On the existence of transversal heteroclinic orbits in discretized dynamical systems

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Abstract

In this paper we prove the existence of transversal heteroclinic orbits for maps that are obtained from one-step methods applied to a continuous dynamical system. It is assumed that the continuous system exhibits a heteroclinic orbit at a specific value of a parameter. While it is known that analytic vector fields lead to exponentially small splittings of separatrices in the discrete system, we analyze here the case of a continuous system that is smooth of finite order only. Assuming that a certain derivative has a jump discontinuity at a specific hyperplane we show that the discretized systems have transversal

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heteroclinic orbits. The essential step in deriving such a result is a refinement of a previously developed error analysis which applies exponential dichotomy and Fredholm techniques to the discretized system.

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1 Introduction

It is well-known that transversal homoclinic orbits for maps entail a certain type of chaotic dynamics, for corresponding results see for example [12, 15, 16, 11] and the references therein. An immediate question then is how to prove the existence of transversal homoclinic orbits for a given map. In this paper we investigate this question for maps that arise from one-step methods applied to continuous dynamical systems for which a connecting orbit, either heteroclinic or homoclinic, is known.

The discretization of continuous dynamical systems near homoclinic and heteroclinic orbits has been analyzed by several authors [6, 17, 1, 18]. A fundamental result was derived by Fiedler and Scheurle in [6]. They consider a homoclinic orbit $\bar{x}(t)$ $(-\infty < t < \infty)$ at $\lambda = \bar{\lambda}$ of a parameterized system

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^k,\ \lambda \in \mathbb{R}^p \tag{1.1}$$

and an associated one-step method

$$x_{n+1} = \psi(x_n, \lambda, \varepsilon), \quad \varepsilon = \text{step} - \text{size}. \tag{1.2}$$

Assuming analyticity of f and ψ and of $\bar{x}(t)$ in some complex strip $|\text{Imt}| < \eta$ they proved that the discrete system has homoclinic orbits $x_n = x_n(\varepsilon)$, $n \in \mathbb{Z}$ that occur in an exponentially narrow sector in the (λ, ε)-plane

$$|\lambda - \tilde{\lambda}(\varepsilon)| \leq c e^{-\eta/\varepsilon}. \tag{1.3}$$

Here $\tilde{\lambda}(\varepsilon) = \bar{\lambda} + O(\varepsilon^d)$ is a smooth function of ε, and d is the order of the one-step method. It is also pointed out in [6] that transversal intersections of the stable and unstable manifolds occur in a generic sense.

The method of proof in [6] is to interpret the one-step method (1.2) as the ε-time map of a non-autonomous, rapidly forced perturbation of (1.1) given by

$$\dot{x} = f(x, \lambda) + \varepsilon^d g(x, \lambda, t/\varepsilon, \varepsilon), \tag{1.4}$$

where $g(x, \lambda, \cdot, \varepsilon)$ is smooth and 1-periodic with respect to the third variable. This system is then analyzed by a combination of Liapunov-Schmidt and Fourier techniques.

In the paper [18] we developed an alternative, somewhat more direct approach to the difference equation (1.2) which applies to general connecting orbits and which also works for vector fields f of finite smoothness. To both equations (1.1) and (1.2) we add a scalar phase condition (cf. (3.3))

$$\Pi(x, \varepsilon, t^*) = 0, \tag{1.5}$$

where t^* acts as a phase parameter. Then we prove the existence of connecting orbits for both systems (1.1),(1.5) and (1.2),(1.5) by using a quantitative implicit function theorem. More precisely, we show that close to the continuous connecting orbit the one-step method has a curve of discrete connecting orbits $x_n(\varepsilon, t^*)$, $n \in \mathbb{Z}$ at parameter values $\lambda(\varepsilon, t^*)$. At fixed step-size ε this curve is parameterized by the phase parameter t^*, and it is closed in the following sense

$$\begin{aligned} x_n(\varepsilon, t^*) &= x_{n-1}(\varepsilon, t^* + \varepsilon), \quad n \in \mathbb{Z}, \\ \lambda(\varepsilon, t^*) &= \lambda(\varepsilon, t^* + \varepsilon). \end{aligned}$$

Furthermore at least two tangencies, characterized by $\frac{\partial \lambda}{\partial t^*} = 0$, occur within this family of connecting orbits (cf. [3] for a numerical illustration of this phenomenon).

In [18] we also derived error estimates of type $O(\varepsilon^d)$ for the differences $x_n(\varepsilon, t^*) - \bar{x}(n\varepsilon + t^*)$ and $\lambda(\varepsilon, t^*) - \bar{\lambda}$ assuming C^{r+1} $(r \geq d)$ smoothness for the function f.

The purpose of this paper is to prove that transversal connecting orbits for the one-step map occur if the function f looses its smoothness at a fixed hyperplane. Transversality is equivalent to finding values t^* at which $\frac{\partial \lambda}{\partial t^*} \neq 0$, see [18]. Our basic tools for achieving this are refined estimates (w.r.t an L_1-norm) for the linearization of the time-ε flow around the sampled connecting orbit $\bar{x}(t^* + n\varepsilon), n \in \mathbb{Z}$. This will be combined with Fredholm operator theory and exponential dichotomy techniques.

In order to avoid complicated discussions of finite smoothness for a general one-step method and to stress the basic idea in the proof, we restrict to the explicit Euler method, i.e. $\psi(x, \lambda, \varepsilon) = x + \varepsilon f(x, \lambda)$. In this case we find an explicit criterion which involves the jump discontinuity of the second derivative (see (**H5**) in section 2) and which guarantees transversal heteroclinic orbits for the Euler mapping. We also find that the splitting distance $\max_{0 \leq t_1, t_2 \leq \varepsilon} |\tilde{\lambda}(t_1) - \tilde{\lambda}(t_2)|$ is at most of the order $O(\varepsilon^2)$.

At the end of section 3 we indicate how this result can be extended to more general one-step methods. In particular the splitting distance is then found to be at most of order $O(\varepsilon^{r+1})$ if $f^{(r+1)}$ is assumed to be piecewise continuous.

In the analytic case, apart from the genericity result in [6], it seems extremely difficult to give explicit criteria that guarantee transversal heteroclinics for (1.2). A substantial theory for analyzing the coefficients of exponentially small terms has been developed by Gelfreich, see [8] for a survey. Compared to our simple criterion (**H5**), however, it is still difficult to verify his assumptions for the discretization problem of connecting orbits.

In the following section 2, we describe our basic assumptions and collect some results from [18] which will be used for the subsequent analysis. Then our main result is proved in section 3. It is based on a refined a-priori estimate and a reparametrization of the connecting orbits summarized in two key lemmata the proof of which will be given in section 4.

2 Basic assumptions

In this section, we introduce the basic regularity conditions for continuous connecting orbits and collect some preliminary results from [18]. Then we describe the finite smoothness assumption which ensures the existence of transversal heteroclinic orbits for the numerical methods.

Consider a parameterized dynamical system

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}^p \tag{2.1}$$

and let $\varphi(x, \lambda, t)$ be its solution flow with $\varphi(x, \lambda, 0) \equiv x$. The parameter λ is used to set up a well-posed equation for connecting orbits. Our assumptions are

(H1) f is C^1 smooth in all variables and its first order derivative is Lipschitz, i.e. there exists a constant $L > 0$ such that for $x,\ y \in \mathbb{R}^k$, $\lambda,\ \mu \in \mathbb{R}^p$

$$\|f_{(x,\lambda)}(x, \lambda) - f_{(x,\lambda)}(y, \mu)\| \leq L(\|x - y\| + |\lambda - \mu|).$$

(H2) $\bar{x}_\pm$ are hyperbolic equilibria of equation (2.1) at $\lambda = \bar{\lambda}$.

According to these assumptions, there exist constants $\lambda_1 > 0$ and a C^1 function $\bar{x}_\pm(\lambda)$ with $\bar{x}_\pm(\bar{\lambda}) = \bar{x}_\pm$, such that for all $|\lambda - \bar{\lambda}| < \lambda_1$, $\bar{x}_\pm(\lambda)$ are hyperbolic equilibria of equation (2.1). By a λ-dependent shift in phase space, we can assume without loss of generality $\bar{x}_\pm(\lambda) \equiv \bar{x}_\pm$ for all $|\lambda - \bar{\lambda}| \leq \lambda_1$.

By $k_{\pm s,u}$, we denote the numbers of stable (resp. unstable) eigenvalues (counting multiplicity) of the matrix $f_x(\bar{x}_\pm, \bar{\lambda})$ and we have that $k_{\pm u,s}$ are constant for all $|\lambda - \bar{\lambda}| < \lambda_1$.

(H3) At the parameter $\lambda = \bar{\lambda}$, equation (2.1) possesses a connecting orbit $\bar{x}(t)$, which satisfies

$$\bar{x}_- = \lim_{t \to -\infty} \bar{x}(t), \quad \bar{x}_+ = \lim_{t \to +\infty} \bar{x}(t).$$

We call $(\bar{x}(\cdot), \bar{\lambda})$ a **connecting orbit pair** (COP for short).

(H4) The COP $(\bar{x}, \bar{\lambda})$ is nondegenerate with respect to parameter λ (cf. [2, Definition 2.1], [10, Definition 3.1] and [18]) in the following sense

(i) the numbers of stable and unstable eigenvalues are related to the number of parameters λ by $p = k + 1 - k_{-u} - k_{+s} = k_{-s} - k_{+s} + 1$,

(ii) any bounded solution $x(t)$, $\mu \in \mathbb{R}^p$ of the equation $\dot{x} = f_x(\bar{x}, \bar{\lambda})x + f_\lambda(\bar{x}, \bar{\lambda})\mu$ satisfies $\mu = 0$ and $x = c\dot{\bar{x}}$ for some $c \in \mathbb{R}$.

A COP $(\bar{x}(\cdot), \bar{\lambda})$ is said to be **regular** if (H1), (H2) and (H4) are satisfied.

Define Banach spaces for $\ell = 0, 1$

$$X^\ell = \{x \in C^\ell(\mathbb{R}, \mathbb{R}^k) : x^{(j)}(t) \text{ is bounded for } j = 0, \cdots, \ell\},$$
$$\|x\|_\ell = \sum_{j=0}^{\ell} \sup_{t \in \mathbb{R}} \|x^{(j)}(t)\|, \text{ with } \|\cdot\| \text{ some norm in } \mathbb{R}^k.$$

Consider the linear operator

$$L : X^1 \to X^0, \quad Lx = \dot{x} - f_x(\bar{x}(\cdot), \bar{\lambda})x, \tag{2.2}$$

and denote its solution operator by $S(t, s)$, i.e. $x(t) = S(t, s)\xi$ solves $Lx = 0$ with $x(s) = \xi$. From the semigroup property we have

$$S(t, s)^{-1} = S(s, t). \tag{2.3}$$

According to [5, 12], [2, Lemma 2.1], L has an exponential dichotomy on $\mathbb{R}^+$ and on $\mathbb{R}^-$. More precisely, we have the following lemma (cf. [13], [10, Lemma2.3] and [4])

Lemma 2.1 [18, Lemma 2.2] *Assume* (H1)–(H4). *Then L has an exponential dichotomy on $\mathbb{R}^\pm$, i.e. there exist data $(K^\pm, \alpha^\pm, P^\pm(t))$, where $K^\pm$, $\alpha^\pm > 0$ are constants, such that for all $t \geq s$ in $\mathbb{R}^\pm$ the following holds*

$$\begin{aligned} S(t, s)P^\pm(s) &= P^\pm(t)S(t, s), && (2.4) \\ \|S(t, s)P^\pm(s)\| &\leq K^\pm e^{-\alpha^\pm(t-s)}, && (2.5) \\ \|S(s, t)(I - P^\pm(t))\| &\leq K^\pm e^{-\alpha^\pm(t-s)}. && (2.6) \end{aligned}$$

From (2.3) and (2.4) we see that the projectors $P^\pm(t)$ are uniquely determined by $P^\pm(0)$ as follows

$$P^\pm(t) = S(t, 0)P^\pm(0)S(0, t). \tag{2.7}$$

Define the adjoint operator of L by

$$L^*: X^1 \to X^0, \quad L^*x = \dot{x} + f_x(\bar{x}(\cdot), \bar{\lambda})^T x,$$

and denote its solution operator by $S^*(t,s)$. Then we have $S^*(t,s) = S(s,t)^T$.

The Fredholm properties are summarized in the following lemma.

Lemma 2.2 [18, Lemma 2.3] *Assume* (H1)–(H4), *then the adjoint operator* L^* *has an exponential dichotomy on* $\mathbb{R}^\pm$ *with data* $(C_0K^\pm, \alpha^\pm, I - P^\pm(t)^T)$, *where* C_0 *only depends on the norm in* $\mathbb{R}^k$. *Moreover,*

$$\mathcal{N}(L) = \{S(t,0)x_0 : x_0 \in \mathcal{R}(P^+(0)) \cap \mathcal{N}(P^-(0))\}, \tag{2.8}$$

$$\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*) + k_{+s} - k_{-s}, \tag{2.9}$$

$$x \in \mathcal{R}(L) \Leftrightarrow \int_{-\infty}^{+\infty} y^T(t)x(t)dt = 0, \ \forall y(\cdot) \in \mathcal{N}(L^*), \tag{2.10}$$

and $L: X^1 \to X^0$ *is Fredholm of index* $k_{+s} - k_{-s} = k_{-u} - k_{+u}$.

With the aid of Lemma 2.2, we have the following characterization of the regularity, see also [2, Proposition 2.1].

Lemma 2.3 [18, Lemma 2.4] *Assume* (H1)–(H3). *Then* $(\bar{x}, \bar{\lambda})$ *is regular if and only if the linear operator* L *has the properties*

(1) $\dim \mathcal{N}(L) = 1$, $\dim \mathcal{N}(L^*) = p$,

(2) the $p \times p$ *matrix*

$$E = \int_{-\infty}^{+\infty} Y^T(t) f_\lambda(\bar{x}(t), \bar{\lambda}) dt$$

is nonsingular, where the columns $y^i(\cdot) \in X^1$ *of*

$$Y(\cdot) = (y^1(\cdot), \ldots, y^p(\cdot))$$

form a basis of $\mathcal{N}(L^*)$.

Using the exponential dichotomy of the operator L^* and Lemma 2.3, we obtain the rate of convergence of $Y(t)$ as $t \to \pm\infty$.

Corollary 2.4 $\|Y(t)\| \le CKe^{-\alpha^{\pm}|t|}$, *as* $t \to \pm\infty$, *with* $K = \max\{K^+, K^-\}$.

In the following we specify our finite smoothness assumptions for the function $f(x,\lambda)$ along the connecting orbit $\bar{x}(t)$. For simplicity, we assume that non-smoothness of the function $f(x,\lambda)$ occurs only in the highest derivative at some hyperplane passing through $\bar{x}(0)$. Let Ω be a small open ball in $\mathbb{R}^k$ centered around $\bar{x}(0)$, and let Σ be the hyper-plane that passes through the point $\bar{x}(0)$ and has normal vector $\vec{n}_0 = \dot{\bar{x}}(0)/\|\dot{\bar{x}}(0)\|$. The region Ω is divided by the hyper-plane Σ into two parts, Ω_- and Ω_+ (see Figure 2.1 below)

$$\Omega_- = \{x \in \bar{\Omega}; \vec{n}_0^T(x - \bar{x}(0)) \le 0\}, \quad \Omega_+ = \{x \in \bar{\Omega}; \vec{n}_0^T(x - \bar{x}(0)) \ge 0\}.$$

We denote the restriction of the function $f(\cdot,\lambda)$ to the region $\Omega_\pm$ by $f^\pm(\cdot,\lambda)$ for all λ. Let $\Sigma_\Omega = \Sigma \cap \bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω.

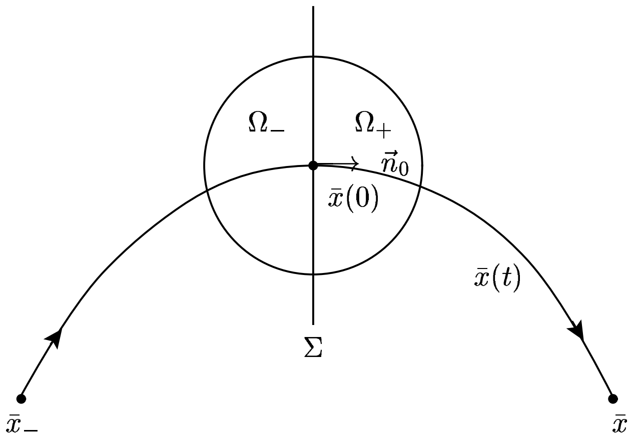

Figure 2.1 *Illustration of the finite smoothness property near the heteroclinic orbit.*

Our main assumption then is

(H5) The connecting orbit $\bar{x}(t)$ $(t \in \mathbb{R})$ intersects the segment Σ_Ω only at the point $\bar{x}(0)$. The second derivatives $f_{x\lambda}$ and $f_{\lambda\lambda}$ are continuous and bounded in the whole space and f_{xx} is continuous and bounded in the region $(\mathbb{R}^k/\Omega) \times \mathbb{R}^p$. Finally, the second derivative $f_{xx}^\pm$ exists and is continuous and bounded in $\Omega_\pm \times \mathbb{R}^p$ and satisfies the following jump condition

$$Y(0)^T(f_{xx}^+(\bar{x}(0),\bar{\lambda}) - f_{xx}^-(\bar{x}(0),\bar{\lambda}))\dot{\bar{x}}(0)^2 \ne 0,$$

where $Y(t)$ is defined in Lemma 2.3.

In order to simplify the analysis in this paper, we restrict ourselves to the explicit Euler method, i.e.

$$x_{n+1} = \psi(x_n, \lambda, \varepsilon) = x_n + \varepsilon f(x_n, \lambda). \tag{2.11}$$

Then $\psi(\cdot, \lambda, \varepsilon)$ satisfies the same finite smoothness assumption in (H5).

As a final prerequisite we will need a lemma on the convergence of Riemann sums which will be used frequently in the next sections.

Lemma 2.5 [18, Lemma 2.6] *Assume $g : \mathbb{R} \to \mathbb{R}^k$ is a continuous function and satisfies the estimate $\|g(t)\| \le Ce^{-\rho|t|}$ for some $C, \rho > 0$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{n=0}^{\infty} g(n\varepsilon + t^*) = \int_0^{\infty} g(t + t^*)dt$$

uniformly for t^* in any compact set.

3 Transversal connecting orbits for systems of finite smoothness

In this section, we first summarize the existence results for heteroclinic orbits under discretization developed from [18]. Then we study the existence of transversal heteroclinic orbits under the finite smoothness property (H5).

Since equation (2.1) is an autonomous system, its solution permits a phase shift. If $x(t)$ is a solution of equation (2.1) passing through a point x_0 at time $t = 0$ then for any value $t^* \in \mathbb{R}$ the function $x(t + t^*)$ is also a solution of equation (2.1) passing through the point x_0 at time $t = -t^*$.

For $J = \mathbb{Z}_\pm$ define the Banach space

$$S_J = \{x_J = (x_n)_{n \in J} : x_n \in \mathbb{R}^k, \|x_J\| = \sup_{n \in J} \|x_n\| < \infty\}.$$

The following operators and functions are used for the investigations in [18].

$$\tilde{\Gamma} : \quad \begin{aligned} & S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R} \to S_{\mathbb{Z}} \\ & (x_{\mathbb{Z}}, \lambda, \varepsilon) \mapsto (x_{n+1} - \psi(x_n, \lambda, \varepsilon))_{n \in \mathbb{Z}}, \end{aligned} \tag{3.1}$$

$$\Gamma : \quad \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R} \to S_{\mathbb{Z}} \\ (x_{\mathbb{Z}}, \lambda, \varepsilon) \mapsto (x_{n+1} - \varphi(x_n, \lambda, \varepsilon))_{n\in\mathbb{Z}}, \end{array} \tag{3.2}$$

$$\Pi : \quad \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x_{\mathbb{Z}}, \varepsilon, t^*) \mapsto \displaystyle\sum_{n=-\infty}^{\infty} \varepsilon \dot{\bar{x}}^T (t^* + n\varepsilon)(x_n - \bar{x}(t^* + n\varepsilon)), \end{array} \tag{3.3}$$

$$\tilde{F} : \quad \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R} \times \mathbb{R} \to S_{\mathbb{Z}} \times \mathbb{R} \\ (x_{\mathbb{Z}}, \lambda, \varepsilon, t^*) \mapsto (\tilde{\Gamma}(x_{\mathbb{Z}}, \lambda, \varepsilon), \Pi(x_{\mathbb{Z}}, \varepsilon, t^*)), \end{array} \tag{3.4}$$

$$F : \quad \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R} \times \mathbb{R} \to S_{\mathbb{Z}} \times \mathbb{R} \\ (x_{\mathbb{Z}}, \lambda, \varepsilon, t^*) \mapsto (\Gamma(x_{\mathbb{Z}}, \lambda, \varepsilon), \Pi(x_{\mathbb{Z}}, \varepsilon, t^*)). \end{array} \tag{3.5}$$

It is clear that the zeroes $x_{\mathbb{Z}}$ of $\tilde{F} = 0$ yield connecting orbits of the discrete map $\psi(\cdot, \lambda, \varepsilon)$ that satisfy the phase condition $\Pi(x_{\mathbb{Z}}, \varepsilon, t^*) = 0$ for some λ, ε and t^*. The basic existence result from [18] is as follows.

Theorem 3.1 [18, Theorem 4.3 and 4.4] *Assume* (H1)–(H4)*. Then there exists a constant* $\varepsilon_0 > 0$*, such that for* $0 < \varepsilon < \varepsilon_0$*,* $t^* \in \mathbb{R}$*, the function* $\tilde{F}$ *defined in* (3.3) *has a unique zero* $(\tilde{x}_{\mathbb{Z}}(\varepsilon, t^*), \tilde{\lambda}(\varepsilon, t^*))$ *in a neighborhood of* $(\bar{x}_{\mathbb{Z}}(\varepsilon, t^*), \bar{\lambda}) = ((\bar{x}(t^* + n\varepsilon))_{n\in\mathbb{Z}}, \bar{\lambda})$*. Moreover,it satisfies the estimates*

$$\begin{aligned} \sup_{n\in\mathbb{Z}} \|\tilde{x}_n(\varepsilon, t^*) - \bar{x}(t^* + n\varepsilon)\| &= O(\varepsilon), \\ \|\tilde{\lambda}(\varepsilon, t^*) - \bar{\lambda}\| &= O(\varepsilon), \end{aligned} \tag{3.6}$$

and has the following periodicity property for all $n \in \mathbb{Z}$

$$\tilde{x}_n(\varepsilon, t^*) = \tilde{x}_{n-1}(\varepsilon, t^* + \varepsilon), \tag{3.7}$$

$$\tilde{\lambda}(\varepsilon, t^*) = \tilde{\lambda}(\varepsilon, t^* + \varepsilon). \tag{3.8}$$

In the next theorem, we state the criterion for detecting the tangential or transversal property of a heteroclinic orbit within the family $\tilde{x}_{\mathbb{Z}}(\varepsilon, t^*)$. To simplify the notation, we remove the explicit dependence on the variable ε in all expressions where no confusion should occur. We also denote by $\dot{\tilde{x}}(t^*)$, $\ddot{\tilde{x}}(t^*)$, $\dot{\lambda}(t^*)$ etc. the derivatives with respect to the phase variable t^*.

Theorem 3.2 [18, Theorem 4.6] *Assume* (H1)–(H4) *and* $0 < \varepsilon < \varepsilon_0$. *If the derivative* $\dot{\tilde{\lambda}}(t_0) \neq 0$, *then* $\tilde{x}_{\mathbb{Z}}(t_0)$ *is a transversal connecting orbit for the map* $\psi(\cdot, \tilde{\lambda}(t_0), \varepsilon)$*; while in case* $\dot{\tilde{\lambda}}(t_0) = 0$, *the orbit* $\tilde{x}_{\mathbb{Z}}(t_0)$ *is nondegenerate and* 1*-tangential for the map* $\psi(\cdot, \tilde{\lambda}(t_0), \varepsilon)$ *with respect to the parameter* λ.

An easy consequence of the periodicity (3.8) and Theorem 3.2 is the following Corollary.

Corollary 3.3 [18, Corollary 4.7] *Assume* (H1)–(H4) *and* $0 < \varepsilon < \varepsilon_0$. *Then there exist at least two nondegenerate* 1*-tangential heteroclinic orbit among the family of heteroclinic orbits* $\tilde{x}_{\mathbb{Z}}(t^*)$ *for the one-step map* $\psi(\cdot, \tilde{\lambda}(t^*), \varepsilon)$.

We are going to verify the transversality condition from Theorem 3.2 by using the finite smoothness assumption (H5). The following two lemmata play a key role in the proof of our main result and we leave their proofs to the next section.

In the following, for any $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$ that is absolutely summable we use the L_1-norm defined by $\|y_{\mathbb{Z}}\|_1 = \sum_{n \in J} \|y_n\|$. For an abbreviation, denoted by $F'(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda})$ the derivative of the function $F(x_{\mathbb{Z}}, \lambda)$ with respect to (x_Z, λ) at the point $(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda})$.

Lemma 3.4 *Assume* (H1)–(H4). *Then there exist constants* $\varepsilon_0 > 0$ *and* $\beta > 0$, *such that for all* $0 < \varepsilon < \varepsilon_0$, $t^* \in \mathbb{R}$ *and* $(v_{\mathbb{Z}}, \mu) \in S_{\mathbb{Z}} \times \mathbb{R}^p$, *there hold*

$$\|y_{\mathbb{Z}}\|_1 + |\omega| \geq \beta \|(v_{\mathbb{Z}}, \mu)\| \tag{3.9}$$

for all summable $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$, *and*

$$\|y_{\mathbb{Z}}\| + |\omega| \geq \beta\varepsilon \|(v_{\mathbb{Z}}, \mu)\| \tag{3.10}$$

for all $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$, *where* $(y_{\mathbb{Z}}, \omega) = \tilde{F}'(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda})(v_{\mathbb{Z}}, \mu)$.

The following Lemma specifies in which way the discrete connecting orbits pass the section Σ_Ω.

Lemma 3.5 *Assume* (H1)–(H5). *Then there exist constants* $t_0^* > 0$, $\varepsilon_0 > 0$ *and* $C > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the functions* $(\tilde{x}_{\mathbb{Z}}(\varepsilon, t^*), \tilde{\lambda}(\varepsilon, t^*))$ *satisfy*

$$\sup_{n\in\mathbb{Z}} \|\dot{\tilde{x}}_n(\varepsilon, t^*) - \dot{\bar{x}}_n(\varepsilon, t^*)\| + |\dot{\tilde{\lambda}}(t^*)| \leq C\varepsilon \quad \text{for all } t^*.$$

Moreover, there exists a continuous function $t^*(\varepsilon)$ *with* $\lim_{\varepsilon\to 0+} t^*(\varepsilon) = 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *we have* $\tilde{x}_0(\varepsilon, t^*(\varepsilon)) \in \Sigma_\Omega$ *but* $\tilde{x}_n(\varepsilon, t^*(\varepsilon)) \notin \Sigma_\Omega$ *for* $n \neq 0$. *Likewise,* $\tilde{x}_n(\varepsilon, t^*) \notin \Sigma_\Omega$ *for all* $n \in \mathbb{Z}$ *if* $t^* \neq t^*(\varepsilon) + \ell\varepsilon$ $(\ell \in \mathbb{Z})$. *Furthermore, we have* $\tilde{x}_0(\varepsilon, t^*) \in \Omega_+$ *if* $0 < t^* - t^*(\varepsilon) < t_0^*$ *and* $\tilde{x}_0(\varepsilon, t^*) \in \Omega_-$ *if* $0 < t^*(\varepsilon) - t^* < t_0^*$.

We are now ready to state and prove our main result.

Theorem 3.6 *Assume* (H1)–(H5). *Then there exists a constant* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *there exists at least one* $\tilde{t}^*$ *such that* $\tilde{x}_{\mathbb{Z}}(\tilde{t}^*)$ *is a transversal heteroclinic orbit for the one-step map* $\psi(\cdot, \tilde{\lambda}(\tilde{t}^*), \varepsilon)$ *from* (2.11) *and we have the estimate*

$$\max_{0\leq t_1, t_2\leq\varepsilon} |\tilde{\lambda}(t_1) - \tilde{\lambda}(t_2)| \geq C\varepsilon^2.$$

Proof. Let the open ball Ω centered at $\bar{x}(0)$ be of radius r_1 and let Ω_2 be the open ball centered at $\bar{x}(0)$ with radius $2r_1$. We may assume $r_1 > 0$ so small that $\bar{x}_\pm \notin \bar{\Omega}_2$. See Figure 3.1 below.

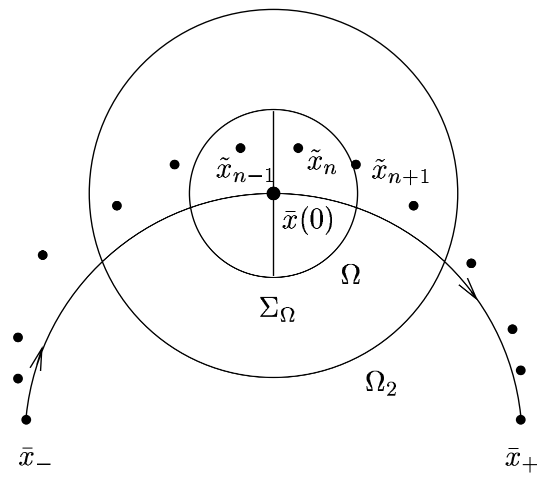

Figure 3.1 *The illustration of positive distance from homoclinic to the segment* Σ_Ω.

For any $0 < \varepsilon < \varepsilon_0$, define $\mathbb{T} = \{t^*; t^* \neq \ell\varepsilon, t^*(\varepsilon) + \ell\varepsilon \text{ for all } \ell \in \mathbb{Z}\}$. We will prove that the COPs $(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))$ are C^2 smooth in $t^* \in \mathbb{T}$.

For any $t^*_+ \in \mathbb{T}$, it follows from Lemma 3.5 that $\tilde{x}_n(t^*_+) \notin \Sigma_\Omega$ for all $n \in \mathbb{Z}$. Since $\lim_{n\to\pm\infty} \tilde{x}_n(t^*_+) = \bar{x}_\pm \notin \Omega_2$ there exist at most finitely many points $\tilde{x}_n(t^*_+) \in \Omega_2$ (cf. Figure 3.1). Hence we find a constant $r_0 \in (0, r_1)$ such that if $|x - \tilde{x}_n(t^*_+)| < r_0$ for some integer $n \in \mathbb{Z}$, then $x \notin \Sigma_\Omega$.

Define the ball $B(t^*_+) \subset S_{\mathbb{Z}}$ by

$$B(t^*_+) = \{x_{\mathbb{Z}} \in S_{\mathbb{Z}}; \|x_{\mathbb{Z}} - \tilde{x}_{\mathbb{Z}}(t^*_+)\| = \sup_{n\in\mathbb{Z}} \|x_n - \tilde{x}_n(t^*_+)\| < r_0\}.$$

Then for any $x_{\mathbb{Z}} \in B(t^*_+)$, we have $x_n \notin \Sigma_\Omega$ and the function $f(x, \lambda)$ is C^2 continuous at (x_n, λ) for all n. From the definition (3.1), we obtain that the function $\tilde{\Gamma}(x_{\mathbb{Z}}, \lambda, \varepsilon)$ is C^2 smooth with respect to $(x_{\mathbb{Z}}, \lambda)$ for $(x_{\mathbb{Z}}, \lambda, \varepsilon) \in B(t^*_+) \times \mathbb{R}^p \times \mathbb{R}^+$. Because $\bar{x}(t^*_+ + n\varepsilon) \notin \Sigma_\Omega$ for all n it follows from equations (3.1) and (3.3) that the function $\Pi(x_{\mathbb{Z}}, \varepsilon, t^*)$ is C^2 smooth with respect to $(x_{\mathbb{Z}}, t^*)$ for all $(x_{\mathbb{Z}}, \varepsilon, t^*) \in B(t^*_+) \times \mathbb{R}^+ \times \mathbb{T}$. Therefore $\tilde{F}(x_{\mathbb{Z}}, \lambda, \varepsilon, t^*)$ is also C^2 smooth with respect to $(x_{\mathbb{Z}}, \lambda, t^*) \in B(t^*_+) \times \mathbb{R}^p \times \mathbb{T}$ for $0 < \varepsilon < \varepsilon_0$.

It has been proved in Theorem 3.1 that $(\tilde{x}_{\mathbb{Z}}(t^*_+), \tilde{\lambda}(t^*_+))$ is a regular solution of the equation $\tilde{F} = 0$ at $t^* = t^*_+$. Applying an implicit function theorem (cf. [18, Lemma 4.2]) to the equation $\tilde{F} = 0$ at the point $(\tilde{x}_{\mathbb{Z}}(t^*_+), \tilde{\lambda}(t^*_+), t^*_+)$, we obtain a constant $0 < h < \varepsilon$ such that $t^* \in \mathbb{T}$ holds for $|t^* - t^*_+| < h$, and the equation $\tilde{F} = 0$ has a unique C^2 smooth solution $(\hat{x}_{\mathbb{Z}}(t^*), \hat{\lambda}(t^*))$ near $(\tilde{x}_{\mathbb{Z}}(t^*_+), \tilde{\lambda}(t^*_+))$, which also satisfies $\hat{x}_{\mathbb{Z}}(t^*) \in B(t^*_+)$. From the uniqueness we know $(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*)) = (\hat{x}_{\mathbb{Z}}(t^*), \hat{\lambda}(t^*))$ for $|t^* - t^*_+| < h$. Therefore the functions $(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))$ are C^2 smooth with respect to $t^* \in \mathbb{T}$.

By the definition of $\mathbb{T}$ there exists a constant $h_0 > 0$ such that $t^* \in \mathbb{T}$ holds for any $0 < |t^* - t^*(\varepsilon)| < h_0$. Thus, the function $\tilde{F}(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*), t^*)$ is C^2 continuous with respect to t^* for $0 < |t^* - t^*(\varepsilon)| < h_0$.

Differentiating the equation $\tilde{F} = 0$ twice with respect to $t^* \in \mathbb{T}$, we obtain for $n \in \mathbb{Z}$

$$\ddot{\tilde{x}}_{n+1}(t^*) - \psi_x(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\ddot{\tilde{x}}_n(t^*) - \psi_\lambda(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\ddot{\tilde{\lambda}}(t^*) = g^1_n(t^*), \quad (3.11)$$

$$\sum_{n=-\infty}^{\infty} \varepsilon \dot{\tilde{x}}_n(t^*)^T \ddot{\tilde{x}}_n(t^*) = g^2(t^*), \tag{3.12}$$

where

$$\begin{aligned} g_n^1(t^*) = & -\varepsilon f_{xx}(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\dot{\tilde{x}}_n(t^*)^2 - 2\varepsilon f_{x\lambda}(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\dot{\tilde{x}}_n(t^*)\dot{\tilde{\lambda}}(t^*) \\ & -\varepsilon f_{\lambda\lambda}(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\dot{\tilde{\lambda}}(t^*)^2, \qquad (3.13) \\ g^2(t^*) = & \sum_{n=-\infty}^{+\infty} \varepsilon \dot{\tilde{x}}_n(t^*)^T \ddot{\bar{x}}_n(t^*) - 2\sum_{-\infty}^{\infty} \varepsilon (f_x(\bar{x}_n(t^*), \bar{\lambda})\dot{\bar{x}}_n(t^*))^T (\dot{\tilde{x}}_n(t^*) - \dot{\bar{x}}_n(t^*)) \\ & - \sum_{n=-\infty}^{\infty} \varepsilon (f_{xx}(\bar{x}_n(t^*), \bar{\lambda})\dot{\bar{x}}_n(t^*)^2 + f_x(\bar{x}_n(t^*), \bar{\lambda})\ddot{\bar{x}}_n(t^*))^T (\tilde{x}_n(t^*) - \bar{x}_n(t^*)). \end{aligned}$$

We will solve equations (3.11) and (3.12) for $(\ddot{\tilde{x}}_n(t^*), \ddot{\tilde{\lambda}}(t^*))$ and study its one-sided limit as $t^* \to t^*(\varepsilon)^{\pm}$. According to the estimate (3.10) in Lemma 3.4, we need to analyze the behavior of $\|g_{\mathbb{Z}}^1(t^*)\|$ and $|g^2(t^*)|$ as $t^* \to t^*(\varepsilon)^{\pm}$.

Notice that terms of different smoothness with respect to $t^* \in \mathbb{R}$ appear in these expressions: the functions $\tilde{x}_n(t^*)$ and $\tilde{\lambda}(t^*)$ are C^1 , $\bar{x}_n(t^*)$ is C^2 and the compositions with the functions $\psi_{x\lambda}$ and $\psi_{\lambda\lambda}$ are C^0 for all $n \in \mathbb{Z}$.

For any $|t^* - t^*(\varepsilon)| < h_0$, we know that the orbit $\tilde{x}_{\mathbb{Z}}(t^*)$ intersects the segment Σ_Ω only at $n = 0$ and $t^* = t^*(\varepsilon)$. For $n \neq 0$ and $|t^* - t^*(\varepsilon)| < h_0$, we have $\tilde{x}_n(t^*) \notin \Sigma_\Omega$ and the function $f_{xx}(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))$ is continuous with respect to t^*. For $n = 0$ and $0 < \pm(t^* - t^*(\varepsilon)) < h_0$ we know $\tilde{x}_0(t^*) \in \Omega_\pm$ by Lemma 3.5 and therefore $f_{xx}(\tilde{x}_0(t^*), \tilde{\lambda}(t^*)) = f_{xx}^{\pm}(\tilde{x}_0(t^*), \tilde{\lambda}(t^*))$. Using condition (H5) we then obtain the existence of the one-sided limits

$$\lim_{t^* \to t^*(\varepsilon)^{\pm}} f_{xx}(\tilde{x}_0(t^*), \tilde{\lambda}(t^*)) = f_{xx}^{\pm}(\bar{x}_0(t^*(\varepsilon)), \bar{\lambda}(t^*(\varepsilon))).$$

Hence the one-sided limit $g_n^1(t^*(\varepsilon)^{\pm}) = \lim_{t^* \to t^*(\varepsilon)^{\pm}} g_n^1(t^*)$ exists uniformly in $n \in \mathbb{Z}$.

In a similar way, we obtain the existence of the one-sided limit

$$\lim_{t^* \to t^*(\varepsilon)^{\pm}} g^2(t^*) = g^2(t^*(\varepsilon)^{\pm}).$$

Rewrite equations (3.11) and (3.12) as an operator equation

$$\tilde{F}'(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))(\ddot{\tilde{x}}_{\mathbb{Z}}(t^*), \ddot{\tilde{\lambda}}(t^*)) = (g_{\mathbb{Z}}^1(t^*), g^2(t^*)). \tag{3.14}$$

It is clear that $\tilde{F}'(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))$ is continuous w.r.t. the max norms for all $t^* \in \mathbb{R}$ and has a bounded inverse (see Lemma 3.4 and [18, Theorem 3.7] for max norms on both sides), therefore has a continuous inverse. Taking one-sided limits in (3.14) we can deduce the existence of

$$\lim_{t^* \to t^*(\varepsilon)^{\pm}} (\ddot{\tilde{x}}_{\mathbb{Z}}(t^*), \ddot{\tilde{\lambda}}(t^*)) = (\ddot{\tilde{x}}_{\mathbb{Z}}(t^*(\varepsilon)^{\pm}), \ddot{\tilde{\lambda}}(t^*(\varepsilon)^{\pm})).$$

Subtracting the two one-sided limits of equation (3.14) from each other we obtain

$$\tilde{F}'(\tilde{x}_{\mathbb{Z}}(t^*(\varepsilon)), \tilde{\lambda}(t^*(\varepsilon)))(v_{\mathbb{Z}}(\varepsilon), \lambda^*(\varepsilon)) = (\varepsilon c^*_{\mathbb{Z}}(\varepsilon), g^*(\varepsilon)) \tag{3.15}$$

where

$$\begin{aligned} v_n(\varepsilon) &= \ddot{\tilde{x}}_n(t^*(\varepsilon)^+) - \ddot{\tilde{x}}_n(t^*(\varepsilon)^-), \\ \lambda^*(\varepsilon) &= \ddot{\tilde{\lambda}}(t^*(\varepsilon)^+) - \ddot{\tilde{\lambda}}(t^*(\varepsilon)^-), \\ \varepsilon c^*_n(\varepsilon) &= g^1_n(t^*(\varepsilon)^+) - g^1_n(t^*(\varepsilon)^-), \\ g^*(\varepsilon) &= g^2(t^*(\varepsilon)^+) - g^2(t^*(\varepsilon)^-). \end{aligned}$$

Notice that by (H5) either $\bar{x}_n(t^*(\varepsilon)) \notin \Sigma_\Omega$ for all $n \in \mathbb{Z}$, then $g^*(\varepsilon) = 0$, or $\bar{x}_N(t^*(\varepsilon)) \in \Sigma_\Omega$ for exactly one $N \in \mathbb{Z}$ and then

$$\begin{aligned} g^*(\varepsilon) &= \varepsilon[(f^-_{xx}(\bar{x}_N(t^*(\varepsilon)), \bar{\lambda}) - f^+_{xx}(\bar{x}_N(t^*(\varepsilon)), \bar{\lambda}))\dot{\bar{x}}_N(t^*(\varepsilon))^2]^T \cdot \\ &\quad (\tilde{x}_N(t^*(\varepsilon)) - \bar{x}_N(t^*(\varepsilon))). \end{aligned}$$

In any case, the estimate $|g^*(\varepsilon)| \leq C\varepsilon^2$ holds due to (3.6).

According to Lemma 3.5 we know that $\tilde{x}_n(t^*(\varepsilon)) \notin \Sigma_\Omega$ for $n \neq 0$, and hence the function $f_{xx}(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))$ is continuous in t^*. Therefore, $c^*_n(\varepsilon) = 0$ for $n \neq 0$. On the other hand

$$c^*_0(\varepsilon) = [f^-_{xx}(\tilde{x}_0(t^*(\varepsilon)), \tilde{\lambda}(t^*(\varepsilon))) - f^+_{xx}(\tilde{x}_0(t^*(\varepsilon)), \tilde{\lambda}(t^*(\varepsilon)))]\dot{\tilde{x}}_0(t^*(\varepsilon))^2 \tag{3.16}$$

Applying the estimates from Theorem 3.1 and Lemma 3.5 we find

$$c^*_0(0) = \lim_{\varepsilon \to 0^+} c^*_0(\varepsilon) = [f^-_{xx}(\bar{x}(0), \bar{\lambda}) - f^+_{xx}(\bar{x}(0), \bar{\lambda})]\dot{\bar{x}}(0)^2, \tag{3.17}$$

hence $\|c^*_{\mathbb{Z}}(\varepsilon)\|_1 \leq C$ for $0 < \varepsilon < \varepsilon_0$.

From the estimate (3.9) in Lemma 3.4 and equation (3.15) it follows that

$$\|v_{\mathbb{Z}}\| + |\lambda^*| \leq \beta^{-1}(\varepsilon\|c^*_{\mathbb{Z}}(\varepsilon)\|_1 + |g^*(\varepsilon)|) \leq C\varepsilon. \tag{3.18}$$

We take the inner product of (3.15) from the left with $(Y^T_{\mathbb{Z}}(\varepsilon), 0)$, where $Y_n(\varepsilon) = Y(t^*(\varepsilon) + (n+1)\varepsilon)$ and Y is the matrix from Lemma 2.3. According to [18, Lemma3.5] we have $Y^T_{\mathbb{Z}}\Gamma_{x_{\mathbb{Z}}}(\bar{x}_{\mathbb{Z}}, \bar{\lambda}) = 0$ (this uses the relation between the solution operator of the linearized equation and its adjoint) and therefore we obtain

$$Y^T_{\mathbb{Z}}(\varepsilon)\tilde{\Gamma}_\lambda(\tilde{x}_{\mathbb{Z}}(t^*(\varepsilon)), \tilde{\lambda}(t^*(\varepsilon)))\lambda^*(\varepsilon) = \varepsilon Y^T_0(\varepsilon)c^*_0(\varepsilon) + Y^T_{\mathbb{Z}}(\varepsilon)A_{\mathbb{Z}}(\varepsilon)v_{\mathbb{Z}}(\varepsilon), \tag{3.19}$$

where

$$A_{\mathbb{Z}}(\varepsilon) = \Gamma_{x_{\mathbb{Z}}}(\bar{x}_{\mathbb{Z}}(t^*(\varepsilon)), \bar{\lambda}) - \tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{x}_{\mathbb{Z}}(t^*(\varepsilon)), \tilde{\lambda}(t^*(\varepsilon))).$$

Clearly we have

$$\|Y^T_{\mathbb{Z}}A_{\mathbb{Z}}v_{\mathbb{Z}}\| = \|\sum_{n\in\mathbb{Z}} Y^T_n A_n v_n\| \leq \sum_{n\in\mathbb{Z}} \|Y^T_n\| \cdot \|\varphi_x(\bar{x}_n, \bar{\lambda}, \varepsilon) - \psi_x(\tilde{x}_n, \tilde{\lambda}, \varepsilon)\| \cdot \|v_n\|.$$

From Corollary 2.4 we obtain

$$\|Y^T_{\mathbb{Z}}\|_1 = \sum_{n\in\mathbb{Z}} \|Y^T_n\| \leq \sum_{n\in\mathbb{Z}} CKe^{-\alpha\varepsilon|n|} \leq C/\varepsilon.$$

Using the estimates (3.6) from Theorem 3.1 and the well known fact that for Euler's method $\|\varphi_x(\cdot,\cdot,\varepsilon) - \psi_x(\cdot,\cdot,\varepsilon)\| = O(\varepsilon^2)$ holds uniformly in compact domains, we have

$$\begin{aligned} & \|\varphi_x(\bar{x}_n, \bar{\lambda}, \varepsilon) - \psi_x(\tilde{x}_n, \tilde{\lambda}, \varepsilon)\| \\ \leq\ & \|\varphi_x(\bar{x}_n, \bar{\lambda}, \varepsilon) - \varphi_x(\tilde{x}_n, \tilde{\lambda}, \varepsilon)\| + \|\varphi_x(\tilde{x}_n, \tilde{\lambda}, \varepsilon) - \psi_x(\tilde{x}_n, \tilde{\lambda}, \varepsilon)\| \\ \leq\ & C\varepsilon^2. \end{aligned} \tag{3.20}$$

Collecting estimates we obtain

$$\|Y^T_{\mathbb{Z}}A_{\mathbb{Z}}v_{\mathbb{Z}}\| \leq \|Y^T_{\mathbb{Z}}\|_1 \cdot \|A_{\mathbb{Z}}\| \cdot \|v_{\mathbb{Z}}\| \leq C\varepsilon^2. \tag{3.21}$$

According to (2.11) and the estimates (3.6) in Theorem 3.1 we get

$$Y_{\mathbb{Z}}^T \tilde{\Gamma}_\lambda(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda}) = \sum_{n \in Z} Y_n^T \psi_\lambda(\tilde{x}_n, \tilde{\lambda}, \varepsilon) = \sum_{n \in Z} Y_n^T (\varepsilon f_\lambda(\bar{x}_n, \bar{\lambda}, \varepsilon) + O(\varepsilon^2))$$

and from Lemma 2.5 it follows that

$$Y_{\mathbb{Z}}^T(\varepsilon) \tilde{\Gamma}_\lambda(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda}) = M + O(\varepsilon), \quad \text{where} \quad M = \int_{-\infty}^{+\infty} Y^T(t) f_\lambda(\bar{x}(t), \bar{\lambda}) dt.$$

The matrix M is nonsingular due to Lemma 2.3 and so is $Y_{\mathbb{Z}}^T \tilde{\Gamma}_\lambda(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda})$ for ε small enough. According to (H5) the vector $d = M^{-1} Y(0)^T c_0^*(0)$ is nontrivial and together with (3.17), (3.21) equation (3.19) leads to the expansion

$$\lambda^*(t^*(\varepsilon)) = \ddot{\tilde{\lambda}}(t^*(\varepsilon)^+) - \ddot{\tilde{\lambda}}(t^*(\varepsilon)^-) = \varepsilon d + o(\varepsilon).$$

In particular $\lambda^*(t^*(\varepsilon)) \neq 0$ for $\varepsilon > 0$ small, and thus $\dot{\tilde{\lambda}}(t^*)$ is strictly monotone for $t^* \geq t^*(\varepsilon)$ or $t^* \leq t^*(\varepsilon)$. This guarantees $\dot{\tilde{\lambda}}(t^*) \neq 0$ on some open t^* interval and hence transversal connecting orbits for the one-step method $\psi(\cdot, \tilde{\lambda}(t^*), \varepsilon)$ according to Theorem 3.2. Finally, the lower estimate in Theorem 3.6 follows from a one-sided Taylor expansion of the function $\tilde{\lambda}(t^*)$ at $t^* = t^*(\varepsilon)$. ■

Remark 3.7 *We concentrated on the explicit Euler method and a simple discontinuity, because the proof then shows clearly how the discontinuity of the continuous system creates discrete transversal heteroclinic orbits for the numerical method. This can be generalized in various ways.*

1) Instead of assumption (H1) *we can assume that the function f is C^r $(r \geq 1)$ smooth, and its $(r+1)$-th order derivatives satisfy the assumption* (H5) *with $f_{xx}^{\pm}(\cdot)\dot{\bar{x}}(0)^2$ replaced by $f_x^{\pm(r+1)}(\cdot)\dot{\bar{x}}(0)^{r+1}$ in the inequality. Then we expect a result with a higher order estimate*

$$\max_{0 \leq t_1, t_2 \leq \varepsilon} |\tilde{\lambda}(t_1) - \tilde{\lambda}(t_2)| \geq C\varepsilon^{r+1}.$$

2) The segment Σ_Ω of discontinuity can be replaced by a smooth manifold and we require that the heteroclinic orbit $\bar{x}(t)$ $(t \in \mathbb{R})$ transversely passes through Σ_Ω exactly once, then Theorem 3.6 *still holds.*

3) One can extend Theorem 3.6 *to other one-step methods by carefully analyzing the discontinuity of the numerical methods and modifying the techniques of our proof. For example, consider the mid-point Euler method*

$$\frac{x_{n+1} - x_n}{\varepsilon} = f(\frac{x_{n+1} + x_n}{2}, \lambda), \tag{3.22}$$

which uniquely determines a one-step method $x_{n+1} = \psi(x_n, \lambda, \varepsilon)$. *In addition assume the discontinuity of* f *occurs on the line* Σ_Ω *as in* (H5). *Then the discontinuity of* ψ *must satisfies* $\frac{1}{2}(x + \psi(x, \lambda, \varepsilon)) \in \Sigma_\Omega$ *which leads to*

$$\vec{n}_0^T \left(\frac{x + \psi(x, \lambda, \varepsilon)}{2} - \bar{x}(0) \right) = 0. \tag{3.23}$$

Solving the coupled equations (3.22) *and* (3.23) *we obtain a discontinuity of* ψ *on some parametrized* $(k-1)$ *dimensional manifold* $\Sigma_\Omega(\lambda, \varepsilon)$ *which in contrast to the case 2 above depends on* λ *and* ε. *Next we show that for any small* $\varepsilon > 0$ *the corresponding heteroclinic manifold* $\tilde{x}_0(\varepsilon, t^*)$ $(t^* \in \mathbb{R})$ *of the map* ψ *uniquely and transversely intersects the manifold* $\Sigma_\Omega(\tilde{\lambda}(\varepsilon, t^*), \varepsilon)$ *at the value* $t^* = t^*(\varepsilon)$, *and the function* $t^*(\varepsilon)$ *is obtained as in* Lemma 3.5. *In this way one can extend* Theorem 3.6 *to the mid-point Euler method.*

4) For general d-th order one-step methods ψ, *e.g. Runge-Kutta methods, the discontinuity of the function* $f_x^{(r+1)}$ *may cause very complicated discontinuity properties of the function* $\psi_x^{(r+1)}$. *We have not followed the technical details of such a generalization, but we think that* Theorem 3.6 *applies to this case as well.*

4 Proof of two Lemmata

In the following we study the difference equation

$$u_{n+1} = A_n u_n, \quad n \in J \tag{4.1}$$

and its perturbation

$$u_{n+1} = (A_n + E_n) u_n, \quad n \in J, \tag{4.2}$$

where $J = \mathbb{Z}_\pm$ as in section 3, $\|A_n^{-1}\| \le C_A$ and $E_J = (E_n)_{n\in J}$ with $E_n \in \mathbb{R}^{k,k}$. We formulate a perturbation lemma with sharp estimates of the data for the exponential dichotomy (see [9, Lemma 1.1.9]).

Lemma 4.1 [9, Lemma 1.1.9] *Assume the difference equation* (4.1) *has an exponential dichotomy on J with data (K, α, P_J). For any given $\tilde\alpha \in (0, \alpha)$, there exist constants $C_*, \delta, K_0, K_1 > 0$ such that the following holds.*

(1) *For any E_J with $\|E_J\|_\infty \le \delta$ the perturbed difference equation* (4.2) *has an exponential dichotomy on J with data $(K_0, \tilde\alpha, Q_J(E_J))$, $Q_n(E_J)$ is a projector with the same rank as P_n and it satisfies the estimate*

$$\|Q_n(E_J) - P_n\| \le C_* \|E_J\|_\infty \quad \text{for } n \in J. \tag{4.3}$$

(2) *The solution operator $\Psi(E_J, n, m)$ of equation* (4.2) *is C^1 smooth for $\|E_J\|_\infty \le \delta$ and satisfies*

$$\begin{aligned} \left\| \frac{\partial}{\partial(E_J)} (\Psi(E_J, n, m) Q_m(E_J)) \right\| &\le K_1 e^{-\tilde\alpha(n-m)} \quad \text{for } n \ge m, \\ \left\| \frac{\partial}{\partial(E_J)} (\Psi(E_J, n, m)(I - Q_m(E_J))) \right\| &\le K_1 e^{-\tilde\alpha(m-n)} \quad \text{for } n \le m. \end{aligned}$$

Remark 4.2 *In the proof of* Lemma 4.1 ([9]) *it is shown that the constants above can be taken as $C_* = 2K\tau_1$, $K_0 = 2K+1$, $K_1 = 4\tau_1$, $\delta = \frac{1}{2}\min\{\frac{1}{C_A}, \frac{1}{\tau_1}\}$ and*

$$\tau_1 = K\left(\frac{1}{e^{\tilde\alpha} - e^{-\alpha}} + \max\left\{\frac{1}{e^{-\tilde\alpha} - e^{-\alpha}}, \frac{1}{e^{\alpha} - e^{\tilde\alpha}}\right\} + \frac{1}{e^{\alpha} - e^{-\tilde\alpha}}\right). \tag{4.4}$$

Proof of Lemma 3.4 Clearly $\bar x_\mathbb{Z}(\varepsilon, t^*)$ is a heteroclinic orbit of the map $\varphi(\cdot, \bar\lambda, t^*)$. The difference equation

$$u_{n+1} = \varphi_x(\bar x_n(\varepsilon, t^*), \bar\lambda, \varepsilon) u_n \tag{4.5}$$

has an exponential dichotomy on $\mathbb{Z}_\pm$ with data $(K^\pm, \varepsilon\alpha^\pm, P^\pm(t^* + n\varepsilon))$ (cf. Lemma 2.1 and [18, Lemma 3.4]). Let

$$E_n(\varepsilon, t^*) = \psi_x(\tilde x_n(\varepsilon, t^*), \tilde\lambda(\varepsilon, t^*), \varepsilon) - \varphi_x(\bar x_n(\varepsilon, t^*), \bar\lambda, \varepsilon).$$

It follows from equation (3.20) that $\|E_n\| \leq C\varepsilon^2$ for some constant $C > 0$. Applying Lemma 4.1 we obtain that the difference equation

$$u_{n+1} = \psi_x(\tilde{x}_n(\varepsilon, t^*), \tilde{\lambda}(\varepsilon, t^*), \varepsilon)u_n = (\varphi_x(\bar{x}_n(\varepsilon, t^*), \bar{\lambda}, \varepsilon) + E_n)u_n \qquad (4.6)$$

has an exponential dichotomy on $\mathbb{Z}_\pm$ with data $(\tilde{K}^\pm, \varepsilon\tilde{\alpha}^\pm, Q^\pm_{\mathbb{Z}_\pm}(\varepsilon, t^*))$. Here $\tilde{K}^\pm = 2K^\pm + 1$, $0 < \tilde{\alpha}^\pm < \alpha^\pm$ and $Q^\pm_{\mathbb{Z}_\pm}(\varepsilon, t^*)$ are projectors with the same rank as $P^\pm(t^* + n\varepsilon)$ and they satisfy the estimate (cf. (4.3))

$$\|Q^\pm_n(\varepsilon, t^*) - P^\pm(t^* + n\varepsilon)\| \leq K^\pm \tau^\pm_1 \|E_{\mathbb{Z}_\pm}\|_\infty,$$

where $\tau^\pm_1$ is defined as in (4.4) replacing α and $\tilde{\alpha}$ by $\alpha\varepsilon$ and $\tilde{\alpha}\varepsilon$, respectively. Direct computation gives $\tau^\pm_1 \leq C/\varepsilon$ for some constant $C > 0$. Thus, there holds

$$\sup_{n\in\mathbb{Z}_\pm} \|P^\pm(t^* + n\varepsilon) - Q^\pm_n(\varepsilon, t^*)\| = O(\varepsilon) \text{ as } \varepsilon \to 0. \qquad (4.7)$$

Denote the solution operator of equation (4.6) by $\Phi(\varepsilon, t^*, n, m)$. According to [18, Lemma 3.3], $S(\varepsilon, t^*, n, m) = S(t^* + n\varepsilon, t^* + m\varepsilon)$ is the solution operator of equation (4.5).

To simplify the notations, sometimes we drop the explicit dependence on ε and/or t^* in the following discussions.

In order to apply the second assertion of Lemma 4.1 to equations (4.5) and (4.6), we relate the quantities Φ, S to those as in Lemma 4.1 by setting $\Psi(E_J, n, m) = \Phi(\varepsilon, t^*, n, m)$ and $\Psi(0, n, m) = S(\varepsilon, t^*, n, m)$.

First we consider the case $J = \mathbb{Z}_+$, by virtue of mean value theorem and the estimates in Lemma 4.1 we obtain for any $n \geq m \geq 0$

$$\begin{aligned} & \|S(n,m)P^+(t^* + m\varepsilon) - \Phi(n,m)Q^+_m\| \\ \leq\; & \sup_{\|\tilde{E}_J\|\leq\delta} \|\frac{\partial}{\partial(\tilde{E}_J)}(\Psi(\tilde{E}_J, n, m)Q^+_m)\| \cdot \|E_J\|_\infty \\ \leq\; & 4\tau^+_1 e^{-(n-m)\tilde{\alpha}\varepsilon}\|E_J\|_\infty \leq C\varepsilon e^{-(n-m)\tilde{\alpha}\varepsilon} \qquad (4.8) \end{aligned}$$

for some constant $C > 0$. Similarly, we obtain for $m \geq n \geq 0$

$$\|S(n,m)(I - P^+(t^* + m\varepsilon)) - \Phi(n,m)(I - Q^+_m)\| \leq C\varepsilon e^{-(m-n)\tilde{\alpha}\varepsilon}. \qquad (4.9)$$

In the case $J=\mathbb{Z}_-$, estimates similar to (4.8) and (4.9) hold.

Consider the linearization of equation $\tilde{F}=0$ at $(\tilde{x}_{\mathbb{Z}},\tilde{\lambda})$. For any given $r_{\mathbb{Z}}\in S_{\mathbb{Z}}$ and $\omega\in\mathbb{R}$ we try to find a solution $(v_{\mathbb{Z}},\mu)$ of the following linear equations

$$\begin{aligned} v_{n+1}-A_nv_n-g_n\mu &= r_n, \quad n\in\mathbb{Z}, &(4.10)\\ \sum_{n=-\infty}^{\infty}\varepsilon\dot{\tilde{x}}_n^Tv_n &= \omega, &(4.11)\end{aligned}$$

where $A_n=\psi_x(\tilde{x}_n,\tilde{\lambda},\varepsilon)$, $g_n=\psi_\lambda(\tilde{x}_n,\tilde{\lambda},\varepsilon)$ and $\dot{\tilde{x}}_n=\dot{\tilde{x}}(t^*+n\varepsilon)$. For any $y_{\mathbb{Z}}\in S_{\mathbb{Z}}$, define

$$\begin{aligned} \bar{v}_n^+(y_{\mathbb{Z}}) &= \sum_{i=1}^{n}\Phi(n,i)Q_i^+y_{i-1}-\sum_{i=n+1}^{\infty}\Phi(n,i)(I-Q_i^+)y_{i-1}, \quad \text{for } n\geq 0,\\ \bar{v}_n^-(y_{\mathbb{Z}}) &= \sum_{-\infty}^{i=n}\Phi(n,i)Q_i^-y_{i-1}-\sum_{i=n+1}^{0}\Phi(n,i)(I-Q_i^-)y_{i-1}, \quad \text{for } n\leq 0.\end{aligned}$$

Then $\bar{v}_{\mathbb{Z}_\pm}^\pm$ are the unique bounded solutions of equation $v_{n+1}=A_nv_n+y_n$ in $S_{\mathbb{Z}_\pm}$ with initial properties $Q_0^+\bar{v}_0^+=0$ and $(I-Q_0^-)\bar{v}_0^-=0$, respectively, and satisfy the following estimates for $\tilde{K}=\max\{\tilde{K}^+,\tilde{K}^-\}$

$$\|\bar{v}_n^\pm(y_{\mathbb{Z}})\|\leq\tilde{K}\|y_{\mathbb{Z}}\|_1, \quad n\in\mathbb{Z}_\pm. \tag{4.12}$$

To obtain the general solution of the inhomogeneous equation (4.10), we add solutions of the homogeneous equation as follows. For $\eta\in\mathcal{R}(Q_0^+)$ and $\xi\in\mathcal{R}(I-Q_0^-)$, let

$$\begin{aligned} v_n^+ &= \Phi(n,0)\eta+\bar{v}_n^+(r_{\mathbb{Z}}+g_{\mathbb{Z}}\mu), \quad n\geq 0,\\ v_n^- &= \Phi(n,0)\xi+\bar{v}_n^-(r_{\mathbb{Z}}+g_{\mathbb{Z}}\mu), \quad n\leq 0.\end{aligned} \tag{4.13}$$

Then we set $v_n=v_n^+$ if $n\geq 0$ and $v_n=v_n^-$ if $n\leq -1$. Therefore, the pair $(v_{\mathbb{Z}},\mu)$ solves equations (4.10) and (4.11) if and only if

$$v_0^+=v_0^-, \quad \sum_{n=-\infty}^{\infty}\varepsilon\dot{\tilde{x}}_n^Tv_n=\omega. \tag{4.14}$$

Equation (4.14) reads

$$\begin{aligned} \xi - \eta + \Omega(\varepsilon, t^*)\mu &= \varrho(\varepsilon, t^*), \\ \Theta(\varepsilon, t^*)\xi + \Lambda(\varepsilon, t^*)\eta + \Xi(\varepsilon, t^*)\mu &= \omega - \delta(\varepsilon, t^*), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \Omega(\varepsilon, t^*) &= \sum_{i=-\infty}^{0} \Phi(0,i)Q_i^- g_{i-1} + \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+)g_{i-1}, \\ \Theta(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \Phi(n,0)(I - Q_0^-), \\ \Lambda(\varepsilon, t^*) &= \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \Phi(n,0)Q_0^+, \\ \Xi(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^-(g_{\mathbb{Z}}) + \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^+(g_{\mathbb{Z}}), \\ \varrho(\varepsilon, t^*) &= -\left(\sum_{i=-\infty}^{0} \Phi(0,i)Q_i^- r_{i-1} + \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+)r_{i-1} \right), \\ \delta(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^-(r_{\mathbb{Z}}) + \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^+(r_{\mathbb{Z}}). \end{aligned}$$

In the following we study the limit properties of the items above. For an illustration we analyze the second item of $\Omega(\varepsilon, t^*)$. The definition of ψ in (2.11) and estimate (3.6) in Theorem 3.1 give $g_i = \varepsilon f_\lambda(\bar{x}_i, \bar{\lambda}) + O(\varepsilon^2)$ and

$$\begin{aligned} & \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+)g_{i-1} \\ = \; & \varepsilon \sum_{i=1}^{\infty} \left[\Phi(0,i)(I - Q_i^+) - S(0, t^* + i\varepsilon)(I - P^+(t^* + i\varepsilon)) \right] f_\lambda(\bar{x}_{i-1}, \bar{\lambda}) \\ & + \varepsilon \sum_{i=1}^{\infty} S(0, t^* + i\varepsilon)(I - P^+(t^* + i\varepsilon))[f_\lambda(\bar{x}_{i-1}, \bar{\lambda}) - f_\lambda(\bar{x}_i, \bar{\lambda})] \\ & + \varepsilon \sum_{i=1}^{\infty} S(0, t^* + i\varepsilon)(I - P^+(t^* + i\varepsilon)) f_\lambda(\bar{x}(t^* + i\varepsilon), \bar{\lambda}) \\ & + \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+) \cdot O(\varepsilon^2) \\ = \; & B_1 + B_2 + B_3 + B_4. \end{aligned}$$

It follows from the assumption (H1) that $f_\lambda(\cdot,\cdot)$ is bounded. Noticing the estimate in (4.9) we obtain

$$\|B_1\| \leq C\varepsilon^2 \sum_{i=1}^{\infty} e^{-i\tilde{\alpha}\varepsilon} \leq C\varepsilon.$$

Since $|f_\lambda(\bar{x}_{i-1},\bar{\lambda}) - f_\lambda(\bar{x}_i,\bar{\lambda})| \leq C|\bar{x}(t^*+(i-1)\varepsilon) - \bar{x}(t^*+i\varepsilon)| \leq C\varepsilon$, together with the estimate in (2.6) we obtain $\|B_{2,4}\| \leq C\varepsilon$. Applying Lemma 2.5 we have

$$\lim_{\varepsilon\to 0} B_3 = \int_0^{\infty} S(0,t^*+t)(I-P^+(t^*+t))f_\lambda(\bar{x}(t^*+t),\bar{\lambda})dt$$

uniformly for t^* in any compact set. Therefore, we have shown

$$\lim_{\varepsilon,t^*\to 0} \sum_{i=1}^{\infty} \Phi(0,i)(I-Q_i^+)g_{i-1} = \int_0^{\infty} S(0,t)(I-P^+(t))f_\lambda(t)dt,$$

where $f_\lambda(t) = f_\lambda(\bar{x}(t),\bar{\lambda})$.

In a similar way, the remaining items in (4.15) can be investigated and we end up with

$$\begin{aligned}
\Omega(0) &= \lim_{\varepsilon,t^*\to 0} \Omega(\varepsilon,t^*) \\
&= \int_{-\infty}^{0} S(0,t)P^-(t)f_\lambda(t)dt + \int_0^{+\infty} S(0,t)(I-P^+(t))f_\lambda(t)dt \\
\Theta(0) &= \lim_{\varepsilon,\, t^*\to 0} \Theta(\varepsilon,t^*) = \int_{-\infty}^{0} \dot{\bar{x}}(t)^T S(t,0)(I-P_0^-)dt, \\
\Lambda(0) &= \lim_{\varepsilon,\, t^*\to 0} \Lambda(\varepsilon,t^*) = \int_0^{\infty} \dot{\bar{x}}(t)^T S(t,0)P_0^+ dt, \\
\Xi(0) &= \lim_{\varepsilon,\, t^*\to 0} \Xi(\varepsilon,t^*) = \int_0^{+\infty} \dot{\bar{x}}^T(t) \left[\int_0^t S(t,s)P^+(s)f_\lambda(t)ds \right. \\
& \quad \left. - \int_t^{+\infty} S(t,s)(I-P^+(s))f_\lambda(s)ds\right] dt \\
& \quad + \int_{-\infty}^{t} \dot{\bar{x}}^T(t) \left[\int_{-\infty}^{t} S(t,s)P^-(s)f_\lambda(s)\, ds \right. \\
& \quad \left. - \int_t^0 S(t,s)(I-P^-(s)f_\lambda(s))ds\right] dt.
\end{aligned}$$

Hence, as ε and t^* small enough the functions $\Omega(\varepsilon,t^*)$, $\Theta(\varepsilon,t^*)$, $\Lambda(\varepsilon,t^*)$ and $\Xi(\varepsilon,t^*)$ are uniformly bounded. From the expression of $(\varrho(\varepsilon,t^*),\delta(\varepsilon,t^*))$ and using Lemma 2.5 we can prove

$$\|(\varrho(\varepsilon,t^*),\delta(\varepsilon,t^*))\| \le C\tilde{K}\|r_{\mathbb{Z}}\|_1 \tag{4.16}$$

for some constant $C > 0$.

Rewrite equation (4.15) in a short form $M^{\varepsilon,t^*}(\xi,\eta,\mu) = (\varrho,\omega+\delta)$, where M^{ε,t^*} is a $(k_{+s}+k_{-s}+p)\times(k+1) = (k+1)\times(k+1)$ matrix. We will prove the matrix M^{ε,t^*} is invertible and its inverse is uniformly bounded for ε and t^* small.

According to the estimates above, we see that M^{ε,t^*} converges to some limit matrix M as $\varepsilon \to 0$ and $t^* \to 0$. We only need to prove that the matrix M is nonsingular. Assume $M(\xi,\eta,\mu) = 0$, where $\eta \in \mathcal{R}(Q_0^+)$, $\xi \in \mathcal{R}(I-Q_0^-)$ and $\mu \in \mathbb{R}^p$. Define

$$\begin{aligned} v^+(t) &= S(t,0)\eta + \int_0^t S(t,s)P^+(s)f_\lambda(s)\mu ds \\ &\quad - \int_t^\infty S(t,s)(I-P^+(s))f_\lambda(t)\mu ds, \quad \text{if } t \ge 0, \\ v^-(t) &= S(t,0)\xi + \int_{-\infty}^t S(t,s)P^-(s)f_\lambda(s)\mu ds \\ &\quad - \int_t^0 S(t,s)(I-P^-(s))f_\lambda(s)\mu ds, \quad \text{if } t \le 0. \end{aligned}$$

Clearly $v^\pm(\cdot)$ is a bounded solution of the following equation in $\mathbb{R}^\pm$

$$\dot{x}(t) - f_x(\bar{x}(t),\bar{\lambda})x(t) = f_\lambda(\bar{x}(t),\bar{\lambda})\mu \tag{4.17}$$

with initial properties $P^+(0)v^+(0) = \eta$ and $(I-P^-(0))v^-(0) = \xi$, respectively.

Let $v(t) = v^+(t)$ if $t \ge 0$ and $v(t) = v^-(t)$ if $t < 0$. Similar to the arguments above for the discrete case, we find that $v(t)$ solves equation (4.17) for $t \in \mathbb{R}$ and satisfies $\int_{-\infty}^{+\infty} \dot{\bar{x}}^T(t)v(t)dt = 0$ iff $M(\xi,\eta,\mu) = 0$. From the nondegenerate assumption (H4) it follows that $\mu = 0$ and $v(t) = c\dot{\bar{x}}(t)$ for some constant c. Therefore $\int_{-\infty}^{+\infty} c\dot{\bar{x}}(t)^T\dot{\bar{x}}(t)dt = 0$ which yields $c = 0$ and implies that $\eta = 0 = \xi$. By now, we have proved that the matrix M is invertible. Then there exist constants $\varepsilon_0 > 0$ and $t_0^* > 0$ such that for all $|\varepsilon| < \varepsilon_0$ and $|t^*| < t_0^*$, the matrices M^{ε,t^*} are nonsingular and their inverses are uniformly bounded. Thus we can uniquely solve the equation (4.15) from which we construct a unique bounded solution $(v_{\mathbb{Z}}, \mu)$ of equations (4.10) and (4.11). Noticing the definition for $\bar{v}^{\pm}_{\mathbb{Z}_\pm}$ in equation (4.13), the estimates in (4.12) and (4.16), we get for $|\varepsilon| < \varepsilon_0$ and $|t^*| < t_0^*$

$$|(v_n, \mu)| \leq C\tilde{K}(\|r_{\mathbb{Z}}\|_1 + |\omega|) \leq \frac{1}{\beta}(\|r_{\mathbb{Z}}\|_1 + |\omega|) \tag{4.18}$$

for some constant $\beta > 0$. If we require $\varepsilon_0 < t_0^*$, using the periodic property with respect to t^* in (3.7) and (3.8), we can extend this estimate (4.18) to $|\varepsilon| < \varepsilon_0$ and $t^* \in \mathbb{R}$. ■

Proof of Lemma 3.5 Differentiating equations $\tilde{F}(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*), t^*) = 0$ and $F(\bar{x}_{\mathbb{Z}}(t^*), \bar{\lambda}, t^*) = 0$ with respect to t^* gives, respectively

$$\begin{aligned} \tilde{F}'(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))(\dot{\tilde{x}}_{\mathbb{Z}}(t^*), \dot{\tilde{\lambda}}(t^*)) + \tilde{F}_{t^*} &= 0, \\ F'(\bar{x}_{\mathbb{Z}}(t^*), \bar{\lambda})(\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0) + F_{t^*} &= 0. \end{aligned}$$

Subtract these two equations from each other we obtain

$$\begin{aligned} & \tilde{F}'(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda})[(\dot{\tilde{x}}_{\mathbb{Z}}(t^*), \dot{\tilde{\lambda}}(t^*)) - (\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0)] \\ = \; & [(F'(\bar{x}_{\mathbb{Z}}, \bar{\lambda}) - \tilde{F}'(\bar{x}_{\mathbb{Z}}, \bar{\lambda})) + (\tilde{F}'(\bar{x}_{\mathbb{Z}}, \bar{\lambda}) - \tilde{F}'(\tilde{x}_{\mathbb{Z}}, \tilde{\lambda}))](\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0) + (F_{t^*} - \tilde{F}_{t^*}) \\ = \; & (A_1 + A_2)(\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0) + A_3. \end{aligned} \tag{4.19}$$

It is clear that $\|\dot{\bar{x}}_n(t^*)\| \leq CKe^{-\alpha\varepsilon|n|}$, then $\sum_{n=-\infty}^{\infty} \|\dot{\bar{x}}_{\mathbb{Z}}(t^*)\| \leq C/\varepsilon$.

According to the definition $\tilde{F}$ in (3.4), F in (3.5) and [18, Lemma 4.1] we find for $\varepsilon > 0$ small

$$\begin{aligned} \|A_1 \cdot (\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0)\| &\leq \sum_{n=-\infty}^{\infty} \|\varphi_x(\bar{x}_n(t^*), \bar{\lambda}) - \psi_x(\bar{x}_n(t^*), \bar{\lambda})\| \cdot \|\dot{\bar{x}}_{\mathbb{Z}}(t^*)\| \\ &\leq \sum_{n=-\infty}^{\infty} C\varepsilon^2 \|\dot{\bar{x}}_{\mathbb{Z}}(t^*)\| \leq C\varepsilon. \end{aligned}$$

It follows from the assumption (H1) that f_x is Lipschitz continuous, then

$$\begin{aligned}
\|A_2 \cdot (\dot{\bar{x}}_{\mathbb{Z}}(t^*), 0)\| &\leq \sum_{n=-\infty}^{\infty} \|\psi_x(\bar{x}_n(t^*), \bar{\lambda}) - \psi_x(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\| \cdot \|\dot{\bar{x}}_n(t^*)\| \\
&\leq \sum_{n=-\infty}^{\infty} \varepsilon \|f_x(\bar{x}_n(t^*), \bar{\lambda}) - f_x(\tilde{x}_n(t^*), \tilde{\lambda}(t^*))\| \cdot \|\dot{\bar{x}}_n(t^*)\| \\
&\leq \sum_{n=-\infty}^{\infty} L\varepsilon(\|\bar{x}_n(t^*) - \tilde{x}_n(t^*)\| + |\bar{\lambda} - \tilde{\lambda}(t^*)|)\|\dot{\bar{x}}_n(t^*)\| \\
&\leq C\varepsilon.
\end{aligned}$$

Noticing the definition of the phase condition Π in (3.3) we have

$$\Pi_{t^*}(\tilde{x}_{\mathbb{Z}}, \varepsilon, t^*) - \Pi_{t^*}(\bar{x}_{\mathbb{Z}}, \varepsilon, t^*) = \sum_{n=-\infty}^{\infty} \varepsilon(f_x(\bar{x}_n(t^*), \bar{\lambda})\dot{\bar{x}}_n(t^*))^T(\tilde{x}_n(t^*) - \bar{x}_n(t^*)).$$

Since the functions $\tilde{\Gamma}$ in (3.1) and Γ in (3.2) do not depend on t^* explicitly, we conclude that for $\varepsilon > 0$ small

$$\|A_3\| \leq \sum_{n=-\infty}^{\infty} C\varepsilon^2 \|\dot{\bar{x}}_n(t^*)\| \leq C\varepsilon.$$

By now we have proved that the right-hand side of equation (4.19) is the order of $O(\varepsilon)$ as $\varepsilon > 0$ small, together with the estimates for $\tilde{F}'$ in Lemma 3.4 we obtain that there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there holds

$$\sup_{n\in\mathbb{Z}} \|\dot{\tilde{x}}_n(t^*) - \dot{\bar{x}}_n(t^*)\| + |\dot{\tilde{\lambda}}| \leq C\varepsilon \quad \text{for } t^* \in \mathbb{R}. \tag{4.20}$$

Next we prove the existence of the function $t^*(\varepsilon)$ by using a qualitative implicit function theorem (cf. [18, Lemma 4.2]). Define a function

$$\rho(t^*, \varepsilon) = \vec{n}_0^T(\tilde{x}_0(\varepsilon, t^*) - \bar{x}(0)).$$

It is easy to see $\lim_{\varepsilon\to 0^+} \rho_{t^*}(0, \varepsilon) = \lim_{\varepsilon\to 0^+} \vec{n}_0^T \dot{\tilde{x}}_0(\varepsilon, 0) = \vec{n}_0^T \dot{\bar{x}}_0(0) > 0$. Let $\sigma = \vec{n}_0^T \dot{\bar{x}}_0(0)/2 > 0$, then $\sigma \leq \rho_{t^*}(0, \varepsilon)$ for $0 < \varepsilon < \varepsilon_0$ (we may reduce ε_0 if necessary). Let $\kappa = \sigma/2$ and because for $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} & |\rho_{t^*}(t^*, \varepsilon) - \rho_{t^*}(0, \varepsilon)| = |\vec{n}_0^T(\dot{\tilde{x}}_0(\varepsilon, t^*) - \dot{\tilde{x}}_0(\varepsilon, 0))| \\ \le\ & |\dot{\tilde{x}}_0(\varepsilon, t^*) - \dot{\bar{x}}_0(\varepsilon, t^*)| + |\dot{\bar{x}}_0(\varepsilon, t^*) - \dot{\bar{x}}_0(\varepsilon, 0)| + |\dot{\bar{x}}_0(\varepsilon, 0) - \dot{\tilde{x}}_0(\varepsilon, 0)| \\ \le\ & C\varepsilon + |f(\bar{x}(t^*), \bar{\lambda}) - f(\bar{x}(0), \bar{\lambda})| \\ \le\ & C(\varepsilon + |t^*|) \end{aligned}$$

then there exists a constant $t_0^* > 0$ such that for $|t^*| < t_0^*$ and $0 < \varepsilon < \varepsilon_0$ (we may reduce ε_0 if necessary) there holds

$$|\rho_{t^*}(t^*, \varepsilon) - \rho_{t^*}(0, \varepsilon)| \le \kappa < \sigma \le |\rho_{t^*}(0, \varepsilon)|.$$

Since $|\rho(0, \varepsilon)| \le |\tilde{x}_0(\varepsilon, t^*) - \bar{x}(0)| \le C\varepsilon$, we know that there exists a constant $\varepsilon_0 > 0$ such that

$$|\rho(0, \varepsilon)| \le (\sigma - \kappa)t_0^* \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Applying a qualitative implicit function theorem of [18, Lemma 4.2] to equation $\rho(t^*, \varepsilon) = 0$, then we obtain a unique continuous solution $t^* = t^*(\varepsilon)$ $(0 < \varepsilon < \varepsilon_0)$, which implies $\tilde{x}_0(\varepsilon, t^*(\varepsilon)) \in \Sigma_\Omega$.

It is easy to see $\lim\limits_{t^* \to 0, \varepsilon \to 0^+} \rho(t^*, \varepsilon) = 0$, which implies $\lim\limits_{\varepsilon \to 0^+} t^*(\varepsilon) = 0$.

For $|t^* - t^*(\varepsilon)| > 0$ small, using Taylor formula we calculate

$$\begin{aligned} \vec{n}_0^T(\tilde{x}_0(\varepsilon, t^*) - \bar{x}(0)) &= \vec{n}_0^T(\tilde{x}_0(\varepsilon, t^*) - \tilde{x}_0(\varepsilon, t^*(\varepsilon))) \\ &= \vec{n}_0^T \dot{\tilde{x}}_0(\varepsilon, \theta(\varepsilon, t^*))(t^* - t^*(\varepsilon)), \end{aligned} \tag{4.21}$$

where $\theta(\varepsilon, t^*)$ is a number between t^* and $t^*(\varepsilon)$. As $t^* \to 0$ and $\varepsilon \to 0^+$, there hold $\theta(\varepsilon, t^*) \to 0$ and $\vec{n}_0^T \dot{\tilde{x}}_0(\varepsilon, \theta(\varepsilon, t^*)) \to \vec{n}_0^T \dot{\bar{x}}_0 > 0$. Therefore the sign of the left-hand side in equation (4.21) is determined only by the sign of $t^* - t^*(\varepsilon)$ and this ensures that $\tilde{x}_0(\varepsilon, t^*)$ locates in the desired region of $\Omega_\pm$ while $\pm(t^* - t^*(\varepsilon)) > 0$ and $|t^*| < t_0^*$.

Now, we start to prove for $n \ne 0$, $\tilde{x}_n(\varepsilon, t^*(\varepsilon)) \notin \Sigma_\Omega$. As preparation we define a manifold

$$M(t_0^*) = \{\bar{x}(t^*); |t^*| \ge t_0^*\} \cup \{\bar{x}_\pm\}$$

which is bounded and compact. Due to the assumption in (H5), the distance between this manifold $M(t_0^*)$ and the segment Σ_Ω is positive. From the estimates in (3.6) and the periodic property in (3.7) and (3.8) it follows that

$$\|\tilde{x}_0(\varepsilon, t^*) - \bar{x}(t^*)\| = O(\varepsilon) \quad \text{for } t^* \in \mathbb{R}.$$

Hence for $\varepsilon > 0$ small and $|t^*| \geq t_0^*$ there holds $\tilde{x}_0(\varepsilon, t^*) \notin \Sigma_\Omega$.

For contradiction we assume there exist $0 < \hat{\varepsilon} < \varepsilon_0$ and $n_0 \neq 0$ such that $\tilde{x}_{n_0}(\hat{\varepsilon}, t^*(\hat{\varepsilon})) \in \Sigma_\Omega$ with $\tilde{x}_{n_0}(\hat{\varepsilon}, t^*(\hat{\varepsilon})) = \tilde{x}_0(\hat{\varepsilon}, t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon})$. If $|t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon}| \geq t_0^*$, then $\tilde{x}_0(\hat{\varepsilon}, t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon}) \notin \Sigma_\Omega$, thus there must be $|t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon}| < t_0^*$, which together with the assumption above, implies $\rho(t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon}, \hat{\varepsilon}) = 0$. From the uniqueness of the solution of the equation $\rho(t^*, \varepsilon) = 0$ it follows that $t^*(\hat{\varepsilon}) + n_0\hat{\varepsilon} = t^*(\hat{\varepsilon})$ which implies either $n_0 = 0$ or $\hat{\varepsilon} = 0$. This is impossible. Hence $\tilde{x}_n(\varepsilon, t^*(\varepsilon)) \notin \Sigma_\Omega$ for $n \neq 0$.

Similarly we can prove that $\tilde{x}_0(\varepsilon, t^*) \notin \Sigma_\Omega$ for $0 < \varepsilon < \varepsilon_0$ and $t^* \neq t^*(\varepsilon) + \ell\varepsilon$ $(\ell \in \mathbb{Z})$. This, together with the periodic property in (3.7), implies that $\tilde{x}_n(\varepsilon, t^*) \notin \Sigma_\Omega$ for all n and $t^* \neq t^*(\varepsilon) + \ell\varepsilon$ $(\ell \in \mathbb{Z})$. ■

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