-1} \lambda^{qi}} \right)^{-1-1/q}.$$

Proof: Using the explicit representation (5) of x_n the proof of (6) follows by induction.

For the solution operator Φ we get for $n \geq m \geq 0$

$$\begin{aligned}
\Phi(n, m) &= \prod_{i=m}^{n-1} g_x(x_i, \lambda, q, b) = \prod_{i=m}^{n-1} \frac{\lambda}{\left(1 + \frac{bq}{\lambda} \frac{\lambda^{qi}}{\frac{bq}{\lambda}\left(\gamma^q + \sum_{j=0}^{i-1} \lambda^{qj}\right)}\right)^{1+1/q}} \\
&= \prod_{i=m}^{n-1} \frac{\lambda\left(\gamma^q + \sum_{j=0}^{i-1} \lambda^{qj}\right)^{1+1/q}}{\left(\gamma^q + \sum_{j=0}^{i} \lambda^{qj}\right)^{1+1/q}} = \frac{\lambda^{n-m}\left(\gamma^q + \sum_{j=0}^{m-1} \lambda^{qj}\right)^{1+1/q}}{\left(\gamma^q + \sum_{j=0}^{n-1} \lambda^{qj}\right)^{1+1/q}}.
\end{aligned}$$

■

For $q = 2$ the fixed point $\xi(\lambda) = 0$ undergoes a pitchfork bifurcation at $\lambda = 1$ (cf. [4], [6]) and for $\lambda > 1$ a new branch of fixed points $\eta_\pm(\lambda) := \pm\sqrt{\frac{\lambda^3-\lambda}{2b}}$ emanates.

The explicit representation (5) shows that we have an exponential rate of convergence for $\lambda < 1$ which turns into a polynomial rate at $\lambda = 1$. For $\lambda > 1$ we have an exponentially fast convergence to one of the new fixed points $\eta_\pm(\lambda)$ depending on the sign of the initial value, see Figure 2.

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