Dynamical systems generated by parabolic equations on the real line

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Abstract

A parabolic reaction-diffusion equation is considered on the real line. An existence theorem is proved in a weighted Sobolev space with a specific weight. Estimates of solutions and their higher derivatives are obtained. The existence of the global attractor in the sense of Babin-Vishik is established.

1 Introduction

The qualitative study of dynamical systems generated by partial differential equations on unbounded domains has been started not long ago. This study was stimulated by Babin and Vishik who gave a definition of an attractor which corresponds to a pair of spaces (see [1], [2]). Later several papers appeared, studying attractors of dynamical systems generated by various partial differential equations and by their discretizations (see [6], [3], [7], [8] for example).

We consider the equation

$$\partial_t u(t,x) = \Delta u(t,x) - f(u(t,x)), \tag{1}$$

$$u|_{t=0} = u_0, \tag{2}$$

with a smooth globally Lipschitz continuous nonlinearity f. In [2], Babin and Vishik have proved the existence of an (H, H_w)-attractor of the dynamical system generated by equation (1) on an unbounded domain, where H is a weighted analog of the Sobolev space W_2^1 and H_w is the same space endowed with the weak topology. In [6], Mielke and Schneider studied the Ginzburg-Landau equation and proved the existence of the global attractor in the case where the two topologies chosen, correspond to the weighted Sobolev norm and a stronger norm called "‘uniformly local"’.

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In this paper, we prove the existence of the attractor for the dynamical system generated by the one-dimensional equation (1)-(2) for the same pair of spaces as in [6]. Our estimates will also be essential in proving the convergence of attractors for discretizations of (1), see the subsequent paper [4].

The structure of the paper is as follows. Section 2 contains the terminology, notation and elementary properties, we later use. The main result of Section 3 is an existence theorem for a solution of equations (1)-(2). Section 3 contains the estimates of solutions of equation (1)-(2) while Section 4 provides estimates of higher derivatives. Finally, in Section 5 we prove the existence of the attractor.

2 Terminology and elementary properties

In this section, we introduce notation and conventions we later use. Also we mention some known facts.
1) Let E be a Banach space. We use the symbol $\|\cdot\|_E$ to denote its norm. The symbol E^* denotes the space of continuous linear functionals on the space E.

Given a number $p \in [1;+\infty]$, choose p' such that $\frac{1}{p}+\frac{1}{p'}=1$ holds.

We define some spaces of mappings:
a) $L_p([0;T];E)$, $T \geq 0$, is the space of mappings $u:[0;T]\to E$ defined for almost all $t\in[0;T]$, such that the function $\|u(\cdot)\|_E$ belongs to the space $L_p([0;T])$, cf. [7].
b) $C_0^\infty((0;T);E)$ is the space of mappings $u \in C^\infty([0;T];E)$ such that $\overline{\operatorname{supp} u} \subset (0;T)$. Similarly,

$$C_0^\infty([0;T);E) := \{u \in C^\infty([0;T];E) : \overline{\operatorname{supp} u} \subset [0;T)\}.$$

2) Let $\Omega \subset \mathbb{R}$ be an open interval. We denote by $H^l(\Omega)$ the usual Sobolev spaces of real valued functions that have local derivatives in $L_2(\Omega)$ up to the order l.

Fix a number $\gamma < 0$ and introduce the weight function $\varphi(x) := (1+|x|^2)^\gamma$.

We denote by $H_{0,\gamma}(\Omega)$ the space of locally integrable functions $u:\Omega\to\mathbb{R}$, defined for almost all $x\in\Omega$, such that

$$\|\varphi^{1/2}u\|_{L_2(\Omega)} = \int\limits_\Omega \varphi(x)|u(x)|^2\,dx < +\infty.$$

The space $H_{0,\gamma}(\Omega)$ is endowed with the scalar product

$$\langle u;v\rangle_{0,\gamma} := \int\limits_\Omega \varphi(x)u(x)v(x)\,dx.$$

We also use the Sobolev spaces $H_{l,\gamma}(\Omega)$ ($l \in \mathbb{N}$) of functions $u \in H_{0,\gamma}(\Omega)$ such that $\mathcal{D}^\alpha u \in H_{0,\gamma}(\Omega)$ for all $\alpha \le l$. The spaces $H_{l,\gamma}(\Omega)$ are endowed with the scalar product

$$\langle u; v\rangle_{l,\gamma} := \sum_{0\le\alpha\le l} \langle \mathcal{D}^\alpha u; \mathcal{D}^\alpha v\rangle_{0,\gamma}.$$

In the following we consider the weight functions $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ for $\varepsilon > 0$.

Lemma 2.1 ***(properties of the functions*** φ_ε ***and spaces*** $H_{l,\gamma}$***)***
There exist constants $C_1 = C_1(\varepsilon, \gamma)$ and $C_2 = C_2(\gamma)$ that are independent of Ω, with the following properties.

(1) If a function u belongs to the space $H_{l,\gamma}(\Omega)(l = 0, 1,$ or $2)$, then $\varphi_\varepsilon^{1/2} u \in H^l(\Omega)$, and the following inequalities hold:

$$C_1^{-1}\|\varphi_\varepsilon^{1/2}u\|_{H^l(\Omega)} \le \|u\|_{l,\gamma} \le C_1\|\varphi_\varepsilon^{1/2}u\|_{H^l(\Omega)}. \tag{3}$$

Note that $H^0(\Omega) := L_2(\Omega)$.

(2) The following inequalities hold:

$$|\varphi_\varepsilon'(x)| \le C_2\varepsilon(\varphi_\varepsilon(x))^{1-(2\gamma)^{-1}} \le C_2\varepsilon\varphi_\varepsilon(x),\ x \in \mathbb{R}. \tag{4}$$

For a proof, we refer to [3].

It follows from Lemma 2.1 that for any fixed ε, the norms with weights φ_ε instead of the weight φ in the spaces $H^l_\gamma(\Omega)$ are equivalent to the original ones.

We also use the Sobolev space $\overset{0}{W}{}^1_2(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ w.r.t. the norm of $W^1_2(\Omega)$. Additional properties can be found in [5]. We denote the corresponding weighted space by $\overset{0}{H}_{1,\gamma}(\Omega)$.

3) Let E and E_0 be Banach spaces, $E \subset E_0$. Given arbitrary elements $f \in E_0^*$ and $v \in E_0$, we denote $\langle f; v\rangle_{E_0} := f(v)$. We will omit the index 'E_0' when it does not lead to a confusion. We call a mapping $w \in L_1([0;T]; E_0)$ an E_0-derivative of a mapping $u \in L_1([0;T]; E)$ if for any mapping $\varphi \in C_0^\infty((0;T); E_0^*)$, the identity

$$\int_0^T \langle\varphi(t); w(t)\rangle\, dt = -\int_0^T \langle\partial_t\varphi(t); u(t)\rangle\, dt \tag{5}$$

holds. We call such a derivative a weak derivative as opposed to the 'strong' Sobolev derivatives.

If $E_0 = E^*$, we call the mapping w the derivative of the mapping u and denote it by $\partial_t u$. Note that the meaning of the introduced symbol depends on the embedding of E into E^*. We fix the embedding later if necessary.

Lemma 2.2 ***(traces and the integration by parts formula)***
Let H be a Hilbert space with a scalar product $\langle\cdot;\cdot\rangle_H$, let V be a Banach space such that $V \subset H \subset V^*$ (in this case, we have the standard embedding of the space V into the space V^* which is the composition of the embeddings $V \hookrightarrow H \hookrightarrow H^* \hookrightarrow V^*$), and let p be a number from the interval $[1;+\infty]$. Let functions $u, v \in L_p([0;T];V) \cap L_\infty([0;T];H)$ have the derivatives $\partial_t u, \partial_t v \in L_{p'}([0;T];V^*)$. Then for any $t \in [0;T]$, the traces $u(t) \in H$ and $v(t) \in H$ are well-defined and the following equality holds:

$$\int_0^T \langle \partial_t u; v\rangle\, dt + \int_0^T \langle \partial_t v; u\rangle\, dt = \langle u(T); v(T)\rangle_H - \langle u(0); v(0)\rangle_H.$$

See [1], p.30 for a proof.

Note that the value of a mapping $u \in L_p([0;T];V)$ at a point is not defined, in general. Below we denote by $u(t)$ the trace of the mapping u at a point t.

We also note that under the conditions of Lemma 2.2, the function $\langle u(t); v(t)\rangle_H$ has a Sobolev derivative

$$\partial_t \langle u(t); v(t)\rangle_H = \langle \partial_t u; v\rangle + \langle \partial_t v; u\rangle$$

of the class $L^1[0;T]$.

4) We use the Gronwall lemma in the following two forms.

Lemma 2.3 ***(Integral Gronwall inequality)*** *Consider functions $f \in W_1^1([0;T])$ and $g : [0;T] \to \mathbb{R}^+$ such that for some constants C_1 and $C_2 \geq 0$, the following inequality holds:*

$$f(t) + g(t) \leq C_1 + C_2 \int_0^t f(\tau)\, d\tau \text{ for all } t \in [0;T].$$

Then the following inequalities hold:

$$f(t) \leq C_1 e^{C_2 t} \quad \text{and} \quad g(t) \leq C_1 e^{C_2 t}.$$

Lemma 2.4 ***(differential Gronwall inequality)***
Let $f \in W_1^1[0,T]$ be a function satisfying the following inequality:

$$f'(t) \leq -\lambda f(t) + C_0,$$

for almost all $t \in [0;T]$, where the constant $\lambda \geq 0$. Then the following inequality holds for all $t \in [0;T]$:

$$f(t) \leq C_0 \lambda^{-1}(1 - e^{-\lambda t}) + f(0) e^{-\lambda t}.$$

A proof for the case $f \in C^1[0;T]$ can be found in [1], p.18. The case $f \in W_1^1[0;T]$ is treated similarly.

3 Existence theorem

Let $\Omega \subset \mathbb{R}$ be an interval. We consider the boundary value problem

$$\partial_t u(t,x) = \Delta u(t,x) - f(u(t,x),t,x), \tag{6}$$

$$u|_{t=0} = u_0, \tag{7}$$

$$u|_{\partial\Omega} = 0, \tag{8}$$

where these equalities are true for almost all $(t,x) \in [0;T] \times \Omega$.

We assume that $f \in C(\mathbb{R} \times [0;T] \times \Omega)$ satisfies the following inequality:

$$|f(u,t,x)| \leq C_f |u| \tag{9}$$

with a constant C_f. We also assume that $u_0 \in H_{0,\gamma}(\Omega)$.

Assume that the function $f = f(u)$ is Lipschitz continuous and satisfies the following dissipativity condition:

$$f(u) \cdot u \geq \lambda_0 |u|^2 - \lambda_1 \tag{10}$$

with some constants $\lambda_0 > 0$ and $\lambda_1 \in \mathbb{R}$ (we work with such functions in Section 5). It is easy to see that such a function f has a zero $v \in \mathbb{R}$. By the change of variable $u := u - v$, we obtain an equation (6)-(8) with the function $\tilde{f}(u) = f(u+v)$ having 0 as a zero. Since the function $\tilde{f}$ is Lipschitz continuous too, it satisfies inequality (9).

We consider two kinds of solutions of equation (6)-(8). A weak solution of equation (6)-(8) is a mapping

$$u \in L_\infty([0;T]; H_{0,\gamma}(\Omega)) \cap L_2([0;T]; \overset{0}{H}_{1,\gamma}(\Omega))$$

having $\partial_t u$ as a weak derivative and satisfies the equalities (6)-(8). Here we embed the right-hand side of equation (6) into the space

$$L_2([0;T]; (H_{1,\gamma}(\Omega))^*)$$

according to the following formula:

$$\int_0^T \langle \Delta u - f; w \rangle dt = -\int_0^T \int_\Omega (u'_x (\varphi w)'_x + \varphi f w) dx dt$$

for all mappings $w \in L_2([0;T]; H_{1,\gamma}(\Omega))$. Note that if the function $\Delta u - f$ belongs to the space $L_2([0;T]; H_{0,\gamma}(\Omega))$ the standard embedding of this function to the space $L_2([0;T]; (H_1(\Omega))^*)$ leads to the same result.

A strong solution of equation (6)-(8) is a mapping

$$u \in L_\infty([0;T]; H_{1,\gamma}(\Omega)) \cap L_2([0;T]; H_{2,\gamma}(\Omega))$$

having $\partial_t u$ as a strong derivative and satisfying the equalities (6)-(8).

We note that any strong solution is also a weak solution. Furthermore, if $\gamma \leq -1/2$ (this is the case considered in Section 5), then the above-mentioned change of variables $u := u - v$ transforms all functional spaces, we use, onto themselves.

For a bounded interval Ω, the following equalities hold:

$$H_{0,\gamma}(\Omega) = L_2(\Omega), \ \overset{0}{H}_{1,\gamma}(\Omega) = \overset{0}{W}{}_2^1(\Omega),$$

and the corresponding norms are equivalent. Hence, the definition of the weak derivative implies that the mapping

$$u \in L_\infty([0;T]; L_2(\Omega)) \cap L_2([0;T]; \overset{0}{W}{}_2^1(\Omega))$$

is a weak solution of equation (6)-(8) if and only if for any mapping

$$v \in L_2([0;T]; \overset{0}{W}{}_2^1(\Omega)) \cap L_\infty([0;T]; L_2(\Omega)), \quad v(T) = 0,$$

having a strong derivative $\partial_t v \in L_2([0;T]; L_2(\Omega))$, the following equality holds:

$$\int_0^t \int_\Omega (u'_x v'_x + f(u)v - u\partial_t v)\, dxdt = \langle u_0; v(0)\rangle_{L_2(\Omega)}.$$

Thus, weak and strong solutions of equation (6)-(8) correspond to the standard weak and strong solutions of a parabolic equation on a bounded domain. The corresponding theory states that under our assumptions, there exists a weak solution of equation (6)-(8). If $u_0 \in H_{1,\gamma}(\Omega)$ is additionally assumed, a strong solution of equation (6)-(8) exists.

Our aim in this section is to prove an existence theorem for strong solutions in case of an unbounded interval Ω. We begin with estimates of solutions. Here and below, we denote by C positive constants that do not depend on the interval Ω, on the initial value u_0 and on a solution u. The value of C may be different in different places.

Lemma 3.1 *Let the interval Ω be bounded. Then the solution $u(t,x)$ of the equation (6)-(8) satisfies the inequalities*

$$\|u(t)\|_{0,\gamma} \leq C\|u_0\|_{0,\gamma}\, e^{Ct}, \ t \in [0;T], \tag{11}$$

$$\int_0^T \|u(t)\|_{1,\gamma}^2\, dt \leq C\|u_0\|_{0,\gamma}^2\, e^{CT}. \tag{12}$$

Note that we do not have to specify whether we consider a weak or a strong solution before Lemma 3.4 since our statements here hold for weak solutions.

Proof.

Fix a number $\varepsilon > 0$. Let us first calculate the $(\overset{0}{W}{}_2^1(\Omega))^*$-derivative $\partial_t'(\varphi_\varepsilon^{1/2}u)$ of the mapping $\varphi_\varepsilon^{1/2}u$ corresponding to the embedding

$$\overset{0}{W}{}_2^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow (\overset{0}{W}{}_2^1(\Omega))^*.$$

For an arbitrary function $v \in H^1([0,T]; \overset{0}{W}{}_2^1(\Omega))$ it follows from the definition of weak derivatives and the embedding of the function $\Delta u - f$ that

$$\begin{aligned}
&\int_0^T \langle \partial_t'(\varphi_\varepsilon^{1/2}u); v\rangle_{\overset{0}{W}{}_2^1(\Omega)} \\
= &-\int_0^T \langle \varphi_\varepsilon^{1/2}u; \partial_t v\rangle_{\overset{0}{W}{}_2^1(\Omega)} \\
= &-\int_0^T \int_\Omega \varphi_\varepsilon^{1/2}u\partial_t v \\
= &-\int_0^T \int_\Omega \varphi u\partial_t(\frac{\varphi_\varepsilon^{1/2}}{\varphi}v) \\
= &\int_0^T \langle \partial_t u; \frac{\varphi_\varepsilon^{1/2}}{\varphi}v\rangle_{H_{1,\gamma}(\Omega)} \\
= &-\int_0^T \int_\Omega (u_x'(\varphi_\varepsilon^{1/2}v)_x' + \varphi_\varepsilon^{1/2}fv).
\end{aligned}$$

Hence,

$$\langle \partial_t'(\varphi_\varepsilon^{1/2}u); v\rangle_{\overset{0}{W}{}_2^1(\Omega)} = -\int_\Omega (u_x'(\varphi_\varepsilon^{1/2}v)_x' + \varphi_\varepsilon^{1/2}fv).$$

After the derivative ∂_t' is calculated we apply Lemma 2.2 to the mappings $u := \varphi_\varepsilon^{1/2}u$ and $v := \varphi_\varepsilon^{1/2}u$ (and spaces $V := \overset{0}{W}{}_2^1(\Omega)$, $H := L_2(\Omega)$) and the interval $[0, t_1]$ and deduce that

$$\begin{aligned}
&\|\varphi_\varepsilon^{1/2}u(t_1)\|_{L_2(\Omega)}^2 - \|\varphi_\varepsilon^{1/2}u(0)\|_{L_2(\Omega)}^2 \\
= &\ 2\int_0^{t_1} \langle \partial_t'(\varphi_\varepsilon^{1/2}u); \varphi_\varepsilon^{1/2}u\rangle_{\overset{0}{W}{}_2^1(\Omega)}\, dt \\
= &-2\int_0^{t_1} \int_\Omega ((\varphi_\varepsilon)' u\, u_x' + \varphi_\varepsilon (u_x')^2 + \varphi_\varepsilon fu)\, dx\, dt. \qquad (13)
\end{aligned}$$

Estimating the right-hand side of (13), we get by (4)

$$\|\varphi_\varepsilon^{1/2}u(t_1)\|^2_{L_2(\Omega)} + 2\int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon (u'_x)^2\,dxdt \tag{14}$$
$$\leq \|\varphi_\varepsilon^{1/2}u(0)\|^2_{L_2(\Omega)} + C\varepsilon\int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon(|u|^2+|u'_x|^2)\,dxdt + C\int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon|u|^2\,dxdt,$$

where the constant C does not depend even on the number ε. We choose the number ε such that $0<\varepsilon<C^{-1}$. Then inequality (14) can be rewritten as

$$\|\varphi_\varepsilon^{1/2}u(t_1)\|^2_{L_2(\Omega)} + \int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon|u'_x|^2\,dxdt \leq \|\varphi_\varepsilon^{1/2}u(0)\|^2_{L_2(\Omega)} + C\int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon|u|^2\,dxdt. \tag{14'}$$

Applying Lemma 2.3, we get the following inequalities:

$$\begin{aligned}\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\Omega)} &\leq C\|\varphi_\varepsilon^{1/2}u(0)\|^2_{L_2(\Omega)}\cdot e^{Ct},\\ \int\limits_0^t \|\varphi_\varepsilon^{1/2}u'_x(\tau)\,\|^2_{L_2(\Omega)}\,d\tau &\leq C\|\varphi_\varepsilon^{1/2}u(0)\|^2_{L_2(\Omega)}\cdot e^{Ct}.\end{aligned}$$

Due to inequality (3), the estimates (11) and (12) are valid.

□

We give some more estimates of solutions which are proved similarly.

Lemma 3.2 *Let the interval Ω be bounded and let the function f be uniformly Lipschitz continuous in u with respect to (t,x) for all $(t,x)\in[0;T]\times\Omega$. Assume that functions u_1 and u_2 are solutions of equation (6) and (8) with initial values $u_1|_{t=0}=u_{01}$ and $u_2|_{t=0}=u_{02}$, respectively, where $u_{01},u_{02}\in H_{0,\gamma}$.*

Then the following inequality holds:

$$\|u_1(t)-u_2(t)\|_{0,\gamma}\leq C\|u_{01}-u_{02}\|_{0,\gamma}\cdot e^{Ct},\ t\in[0;T].$$

Proof. We note that if f is uniformly Lipschitz continuous in u with respect to (t,x) for all $(t,x)\in[0;T]\times\Omega$, then f has the Sobolev derivative $f'_u\in L_\infty(\mathbb{R}\times[0;T]\times\Omega)$.

Let us write equations (6) and (8) for the solutions u_1 and u_2 and subtract the latter equations from the former ones:

$$\begin{aligned}\partial_t(u_1-u_2)(t,x) &= \Delta(u_1-u_2)(t,x)+(f(u_1(t,x),t,x)-f(u_2(t,x),t,x)),\\ (u_1-u_2)|_{\partial\Omega} &= 0.\end{aligned} \tag{15}$$

We result in an equation of the form (6)-(8) with the function

$$f(u,t,x) := u\int_0^1 f'_u(\theta u_1(t,x)+(1-\theta)u_2(t,x),t,x)d\theta$$

for the mapping $u := u_1 - u_2$ with the initial value $(u_1-u_2)|_{t=0} = u_{01}-u_{02}$. We note that such a function f satisfies condition (9). Hence, we obtain the required inequality from Lemma 3.1.

□

Lemma 3.3 *Let $\gamma < -1/2$ and let Ω be a bounded interval. Assume that the function f satisfies the estimate*

$$f(u,t,x)\cdot u \geq \lambda_0|u|^2 - \lambda_1. \tag{16}$$

Then the following inequality holds:

$$\|u(t)\|^2_{0,\gamma} \leq C(1+\|u_0\|^2_{0,\gamma}\, e^{-\frac{\lambda_0}{2}t}),\ t\in[0;T],$$

where the constant C does not depend on T.

Proof. Fix a number $\varepsilon > 0$. According to Lemma 2.2 applied to the mappings $u := \varphi_\varepsilon^{1/2}u$ and $v := \varphi_\varepsilon^{1/2}u$, the function $\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\mathbb{R})}$ is of class $W^1_1([0;T])$ and its Sobolev derivative $\partial_t\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\mathbb{R})}$ is equal to $2\langle\partial'_t(\varphi_\varepsilon^{1/2}u);\varphi_\varepsilon^{1/2}u\rangle_{\overset{0}{W}{}^1_2(\Omega)}$, where the symbol ∂'_t was defined in the proof of Lemma 3.1.
Taking into account the expression for the derivative $\partial'_t(\varphi_\varepsilon^{1/2}u)$ calculated in the proof of Lemma 3.1, we see that

$$\partial_t\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\mathbb{R})} = -2\int_\Omega((\varphi_\varepsilon)'u\,u'_x + \varphi_\varepsilon(u'_x)^2 + \varphi_\varepsilon f u). \tag{17}$$

Estimating the right-hand side of (17) and taking into account the inequalities (4) and (16), we obtain the following estimate:

$$\begin{aligned}&\partial_t\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\mathbb{R})} + 2\int_\Omega \varphi_\varepsilon(u'_x)^2\,dx\\ \leq\ & C\varepsilon\int_\Omega\varphi_\varepsilon(|u|^2+|u'_x|^2)\,dxdt - \lambda_0\int_\Omega\varphi_\varepsilon|u|^2\,dxdt + \lambda_1\int_{\mathbb{R}}\varphi_\varepsilon\,dx,\end{aligned} \tag{18}$$

where the constant C does not depend on the number ε. We choose the number ε such that $0<\varepsilon<C^{-1}\min\{1;\lambda_0/2\}$. Then inequality (18) can be rewritten as

$$\partial_t\|\varphi_\varepsilon u(t)\|^2_{L_2(\mathbb{R})} + \int_\Omega\varphi_\varepsilon(u'_x)^2\,dx \leq -\frac{\lambda_0}{2}\int_\Omega\varphi_\varepsilon|u|^2\,dxdt + \lambda_1\int_{\mathbb{R}}\varphi_\varepsilon\,dx.$$

Applying Lemma 2.4, we deduce the inequality

$$\|\varphi_\varepsilon^{1/2}u(t)\|^2_{L_2(\Omega)} \le \|\varphi_\varepsilon^{1/2}u(0)\|^2_{L_2(\Omega)} \cdot e^{-\frac{\lambda_0}{2}t} + \frac{2C}{\lambda_0}(1 - e^{-\frac{\lambda_0}{2}t}).$$

The latter inequality and inequality (3) imply the required estimate.

□

Below in this section, we assume that the initial value u_0 belongs to the space $H_{1,\gamma}(\Omega)$ and consider only strong solutions of equation (6)-(8).

Lemma 3.4 *Let the interval* Ω *be bounded. Then any solution* $u \in L_\infty([0;T]; H_{1,\gamma}(\Omega)) \cap L_2([0;T]; H_{2,\gamma}(\Omega))$ *of equation (6)-(8) satisfies the following estimates:*

$$\begin{aligned}
\|u(t)\|_{1,\gamma} &\le C\|u_0\|_{1,\gamma}\, e^{Ct}, \qquad t \in [0;T], &(19)\\
\int_0^T \|u(t)\|^2_{2,\gamma}\, dt &\le C\|u_0\|^2_{1,\gamma}\, e^{CT}, &(20)\\
\int_0^T \|\partial_t u(t)\|^2_{0,\gamma}\, dt &\le C\|u_0\|^2_{1,\gamma}\, e^{CT}. &(21)
\end{aligned}$$

Proof. Fix $\varepsilon > 0$. Note that

$$\begin{aligned}
u'_x &\in L_\infty([0;T]; H_{0,\gamma}(\Omega)) \cap L_2([0;T]; H_{1,\gamma}(\Omega))\\
&= L_\infty([0;T]; L_2(\Omega)) \cap L_2([0;T]; H^1(\Omega))
\end{aligned}$$

and

$$\partial_t u \in L_2([0;T]; L_2(\Omega)).$$

Let us first calculate the derivative $\partial'_t(\varphi_\varepsilon^{1/2}u'_x)$, where the symbol ∂'_t was defined in the proof of Lemma 3.1. Since for any function $v \in H^1([0,T]; \overset{0}{W}{}^1_2(\Omega))$

the equalities

$$
\begin{aligned}
& \int_0^T \langle \partial_t'(\varphi_\varepsilon^{1/2} u_x'); v\rangle_{\overset{0}{W}{}_2^1(\Omega)} \\
= & -\int_0^T \langle \varphi_\varepsilon^{1/2} u_x'; \partial_t v\rangle_{\overset{0}{W}{}_2^1(\Omega)} \\
= & -\int_0^T \int_\Omega \varphi_\varepsilon^{1/2} u_x' \partial_t v \\
= & -\int_0^T \int_\Omega u \partial_t (\varphi_\varepsilon^{1/2} v)_x' \\
= & -\int_0^T \int_\Omega \partial_t u (\varphi_\varepsilon^{1/2} v)_x'
\end{aligned}
$$

hold, we conclude that

$$
\langle \partial_t'(\varphi_\varepsilon^{1/2} u_x'); v\rangle_{\overset{0}{W}{}_2^1(\Omega)} = -\int_\Omega \partial_t u (\varphi_\varepsilon^{1/2} v)_x'.
$$

Now we apply Lemma 2.2 to the mappings $\varphi_\varepsilon^{1/2} u_x'$ and $\varphi_\varepsilon^{1/2} u_x'$ and the interval $[0; t_1]$. According to this lemma,

$$
\begin{aligned}
& \|\varphi_\varepsilon^{1/2} u_x'(t_1)\|_{L_2(\Omega)}^2 - \|\varphi_\varepsilon^{1/2} u_x'(0)\|_{L_2(\Omega)}^2 \\
= & \; 2\int_0^{t_1} \langle \partial_t'(\varphi_\varepsilon^{1/2} u_x'); \varphi_\varepsilon^{1/2} u_x'\rangle_{\overset{0}{W}{}_2^1(\Omega)} \, dt \\
= & -2\int_0^{t_1} \int_\Omega (\Delta u - f)(\varphi_\varepsilon' u_x' + \varphi_\varepsilon \Delta u).
\end{aligned}
\tag{22}
$$

We estimate the right-hand side of (22) taking (4) into account

$$
\begin{aligned}
& \|\varphi_\varepsilon^{1/2} u_x'(t_1)\|_{L_2(\Omega)}^2 + 2\int_0^{t_1} \int_\Omega \varphi_\varepsilon |\Delta u|^2 \\
\leq & \; \|\varphi_\varepsilon^{1/2} u_x'(0)\|_{L_2(\Omega)}^2 + C\varepsilon \int_0^{t_1} \int_\Omega \varphi_\varepsilon (|\Delta u|^2 + |u_x'|^2) + D(\varepsilon) \int_0^{t_1} \int_\Omega \varphi_\varepsilon |u|^2,
\end{aligned}
\tag{23}
$$

where the constant C does not depend even on the number ε and the constant $D(\varepsilon) \geq 0$. Choosing the number ε such that $\varepsilon < C^{-1}$, we deduce from inequality (23) that

$$\begin{aligned} &\|\varphi_\varepsilon^{1/2} u_x'(t_1)\|^2_{L_2(\Omega)} + \int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon |\Delta u|^2\, dxdt \\ \leq\ & |\varphi_\varepsilon^{1/2} u_x'(0)\|^2_{L_2(\Omega)} + C\int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon(|u_x'|^2 + |u|^2)\, dxdt. \end{aligned} \tag{24}$$

We add inequalities (24) and (14′):

$$\|\varphi_\varepsilon^{1/2} u_x'(t_1)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(t_1)\|^2_{L_2(\Omega)} + \int\limits_0^{t_1}\int\limits_\Omega \varphi_\varepsilon(|\Delta u|^2 + |u_x'|^2)\, dxdt \leq$$

$$\leq \|\varphi_\varepsilon^{1/2} u_x'(0)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(0)\|^2_{L_2(\Omega)} +$$

$$+C\int\limits_0^{t_1}(\|\varphi_\varepsilon^{1/2} u_x'(t)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\Omega)})\, dt.$$

Applying Lemma 2.3, we see that

$$\|\varphi_\varepsilon^{1/2} u_x'(t)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\Omega)} \leq$$

$$\leq C(\|\varphi_\varepsilon^{1/2} u_x'(0)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(0)\|^2_{L_2(\Omega)}) \cdot e^{Ct}$$

and

$$\int\limits_0^t(\|\varphi_\varepsilon^{1/2}\Delta u(\tau)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2}|u_x'|\,\|^2_{L_2(\Omega)})\, d\tau \leq$$

$$\leq C(\|\varphi_\varepsilon^{1/2} u_x'(0)\|^2_{L_2(\Omega)} + \|\varphi_\varepsilon^{1/2} u(0)\|^2_{L_2(\Omega)}) \cdot e^{Ct}$$

hold for all $t \in [0;T]$. Due to inequality (3), we obtain the estimates (19) and (20).

Estimate (21) is deduced in the following way:

$$\begin{aligned} \int\limits_0^T \|\partial_t u(t)\|^2_{0,\gamma}\, dt &= \int\limits_0^T \|\Delta u(t,\cdot) - f(u(t,\cdot),t,\cdot)\|^2_{0,\gamma}\, dt \\ &\leq 2\int\limits_0^T \|\Delta u(t,\cdot)\|^2_{0,\gamma}\, dt + C\int\limits_0^T \|u(t,\cdot)\|^2_{0,\gamma}\, dt \\ &\leq C\int\limits_0^T \|u(t,\cdot)\|^2_{2,\gamma}\, dt \leq C\cdot\|u_0\|^2_{1,\gamma}\cdot e^{CT}. \end{aligned}$$

□

Lemma 3.5 *Let the interval Ω be bounded and let the mapping f be uniformly Lipschitz continuous in u with respect to (t,x) for all $(t,x) \in [0;T]\times\Omega$. Take $u_{01}, u_{02} \in H_{1,\gamma}(\Omega)$. Then the solutions*

$$u_1, u_2 \in L_\infty([0;T]; H_{1,\gamma}(\Omega)) \cap L_2([0;T]; H_{2,\gamma}(\Omega))$$

of equation (6) and (8) with the initial values $u_1|_{t=0} = u_{01}$ and $u_2|_{t=0} = u_{02}$ satisfy the following inequality:

$$\|u_1(t) - u_2(t)\|_{1,\gamma} \leq C\|u_{01} - u_{02}\|_{1,\gamma}\, e^{Ct}, \qquad t \in [0;T].$$

Proof. Similar to the proof of Lemma 3.2 the statement follows from Lemma 3.4.

□

Theorem 3.6 ***(existence of solution on an unbounded interval)*** *There exists a solution*

$$u \in L_2([0;T]; H_{2,\gamma}(\mathbb{R})) \cap L_\infty([0;T]; H_{1,\gamma}(\mathbb{R}))$$

of equation (6)-(8) with $\Omega = \mathbb{R}$. This solution satisfies all estimates from the Lemmas 2.1-2.4.

Proof. We define for any number $R \geq 1$ a smooth cut-off function Ψ_R, such that the following conditions are satisfied:

- $0 \leq \Psi_R(\xi) \leq 1$ for all $\xi \in \mathbb{R}$,
- $\Psi_R(\xi) = 1$ if $|\xi| \leq (R-1)^2$,
- $\Psi_R(\xi) = 0$ if $|\xi| \geq R^2$,
- $|\Psi_R^{(k)}(\xi)| \leq C_k$ for all $\xi \in \mathbb{R}$.

Further, we define $\Omega_R := (-R;R)$ and $u_{0,R}(x) := \Psi_R(|x|^2)u_0(x)$. Let

$$\hat{u}_R \in L_2([0;T]; H_{2,\gamma}(\Omega_R)) \cap L_\infty([0;T]; H_{1,\gamma}(\Omega_R))$$

be a solution of the equations (6)-(8) with $u_0 := u_{0,R}$ on the bounded interval Ω_R. Since the interval is bounded, this solution exists and is unique.

Note that $\|u_{0,R}\|_{l,\gamma} \leq C\|u_0\|_{l,\gamma}$ $(l \leq 1)$. Hence, for all $R \geq 1$, the norms of the mappings $\hat{u}_R$ in the corresponding spaces

$$L_2([0;T]; H_{2,\gamma}(\Omega_R)) \cap L_\infty([0;T]; H_{1,\gamma}(\Omega_R))$$

are uniformly bounded with respect to R.

We define

$$u_R(t,x) := \Psi_R(|x|^2) \cdot \begin{cases} \hat{u}_R(t,x) & \text{if} |x| \le R, \\ 0 & \text{otherwise.} \end{cases}$$

The norms of the mappings u_R in the space

$$L_2([0;T]; H_{2,\gamma}(\mathbb{R})) \cap L_\infty([0;T]; H_{1,\gamma}(\mathbb{R}))$$

are also uniformly bounded with respect to R. Similarly, the norms of the mappings $\partial_t u_R$ from Lemma 3.4 are uniformly bounded with respect to R. It follows that there exists a sequence $u_j := u_{R_j}$, $j \in \mathbb{N}$, such that $R_j \to +\infty$ as $j \to +\infty$ with the following properties: the sequence $\{u_j\}_{j\in\mathbb{N}}$ is weakly convergent in the space $L_2([0;T]; H_{2,\gamma}(\mathbb{R}))$ and weak-* convergent in the space $L_\infty([0;T]; H_{1,\gamma}(\mathbb{R}))$, and the sequences $\{\partial_t u_j\}_{j\in\mathbb{N}}$ and $\{f(u_j(\cdot,\cdot),\cdot,\cdot)\}_{j\in\mathbb{N}}$ are weakly convergent in the space $L_2([0;T]; H_{0,\gamma}(\mathbb{R}))$.

We denote the limit of the sequence $\{u_j\}_{j\in\mathbf{N}}$ by $u_\infty = u_\infty(t,x)$. Passing to the limit in equation (5), we see that $\partial_t u_j \to \partial_t u_\infty$ as $j \to +\infty$. Furthermore, since $u_j \to u_\infty$ as $j \to \infty$, we see that $u_j(t,x) \to u_\infty(t,x)$ as $j \to +\infty$ weakly in the space $H^1_{loc}(\mathbb{R} \times [0,T])$, and hence, strongly in the space $L_{2,loc}(\mathbb{R} \times [0,T])$. Since the function f is Lipschitz continuous in u, the functions $f(u_j(\cdot,\cdot),\cdot,\cdot)$ also converge to the function $f(u_\infty(\cdot,\cdot),\cdot,\cdot)$ in the space $L_{2,loc}(\mathbb{R} \times [0,T])$. Finally, passing to the limit in equation (6), we obtain the equality $\partial_t u_\infty = \Delta u_\infty - f(u_\infty(\cdot,\cdot),\cdot,\cdot)$.

Now we prove the equality $u_\infty|_{t=0} = u_0$. Since the sequences of mappings $\{\partial_t u_j\}_{j\in\mathbb{N}}$ and $\{u_j\}_{j\in\mathbb{N}}$ are weakly convergent in the space

$$L_2([0;T]; H_{0,\gamma}(\mathbb{R})),$$

$u_j \to u_\infty$ weakly as $j \to +\infty$ in the space $W^1_2([0;T]; H_{0,\gamma}(\mathbb{R}))$. It follows that the traces $\{u_j|_{t=0}\}_{j\in\mathbb{N}}$ converge to the trace $u_\infty|_{t=0}$ in the space $H_{0,\gamma}$. On the other hand, since $u_j|_{t=0} = \Psi^2_{R_j}(|\cdot|^2)u_0$, it follows from the Lebesgue theorem, that the functions $u_j|_{t=0}$ converge to the function u_0 as $j \to +\infty$. Thus, $u_\infty|_{t=0} = u_0$.

Finally, the conclusions of the Lemmas 2.1-2.4 hold for solutions on the interval $\mathbb{R}$, due to the weak upper semi-continuity of the norms in spaces $L_p([0;T]; H_{l,\gamma}(\mathbb{R}))$.

□

Theorem 3.7 *Equations (6)-(8) where* $\Omega = \mathbb{R}$ *generate a semi-flow* $\{S_t\}_{t\ge0}$ *in the space* $H_{1,\gamma}(\mathbb{R})$. *If the mapping* f *is uniformly Lipschitz continuous in* u *with respect to (t,x) for all* $(t,x) \in [0;T] \times \Omega$, *then all the mappings* S_t, $t \ge 0$, *are continuous in this space.*

Proof. It follows from Theorem 3.6 and Lemma 3.5 that there exist mappings

$$S_t : H_{1,\gamma}(\mathbb{R}) \to H_{1,\gamma}(\mathbb{R}), t \geq 0,$$

such that for any function $u_0 \in H_{1,\gamma}(\mathbb{R})$, the mapping $u(x,t) := S_t u_0$ is a solution of the equations (6)-(8) in the space

$$L_\infty([0;T]; H_{1,\gamma}(\mathbb{R})) \cap L_2([0;T]; H_{2,\gamma}(\mathbb{R})).$$

The continuity of the mappings S_t follows from Lemma 3.5.

□

4 Estimates of higher derivatives

Here and below, we assume that $\Omega = \mathbb{R}$ and consider the same boundary value problem as in Section 3:

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x) - f(u(t,x),t,x), && (25) \\ u|_{t=0} &= u_0 \,. && (26) \end{aligned}$$

We assume that the mapping $f \in C(\mathbb{R} \times [0;T] \times \mathbb{R})$ satisfies the estimate (9).

Lemma 4.1 *Let u be a strong solution of equation (25) (we assume nothing about its initial value). Then for any numbers $n \in \mathbb{N}$ and $t \in [0;T]$, the following estimates hold:*

$$\begin{aligned} t^n \|u(t)\|^2_{0,\gamma} &\leq C \int_0^t \tau^{n-1} \|u(\tau)\|^2_{0,\gamma} \, d\tau, && (27) \\ \int_0^t \tau^n \|u(\tau)\|^2_{1,\gamma} \, d\tau &\leq C \int_0^t \tau^{n-1} \|u(\tau)\|^2_{0,\gamma} \, d\tau, && (28) \\ t^n \|u(t)\|^2_{1,\gamma} &\leq C \int_0^t \tau^{n-1} \|u(\tau)\|^2_{1,\gamma} \, d\tau, && (29) \\ \int_0^t \tau^n \|u(\tau)\|^2_{2,\gamma} \, d\tau &\leq C \int_0^t \tau^{n-1} \|u(\tau)\|^2_{1,\gamma} \, d\tau, && (30) \\ \int_0^t \tau^n \|\partial_t u(\tau)\|^2_{0,\gamma} \, d\tau &\leq C \int_0^t \tau^{n-1} \|u(\tau)\|^2_{1,\gamma} \, d\tau. && (31) \end{aligned}$$

Proof. We proceed similarly to the proofs of Lemmata 3.1 and 3.4. For an arbitrary number $\varepsilon > 0$, it follows from Lemma 2.2 applied to the mappings $u := \varphi_\varepsilon^{1/2} u$ and $v := t^n \varphi_\varepsilon^{1/2} u$ and the interval $[0; t]$ that

$$\begin{aligned}
& t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})} \\
= \;& \int_0^t \langle \varphi_\varepsilon^{1/2} \partial_t u(\tau); \tau^n \varphi_\varepsilon^{1/2} u(\tau) \rangle + \int_0^t \langle \varphi_\varepsilon^{1/2} \partial_\tau (\tau^n u(\tau)); \varphi_\varepsilon^{1/2} u(\tau) \rangle \\
= \;& 2 \int_0^t \langle \varphi_\varepsilon^{1/2} \partial_t u(\tau); \tau^n \varphi_\varepsilon^{1/2} u(\tau) \rangle + n \int_0^t \langle \varphi_\varepsilon^{1/2} \tau^{n-1} u(\tau); \varphi_\varepsilon^{1/2} u(\tau) \rangle \\
= \;& 2 \int_0^t \langle \varphi_\varepsilon^{1/2} (\Delta u(\tau) - f(u(\tau), \tau, \cdot)); \tau^n \varphi_\varepsilon^{1/2} u(\tau) \rangle \\
& + n \int_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})} \\
= \;& -2 \int_0^t \int_{\mathbb{R}} \tau^n \varphi_\varepsilon (|u'_x(\tau, x)|^2 + f(u(\tau, x), \tau, x) u(\tau, x)) \\
& -2 \int_0^t \int_{\mathbb{R}} \tau^n (\varphi_\varepsilon)' u u'_x + n \int_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})}. \qquad (32)
\end{aligned}$$

We estimate the right-hand side of equality (32) as follows:

$$\begin{aligned}
& t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})} + 2 \int_0^t \int_{\mathbb{R}} \tau^n \varphi_\varepsilon |u'_x|^2 \\
\le \;& C \int_0^t \int_{\mathbb{R}} \tau^n \varphi_\varepsilon |u|^2 + C\varepsilon \int_0^t \int_{\mathbb{R}} \tau^n \varphi_\varepsilon (|u|^2 + |u'_x|^2) \\
& + n \int_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})}.
\end{aligned}$$

As usual, we choose $\varepsilon < C^{-1}$ and deduce the following inequality:

$$\begin{aligned}
& t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} \tau^n \varphi_\varepsilon |u'_x|^2 \\
\le \;& C \int_0^t \tau^n \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})} + n \int_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})}.
\end{aligned}$$

Adding the term $\int\limits_0^t \tau^n \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})}$ to both sides of the latter inequality gives

$$\begin{aligned} & t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})} + \int\limits_0^t \int\limits_{\mathbb{R}} \tau^n \varphi_\varepsilon (|u'_x|^2 + |u|^2) \\ \leq \; & C \int\limits_0^t \tau^n \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})} + n \int\limits_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})}. \end{aligned} \tag{33}$$

Finally, applying Lemma 2.3 to the functions $t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})}$ and

$$\int\limits_0^t \int\limits_{\mathbb{R}} \tau^n \varphi_\varepsilon (|u'_x|^2 + |u|^2),$$

we deduce from inequality (33) that

$$\begin{aligned} t^n \|\varphi_\varepsilon^{1/2} u(t)\|^2_{L_2(\mathbb{R})} & \leq n \int\limits_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})} \cdot e^{Ct}, \\ \int\limits_0^t \int\limits_{\mathbb{R}} \tau^n \varphi_\varepsilon (|u'_x|^2 + |u|^2) & \leq n \int\limits_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u(\tau)\|^2_{L_2(\mathbb{R})} \cdot e^{Ct} \end{aligned}$$

hold for all $t \in [0; T]$.
The inequalities (27) and (28) follow from Lemma 2.1.

Now we apply Lemma 2.2 to the mappings $u := \varphi_\varepsilon^{1/2} u'_x$; $v := t^n \varphi_\varepsilon^{1/2} u'_x$ (they satisfy the conditions of Lemma 2.2 as was shown in the proof of Lemma 3.4) and the interval $[0; t]$. We obtain the following equalities:

$$\begin{aligned} & t^n \|\varphi_\varepsilon^{1/2} u'_x(t)\|^2_{L_2(\mathbb{R})} \\ = \; & 2 \int\limits_0^t \langle \varphi_\varepsilon^{1/2} \partial_t u'_x(\tau); \tau^n \varphi_\varepsilon^{1/2} u'_x(\tau) \rangle + n \int\limits_0^t \tau^{n-1} \langle \varphi_\varepsilon^{1/2} u'_x(\tau); \varphi_\varepsilon^{1/2} u'_x(\tau) \rangle \\ = \; & 2 \int\limits_0^t \int\limits_{\mathbb{R}} \tau^n \partial_t u(\tau, x) (\varphi_\varepsilon u'_x)'_x(\tau, x) + n \int\limits_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u'_x(\tau)\|^2_{L_2(\mathbb{R})} \\ = \; & -2 \int\limits_0^t \int\limits_{\mathbb{R}} \tau^n (\Delta u(\tau, x) - f(u(\tau, x), \tau, x)) ((\varphi_\varepsilon)'_x u'_x(\tau, x) + \varphi_\varepsilon \Delta u(\tau, x)) \\ & + n \int\limits_0^t \tau^{n-1} \|\varphi_\varepsilon^{1/2} u'_x(\tau)\|^2_{L_2(\mathbb{R})}. \end{aligned} \tag{34}$$

We estimate the right-hand side of (34):

$$
\begin{aligned}
& t^n\|\varphi_\varepsilon^{1/2}u'_x(t)\|^2_{L_2(\mathbb{R})} + 2\int_0^t\int_{\mathbb{R}} \tau^n\varphi_\varepsilon|\Delta u(\tau,x)|^2 \\
\le\ & C\varepsilon\int_0^t\int_{\mathbb{R}} \tau^n\varphi_\varepsilon(|\Delta u|^2+|u'_x|^2) + D(\varepsilon)\int_0^t\int_{\mathbb{R}} \tau^n\varphi_\varepsilon|u|^2 \\
& +n\int_0^t \tau^{n-1}\|\varphi_\varepsilon^{1/2}u'_x(\tau)\|^2_{L_2(\mathbb{R})}.
\end{aligned}
$$

We choose $\varepsilon < C^{-1}$ and deduce from the latter inequality that

$$
\begin{aligned}
& t^n\|\varphi_\varepsilon^{1/2}u'_x(t)\|^2_{L_2(\mathbb{R})} + \int_0^t\int_{\mathbb{R}} \tau^n\varphi_\varepsilon|\Delta u(\tau,x)|^2 \\
\le\ & C\int_0^t\int_{\mathbb{R}} \tau^n\varphi_\varepsilon(|u|^2+|u'_x|^2) + n\int_0^t \tau^{n-1}\|\varphi_\varepsilon^{1/2}u'_x(\tau)\|^2_{L_2(\mathbb{R})}.
\end{aligned}
$$

The addition of inequality (33) to the latter inequality gives

$$
\begin{aligned}
& t^n\|\varphi_\varepsilon^{1/2}u(t)\|^2_{H_1(\mathbb{R})} + \int_0^t \tau^n\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_2(\mathbb{R})} \\
\le\ & C\int_0^t \tau^n\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_1(\mathbb{R})} + n\int_0^t \tau^{n-1}\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_1(\mathbb{R})}.
\end{aligned}
$$

By Lemma 2.3, it follows that

$$
\begin{aligned}
t^n\|\varphi_\varepsilon^{1/2}u(t)\|^2_{H_1(\mathbb{R})} &\le n\int_0^t \tau^{n-1}\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_1(\mathbb{R})}\cdot e^{Ct}, \\
\int_0^t \tau^n\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_2(\mathbb{R})} &\le n\int_0^t \tau^{n-1}\|\varphi_\varepsilon^{1/2}u(\tau)\|^2_{H_1(\mathbb{R})}\cdot e^{Ct}
\end{aligned}
$$

hold for all $t\in[0;T]$. The estimates (29) and (30) follow now from Lemma 2.1.

Finally, estimate (31) can be deduced in the following way:

$$\begin{aligned}\int_0^t \tau^n \|\partial_t u(\tau)\|_{0,\gamma}^2 &= \int_0^t \tau^n \|\Delta u(\tau) - f(u(\tau),\tau,\cdot)\|_{0,\gamma}^2 \\ &\leq 2\int_0^t \tau^n \|\Delta u(\tau)\|_{0,\gamma}^2 + C\int_0^t \tau^n \|u(\tau)\|_{0,\gamma}^2 \\ &\leq (C+CT)\int_0^t \tau^{n-1}\|u(\tau)\|_{1,\gamma}^2.\end{aligned}$$

□

We define, for an arbitrary mapping $v : [0;T] \to H_{0,\gamma}(\mathbb{R})$ and a number $h > 0$, a mapping $v^h : [0; T-h] \to H_{0,\gamma}(\mathbb{R})$ by the formula

$$v^h(t) := \frac{v(t+h) - v(t)}{h}.$$

Lemma 4.2 *Let a mapping $v \in L_p([0;t]; H_{l,\gamma}(\mathbb{R}))$ $(p \in [2;+\infty], l \leq 2)$ satisfy the following estimates for any number $h > 0$:*

$$\int_\delta^{t-h} a(\tau)\|v^h(\tau)\|_{H_{l,\gamma}(\mathbb{R})}^p \leq C_v \quad \textit{for } p < +\infty$$

and

$$\operatorname*{esssup}_{\tau\in[\delta;t-h]} a(\tau)\|v^h(\tau)\|_{H_{l,\gamma}(\mathbb{R})} \leq C_v \quad \textit{for } p = +\infty\,, \tag{35}$$

where $\delta \geq 0$, the function $a : [\delta; t] \to \mathbb{R}$ satisfies the inequality $A_0 \geq a(\tau) \geq a_0 > 0$, and the constant C_v does not depend on the number h. Then there exists a strong derivative $\partial_t v \in L_p([\delta;t]; H_{l,\gamma}(\mathbb{R}))$ and the following estimates hold:

$$\int_\delta^{t} a(\tau)\|\partial_t v(\tau)\|_{H_{l,\gamma}(\mathbb{R})}^p \leq C_v \quad \textit{for } p < +\infty$$

and

$$\operatorname*{esssup}_{\tau\in[\delta;t]} a(\tau)\|\partial_t v(\tau)\|_{H_{l,\gamma}(\mathbb{R})} \leq C_v \quad \textit{for } p = +\infty\,. \tag{36}$$

Proof. We prove the lemma in the case $\delta = 0$ since the remaining case is considered similarly.

Let $w \in C_0^\infty((0;T);H_{0,\gamma}(\mathbb{R}))$ be an arbitrary mapping such that $\operatorname{supp} w \subset [h;t-2h]$. Then the following equality holds:

$$\begin{aligned}\int_0^{t-h} \langle v^h(\tau); w(\tau)\rangle_{H_{0,\gamma}(\mathbb{R})} &= \int_0^{t-h} \left\langle \frac{v(t+h)-v(t)}{h}; w(t)\right\rangle_{H_{0,\gamma}(\mathbb{R})} \\ &= \int_h^{t-h} \left\langle v(t); \frac{w(\tau-h)-w(\tau)}{h}\right\rangle_{H_{0,\gamma}(\mathbb{R})} \\ &\quad + \frac{1}{h}\int_{t-h}^{t} \langle v(t); w(t-h)\rangle_{H_{0,\gamma}(\mathbb{R})} \\ &\quad - \frac{1}{h}\int_0^h \langle v(t); w(t)\rangle_{H_{0,\gamma}(\mathbb{R})}.\end{aligned}$$

The last two terms are equal to zero. Thus,

$$\int_0^{t-h} \langle v^h(\tau); w(\tau)\rangle_{H_{0,\gamma}(\mathbb{R})} = -\int_h^{t-h} \langle v(\tau); w^h(\tau-h)\rangle_{H_{0,\gamma}(\mathbb{R})}. \tag{37}$$

We introduce a mapping $\hat{v}^h : [0;T] \to H_{l,\gamma}(\mathbb{R})$ by the formula

$$\hat{v}^h(\tau) := \begin{cases} v^h(\tau) & \text{if } \tau \in [0;t-h], \\ 0 & \text{if } \tau \in (t-h;t] \end{cases}$$

It follows from inequality (35) that the mappings $\hat{v}^h$ are uniformly bounded in the space $L_p([0;t];H_{l,\gamma}(\mathbb{R}))$. Hence, there exists a sequence $h_k \underset{k\to+\infty}{\to} 0$ such that $\hat{v}^{h_k} \underset{k\to+\infty}{\rightharpoondown} \hat{v} \in L_p([0;t];H_{l,\gamma}(\mathbb{R}))$ (i.e. converges weakly) in the space $L_p([0;t];H_{l,\gamma}(\mathbb{R}))$. It follows from (37) that

$$\int_0^t \langle \hat{v}(\tau); w(\tau)\rangle = -\int_0^t \langle v(\tau); \partial_t w(\tau)\rangle. \tag{38}$$

Since equality (38) does not depend on the step size $h > 0$, this equality holds for all mappings $w \in C_0^\infty((0;T);H_{0,\gamma}(\mathbb{R}))$, i.e., the mapping $\hat{v}$ is the strong derivative of the mapping v. Inequality (36) follows from the weak upper semi-continuity of the norm.

$\square$

Lemma 4.3 *Let a mapping u be a strong solution of equation (25) and let the mapping $f = f(u)$ be Lipschitz continuous. Then there exist the mappings*

$$\partial_t u'_x\,, \quad \partial_t \Delta u\,, \quad \partial_t^2 u\,,$$

which are the corresponding strong derivatives of the mapping u on any interval $[\delta; T]$, where $\delta > 0$. The following inequalities hold for any number $m \geq 0$:

$$t^{m+1}\|\partial_t u(t)\|^2_{0,\gamma} \;\leq\; C\int_0^t \tau^m\|\partial_t u(\tau)\|^2_{0,\gamma}\,d\tau, \tag{39}$$

$$\int_0^t \tau^{m+1}\|\partial_t u(\tau)\|^2_{1,\gamma}\,d\tau \;\leq\; C\int_0^t \tau^m\|\partial_t u(\tau)\|^2_{0,\gamma}\,d\tau, \tag{40}$$

$$t^{m+2}\|\partial_t u(t)\|^2_{1,\gamma} \;\leq\; C\int_0^t \tau^m\|\partial_t u(\tau)\|^2_{0,\gamma}\,d\tau, \tag{41}$$

$$\int_0^t \tau^{m+2}\|\partial_t u(\tau)\|^2_{2,\gamma}\,d\tau \;\leq\; C\int_0^t \tau^m\|\partial_t u(\tau)\|^2_{0,\gamma}\,d\tau, \tag{42}$$

$$\int_0^t \tau^{m+2}\|\partial_t^2 u(\tau)\|^2_{0,\gamma}\,d\tau \;\leq\; C\int_0^t \tau^m\|\partial_t u(\tau)\|^2_{0,\gamma}\,d\tau \tag{43}$$

for all $t \in (0; T]$.

Proof. Similarly to proof of Lemma 3.2, we show that the mapping u^h satisfies the following equation for any step size $h > 0$:

$$\partial_t u^h(t,x) = \Delta u^h(t,x) - f_h(u^h(t,x),t,x), \tag{44}$$

where

$$f_h(v,t,x) = v\int_0^1 f'_v((1-\theta)u(t,x) + \theta u(t+h,x))d\theta.$$

Since all the mappings f_h satisfy inequality (9), estimates of Lemma 4.1 hold for all the mappings u^h.

Now we estimate the expression $\int\limits_0^t \tau^m \|u^h(\tau)\|_{0,\gamma}^2 d\tau$:

$$\begin{aligned}
\int\limits_0^t \tau^m \|u^h(\tau)\|_{0,\gamma}^2 &= \int\limits_0^t \int\limits_{\mathbb{R}} \tau^m \varphi |u^h(\tau)|^2 = \int\limits_0^t \int\limits_{\mathbb{R}} \tau^m \varphi | \int\limits_0^1 \partial_t u(\tau+\theta h) d\theta|^2 \\
&\leq \int\limits_0^t \int\limits_{\mathbb{R}} \tau^m \varphi \int\limits_0^1 |\partial_t u(\tau + h\theta)|^2 d\theta \\
&\leq \int\limits_0^1 \int\limits_0^t \int\limits_{\mathbb{R}} (\tau+h\theta)^m \varphi |\partial_t u(\tau+h\theta)|^2 d\theta \\
&\leq \int\limits_0^{t+h} \int\limits_{\mathbb{R}} \tau^m \varphi |\partial_t u(\tau)|^2 = \int\limits_0^{t+h} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2
\end{aligned}$$

if $t+h \leq T$. Hence, denoting $t_1 := t+h$ and applying Lemma 4.1, we obtain the following inequalities:

$$t^{m+1} \|u^h(t)\|_{0,\gamma}^2 \leq C \int\limits_0^{t_1} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2, \tag{45}$$

$$\int\limits_0^t \tau^{m+1} \|u^h(\tau)\|_{1,\gamma}^2 \, d\tau \leq C \int\limits_0^{t_1} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2, \tag{46}$$

for all $t \in [0; t_1 - h]$. Further, from inequality (46) and Lemma 4.1 we deduce that

$$t^{m+2} \|u^h(t)\|_{1,\gamma}^2 \leq C \int\limits_0^{t_1} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2, \tag{47}$$

$$\int\limits_0^t \tau^{m+2} \|u^h(\tau)\|_{2,\gamma}^2 \, d\tau \leq C \int\limits_0^{t_1} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2, \tag{48}$$

$$\int\limits_0^t \tau^{m+2} \|\partial_t u^h(\tau)\|_{0,\gamma}^2 \, d\tau \leq C \int\limits_0^{t_1} \tau^m \|\partial_t u(\tau)\|_{0,\gamma}^2 \tag{49}$$

hold for all $t \in [0; t_1 - h]$.

We choose $\delta > 0$. Inequalities (39) and (41) for $\delta < t \leq T$ follow from inequalities (45) and (47) and from Lemma 4.2. Hence, they hold for all $t \in (0; T]$.

Applying Lemma 4.2 we deduce from inequalities (46), (48), and (49) that

$$\begin{aligned}\int\limits_{\delta}^{t_1} \tau^{m+1}\|\partial_t u(\tau)\|^2_{1,\gamma} &\leq C\int\limits_{0}^{t_1} \tau^{m}\|\partial_t u(\tau)\|^2_{0,\gamma},\\ \int\limits_{\delta}^{t_1} \tau^{m+2}\|\partial_t u(\tau)\|^2_{2,\gamma} &\leq C\int\limits_{0}^{t_1} \tau^{m}\|\partial_t u(\tau)\|^2_{0,\gamma},\\ \int\limits_{\delta}^{t_1} \tau^{m+2}\|\partial_t^2 u(\tau)\|^2_{0,\gamma} &\leq C\int\limits_{0}^{t_1} \tau^{m}\|\partial_t u(\tau)\|^2_{0,\gamma}\end{aligned}$$

hold for all $t \in (\delta; T]$.

Inequalities (40), (42) and (43) follow now from the Lebesgue theorem.

□

Corollary 4.4 *With the assumptions of Lemma 4.3, it follows from Lemma 4.3 and Lemma 3.4 that for any strong solution u of the equation (25)-(26), the following estimates hold:*

$$\begin{aligned} t\|\partial_t u(t)\|^2_{0,\gamma} &\leq C\|u_0\|^2_{1,\gamma}, &(50)\\ \int\limits_0^t \tau\|\partial_t u(\tau)\|^2_{1,\gamma}\,d\tau &\leq C\|u_0\|^2_{1,\gamma}, &(51)\\ t^2\|\partial_t u(t)\|^2_{1,\gamma} &\leq C\|u_0\|^2_{1,\gamma}, &(52)\\ \int\limits_0^t \tau^2\|\partial_t u(\tau)\|^2_{2,\gamma}\,d\tau &\leq C\|u_0\|^2_{1,\gamma}, &(53)\\ \int\limits_0^t \tau^2\|\partial_t^2 u(\tau)\|^2_{0,\gamma}\,d\tau &\leq C\|u_0\|^2_{1,\gamma} &(54)\end{aligned}$$

for all $t \in (0; T]$.

We conclude this section by proving an estimate of a different type.

Lemma 4.5 *Assume that a mapping $f(u,t,x) = f(u)$ is Lipschitz continuous twice differentiable, and satisfies estimate (9) and the estimate $|f''(u)| \leq C_f$ with some constant C_f for all $u \in \mathbb{R}$. Then, for any strong solution u of equation (25)-(26) and for any number $\delta > 0$, the inclusion $\Delta u \in L_2([\delta; T]; H_{2,\gamma}(\mathbb{R}))$ and the inequality*

$$\int\limits_0^t \tau^2\|\Delta u(\tau)\|^2_{2,2\gamma} \leq C(\|u_0\|^4_{1,\gamma} + \|u_0\|^2_{1,\gamma}) \tag{55}$$

hold for all $t \in [0;T]$.

Proof. Since $\Delta u = \partial_t u - f(u)$, it is sufficient to prove that

$$\int\limits_0^t \tau^2 \|f(u(\tau))\|^2_{2,2\gamma} \le C(\|u_0\|^4_{1,\gamma} + \|u_0\|^2_{1,\gamma}). \tag{56}$$

Let us estimate the mapping $f(u)$:

$$\int\limits_0^t \tau^2 \|f(u(\tau))\|^2_{0,2\gamma} \le T^2 C_f^2 \int\limits_0^t \|u(\tau)\|^2_{0,\gamma}.$$

We estimate the derivative of the mapping $f(u)$ as follows:

$$\int\limits_0^t \tau^2 \|(f(u))'_x(\tau)\|^2_{0,2\gamma} \le \int\limits_0^t \tau^2 \|f'(u(\tau)) \cdot u'_x(\tau)\|^2_{0,\gamma} \le T^2 C \int\limits_0^t \|u'_x(\tau)\|^2_{0,\gamma}.$$

Finally, we estimate the second derivative of the mapping $f(u)$ applying the Sobolev embedding theorem since the constant in it does not depend on $R \ge 1$:

$$\begin{aligned}
&\int\limits_0^t \tau^2 \|\Delta(f(u))(\tau)\|^2_{0,2\gamma} \\
\le\ & T \int\limits_0^t \int\limits_{\mathbb{R}} \tau\varphi^2 (f'(u(\tau,x))\Delta u(\tau,x) + f''(u(\tau,x))|u'_x(\tau,x)|^2)^2 \\
\le\ & C \int\limits_0^t \int\limits_{\mathbb{R}} \tau\varphi|\Delta u(\tau,x)|^2 + C \int\limits_0^t \int\limits_{\mathbb{R}} \tau\varphi^2 |u'_x|^4 \\
\le\ & C \int\limits_0^t \|u(\tau)\|^2_{2,\gamma} + C \int\limits_0^t \int\limits_{\mathbb{R}} \varphi|u'_x|^2 \cdot \operatorname*{esssup}_{\substack{x\in\mathbb{R} \\ \tau\in[0;T]}} \tau|\varphi^{1/2}(x) u'_x(\tau,x)|^2.
\end{aligned}$$

Let Ψ_1 be a cut-off function from the proof of theorem 3.6. Then the Sobolev embedding theorem and lemma 2.1 give

$$\begin{aligned}
&\int_0^t \tau^2 \|\Delta(f(u))(\tau)\|_{0,2\gamma}^2 \\
\leq\ & C\int_0^t \|u(\tau)\|_{2,\gamma}^2 \\
&+C\int_0^t \|u'_x(\tau)\|_{0,\gamma}^2 \cdot \operatorname*{esssup}_{\tau\in[0;T]} \sup_{y\in\mathbb{R}} \operatorname*{esssup}_{x\in[y-1,y+1]} \tau|\varphi^{1/2}(x)\Psi_1(x-y)u'_x(\tau,x)|^2 \\
\leq\ & C\int_0^t \|u(\tau)\|_{2,\gamma}^2 \\
&+C\int_0^t \|u'_x(\tau)\|_{0,\gamma}^2 \cdot \operatorname*{esssup}_{\tau\in[0;T]} \sup_{y\in\mathbb{R}} \tau\|\varphi^{1/2}\Psi_1(\cdot-y)u'_x(\tau)\|_{H^1[y-1;y+1]}^2 \\
\leq\ & C\int_0^t \|u(\tau)\|_{2,\gamma}^2 + C\int_0^t \|u'_x(\tau)\|_{0,\gamma}^2 \cdot \operatorname*{esssup}_{\tau\in[0;T]} \tau \sup_{y\in\mathbb{R}} \|\Psi_1(\cdot-y)u'_x(\tau)\|_{1,\gamma}^2 \\
\leq\ & C\int_0^t \|u(\tau)\|_{2,\gamma}^2 + C\int_0^t \|u'_x(\tau)\|_{0,\gamma}^2 \cdot \operatorname*{esssup}_{\tau\in[0;T]} \tau\|u(\tau)\|_{2,\gamma}^2.
\end{aligned}$$

Now it follows from Lemma 3.4 that

$$\int_0^t \tau^2\|f(u(\tau))\|_{2,2\gamma}^2 \leq C\|u_0\|_{1,\gamma}^2(1+\operatorname*{esssup}_{\tau\in[0;T]} \tau\|u(\tau)\|_{2,\gamma}^2). \tag{57}$$

We estimate the last term of the right-hand side of the latter inequality applying Lemma 3.4 and Corollary 4.4 as follows:

$$\begin{aligned}
\operatorname*{esssup}_{\tau\in[0;T]} \tau\|u(\tau)\|_{2,\gamma}^2 &\leq \operatorname*{esssup}_{\tau\in[0;T]} \tau\|u(\tau)\|_{1,\gamma}^2 + \operatorname*{esssup}_{\tau\in[0;T]} \tau\|\Delta u(\tau)\|_{0,\gamma}^2 \\
&\leq C\operatorname*{esssup}_{\tau\in[0;T]} \|u(\tau)\|_{1,\gamma}^2 + 2\operatorname*{esssup}_{\tau\in[0;T]} \tau\|\partial_t u(\tau)\|_{0,\gamma}^2 \\
&\quad +2\operatorname*{esssup}_{\tau\in[0;T]} \tau\|f(u(\tau))\|_{0,\gamma}^2 \\
&\leq C\|u_0\|_{1,\gamma}^2 + C\operatorname*{esssup}_{\tau\in[0;T]} \|u(\tau)\|_{0,\gamma}^2 \\
&\leq C\|u_0\|_{1,\gamma}^2.
\end{aligned}$$

Inequality (56) follows from the last inequality and inequality (57).

□

5 Existence of the attractor

In this section, we consider a particular case of the boundary value problem (6)-(8); we assume that $f = f(u)$ and $\Omega = \mathbb{R}$:

$$\begin{aligned} \partial_t u &= \Delta u - f(u), \qquad (58)\\ u|_{t=0} &= u_0. \qquad (59) \end{aligned}$$

We assume that the function f is Lipschitz continuous and satisfies the dissipativity condition (10). We also assume that the initial value $u_0 \in H_{0,\gamma}(\mathbb{R})$ with $\gamma < -1/2$.

As it was shown in Section 3, we can change the variable $u := u + c$ such that equation (58) transforms into an equation of the same type but with the function $\tilde{f} = f(u - c)$ satisfying inequality (9). Since the problems considered in this section do not depend on such a change, we assume that the function f satisfies inequality (9). Then it follows from the results of Section 3 that the equations (58)-(59) define a continuous semi-flow $\{S_t\}_{t\geq 0}$ on the space $H_{1,\gamma}(\mathbb{R})$.

We state now some properties of the semi-flow $\{S_t\}$ which follow from the results of Section 3.

We introduce the shift operator $T_y : H_{1,\gamma}(\mathbb{R}) \to H_{1,\gamma}(\mathbb{R})$ as follows:

$$(T_y u)(x) := u(x + y).$$

Then the following statement holds.

Lemma 5.1 *For any $t \geq 0$ and $y \in \mathbb{R}$, the identity $S_t T_y = T_y S_t$ holds.*

Proof. Take a function $u_0 \in H_{1,\gamma}(\mathbb{R})$. Then the mapping $u(t,x) := S_t u_0(x)$ is a solution of (58)–(59). Consider the mapping $v(t,x) := T_y(S_t u_0)(x) = (S_t u_0)(x + y)$.

The following equalities hold:

$$\begin{aligned} \partial_t v(t,x) &= \partial_t S_t u_0(x+y) = (\Delta S_t u_0)(x+y) - f(S_t u_0(x+y)) \\ &= \Delta_x(S_t u_0(x+y)) - f(S_t u_0(x+y)) = \Delta v(t,x) - f(v(t,x)). \end{aligned}$$

Hence, the mapping v is a solution of (58)-(59) with the initial value $v|_{t=0} = T_y u_0$. Thus, by uniqueness we get $v = S_t T_y u_0$.

$\square$

We define the space

$$H_{1,u}(\mathbb{R}) := \{u \in H_{1,\gamma}(\mathbb{R}) : \|u\|_{1,u} < +\infty \text{ and } \|T_y u - u\|_{1,u} \underset{y\to 0}{\to} 0\}$$

with the norm $\|u\|_{1,u} := \sup\limits_{y\in\mathbb{R}} \|T_y u\|_{1,\gamma}$.

We note that for any function $u_0 \in H_{1,u}(\mathbb{R}) \subset H_{1,\gamma}(\mathbb{R})$, the solution $u(t,x)$ of (58)-(59) satisfies all the estimates of Lemmata 3.1 - 3.5 with the norm $\|\cdot\|_{1,u}$ instead of $\|\cdot\|_{1,\gamma}$. For instance,

$$\begin{aligned}\|S_t u_0\|_{1,u} &= \sup_{y\in\mathbb{R}} \|T_y S_t u_0\|_{1,\gamma} = \sup_{y\in\mathbb{R}} \|S_t T_y u_0\|_{1,\gamma} \le C \sup_{y\in\mathbb{R}} \|T_y u_0\|_{1,\gamma} \\ &= C\|u_0\|_{1,u} < +\infty.\end{aligned}$$

Hence, $\{S_t\}_{t\ge 0}$ is also a continuous semi-flow on the space $H_{1,u}(\mathbb{R})$.

We also note that the estimates of Lemma 4.1 hold for any solution u of (58)-(59) with $u_0 \in H_{1,u}(\mathbb{R})$.

Lemma 5.2 *The semi-flow $\{S_t\}$ has a bounded absorbing set A_1 in the space $H_{1,\gamma}(\mathbb{R})$ and a nonempty, bounded, and positively invariant absorbing set A_{u1} in the space $H_{1,u}(\mathbb{R})$.*

Proof. By Lemma 3.3, there exists a constant $R \in \mathbb{R}$ such that the set

$$A := \{v \in H_{1,\gamma}(\mathbb{R}) : \|v\|_{H_{0,\gamma}(\mathbb{R})} \le R\}$$

is a positively invariant absorbing set for the semi-flow $\{S_t\}$ in the space $H_{1,\gamma}(\mathbb{R})$. Similarly,

$$A_u := \{v \in H_{1,u}(\mathbb{R}) : \sup_{y\in\mathbb{R}} \|T_y v\|_{H_{0,\gamma}(\mathbb{R})} \le R\}$$

is an absorbing set in the space $H_{1,u}(\mathbb{R})$. The sets $A_1 := S_1 A$ and $A_{u1} := S_1 A_u$ are absorbing in the corresponding spaces as well. Now we claim that these sets are bounded in the corresponding spaces.

Let $u_0 \in A_1$. Then $u(t,x) := S_t u_0(x)$ is a solution of (58)-(59). Hence, by Lemma 4.1 (inequalities (29) with $t = 1$ and $n = 1$) and (12) we get

$$\|u(1,\cdot)\|_{1,\gamma}^2 \le C \int_0^1 \|u(\tau,\cdot)\|_{1,\gamma}^2 \le C \cdot \|u(0,\cdot)\|_{0,\gamma}^2 \le CR. \tag{60}$$

If u_0 also belongs to the set A_{u1}, then estimate (60) holds with the norm $\|\cdot\|_{1,u}$ instead of $\|\cdot\|_{1,\gamma}$.

Thus, both sets A_1 and A_{u1} are bounded in the corresponding spaces. $\square$

We introduce the space

$$H_{2,u}(\mathbb{R}) = \{u \in H_{2,\gamma}(\mathbb{R}) : \|u\|_{2,u} := \sup_{y\in\mathbb{R}} \|T_y u\|_{2,\gamma} < +\infty\}.$$

Lemma 5.3 *The space $H_{2,u}(\mathbb{R})$ is a subspace of the space $H_{1,\gamma}(\mathbb{R})$, and the embedding operator is compact.*

Proof. Consider a sequence $\{u_n\}_{n=1}^{+\infty}$ that is bounded in the space $H_{2,u}(\mathbb{R})$, i.e.

$$\sup_{\substack{n\geq 1\\ y\in\mathbb{R}}} \|T_y u_n\|_{2,\gamma} \leq M.$$

Let ψ_R be the cut-off function introduced in the proof of Theorem 3.6. Then the functions $\{\psi_1 u_n\}_{n=1}^{+\infty}$ are bounded in the space $H^2[-1;1]$. Since the interval $[-1;1]$ is a starlike set we deduce from the Sobolev embedding theorem the existence of a subsequence $\{\psi_1 u_n^{(1)}\}_{n=1}^{+\infty}$ of $\{\psi_1 u_n\}_{n=1}^{+\infty}$ which is convergent in the space $H^1[-1;1]$.

Similarly, we can choose from the sequence $\{u_n^{(1)}\}_{n=1}^{+\infty}$ a subsequence $\{u_n^{(2)}\}_{n=1}^{+\infty}$ such that $\{\psi_2 u_n^{(2)}\}_{n=1}^{+\infty}$ converges in the space $H^1[-2;2]$. Operating further in the same way, we obtain a sequence $\{u_n^{(m)}\}_{n,m=1}^{+\infty}$. We denote $v_n := u_n^{(n)}$ for $n\in\mathbb{N}$. Obviously, the sequence $\{v_n\}_{n=1}^{+\infty}$ is a subsequence of $\{u_n\}_{n=1}^{+\infty}$ and for any $R\geq 1$, the sequence $\{\psi_R v_n\}_{n=1}^{+\infty}$ converges in the space $H^1[-R;R]$. Hence, there exists a function $v:\mathbb{R}\to\mathbb{R}$ such that for any $R\geq 1$, the sequence $\{v_n\}_{n=1}^{+\infty}$ converges towards the function v in the space $H^1[-R+1;R-1]$.

We claim that the sequence $\{v_n\}_{n=1}^{+\infty}$ converges towards the function v in the space $H^1(\mathbb{R})$. Fix a number $\varepsilon>0$. There exists a positive number θ such that $\varphi(x)>1/2$ on the interval $(-\theta;\theta)$. For an arbitrary number $k\in\mathbb{N}$, the following estimates follow from Lemma 2.1:

$$\begin{aligned}
\int_{(k-1)\theta}^{(k+1)\theta}\varphi(x)|v_n(x)|^2dx &\leq \varphi((k-1)\theta)\int_{(k-1)\theta}^{(k+1)\theta}|v_n(x)|^2dx\\
&= \varphi((k-1)\theta)\int_{-\theta}^{\theta}|T_{k\theta}v_n(x)|^2dx\\
&\leq 2\varphi((k-1)\theta)\int_{-\theta}^{\theta}\varphi(x)|T_{k\theta}v_n(x)|^2dx\\
&\leq 2M\varphi((k-1)\theta)\leq CM\varphi(k\theta);
\end{aligned}$$

hence,

$$\|v_n\|^2_{H_{1,\gamma}(\theta(2k-1);+\infty)} = \int_{\theta(2k-1)}^{+\infty}\varphi(x)|v_n(x)|^2dx \leq CM\sum_{i=k}^{+\infty}\varphi(2i\theta), \qquad (61)$$

and similarly,

$$\|v_n\|^2_{H_{1,\gamma}(-\infty;-\theta(2k-1))} = \int_{-\infty}^{-\theta(2k-1)}\varphi(x)|v_n(x)|^2dx \leq CM\sum_{i=-\infty}^{-k}\varphi(2i\theta). \quad (62)$$

The same estimates hold also for the function v since

$$\int_{-\theta}^{\theta} \varphi(x)|T_{k\theta}v_n(x)|^2dx \underset{n\to+\infty}{\to} \int_{-\theta}^{\theta} \varphi(x)|T_{k\theta}v(x)|^2dx$$

for any $k \in \mathbb{N}$.

Since $\gamma < -1/2$, the sum $\sum\limits_{i=-\infty}^{+\infty} \varphi(2i\theta)$ is finite. Thus, there exist a number k such that

$$\sum_{i=-\infty}^{-k} \varphi(2i\theta) + \sum_{i=k}^{+\infty} \varphi(2i\theta) < \frac{\varepsilon}{4CM}.$$

Note that the norms of the spaces $H^1[-2k\theta; 2k\theta]$ and $H_{1,\gamma}[-2k\theta; 2k\theta]$ are equivalent, therefore a function v_n exists such that $\|v_n - v\|_{H_{1,\gamma}[-2k\theta;2k\theta]} < \frac{\varepsilon}{2}$. Thus, the following inequality is a consequence of (61) and (62):

$$\begin{aligned}
\|v_n - v\|_{H_{1,\gamma}(\mathbb{R})} &\leq \|v_n\|_{H_{1,\gamma}(-\infty;-2k\theta)} + \|v_n\|_{H_{1,\gamma}(2k\theta;+\infty)} \\
&\quad + \|v\|_{H_{1,\gamma}(-\infty;-2k\theta)} + \|v\|_{H_{1,\gamma}(2k\theta;+\infty)} \\
&\quad + \|v_n - v\|_{H_{1,\gamma}(-2k\theta;2k\theta)} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

□

Now we apply a standard theorem of the existence of an attractor.

Theorem 5.4 *Given Banach spaces $Z_u \subset Z_\rho$, let a semi-flow $\{S_t\}_{t\geq 0}$ satisfy the following conditions:*

(c1) it is translationally invariant (i.e., $T_yS_t = S_tT_y$ for any $y \in \mathbb{R}$ and $t \geq 0$) and continuous in the spaces Z_ρ and Z_u;

(c2) it has a nonempty, bounded, and positively invariant absorbing set **B** *in Z_u;*

(c3) for any $B \subset$ **B** *there exists $\tau > 0$ such that the set $S_\tau(B)$ is precompact in Z_ρ.*

Then there exists a $(Z_u; Z_\rho)$-attractor of the semi-flow $\{S_t\}_{t\geq 0}$.

For a proof, see [6].

Theorem 5.5 *With our conditions, the semi-flow $\{S_t\}_{t\geq 0}$ has a $(H_{1,u}(\mathbb{R}); H_{1,\gamma}(\mathbb{R}))$-attractor.*

Proof. Conditions (c1) and (c2) of Theorem 5.4 are established in Lemmata 5.1 and 5.2.

Condition (c3) of Theorem 5.4 is also satisfied since it follows from Corollary 4.4, that the mapping S_1 maps continuously the space $H_{1,u}(\mathbb{R})$ to the space $H_{2,u}(\mathbb{R})$. Thus, Lemma 5.3 implies that for any bounded subset $B \subset H_{1,u}(\mathbb{R})$, the set $S_1(B)$ is precompact in the space $H_{1,\gamma}(\mathbb{R})$.

$\square$

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