Generalized Hopf bifurcation for non-smooth planar dynamical systems

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Abstract

In this paper, we study the existence of periodic orbits bifurcating from stationary solutions in a non-smooth planar dynamical system. This phenomenon is interpreted as generalized Hopf bifurcation. In the case of smoothness, Hopf bifurcation is characterized by a pair of complex conjugate eigenvalues crossing through the pure imaginary axis. This method does not apply to a non-smooth system due to the lack of linearization. In fact, the generalized Hopf bifurcation is determined by interactions between the discontinuity of the system and the eigenstructure of each subsystem. We combine a geometrical method and an analytical method to investigate the generalized Hopf bifurcation. The bifurcating periodic orbits are obtained by studying the fixed points of return maps.

Keywords: non-smooth dynamical system, generalized Hopf bifurcation, periodic orbit.

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1 Introduction

Our research interests emanate from a practical problem which models a brake system provided by K. Popp (1999, personal communication). This problem is described by a 6-dimensional piecewise smooth system of the form

$$\dot{x} = \begin{cases} f^{+}(x,\lambda) & \text{if} \quad x_1 > 0, \\ f^{-}(x,\lambda) & \text{if} \quad x_1 < 0, \end{cases}$$

where $x \in \mathbb{R}^6$ and $\lambda \in \mathbb{R}^p$ is a parameter. Assume $f^{\pm}(0,\lambda) \equiv 0$. The main purpose of this subject is to study the existence of bounded solutions or periodic solutions near stationary points.

In the case of smoothness, it is an efficient way to study the existence of periodic orbits by means of Hopf bifurcation. Here, a pair of complex eigenvalues is assumed to exist and cross transversally the imaginary axis. Applying center manifold theory, a high dimensional system can be reduced to a planar system. Then the existence of periodic orbits follows from studying Hopf bifurcation, see e.g. [14]. In the case of non-smoothness, there is no straightforward way to extend the Hopf bifurcation theory to a high dimensional system. In fact, there is no result at hand on the existence of invariant manifolds for high dimensional non-smooth system near a stationary point located at the discontinuity. This means that the reduction for a non-smooth system from high dimension to low dimension so far is not available.

Understanding Hopf bifurcation for smooth planar systems plays an important role when studying high dimensional systems. Despite the lack of a suitable reduction procedure in the non-smooth case it seems worthwhile to study the emergence of periodic orbits for a non-smooth planar system.

In the papers [15, 12, 20, 19] sufficient conditions are given that guarantee the emergence of a branch of periodic orbits from a stationary solution in a piecewise smooth system. This phenomenon is called **generalized Hopf bifurcation**. The condition combines the eigenstructure of the smooth subsystems with the behavior of the vector field on the line of discontinuity. It is shown in [19] that in the corner case with several lines of discontinuity a rich bifurcation behavior with complicated dynamics can occur.

In this paper we prove a theorem on generalized Hopf bifurcation that treats a more general situation than in [15, 12].

To be more specific we consider a piecewise smooth planar system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} f^{+}(x, y, \lambda), & x > 0 \\ f^{-}(x, y, \lambda), & x < 0 \end{cases} \tag{1.1}$$

where

$$f^{\pm}(x, y, \lambda) = A^{\pm}(\lambda)(x, y)^{T} + g^{\pm}(x, y, \lambda) \tag{1.2}$$

with $|g^{\pm}(x, y, \lambda)| = \mathcal{O}(x^2 + y^2)$ as $(x, y) \to 0$. Assume that the matrix $A^{\pm}$ has a pair of complex eigenvalues $\alpha^{\pm}(\lambda) \pm i\omega^{\pm}(\lambda)$, where $i^2 = -1$. In the case of smoothness, by proper choice of coordinate the matrix $A^{\pm}(\lambda)$ can be assumed to be in normal form

$$A^{\pm}(\lambda) = \begin{pmatrix} \alpha^{\pm}(\lambda) & \omega^{\pm}(\lambda) \\ -\omega^{\pm}(\lambda) & \alpha^{\pm}(\lambda) \end{pmatrix}. \tag{1.3}$$

We show that the classical condition $\alpha^{\pm}(0) = 0$, $\frac{d}{d\lambda}\alpha^{\pm}(0) \neq 0$ and $\omega(0) > 0$ for Hopf bifurcation in the smooth case can be replaced by

$$B(0) = 1, \quad B'(0) \neq 0,$$

where $B(\lambda)$ is the bifurcation function

$$B(\lambda) = \exp\left[\pi\left(\frac{\alpha^{+}(\lambda)}{\omega^{+}(\lambda)} + \frac{\alpha^{-}(\lambda)}{\omega^{-}(\lambda)}\right)\right].$$

This function exactly determines the Poincaré map on the y-axis in the piecewise linear case.

In contrast to [15, 12, 20, 19] we do not assume that the linearization on both sides of the y-axis can be transformed to normal form (1.3) simultaneously.

The paper is organized as follows. In section 2, we discuss the systems and basic assumptions. In section 3, we study the existence of a family of periodic orbits for the piecewise linear system and we also derive the function $B(\lambda)$ which is used to characterize the generalized Hopf bifurcation. In section 4, we state and prove the main results of this paper. Finally in section 5 we study an example to show the return map and its fixed points.

2 Basic assumptions

In this section we first introduce a class of non-smooth planar systems which will be analyzed in this paper, and then discuss some basic assumptions.

Let Ω be an open disk with radius $r > 0$ centered about the origin and let Λ be an open interval containing 0 in its interior. Define the semi-disk $\Omega^{\pm} = \Omega \cap \{(x, y)^T : \pm x > 0\}$ and denote its closure by $\overline{\Omega^{\pm}}$.

Throughout this paper we use the functions $K^{\pm} = (K_1^{\pm}, K_2^{\pm})$ defined on $\Omega \times \Lambda$ to describe the vector fields.

Consider a non-smooth planar parameter dependent dynamical system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} K_1(x, y, \lambda) \\ K_2(x, y, \lambda) \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \Omega, \quad \lambda \in \Lambda, \tag{2.1}$$

where the function $K = (K_1, K_2)^T : \Omega \times \Lambda \to \mathbb{R}^2$ is piece-wise defined as

$$K(x, y, \lambda) = \begin{cases} K^+(x, y, \lambda), & x > 0, \\ K^-(x, y, \lambda), & x < 0. \end{cases}$$

Our assumptions are:

(H1) The function $K^{\pm}(x, y, \lambda)$ is C^k-smooth ($k \geq 2$) for $(x, y, \lambda) \in \Omega \times \Lambda$.

(H2) $K^{\pm}(0, 0, \lambda) \equiv 0$ for all $\lambda \in \Lambda$.

Assumption (H1) specifies that the discontinuity only appears on the y-axis. (H2) ensures that the origin is always a stationary point for the system (2.1). It results from the assumptions (H1) and (H2) that the function $K^{\pm}$ has the following form by Taylor expansion at $x = y = 0$

$$K^{\pm}(x, y, \lambda) = A^{\pm}(\lambda)(x, y)^T + (g_1^{\pm}(x, y, \lambda), g_2^{\pm}(x, y, \lambda))^T, \qquad (2.2)$$

where $A^{\pm}(\lambda)$ is a 2×2 matrix, the high order term $g^{\pm} = (g_1^{\pm}, g_2^{\pm})^T$ is C^k smooth and satisfies $|g_{1,2}^{\pm}| = \mathcal{O}(x^2 + y^2)$ as $(x, y) \to 0$ uniformly for $\lambda \in \Lambda$.

Remark 2.1 *The equation* (2.2) *allows to define a piecewise linearization for equation* (2.1)*:* $(\dot{x}, \dot{y})^T = A^{\pm}(\lambda)(x, y)^T$. *This equation will play an important role in the following discussions.*

(H3) The matrix $A^{\pm}(\lambda)$ possesses a pair of complex eigenvalues $\alpha^{\pm}(\lambda) \pm i\omega^{\pm}(\lambda)$ with $\omega^{\pm}(0) > 0$.

Remark 2.2 *1) For smooth system, the assumptions*

$$\alpha^{\pm}(0) = 0, \quad \frac{d}{d\lambda}\alpha^{\pm}(0) \neq 0$$

are assumed in order to study the Hopf bifurcation. For the non-smooth system (2.1)*, the singularities at the origin depend not only on the eigenvalue structures of the matrix A^- and A^+, but also on the discontinuity of the system. Therefore, the conditions which are used to characterize the generalized Hopf bifurcation are more complicated.*

2) It follows from assumption (H3) *that there exists a constant $\omega^* > 0$ such that $\omega^{\pm}(\lambda) \geq \omega^*$ for $\lambda \in \Lambda$, where the interval Λ may be shrunk if necessary.*

According to assumption (H3) there exists an invertible matrix $Q^{\pm}(\lambda)$ such that

$$A^{\pm}(\lambda) = \begin{pmatrix} a_{11}^{\pm}(\lambda) & a_{12}^{\pm}(\lambda) \\ a_{21}^{\pm}(\lambda) & a_{22}^{\pm}(\lambda) \end{pmatrix} = Q^{\pm}(\lambda)^{-1} \begin{pmatrix} \alpha^{\pm}(\lambda) & \omega^{\pm}(\lambda) \\ -\omega^{\pm}(\lambda) & \alpha^{\pm}(\lambda) \end{pmatrix} Q^{\pm}(\lambda). \quad (2.3)$$

In this paper, we study period orbits that transversally cross through the y-axis and circle round the origin. For this purpose we require:

(H4) $a_{12}^{\pm}(0) > 0$. Or **(H4′)** $a_{12}^{\pm}(0) < 0$**.**

In a standard way we may extend system (4.1) to a differential inclusion ([6, 9])

$$(\dot{x}(t), \dot{y}(t))^T \in \mathcal{K}(x(t), y(t), \lambda) \quad (2.4)$$

by setting $\mathcal{K} : \Omega \times \Lambda \to 2^{\mathbb{R}^2}$ and

$$\mathcal{K}(x, y, \lambda) = \begin{cases} \big\{K^{\pm}(x, y, \lambda)\big\}, & \text{if } (x, y) \in \Omega^{\pm}, \\ \big\{sK^{+}(0, y, \lambda) + (1 - s)K^{-}(0, y, \lambda) : 0 \le s \le 1\big\}, & \\ & \text{if } (0, y) \in \Omega. \end{cases}$$

In a small neighborhood of the origin, maximal existence and uniqueness in forward time of a solution of the differential inclusion (2.4) are guaranteed by the assumptions (H1)–(H3), (H4) or (H4′) (cf. [6, 9]).

The assumptions (H4) and (H4′) assure that the flow of equation (2.1) crosses the y-axis clockwise and counter-clockwise (cf. Figures 2.1 and 2.2). These two assumptions are equivalent to each other by reversing time. Therefore, it is sufficient to work with (H4).

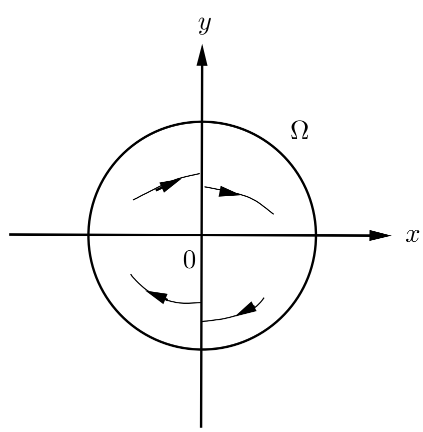

Figure 2.1 *The evolutionary direction of flow under* (H4).

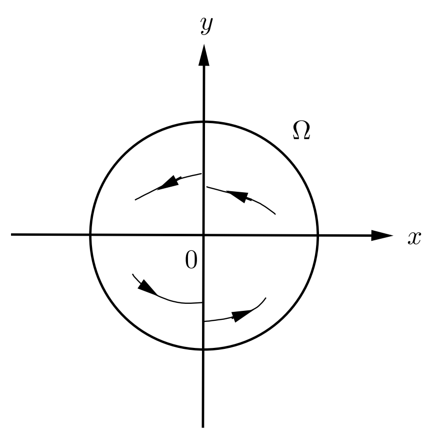

Figure 2.2 *The evolutionary direction of flow under* (H4′).

3 Properties of a piecewise linearization

In this section we study the following piecewise linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} A^{+}(\lambda)\begin{pmatrix} x \\ y \end{pmatrix}, & x > 0, \\ A^{-}(\lambda)\begin{pmatrix} x \\ y \end{pmatrix}, & x < 0. \end{cases} \tag{3.1}$$

We will prove that (3.1) possesses a family of continuous periodic orbits. Since we can explicitly solve each subsystem of equation (3.1), we can set up a Poincaré map on the y-axis for each subsystem. Then we define a return map for equation (3.1) on the y-axis by composing these two Poincaré maps. Finally, we study the periodic orbits of equation (3.1) by searching for fixed points of the return map.

Let $\Pi^{\pm} = \{(x, y)^T : \pm x > 0\}$ and $M^{\pm} = \{(0, y)^T : \pm y > 0\}$. Consider the linear flow

$$\varphi^{\pm}(\lambda, t) = e^{\alpha^{\pm}(\lambda)t} \begin{pmatrix} \cos(\omega^{\pm}(\lambda)t) & \sin(\omega^{\pm}(\lambda)t) \\ -\sin(\omega^{\pm}(\lambda)t) & \cos(\omega^{\pm}(\lambda)t) \end{pmatrix}.$$

Then, the function $\Phi^{\pm}(x_0, y_0, \lambda, t) = Q^{\pm}(\lambda)^{-1}\varphi^{\pm}(\lambda, t)Q^{\pm}(\lambda)(x_0, y_0)^T$ solves equation $(\dot{x}, \dot{y})^T = A^{\pm}(\lambda)(x, y)^T$ with initial value (x_0, y_0) at $t = 0$.

For any $(0, y_0^{\pm})^T \in M^{\pm}$ and as time t increases, the flow $\Phi^{\pm}(0, y_0^{\pm}, \lambda, t)$ will leave the set $\overline{\Pi^{\mp}}$ through $M^{\pm}$ transversally and enter the set $\overline{\Pi^{\pm}}$, cf. Figure 3.1. Furthermore, after finite time this flow will transversally reach the line $M^{\mp}$ again. Hence, we can define a Poincaré map

$$P^{\pm} : \mathbb{R}^{\pm} \times \Lambda \to \mathbb{R}^{\mp}$$

such that $(0, P^{\pm}(y, \lambda))^T \in M^{\mp}$ denotes the first exit point of $\overline{\Pi^{\pm}}$ for initial value $(0, y)^T \in M^{\pm}$ under the evolution of the flow $\Phi^{\pm}(x, y, \lambda, t)$. Further, we denote the corresponding minimal passage time of the flow inside $\overline{\Pi^{\pm}}$ by $T^{\pm}(y, \lambda)$.

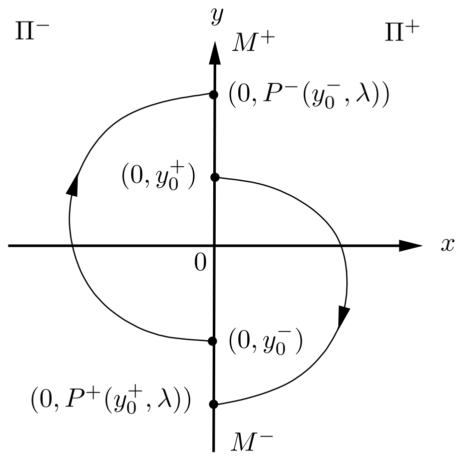

Figure 3.1 *Illustrations of the Poincaré map for each linear subsystem of* (3.1)*.*

Lemma 3.1 *Assume* (H1)–(H4)*. Then, for* $y \in \mathbb{R}^{\pm}$ *and* $\lambda \in \Lambda$ *we have:*

1) The function $T^{\pm}$ *is independent of* y *and satisfies*

$$T^{\pm}(y, \lambda) \equiv \frac{\pi}{\omega^{\pm}(\lambda)}.$$

2) The function $P^{\pm}$ *can be expressed as*

$$P^{\pm}(y, \lambda) = y e_2^T Q^{\pm}(\lambda)^{-1} \varphi^{\pm}(\lambda, T^{\pm}(y, \lambda)) Q^{\pm}(\lambda) e_2, \tag{3.2}$$

where $e_2 = (0, 1)^T$.

Proof. According to the definition of $T^{\pm}(y, \lambda)$ we have

$$\Phi^{\pm}(0, y, \lambda, T^{\pm}(y, \lambda)) \in M^{\mp}, \quad \text{for } (0, y)^T \in M^{\pm}.$$

This means that the first coordinate of $\Phi^{\pm}(0, y, \lambda, T^{\pm}(y, \lambda))$ vanishes, i.e.

$$e_1^T \Phi^{\pm}(0, y, \lambda, T^{\pm}(y, \lambda)) = 0, \tag{3.3}$$

where $e_1 = (1, 0)^T$. Assume

$$Q^{\pm}(\lambda) = \begin{pmatrix} q_{11}^{\pm}(\lambda) & q_{12}^{\pm}(\lambda) \\ q_{21}^{\pm}(\lambda) & q_{22}^{\pm}(\lambda) \end{pmatrix}, \quad Q^{\pm}(\lambda)^{-1} = \begin{pmatrix} \bar{q}_{11}^{\pm}(\lambda) & \bar{q}_{12}^{\pm}(\lambda) \\ \bar{q}_{21}^{\pm}(\lambda) & \bar{q}_{22}^{\pm}(\lambda) \end{pmatrix}.$$

To simplify the notations, sometimes we drop the explicit dependence on the parameter λ in the following evaluations.

$$\begin{aligned} & e_1^T\Phi^\pm(0,y,\lambda,t) \\ = \; & e_1^T Q^\pm(\lambda)^{-1}\varphi^\pm(\lambda,t)Q^\pm(\lambda)(0,y)^T = \big(\bar q_{11}^\pm,\bar q_{12}^\pm\big)\varphi^\pm(\lambda,t)\big(q_{12}^\pm,q_{22}^\pm\big)^T y \\ = \; & y e^{\alpha^\pm t}\big((\bar q_{11}^\pm q_{12}^\pm+\bar q_{12}^\pm q_{22}^\pm)\cos(\omega^\pm t)+(\bar q_{11}^\pm q_{22}^\pm-\bar q_{12}^\pm q_{12}^\pm)\sin(\omega^\pm t)\big). \end{aligned}$$

It follows from $Q^\pm(\lambda)^{-1}\cdot Q^\pm(\lambda)\equiv I$ that $\bar q_{11}^\pm q_{12}^\pm+\bar q_{12}^\pm q_{22}^\pm\equiv 0$ for all $\lambda\in\Lambda$. Due to (2.3) we compute

$$a_{12}^\pm=\alpha^\pm(\bar q_{11}^\pm q_{12}^\pm+\bar q_{12}^\pm q_{22}^\pm)+\omega^\pm(\bar q_{11}^\pm q_{22}^\pm-\bar q_{12}^\pm q_{12}^\pm)=\omega^\pm(\bar q_{11}^\pm q_{22}^\pm-\bar q_{12}^\pm q_{12}^\pm). \quad (3.4)$$

Assumption (H4) implies $\bar q_{11}^\pm q_{22}^\pm-\bar q_{12}^\pm q_{12}^\pm>0$. Then the first exist time is the smallest positive solution of $\sin(\omega^\pm(\lambda)t)=0$, which leads to the first assertion.

$P^\pm(y,\lambda)$ equals to the second coordinate of $\Phi^\pm(0,y,\lambda,T^\pm(y,\lambda))$, which gives the expression (3.2). ■

Remark 3.2 *Clearly, we can exactly express the matrix* $Q^\pm(\lambda)^{-1}$ *in terms of the elements of the matrix* $Q^\pm(\lambda)$. *Then equation* (3.4) *becomes*

$$a_{12}^\pm=\frac{\omega^\pm}{\det Q^\pm}((q_{12}^\pm)^2+(q_{22}^\pm)^2).$$

Therefore, the assumption (H4) *is equivalent to that the transformation matrix* $Q^\pm$ (cf. (2.3)) *has positive orientation,* i.e., $\det Q^\pm>0$.

By $\Phi(x,y,\lambda,t)$ we denote the solution flow of equation (3.1) which is defined as follows. If $(x,y)\in\Omega^\pm$, define $\Phi(x,y,\lambda,t)=\Phi^\pm(x,y,\lambda,t)$ for all t such that $\Phi^\pm(x,y,\lambda,t)\in\Omega^\pm$. If $x=0$, by assumption (H4) there exists a number $0<t_0<\min\{\frac{\pi}{\omega^+(\lambda)},\frac{\pi}{\omega^-(\lambda)}\}$ such that we can define

$$\Phi(0,y,\lambda,t)=\begin{cases}\Phi^+(0,y,\lambda,t), & \text{for} \quad y>0, \quad 0<t<t_0,\\ \Phi^-(0,y,\lambda,t), & \text{for} \quad y>0, \quad -t_0<t<0,\\ (0,0), & \text{for} \quad y=0, \quad |t|<t_0,\\ \Phi^-(0,y,\lambda,t), & \text{for} \quad y<0, \quad 0<t<t_0,\\ \Phi^+(0,y,\lambda,t), & \text{for} \quad y<0, \quad -t_0<t<0.\end{cases} \quad (3.5)$$

Since (3.1) is an autonomous system the flow $\Phi(x, y, \lambda, t)$ is well defined for t on some interval $J = J(x, y, \lambda)$ (a local dynamical system in the sense of [1, Ch. 10]).

Expand the definition for the functions $P^{\pm}$ and $T^{\pm}$ to $y = 0$ by setting

$$P^{\pm}(0, \lambda) \equiv 0, \quad T^{\pm}(0, \lambda) \equiv \frac{\pi}{\omega^{\pm}(\lambda)}.$$

Then Lemma 3.1 implies that the extended functions $P^{\pm}$ and $T^{\pm}$ are sufficiently smooth in $(y, \lambda) \in (\{0\} \cup \mathbb{R}^{\pm}) \times \Lambda$.

Now, we define the return map $P(y, \lambda)$ and the time map $T(\lambda)$ for the system (3.1) by setting

$$\begin{aligned} P(y, \lambda) &= \begin{cases} P^{-}(P^{+}(y, \lambda), \lambda), & y > 0, \\ 0, & y = 0, \\ P^{+}(P^{-}(y, \lambda), \lambda), & y < 0, \end{cases} \\ T(\lambda) &= \frac{\pi}{\omega^{+}(\lambda)} + \frac{\pi}{\omega^{-}(\lambda)}. \end{aligned} \tag{3.6}$$

Also define a bifurcation function as

$$B(\lambda) = e_2^T Q^{+}(\lambda)^{-1} \varphi^{+}(\lambda, \frac{\pi}{\omega^{+}(\lambda)})) Q^{+}(\lambda) e_2 \cdot e_2^T Q^{-}(\lambda)^{-1} \varphi^{-}(\lambda, \frac{\pi}{\omega^{-}(\lambda)}) Q^{-}(\lambda) e_2.$$

Direct computations give

Proposition 3.3 *Assume* (H1)-(H4). *Then for* $\lambda \in \Lambda$

$$\begin{aligned} B(\lambda) &= \exp\left[\pi\left(\frac{\alpha^{+}(\lambda)}{\omega^{+}(\lambda)} + \frac{\alpha^{-}(\lambda)}{\omega^{-}(\lambda)}\right)\right], \\ P(y, \lambda) &= B(\lambda) y, \quad \textit{for } y \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Our final assumption is

(H5) $\dfrac{\alpha^{+}(0)}{\omega^{+}(0)} + \dfrac{\alpha^{-}(0)}{\omega^{-}(0)} = 0, \ \left(\dfrac{\alpha^{+}(\lambda)}{\omega^{+}(\lambda)} + \dfrac{\alpha^{-}(\lambda)}{\omega^{-}(\lambda)}\right)'\Bigg|_{\lambda=0} \neq 0.$

Remark 3.4 *1) For the case in which both* A^{+} *and* A^{-} *are in normal form formula* (3.7) *already appears in* [15, 12].

2) Our bifurcation function applies to smooth dynamical systems. If the equation (2.1) *is smooth, formula* (3.7) *becomes*

$$B(\lambda) = e^{\frac{2\pi\alpha(\lambda)}{\omega(\lambda)}}.$$

This leads to the standard assumption for Hopf bifurcation.

The following theorem follows from the fact that the fixed points of the returned map $P(\cdot, \lambda)$ correspond to the continuous periodic orbits of the piece-wise linear system (3.1).

Theorem 3.5 *Assume* (H1)–(H5)*.*

(i) *There is a non-trivial periodic orbit for system* (3.1) *iff* $\lambda = 0$ *or* $B(\lambda) = 1$*.*

(ii) *For* $\lambda = 0$ *there is a family of periodic orbits surrounding the origin with period* $\frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$*.*

(iii) *The stationary solution* $(0, 0)$ *of system* (3.1) *is asymptotically stable if* $|B(\lambda)| < 1$ *and unstable if* $|B(\lambda)| > 1$*.*

4 The generalized Hopf bifurcation

In this section we study the generalized Hopf bifurcation for a nonlinear piecewise smooth system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} K_1(x, y, \lambda) \\ K_2(x, y, \lambda) \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \Omega, \quad \lambda \in \Lambda, \tag{4.1}$$

where the function $K = (K_1, K_2)$ is defined as in equation (2.1) and satisfies the assumptions (H1)–(H5).

In the case of smoothness (cf. [1, 4]), the Hopf bifurcation is investigated via studying fixed points of a Poincaré map. Here, we use a similar strategy to study the generalized Hopf bifurcation. The local behavior of the flow of each subsystem of (4.1) near the origin is studied in [2, §8]. These properties ensure that we can define a Poincaré map on each half y-axis for each subsystem of (4.1). A return map is then defined as the composition of these two Poincaré maps. We search for the bifurcating periodic orbits of system (4.1) by studying the nontrivial fixed points of the return map.

By $\Psi^{\pm}(x, y, \lambda, t)$ we denote the solution flow of each subsystem of (4.1). Now we study a Poincaré map $\tilde{P}^{+}$ for the subsystem of (4.1) when $x \geq 0$.

Let $\tilde{M}^{\pm} = M^{\pm} \cap \Omega$. According to [2] there exist constants $0 < \delta^{+} < r$, $\tau^{+} > 0$ and a time function $\tilde{T}^{+} : (0, \delta^{+}) \times \Lambda \to \mathbb{R}^{+}$ such that the following properties hold: for all $0 < y < \delta^{+}$ and $\lambda \in \Lambda$ the flow $\Phi^{+}(0, y, \lambda, t) \in \Omega$ for $0 \leq t \leq \tau^{+}$. Furthermore, this flow has a transversal intersection with $\tilde{M}^{-}$ at time $t = \tilde{T}^{+}(y, \lambda)$ and for $0 < t < \tilde{T}^{+}(y, \lambda)$ it locates inside the right-hand half plane. The Poincaré map is then defined as the y-coordinate of the intersection point

$$\tilde{P}^{+} : \begin{array}{rcl} (0, \delta^{+}) \times \Lambda & \to & \mathbb{R}^{-}, \\ (y, \lambda) & \mapsto & e_2^T \Psi^{+}(0, y, \lambda, \tilde{T}^{+}(y, \lambda)). \end{array}$$

Clearly, if the nonlinear term $g^{\pm}$ vanishes, i.e. $g^{\pm} \equiv 0$, we have

$$\tilde{P}^{+}(y, \lambda) = P^{+}(y, \lambda), \quad \tilde{T}^{+}(y, \lambda) = T^{+}(y, \lambda).$$

In a similar way we define a Poincaré map and a time map for the subsystem of (4.1) when $x \leq 0$:

$$\tilde{P}^{-} : (-\delta^{-}, 0) \times \Lambda \to \mathbb{R}, \quad \tilde{T}^{-} : (-\delta^{-}, 0) \times \Lambda \to \mathbb{R}$$

where $\delta^{-} > 0$ is a small constant.

It is easy to see that there exists a constant $0 < \delta < \delta^{\pm}$ such that $\tilde{P}^{+}(y, \lambda) \in (-\delta^{-}, 0)$ for $0 < y < \delta$ and simultaneously $\tilde{P}^{-}(y, \lambda) \in (0, \delta^{+})$ for $-\delta < y < 0$. According to [11, 4, 10, 1], the functions $\tilde{P}^{\pm}$ and $\tilde{T}^{\pm}$ are smooth for $0 < \pm y < \delta^{\pm}$ and $\lambda \in \Lambda$.

Extend the definition of $\tilde{P}^{\pm}$ and $\tilde{T}^{\pm}$ to the origin $y = 0$ by setting $\tilde{P}^{\pm}(0, \lambda) \equiv 0$ and $\tilde{T}^{\pm}(0, \lambda) = \frac{\pi}{\omega^{\pm}(\lambda)}$. The next lemma follows from the fact that the flow $\Psi^{\pm}(x, y, \lambda, t)$ smoothly depends on initial data (x, y, λ) [11, 1].

Lemma 4.1 *Assume* (H1)–(H4).

1) The functions $\tilde{P}^{\pm}$ and $\tilde{T}^{\pm}$ are continuous for $0 \leq \pm y < \delta^{\pm}$ and $\lambda \in \Lambda$.

2) As $y \to 0$ and $y > 0$, the one-sided limits of the derivatives $\tilde{P}_z^{+}$ and $\tilde{T}_z^{+}$, $z \in \{y, \lambda, y\lambda, \lambda y\}$ exist, which are denoted by $\tilde{P}_z^{+}(0^{+}, \lambda)$ and $\tilde{T}_z^{+}(0^{+}, \lambda)$, respectively. Furthermore, $\tilde{P}_{y\lambda}^{+}(y, \lambda) = \tilde{P}_{\lambda y}^{+}(y, \lambda)$ for $0 \leq y < \delta$.

3) As $y \to 0$ and $y < 0$, the one-sided limits of the derivatives $\tilde{P}_z^{-}$ and $\tilde{T}_z^{-}$, $z \in \{y, \lambda, y\lambda, \lambda y\}$ exist, which are denoted by $\tilde{P}_z^{-}(0^{-}, \lambda)$ and $\tilde{T}_z^{-}(0^{-}, \lambda)$, respectively. Furthermore, $\tilde{P}_{y\lambda}^{-}(y, \lambda) = \tilde{P}_{\lambda y}^{-}(y, \lambda)$ for $-\delta < y \leq 0$.

Define a return map $\tilde{P} : (-\delta, \delta) \times \Lambda \to \mathbb{R}$ and a time map $\tilde{T} : (-\delta, \delta) \times \Lambda \to \mathbb{R}$ for the system (4.1) by

$$\tilde{P}(y, \lambda) = \begin{cases} \tilde{P}^-(\tilde{P}^+(y, \lambda), \lambda), & 0 < y < \delta, \\ 0, & y = 0, \\ \tilde{P}^+(\tilde{P}^-(y, \lambda), \lambda), & -\delta < y < 0, \end{cases} \tag{4.2}$$

$$\tilde{T}(y, \lambda) = \begin{cases} \tilde{T}^+(y, \lambda) + \tilde{T}^-(\tilde{P}^+(y, \lambda), \lambda), & 0 < y < \delta, \\ T(\lambda), & y = 0, \\ \tilde{T}^-(y, \lambda) + \tilde{T}^+(\tilde{P}^-(y, \lambda), \lambda), & -\delta < y < 0. \end{cases} \tag{4.3}$$

Lemma 4.2 *Assume* (H1)–(H4).

1) The time map $\tilde{T}(y, \lambda)$ is continuous for $|y| < \delta$ and $\lambda \in \Lambda$.

2) The return map $\tilde{P}$ together with its derivatives $\tilde{P}_z$, $z \in \{y, \lambda, y\lambda, \lambda y\}$ is continuous and satisfies $\tilde{P}_{y\lambda}(y, \lambda) = \tilde{P}_{\lambda y}(y, \lambda)$ for $|y| < \delta$ and $\lambda \in \Lambda$. Furthermore,

$$\frac{\partial \tilde{P}}{\partial \lambda}(0, \lambda) \equiv 0, \quad \frac{\partial \tilde{P}}{\partial y}(0, \lambda) = B(\lambda). \tag{4.4}$$

Proof. The first assertion is a straightforward corollary of Lemma 4.1. The second assertion is obviously true for $0 < |y| < \delta$. We only need to prove the case of $y = 0$ and equation (4.4).

For any $0 < y < \delta$, we have $\tilde{P}^+(y, \lambda) = e_2^T \Psi^+(0, y, \lambda, \tilde{T}^+(y, \lambda))$, differentiating this function with respect to y gives

$$\tilde{P}_y^+(y, \lambda) = e_2^T \Psi_y^+\big(0, y, \lambda, \tilde{T}^+(y, \lambda)\big) + e_2^T \frac{d}{dt} \Psi^+\big(0, y, \lambda, \tilde{T}^+(y, \lambda)\big) \tilde{T}_y^+(y, \lambda). \tag{4.5}$$

Clearly $\frac{d}{dt}\Psi^+(0, 0, \lambda, t) = f^+(0, 0, \lambda) \equiv 0$ for all t and the one-sided limit of the function $\tilde{T}_y^+(y, \lambda)$ exists as $y \to 0$ and $y > 0$. Let $y \to 0$ with $y > 0$ in equation (4.5), we obtain

$$\tilde{P}_y^+(0^+, \lambda) = e_2^T \Psi_y^+\big(0, 0, \lambda, \tilde{T}^+(0^+, \lambda)\big). \tag{4.6}$$

For any $(x, y) \in \Omega$, the flow $\Psi^+(x, y, \lambda, t)$ satisfies

$$\frac{d}{dt} \Psi^+(x, y, \lambda, t) = A^+(\lambda) \Psi^+(x, y, \lambda, t) + g^+\big(\Psi^+(x, y, \lambda, t), \lambda\big). \tag{4.7}$$

Differentiate equation (4.7) with respect to y

$$\frac{d}{dt}\Psi_y^+(x,y,\lambda,t) = A^+(\lambda)\Psi_y^+(x,y,\lambda,t) + g_{(x,y)}^+\big(\Psi^+(x,y,\lambda,t),\lambda\big)\Psi_y^+(x,y,\lambda,t). \tag{4.8}$$

Let $(x,y)=(0,0)$ in equation (4.8) and then we obtain

$$\frac{d}{dt}\Psi_y^+(0,0,\lambda,t) = A^+(\lambda)\Psi_y^+(0,0,\lambda,t). \tag{4.9}$$

The equality $\Psi^+(x,y,\lambda,0)\equiv(x,y)^T$ yields $\Psi_y^+(0,0,\lambda,0)=(0,1)^T$. Solving equation (4.9) with this initial condition leads to

$$\Psi_y^+(0,0,\lambda,t) = Q^+(\lambda)^{-1}\Phi^+(0,0,\lambda,t)Q^+(\lambda)e_2. \tag{4.10}$$

It follows from the equation (4.6) and (4.10) that

$$\tilde{P}_y^+(0^+,\lambda) = e_2^T Q^+(\lambda)^{-1}\Phi^+\big(0,0,\lambda,T^+(0^+,\lambda)\big)Q^+(\lambda)e_2.$$

In a similar way we can prove

$$\tilde{P}_y^-(0^-,\lambda) = e_2^T Q^-(\lambda)^{-1}\Phi^-\big(0,0,\lambda,T^-(0^-,\lambda)\big)Q^-(\lambda)e_2.$$

Therefore, we obtain

$$\begin{aligned}\tilde{P}_y(0^+,\lambda) &= \tilde{P}_y^-(0^-,\lambda)\tilde{P}_y^+(0^+,\lambda) = B(\lambda)\\ &= \tilde{P}_y^+(0^+,\lambda)\tilde{P}_y^-(0,^-,\lambda) = \tilde{P}_y(0^-,\lambda).\end{aligned} \tag{4.11}$$

Thus, the function $\tilde{P}_y(y,\lambda)$ is continuous at $y=0$. Similarly, we can prove that $\tilde{P}_z(y,\lambda)$ is continuous at $y=0$ for $z\in\{\lambda, y\lambda, \lambda y\}$.

Obviously, $\tilde{P}_{y\lambda}(y,\lambda)=\tilde{P}_{\lambda y}(y,\lambda)$ for $0<|y|<\delta$. The property $\tilde{P}_{y\lambda}(0,\lambda)=\tilde{P}_{\lambda y}(0,\lambda)$ results from the continuity of the function $\tilde{P}_z$, $z\in\{y\lambda,\lambda y\}$ at $y=0$.

$\tilde{P}(0,\lambda)\equiv 0$ implies the first assertion of equation (4.4), and the equation (4.11) leads to the second assertion. ■

Remark 4.3 *We can prove that the time map $\tilde{T}$ is not differentiable at $y=0$ by direct evaluation.*

The nontrivial fixed points of the return map $\tilde{P}(\cdot,\lambda)$ correspond to the nontrivial periodic solutions of system (4.1). Instead of searching for fixed points of $\tilde{P}(\cdot,\lambda)$ we study zeroes of a distance function $V(y,\lambda) := \tilde{P}(y,\lambda) - y$. Separate the trivial solution $y \equiv 0$ by considering the equivalent function $\tilde{V} : (-\delta,\delta)\times\Lambda \to \mathbb{R}$ defined by

$$\tilde{V}(y,\lambda) = \int_0^1 \frac{\partial V}{\partial y}(sy,\lambda)ds.$$

It follows from Lemma 4.2 that the function $\tilde{V}(y,\lambda)$ together with its derivative $\tilde{V}_\lambda(y,\lambda)$ is continuous for $|y|<\delta$ and $\lambda\in\Lambda$. Equation (4.4) implies

$$\tilde{V}(0,\lambda) = B(\lambda) - 1, \quad \tilde{V}_\lambda(0,\lambda) = B_\lambda(\lambda).$$

Applying an appropriate version of the implicit function theorem [5] to the equation $\tilde{V}(y,\lambda) = 0$ near $(y,\lambda) = (0,0)$, we obtain existence and local uniqueness of the nonzero solutions for $\tilde{P}(y,\lambda) = y$. This leads to our main results.

Theorem 4.4 *Assume* (H1)–(H5). *At $\lambda = 0$ there bifurcates a continuous branch of periodic orbits from the origin;* i.e. *there is a constant $\delta_0 > 0$ and a uniquely determined continuous function $\lambda^* : (-\delta_0,\delta_0) \to \mathbb{R}$ satisfying $\lambda^*(0) = 0$ such that for each $y \in (-\delta_0,\delta_0)$ there is a periodic orbit $\gamma^*(y)$ of equation* (4.1) *passing through $(0,y)$ at the parameter $\lambda = \lambda^*(y)$ with period $\tilde{T}(y,\lambda^*(y))$. The function $\tilde{T}$ is continuous and satisfies $\tilde{T}(0,0) = \frac{\pi}{\omega^+(0)} + \frac{\pi}{\omega^-(0)}$. Moreover there is no other periodic orbit of system* (4.1) *locally near $x = y = 0$ and $\lambda = 0$.*

Proof. Using (4.4) we select a neighborhood $\mathcal{U} := [-\delta_0,\delta_0]^2 \times \Lambda_0$ such that $\frac{\partial \tilde{P}}{\partial y}(y,\lambda) > 0$ for $|y| < \delta_0$ and $\lambda \in \Lambda_0$. Then for any given $\lambda \in \Lambda_0$ the function $\tilde{P}(y,\lambda)$ is strictly monotone in $y \in [-\delta_0,\delta_0]$.

For a contradiction we assume that there is a parameter $\lambda_0 \in \Lambda_0$ at which, besides the bifurcating periodic orbit, the system (4.1) possesses a periodic orbit in $[-\delta_0,\delta_0]^2$ excluding the original point inside its interior. According to [2, §8 Lemma 4], this periodic orbit can not entirely lie inside the right-hand or the left-hand half plane. Therefore, it at least intersects the y-axis at one point, say y_0. Without loss of generality we assume $y_0 > 0$. Then the next two intersection points with the y-axis are of the form $(0, y_1)$, $y_1 < 0$ and $(0, \tilde{P}(y_0, \lambda_0))$, $\tilde{P}(y_0, \lambda_0) \in (0, \delta_0]$. Now, if $\tilde{P}(y_0, \lambda_0) = y_0$, by local uniqueness of the periodic obit above we have $\lambda_0 = \lambda^*(y_0)$ and the given orbit coincide with $\gamma^*(y_0)$. If $\tilde{P}(y_0, \lambda_0) \neq y_0$, then due to the monotonicity of $\tilde{P}(\cdot, \lambda_0)$ we obtain $\tilde{P}^n(y_0, \lambda_0) \neq y_0$, $\forall n \geq 1$ which contradicting the periodicity of the given orbit. ■

Remark 4.5 *1) In the case of smoothness, the direction of the bifurcation is determined by the shape of the function $\lambda = \lambda^*(y)$, which is usually given by the derivatives of higher order* ([16]). *In the case of non-smoothness, this approach is not available since the bifurcation function λ^* is only continuous.*

2) In certain cases results on one-sided stability for the bifurcating periodic orbit can be given. Using results in [13] *it can be shown that the stability properties* (iii) *in* Theorem 3.5 *of the stationary solution for the piecewise linear system* (3.1) *carry over to the stationary solution for the full system* (4.1). *Then these stability properties of the stationary solution together with the uniqueness in* Theorem 4.4 *imply that the bifurcating periodic orbit through* $(0, y, \lambda)$ *is asymptotically stable from the interior if $|B(\lambda)| > 1$ and unstable if $|B(\lambda)| < 1$.*

3) An alternative way of proving Theorem 4.4 *is to pretransform the given system with piecewise linear transformations that leave y-axis invariant and to apply the Poincaré technique afterwards. We have not carried out the details of such an approach.*

5 An example

In this section we illustrate the return map $\tilde{P}$ by an example. Consider a model which is approximated by a mass moving horizontally along a surface under the action of two nonlinear springs (cf. Figure 5.1). By u we denote the position of the mass such that $u = 0$ represents the state that both springs are unloaded. We assume that if the displacement is positive, i.e. $u > 0$, only the spring 1 takes action on the mass and the friction force between the mass and the surface is $b^+(\lambda)\dot{u}$, where λ is a control parameter. If the displacement is negative, i.e. $u < 0$, only the spring 2 takes action on the mass and we assume that the surface is smooth such that the friction force with the mass can be neglected. Meanwhile, we also assume that there exists an external force field exciting the mass with an extra force $b^{-}(\lambda)\dot{u}$, where $b^{-}(\lambda) < 0$ when $u < 0$. Then the equation of motion can be expressed as

$$m\ddot{u} + b^{\pm}(\lambda)\dot{u} + (a^{\pm} + f^{\pm}(u))u = 0, \quad \pm u > 0, \tag{5.1}$$

where $a^{\pm}+f^{\pm}(u)$ is the coefficient of the nonlinear springs with $|f(u)| = \mathcal{O}(u)$ as $u \to 0$.

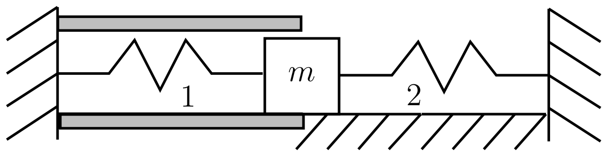

Figure 5.1 *A mass controlled by two springs.*

Without loss of generality we assume $m = 1$. Rewrite equation (5.1) as a planar system by setting $x = u$ and $y = \dot{u}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -b^{\pm}(\lambda)y - a^{\pm}x - f^{\pm}(x)x \end{pmatrix}, \quad \pm x > 0. \tag{5.2}$$

Clearly $(0, 0)$ is always a stationary solution. The piecewise linearization of the system (5.1) at the origin is determined by a matrix

$$A^{\pm}(\lambda) = \begin{pmatrix} 0 & 1 \\ -a^{\pm} & -b^{\pm}(\lambda) \end{pmatrix}.$$

The corresponding eigenvalues $\alpha^{\pm}(\lambda) \pm i\omega^{\pm}(\lambda)$ are given by

$$\alpha^{\pm}(\lambda) = -\frac{1}{2}b^{\pm}(\lambda), \quad \omega^{\pm}(\lambda) = \frac{1}{2}\sqrt{4a^{\pm} - b^{\pm}(\lambda)^2}.$$

For simplicity we assume $b^{\pm}(\lambda) = b_0^{\pm} + b_1^{\pm}\lambda$ and $f^{\pm}(x) = \beta^{\pm}x^2$. Then the first assertion of condition (H5) is equivalent to

$$b_0^{-} = -\sqrt{\frac{a^{-}}{a^{+}}} \cdot b_0^{+}. \tag{5.3}$$

As an example we choose $a^{+} = 0.1$, $a^{-} = 0.2$, $b_0^{+} = 0.05$ and $b_0^{-} = -\frac{\sqrt{2}}{20}$, which satisfy the equation (5.3). Taking $b_1^{\pm} = 1$, then the second assertion of assumption (H5) holds. Let $\beta^{\pm} = 1$ to fix the high order terms. We have computed numerically the return map $\tilde{P}(\cdot, \lambda)$ at different parameters $\lambda = 0.05, 0, -0.4$ (cf. Figure 5.2). For $\lambda > 0$ the graph of the return map $\tilde{P}(\cdot, \lambda)$ and the line $\tilde{P} = y$ have a transversal intersection at the origin. At $\lambda = 0$ they only have a tangential contact at the origin. For $\lambda < 0$, besides the origin they have two transversal intersection points which correspond to the bifurcating periodic orbits.

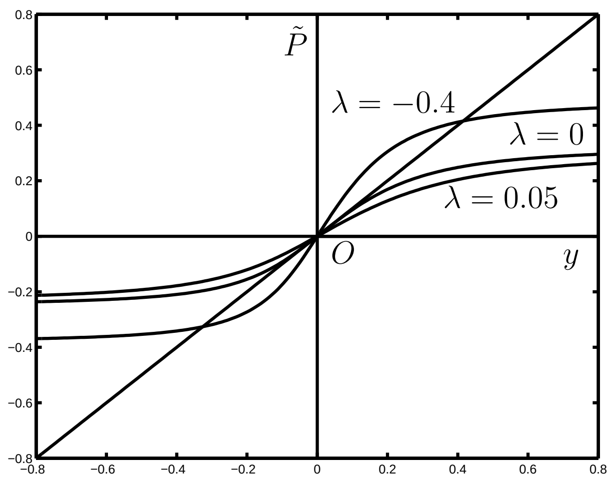

Figure 5.2 *The return maps $\tilde{P}(\cdot, \lambda)$ versus y at different λ-values.*

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