Interpolation and projection operators in weighted function spaces on the real line

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Abstract

In the study of parabolic differential equations on the real line and their discretizations, various weighted Sobolev spaces appear. In this paper, we establish some relations between these spaces. The main applications of these relations are in the study of convergence of discretized attractors towards exact ones.

1 Introduction

The study of dynamical systems generated by partial differential equations on unbounded domains has been stimulated by Babin and Vishik who gave a definition of an attractor which corresponds to a pair of spaces (see [1],[2]). The authors of [6] proposed appropriate weighted Sobolev spaces for this purpose and derived existence results for equations of Ginzburg-Landau type. Following this choice of norms, the papers [3], [5] studied attractors of parabolic reaction diffusion equations of the line and their discretization. When analyzing the convergence of discretized attractors (see [4]) it turns out that specific properties of second order operators on these weighted function spaces are needed.

The purpose of this paper is to derive these important properties for the differential operator as well as for its discretization.

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2 Weighted spaces

Let us first summarize some notation and elementary properties from [3], [5].

Let $d > 0$ be an arbitrary number. We use the following notation: if $u = \{u_k : k \in \mathbb{Z}\}$ is a sequence and $v(x), x \in \mathbb{R}$ is a function, then

$$\begin{aligned}
(\partial_+ u)_k &= (u_{k+1} - u_k)/d, \\
(\partial_- u)_k &= (u_k - u_{k-1})/d, \\
(\partial_+ v)(x) &= (v(x+d) - v(x))/d, \\
(\partial_- v)(x) &= (v(x) - v(x-d))/d, \\
(T_y u)_k &= u_{k+y}, \quad y \in \mathbb{Z}, \\
(T_y v)(x) &= v(x+y), \quad y \in \mathbb{R}.
\end{aligned}$$

Let $\gamma > 0$ be an arbitrary number. We introduce the weight function $\rho(x) := (1+x^2)^{-\gamma}$. For the forthcoming analysis, we need the following properties of the function ρ cf. [5], [3]:

$$\begin{aligned}
|\rho'(x)| &\leq C\rho(x), \\
\rho(x+y) &\leq e^{C|y|}\rho(x), \\
|\rho(x+y) - \rho(x)| &\leq (e^{C|y|} - 1)\rho(x),
\end{aligned}$$

where $C = C(\gamma)$ is a positive constant that does not depend on x.

We denote by $\mathcal{H}_{0,\gamma}(\mathbb{R})$ the space of functions $u : \mathbb{R} \to \mathbb{R}$, defined for almost all $x \in \mathbb{R}$, such that $\|\rho^{1/2}u\|_{L_2(\mathbb{R})} = \int\limits_{\mathbb{R}} \rho(x)|u(x)|^2\,dx < +\infty$. The space $\mathcal{H}_{0,\gamma}(\mathbb{R})$ is endowed with a scalar product $\langle u; v\rangle_{0,\gamma} := \int\limits_{\mathbb{R}} \rho(x)u(x)v(x)\,dx$. We also use the spaces $\mathcal{H}_{l,\gamma}(\mathbb{R})(l \in \mathbb{N})$ of functions $u \in \mathcal{H}_{0,\gamma}(\mathbb{R})$ such that $\mathcal{D}^\alpha u \in \mathcal{H}_{0,\gamma}(\mathbb{R})$ for all $\alpha \leq l$. The spaces $\mathcal{H}_{l,\gamma}(\mathbb{R})$ are endowed with a scalar product $\langle u; v\rangle_{l,\gamma} := \sum\limits_{0 \leq \alpha \leq l} \langle \mathcal{D}^\alpha u; \mathcal{D}^\alpha v\rangle_{0,\gamma}$.

The uniform space $\mathcal{H}_{0,u}$ is the space of functions $u \in \mathcal{H}_{0,\gamma}(\mathbb{R})$ such that the norm

$$\|u\|_{0,u} := \sup_{y \in \mathbb{R}} \|T_y u\|_{0,\gamma}$$

is finite and

$$\|T_y u - u\|_{0,u} \underset{y \to 0}{\to} 0.$$

Similarly, the uniform space $\mathcal{H}_{1,u}$ is the space of functions $u \in \mathcal{H}_{1,\gamma}(\mathbb{R})$ such that $u \in \mathcal{H}_{0,u}$ and $\mathcal{D}u \in \mathcal{H}_{0,u}$. The space $\mathcal{H}_{1,u}$ is endowed with the norm $\|u\|_{1,u}^2 := \|u\|_{0,u}^2 + \|\mathcal{D}u\|_{0,u}^2$.

We say that a set $B \subset \mathcal{H}_{0,u}$ is strongly bounded if it is bounded and the following relation holds:

$$\sup_{u \in B} \|T_y u - u\|_{0,u} \underset{y \to 0}{\to} 0.$$

Similarly, a bounded subset $B \subset \mathcal{H}_{1,u}$ is called strongly bounded if

$$\sup_{u \in B} \|T_y u - u\|_{1,u} \underset{y \to 0}{\to} 0.$$

Lemma 2.1 ***(compactness criterion)***
If a set $B \subset \mathcal{H}_{0,u}$ is strongly bounded, then this set is precompact in the space $\mathcal{H}_{0,\gamma}(\mathbb{R})$.

Proof. In the proof, we use the following notation. If A and B are subsets of the space $\mathcal{H}_{0,\gamma}(\mathbb{R})$, then

$$\mathrm{dev}(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_{0,\gamma}.$$

We fix $\varepsilon > 0$. We first reduce the problem to the case of functions with compact support. There is a number $\theta > 0$ such that $\rho(x) \geq 1/2$ for all $x \in [-\theta; \theta]$. Hence for any function $u \in B$ and number $y \in \mathbb{R}$ the following inequality holds:

$$\begin{aligned} \int_{y-\theta}^{y+\theta} |u(x)|^2 dx &= \int_{-\theta}^{\theta} |(T_y u)(x)|^2 dx \leq 2 \int_{-\theta}^{\theta} \rho(x) |(T_y u)(x)|^2 dx \\ &\leq 2 \int_{\mathbb{R}} \rho(x) |(T_y u)(x)|^2 dx \leq 2\|u\|_{0,u}^2 \leq C, \end{aligned} \tag{1}$$

where the constant C does not depend on u and y.

It follows from inequality (1) that for any number $k \in \mathbb{Z}$,

$$\begin{aligned} \int_{-\infty}^{-k\theta} \rho(x) |u(x)|^2 dx &= \sum_{j=-\infty}^{-k} \int_{(j-1)\theta}^{j\theta} \rho(x) |u(x)|^2 dx \\ &\leq C \sum_{j=-\infty}^{-k} \int_{(j-1)\theta}^{j\theta} \rho(j\theta) |u(x)|^2 dx \leq C \sum_{j=-\infty}^{-k} \rho(j\theta). \end{aligned}$$

Similarly,

$$\int_{k\theta}^{+\infty} \rho(x) |u(x)|^2 dx \leq C \sum_{j=k}^{+\infty} \rho(j\theta).$$

Thus, there exists a number $k_0 = k_0(B)$ such that both integrals $\int\limits_{-\infty}^{-k_0\theta} \rho(x)|u(x)|^2 dx$ and $\int\limits_{k_0\theta}^{+\infty} \rho(x)|u(x)|^2 dx$ are less than ε.

Let χ be a continuous function with support $\operatorname{supp}\chi \subset [-k_0\theta - 1; k_0\theta + 1]$ such that $\chi(x) = 1$ for all $x \in [-k_0\theta; k_0\theta]$ and $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}$. We denote $B_1 := \{\chi u : u \in B\}$. Then for all functions $v \in B_1$ we get $\operatorname{supp} v \subset [-k_0\theta - 1; k_0\theta + 1]$. It follows that for any functions $u \in B$ and $v = \chi u \in B_1$, the following inequalities hold:

$$\begin{aligned} \|u - v\|_{0,\gamma}^2 &= \int\limits_{\mathbb{R}} \rho(x)|u(x)(1 - \chi(x))|^2 dx \\ &\leq \int\limits_{-\infty}^{-k_0\theta} \rho(x)|u(x)|^2 dx + \int\limits_{k_0\theta}^{+\infty} \rho(x)|u(x)|^2 dx \leq 2\varepsilon. \end{aligned}$$

Thus,

$$\operatorname{dev}(B, B_1) \leq 2\varepsilon.$$

On the other hand, the set B_1 is still strongly bounded. Indeed, the following estimates hold for all functions $u \in B$ and $v = \chi u \in B_1$ and numbers $y, z \in \mathbb{R},\ |z| \leq 1$:

$$\|T_y v\|_{0,\gamma}^2 = \int\limits_{\mathbb{R}} \rho(x)|u(x+y)\chi(x+y)|^2 dx \leq \int\limits_{\mathbb{R}} \rho(x)|u(x+y)|^2 = \|T_y u\|_{0,\gamma}^2$$

and

$$\begin{aligned} &\|T_y(T_z v - v)\|_{0,\gamma}^2 \\ = &\int\limits_{\mathbb{R}} \rho(x)|u(x+y+z)\chi(x+y+z) - u(x+y)\chi(x+y)|^2 dx \\ \leq &\ 2\int\limits_{\mathbb{R}} \rho(x)|u(x+y+z)\chi(x+y+z) - u(x+y)\chi(x+y+z)|^2 dx \\ &+2\int\limits_{\mathbb{R}} \rho(x)|u(x+y)\chi(x+y+z) - u(x+y)\chi(x+y)|^2 dx \\ \leq &\ 2\int\limits_{\mathbb{R}} \rho(x)|u(x+y+z) - u(x+y)|^2 dx \\ &+2 \sup_{\substack{x_1, x_2 \in [-k_0\theta - 2; k_0\theta + 2] \\ |x_1 - x_2| \leq |z|}} |\chi(x_1) - \chi(x_2)|^2 \cdot \int\limits_{\mathbb{R}} \rho(x)|u(x+y)|^2 dx \end{aligned}$$

$$\leq \; 2\|T_y(T_z u - u)\|^2_{0,\gamma} + 2 \sup_{\substack{x_1,x_2 \in [-k_0\theta-2;k_0\theta+2] \\ |x_1-x_2|\leq |z|}} |\chi(x_1) - \chi(x_2)|^2 \cdot \|T_y u\|^2_{0,\gamma}$$

$$\underset{z\to 0}{\to} 0.$$

The convergence is uniform in $u \in B$ since the set B is strongly bounded and the function $\chi(x)$ is uniformly continuous on the compact set $[-k_0\theta - 2; k_0\theta + 2]$.

Now we apply the following theorem, cf. [7], [8].

Theorem 2.2 *Let $\Omega \subset \mathbb{R}$ be a bounded interval. Then the set $\mathcal{M} \subset L_p(\Omega)$, $p \in [1, +\infty)$ is precompact if and only if it is bounded in the space $L_p(\Omega)$ and equicontinuous with respect to the shift in the space $L_p(\Omega)$ that is*

$$\sup_{u\in\mathcal{M}} \|T_y u - u\|_{L_p(\mathbb{R})} \underset{y\to 0}{\to} 0,$$

where functions u are defined as zero on the set $\mathbb{R}\backslash\Omega$.

We note that for a bounded set $\Omega \subset \mathbb{R}$ and functions $f \in L_p(\Omega)$ the norms $\|f\|_{L_p(\Omega)}$ and $\|f\|_{\mathcal{H}_{0,u}}$ are equivalent. Hence the strongly bounded set B_1 satisfies the conditions of Theorem 2.2 and it is precompact in both spaces $L_p(\Omega)$ endowed with the norm $\|\cdot\|_{\mathcal{H}_{0,u}}$ and $\mathcal{H}_{0,\gamma}(\mathbb{R})$. Consequently, there exists a finite set B_2 such that $dev(B_2, B_1) < \varepsilon$.

Thus we obtain

$$dev(B_2, B) \leq dev(B_2, B_1) + dev(B_1, B) \leq 3\varepsilon,$$

which implies that the set B is also precompact.

Lemma 2.1 is proved. □

Corollary 2.3 *If a set $B \subset \mathcal{H}_{1,u}$ is strongly bounded then it is precompact in the space $\mathcal{H}_{1,\gamma}(\mathbb{R})$.*

Proof. By Lemma 2.1 both B and $B' := \{u' : u \in B\}$ are precompact subsets of the space $\mathcal{H}_{0,\gamma}(\mathbb{R})$ since they are strongly bounded in the space $\mathcal{H}_{0,u}$. Then for any sequence $\{u_n\}_{n=1}^{+\infty} \subset B$ there exist a subsequence $\{v_n\}_{n=1}^{+\infty}$ such that both sequences $\{v_n\}_{n=1}^{+\infty}$ and $\{v_n'\}_{n=1}^{+\infty}$ are convergent in the space $\mathcal{H}_{0,\gamma}(\mathbb{R})$. Let v and w be their limits. Then passing to the limit in the corresponding integral identity it is easy to see that $w = v'$. Hence $v_n \to v$ as $n \to +\infty$ in the space $\mathcal{H}_{1,\gamma}(\mathbb{R})$. □

3 Discrete weighted spaces

Consider the following discrete spaces. The Hilbert space $H_{0,d,\gamma}$ is the space of sequences $u = \{u_k\}_{k\in\mathbb{Z}}$ such that the squared norm $\|u\|_{0,\gamma}^2 := d \sum_{k\in\mathbb{Z}} \rho(kd)|u_k|^2$ is finite. Similarly, Hilbert spaces $H_{1,d,\gamma}$ and $H_{2,d,\gamma}$ are the spaces of sequences $u \in H_{0,d,\gamma}$ endowed with the norms

$$\|u\|_{1,\gamma}^2 := \|u\|_{0,\gamma}^2 + \|\partial_+ u\|_{0,\gamma}^2$$

and

$$\|u\|_{2,\gamma}^2 := \|u\|_{0,\gamma}^2 + \|\partial_+ u\|_{0,\gamma}^2 + \|\partial_+\partial_+ u\|_{0,\gamma}^2,$$

respectively.

The uniform discrete space $H_{0,d,u}$ is the space of sequences $u \in H_{0,d,\gamma}$ such that the norm

$$\|u\|_{0,u} := \sup_{y\in\mathbb{Z}} \|T_y u\|_{0,\gamma}$$

is finite. The uniform discrete spaces $H_{1,d,u}$ and $H_{2,d,u}$ are the spaces of sequences $u \in H_{0,d,u}$ such that $\partial_+ u \in H_{0,d,u}$ (and $\partial_+\partial_+ u \in H_{0,d,u}$ for $H_{2,d,u}$) endowed with the norms

$$\|u\|_{1,d,u}^2 := \|u\|_{0,d,u}^2 + \|\partial_+ u\|_{0,d,u}^2$$

and

$$\|u\|_{2,d,u}^2 := \|u\|_{0,d,u}^2 + \|\partial_+ u\|_{0,d,u}^2 + \|\partial_+\partial_+ u\|_{0,d,u}^2,$$

respectively.

We embed the space $H_{0,d,\gamma}$ into $\mathcal{H}_{0,\gamma}(\mathbb{R})$ as follows. Consider $k \in \mathbb{Z}$ and a function $\omega_k(x), x \in \mathbb{R}$, such that

$$\omega_k(x) = \begin{cases} (x-(k-1)d)/d, & x \in [(k-1)d, kd], \\ ((k+1)d - x)/d, & x \in [kd, (k+1)d], \\ 0 & \text{otherwise.} \end{cases}$$

With these hat functions we define the interpolation operator
$\mathcal{T} : H_{0,d,\gamma} \to \mathcal{H}_{0,\gamma}(\mathbb{R})$ by

$$\mathcal{T}\{u_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} \omega_k(x) u_k. \tag{2}$$

Lemma 3.1 ***(properties of the operator $\mathcal{T}$)***
Assume that $d < 1$. Then there exist positive constants c and C independent of d such that:

(a) for any sequence $u \in H_{0,d,\gamma}$,

$$c\|u\|_{0,\gamma} \leq \|\mathcal{T}u\|_{0,\gamma} \leq C\|u\|_{0,\gamma}; \tag{3}$$

(b) for any sequence $u \in H_{0,d,u}$,

$$c\|u\|_{0,d,u} \leq \|\mathcal{T}u\|_{0,u} \leq C\|u\|_{0,d,u}; \tag{4}$$

(c) for any sequence $u \in H_{1,d,\gamma}$,

$$c\|u\|_{1,\gamma} \leq \|\mathcal{T}u\|_{1,\gamma} \leq C\|u\|_{1,\gamma}; \tag{5}$$

(d) for any sequence $u \in H_{1,d,u}$,

$$c\|u\|_{1,d,u} \leq \|\mathcal{T}u\|_{1,u} \leq C\|u\|_{1,d,u}; \tag{6}$$

(e) for any sequence $u \in H_{1,d,u}$ and any number $z \in \mathbb{R}$, $|z| < 1$,

$$\|T_z\mathcal{T}u - \mathcal{T}u\|_{0,u} \leq C|z|\|u\|_{1,d,u}; \tag{7}$$

(f) for any sequence $u \in H_{2,d,u}$ and any number $z \in \mathbb{R}$, $|z| < 1$,

$$\|T_z\mathcal{T}u - \mathcal{T}u\|_{1,u} \leq C(|z| + 3|z|^2)^{1/2}\|u\|_{2,d,u}. \tag{8}$$

Proof.

(a) We fix numbers $j \in \mathbb{Z}$ and $0 < \varepsilon \leq 1/2$. For any number $x \in [(j-\varepsilon)d; (j+\varepsilon)d]$, the following inequalities hold:

$$\begin{aligned} \omega_j(x) &\geq 1 - \varepsilon, \\ \omega_{j\pm 1}(x) &\leq \varepsilon, \\ \omega_k(x) &= 0, \quad |k - j| \geq 2. \end{aligned}$$

Hence, for the same x:

$$\begin{aligned} \Big(\sum_{k\in\mathbb{Z}} u_k\omega_k\Big)^2(x) &= \Big(u_{j-1}\omega_{j-1}(x) + u_j\omega_j(x) + u_{j+1}\omega_{j+1}(x)\Big)^2 \\ &= u_j^2\omega_j^2(x) + 2u_j\omega_j(x)u_{j-1}\omega_{j-1}(x) \\ &\quad +2u_j\omega_j(x)u_{j+1}\omega_{j+1}(x) + u_{j-1}^2\omega_{j-1}^2(x) + u_{j+1}^2\omega_{j+1}^2(x) \\ &\geq u_j^2\omega_j^2(x) - \frac{1}{3}u_j^2\omega_j^2(x) - \frac{1}{3}u_j^2\omega_j^2(x) - 3u_{j-1}^2\omega_{j-1}^2(x) \\ &\quad +u_{j-1}^2\omega_{j-1}^2(x) - 3u_{j+1}^2\omega_{j+1}^2(x) + u_{j+1}^2\omega_{j+1}^2(x) \\ &= \frac{1}{3}u_j^2\omega_j^2(x) - 2u_{j-1}^2\omega_{j-1}^2(x) - 2u_{j+1}^2\omega_{j+1}^2(x) \\ &\geq \frac{(1-\varepsilon)^2}{3}u_j^2 - 2\varepsilon^2u_{j-1}^2 - 2\varepsilon^2u_{j+1}^2. \end{aligned} \tag{9}$$

Finally, we estimate the norm $\|\mathcal{T}u\|_{0,\gamma}$ as follows:

$$
\begin{aligned}
\|\mathcal{T}u\|_{0,\gamma}^2 &= \int_{\mathbb{R}} \rho(x)\Big(\sum_{k\in\mathbb{Z}} u_k\omega_k\Big)^2(x)dx \\
&\geq \sum_{j\in\mathbb{Z}} \int_{(j-\varepsilon)d}^{(j+\varepsilon)d} \rho(x)\Big(\sum_{k\in\mathbb{Z}} u_k\omega_k\Big)^2(x)dx \\
&\overset{\text{from (9)}}{\geq} \sum_{j\in\mathbb{Z}} \Big(\frac{(1-\varepsilon)^2}{3}u_j^2 - 2\varepsilon^2 u_{j-1}^2 - 2\varepsilon^2 u_{j+1}^2\Big) \int_{(j-\varepsilon)d}^{(j+\varepsilon)d} \rho(x)dx.
\end{aligned}
$$

We rearrange terms and get

$$
\begin{aligned}
\|\mathcal{T}u\|_{0,\gamma}^2 &\geq \sum_{j\in\mathbb{Z}} \Big(\frac{(1-\varepsilon)^2}{3} \int_{(j-\varepsilon)d}^{(j+\varepsilon)d} \rho(x)dx - 2\varepsilon^2 \int_{(j-1-\varepsilon)d}^{(j-1+\varepsilon)d} \rho(x)dx \\
&\qquad -2\varepsilon^2 \int_{(j+1-\varepsilon)d}^{(j+1+\varepsilon)d} \rho(x)dx\Big) u_j^2 \\
&= \sum_{j\in\mathbb{Z}} \int_{(j-\varepsilon)d}^{(j+\varepsilon)d} \Big(\frac{(1-\varepsilon)^2}{3}\rho(x) - 2\varepsilon^2\rho(x-d) - 2\varepsilon^2\rho(x+d)\Big)dx \cdot u_j^2 \\
&\geq \sum_{j\in\mathbb{Z}} \Big(\frac{(1-\varepsilon)^2}{3} - 4C\varepsilon^2\Big) \int_{(j-\varepsilon)d}^{(j+\varepsilon)d} \rho(x)dx \cdot u_j^2 \\
&\geq 2Cd\varepsilon\Big(\frac{(1-\varepsilon)^2}{3} - 4C\varepsilon^2\Big) \sum_{j\in\mathbb{Z}} \rho(jd)u_j^2 \\
&= 2C\varepsilon\Big(\frac{(1-\varepsilon)^2}{3} - 4C\varepsilon^2\Big) \|u\|_{0,\gamma}^2. \qquad (10)
\end{aligned}
$$

Note that in the previous estimates we may choose the constant C independently of d due to the inequality $d < 1$. We take $\varepsilon = (1 + \sqrt{24C})^{-1}$ and deduce from estimate (10) the first inequality of (3).

Now we prove the second inequality of (3). We first estimate the value $\mathcal{T}u(x)$, $x \in \mathbb{R}$ in the following way:

$$
\begin{aligned}
(\mathcal{T}u)^2(x) &= \Big(\sum_{k\in\mathbb{Z}} u_k\omega_k(x)\Big)^2 \\
&= \sum_{k\in\mathbb{Z}} u_k\omega_k(x)\Big(u_{k-1}\omega_{k-1}(x) + u_k\omega_k(x) + u_{k-1}\omega_{k-1}(x)\Big) \\
&\leq \sum_{k\in\mathbb{Z}} \Big(u_k^2\omega_k^2(x) + \frac{1}{2}u_{k-1}^2\omega_{k-1}(x)\omega_k(x) + \frac{1}{2}u_k^2\omega_{k-1}(x)\omega_k(x) \\
&\qquad + \frac{1}{2}u_k^2\omega_k(x)\omega_{k+1}(x) + \frac{1}{2}u_{k+1}^2\omega_k(x)\omega_{k+1}(x)\Big).
\end{aligned}
$$

We rearrange terms and get

$$
(\mathcal{T}u)^2(x) \leq \sum_{k\in\mathbb{Z}} u_k^2\omega_k(x)\Big(\omega_{k-1}(x) + \omega_k(x) + \omega_{k+1}(x)\Big) = \sum_{k\in\mathbb{Z}} u_k^2\omega_k(x).
$$

We deduce from the last estimate that

$$
\begin{aligned}
\|\mathcal{T}u\|_{0,\gamma}^2 &= \int_{\mathbb{R}} \rho(x)(\mathcal{T}u)^2(x)dx \leq \int_{\mathbb{R}} \rho(x)\sum_{k\in\mathbb{Z}} u_k^2\omega_k(x)dx \\
&= \sum_{k\in\mathbb{Z}} u_k^2 \int_{(k-1)d}^{(k+1)d} \rho(x)\omega_k(x)dx \leq Cd\sum_{k\in\mathbb{Z}} \rho(kd)u_k^2 = C\|u\|_{0,\gamma}^2.
\end{aligned}
$$

Statement (a) is proved.

(b) It suffices to estimate the function $T_z\mathcal{T}u$ for numbers $z = kd$ with integer k since there exist positive constants c and C such that

$$
c\|v\|_{0,\gamma} \leq \|T_zv\|_{0,\gamma} \leq C\|v\|_{0,\gamma}
$$

holds for any function $v \in \mathcal{H}_{0,\gamma}(\mathbb{R})$ and number z, $|z| \leq d$. For the numbers $z = kd$ with some $k \in \mathbb{Z}$, the following equality holds:

$$
T_z\mathcal{T}u = \mathcal{T}T_zu.
$$

Hence, estimate (4) follows from estimate (3).

(c) Since for any number $x \in [kd;(k+1)d]$, $k \in \mathbb{Z}$, the derivative satisfies $(\mathcal{T}u)'(x) = (\partial_+u)_k$, the following inequalities hold:

$$
\|\partial_+u\|_{0,\gamma}^2 \;=\; d\sum_{k\in\mathbb{Z}} \rho(kd)(\partial_+u)_k^2 \leq C\sum_{k\in\mathbb{Z}} \int_{kd}^{(k+1)d} \rho(x)dx(\partial_+u)_k^2
$$

$$
\begin{aligned}
&= C\sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)((\mathcal{T}u)'(x))^2 dx = C\|(\mathcal{T}u)'\|^2_{0,\gamma};\\
\|(\mathcal{T}u)'\|^2_{0,\gamma} &= \sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)((\mathcal{T}u)'(x))^2 dx\\
&= \sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)dx(\partial_+ u)^2_k\\
&\le Cd\sum_{k\in\mathbb{Z}} \rho(kd)(\partial_+ u)^2_k = C\|\partial_+ u\|^2_{0,\gamma}.
\end{aligned}
$$

Therefore inequality (5) follows from the last two inequalities and inequality (3).

(d) Since there exist positive constants c and C such that

$$
c\|v\|_{1,\gamma} \le \|T_z v\|_{1,\gamma} \le C\|v\|_{1,\gamma}
$$

for any function $v \in \mathcal{H}_{1,\gamma}(\mathbb{R})$ and number z, $|z| \le d$, inequality (6) follows immediately from inequality (5).

(e) Let y and z be any real numbers. We consider the case $z > 0$ since the proof for $z < 0$ is similar. Then the following estimates hold for $v = \mathcal{T}u$ and any $x \in \mathbb{R}$:

$$
\begin{aligned}
|T_y(T_z v - v)(x)|^2 &= |v(x+y+z) - v(x+y)|^2\\
&= |z|^2 \Big| \int\limits_0^1 v'(x+y+z\theta)d\theta \Big|^2\\
&\le |z|^2 \int\limits_0^1 |v'(x+y+z\theta)|^2 d\theta.
\end{aligned}
$$

Hence, we obtain estimate (7) in the following way:

$$
\begin{aligned}
\|T_y(T_z v - v)\|^2_{0,\gamma} &= \int\limits_{\mathbb{R}} \rho(x)|T_y(T_z v - v)(x)|^2 dx\\
&\le |z|^2 \int\limits_{\mathbb{R}} \rho(x) \int\limits_0^1 |v'(x+y+z\theta)|^2 d\theta dx
\end{aligned}
$$

$$= |z|^2 \int_0^1 \int_{\mathbb{R}} \rho(x)|v'(x+y+z\theta)|^2 dx d\theta$$

$$= |z|^2 \int_0^1 \|T_{y+z\theta}v'\|_{0,\gamma}^2 d\theta \overset{\text{(by (6))}}{\leq} C|z|^2\|u\|_{1,d,u}^2.$$

(f) We denote $v := \mathcal{T}u$. We consider the case $z > 0$ since the proof for $z < 0$ is similar. We can represent the number z as $z = (j+\theta)d$ for some numbers $j \in \mathbb{Z}$, $j \geq 0$, and $\theta \in [0,1)$. Since for any $k \in \mathbb{Z}$ and $x \in [kd;(k+1)d]$, the derivative $v'(x)$ equals $(\partial_+ u)_k$, we proceed as follows:

$$\begin{aligned}
&\|(T_y(T_z v - v))'\|_{0,\gamma}^2 \\
= &\int_{\mathbb{R}} \rho(x)|v'(x+y+z) - v'(x+y)|^2 dx \\
= &\sum_{k\in\mathbb{Z}} \Big(\int_{kd-y}^{(k+1-\theta)d-y} \rho(x)|v'(x+y+z) - v'(x+y)|^2 dx \\
&+ \int_{(k+1-\theta)d-y}^{(k+1)d-y} \rho(x)|v'(x+y+z) - v'(x+y)|^2 dx \Big) \\
= &\sum_{k\in\mathbb{Z}} \Big(|\partial_+ u_{j+k} - \partial_+ u_k|^2 \int_{kd-y}^{(k+1-\theta)d-y} \rho(x) dx \\
&+ |\partial_+ u_{j+k+1} - \partial_+ u_k|^2 \int_{(k+1-\theta)d-y}^{(k+1)d-y} \rho(x) dx \Big) \\
= &\sum_{k\in\mathbb{Z}} \Big(\Big| d \sum_{m=k}^{j+k-1} \partial_+\partial_+ u_m \Big|^2 \int_{kd-y}^{(k+1-\theta)d-y} \rho(x) dx \\
&+ \Big| d \sum_{m=k}^{j+k} \partial_+\partial_+ u_m \Big|^2 \int_{(k+1-\theta)d-y}^{(k+1)d-y} \rho(x) dx \Big) \\
\leq &\, d^2 \sum_{k\in\mathbb{Z}} \Big(j \sum_{m=k}^{j+k-1} |\partial_+\partial_+ u_m|^2 \int_{kd-y}^{(k+1-\theta)d-y} \rho(x) dx
\end{aligned}$$

$$+(j+1)\sum_{m=k}^{j+k}|\partial_+\partial_+u_m|^2\int\limits_{(k+1-\theta)d-y}^{(k+1)d-y}\rho(x)dx\Big)$$

$$= d^2j\sum_{m\in\mathbb{Z}}|\partial_+\partial_+u_m|^2\sum_{k=m+1-j}^{m}\int\limits_{kd-y}^{(k+1-\theta)d-y}\rho(x)dx$$

$$+d^2(j+1)\sum_{m\in\mathbb{Z}}|\partial_+\partial_+u_m|^2\sum_{k=m-j}^{m}\int\limits_{(k+1-\theta)d-y}^{(k+1)d-y}\rho(x)dx. \qquad (11)$$

Let $[y/d]$ be the integer part of the number y/d. Then $k-[y/d]$ is an integer number such that $(k-[y/d])d\in(kd-y;(k+1)d-y]$. This means that $\rho(x)\le C\rho((k-[y/d])d)$ for any number $x\in[kd-y;(k+1)d-y)$, where the constant C depends only on γ in the definition of the function ρ. Hence, we deduce from inequality (11) that

$$\begin{aligned}
&\|(T_y(T_zv-v))'\|^2_{0,\gamma}\\
\le\;& Cd^2j\sum_{m\in\mathbb{Z}}|\partial_+\partial_+u_m|^2\sum_{k=m+1-j}^{m}d(1-\theta)\rho((k-[y/d])d)\\
&+Cd^2(j+1)\sum_{m\in\mathbb{Z}}|\partial_+\partial_+u_m|^2\sum_{k=m-j}^{m}d\theta\rho((k-[y/d])d)\\
\overset{k:=k-m}{=}\;& Cd^3j(1-\theta)\sum_{k=1-j}^{0}\sum_{m\in\mathbb{Z}}\rho((m+k-[y/d])d)|\partial_+\partial_+u_m|^2\\
&+Cd^3(j+1)\theta\sum_{k=-j}^{0}\sum_{m\in\mathbb{Z}}\rho((m+k-[y/d])d)|\partial_+\partial_+u_m|^2\\
\overset{m:=m+k-[y/d]}{=}\;& Cd^2j(1-\theta)\sum_{k=1-j}^{0}d\sum_{m\in\mathbb{Z}}\rho(md)|\partial_+\partial_+u_{m-k+[y/d]}|^2\\
&+Cd^2(j+1)\theta\sum_{k=-j}^{0}d\sum_{m\in\mathbb{Z}}\rho(md)|\partial_+\partial_+u_{m-k+[y/d]}|^2\\
\le\;& Cd^2\Big(j^2(1-\theta)+(j+1)^2\theta\Big)\|u\|^2_{2,d,u}.
\end{aligned}$$

Finally, we estimate

$$\begin{aligned} d^2\Big(j^2(1-\theta)+(j+1)^2\theta\Big) &= d^2(j^2+(2j+1)\theta) \le z^2+(2z+1)z \\ &= 3z^2+z. \end{aligned}$$

□

4 Properties of projectors

In this section, we consider the subspace $\mathcal{V}_d := \mathcal{T}(H_{0,d,\gamma})$. It follows from Lemma 3.1, (a) that $\mathcal{V}_d$ is a closed subspace of $\mathcal{H}_{0,\gamma}(\mathbb{R})$. We first prove a useful lemma about this subspace.

Lemma 4.1 *There exists a constant $C > 0$ independent of d such that for any $v \in \mathcal{V}_d$, the following inequality holds:*

$$\|v\|_{1,\gamma} \le C d^{-1} \|v\|_{0,\gamma}.$$

Proof. Assume that $v \in \mathcal{V}_d$. Since the operator $\mathcal{T}$ is also an isomorphism of the discrete space $H_{1,d,\gamma}$ and the space $\mathcal{V}_d$ endowed with the norm $\|\cdot\|_{1,\gamma}$, the following inequality holds:

$$\|v\|_{1,\gamma} \le C\|\mathcal{T}^{-1}v\|_{1,d,\gamma} \le Cd^{-1}\|\mathcal{T}^{-1}v\|_{0,d,\gamma} \le Cd^{-1}\|v\|_{0,\gamma}.$$

□

We consider the following two projectors onto the space $\mathcal{V}_d$. The first one is the orthogonal projector P_d onto $\mathcal{V}_d$ in the space $\mathcal{H}_{0,\gamma}(\mathbb{R})$. The second projector Q_d is defined on the space $\mathcal{H}_{1,\gamma}(\mathbb{R})$ by the formula:

$$Q_d u = \mathcal{T}\{u(kd)\}_{k\in\mathbb{Z}}.$$

Lemma 4.2 *The mapping Q_d is well-defined on the space $\mathcal{H}_{1,\gamma}(\mathbb{R})$ and is a projector onto the subspace $\mathcal{V}_d$.*

Proof. Fix $u \in \mathcal{H}_{1,\gamma}(\mathbb{R})$. It is sufficient to prove that the sequence $\{u(kd)\}_{k\in\mathbb{Z}}$ lies in the discrete space $H_{0,d,\gamma}$ since in this case the function $\mathcal{T}\{u(kd)\}_{k\in\mathbb{Z}}$ belongs to the space $\mathcal{V}_d$ and the mapping Q_d is a projector since $Q_d u(kd) = u(kd)$ for all k.

It suffices to prove that $d\sum\limits_{k\in\mathbb{Z}}\rho(kd)|u(kd)|^2<+\infty$. Let us estimate this sum using the Hölder inequality

$$
\begin{aligned}
& d\sum_{k\in\mathbb{Z}}\rho(kd)|u(kd)|^2 \\
= \;& d\sum_{k\in\mathbb{Z}}\rho(kd)\left|\frac{1}{d}\int\limits_{kd}^{(k+1)d}u(x)dx+\frac{1}{d}\int\limits_{kd}^{(k+1)d}(u(kd)-u(x))dx\right|^2 \\
\leq \;& d\frac{2}{d^2}\sum_{k\in\mathbb{Z}}\rho(kd)\left|\int\limits_{kd}^{(k+1)d}u(x)\right|^2+d\frac{2}{d^2}\sum_{k\in\mathbb{Z}}\rho(kd)\left|\int\limits_{kd}^{(k+1)d}\int\limits_{kd}^{x}u'(x_1)dx_1dx\right|^2 \\
\leq \;& 2\sum_{k\in\mathbb{Z}}\rho(kd)\int\limits_{kd}^{(k+1)d}|u(x)|^2dx+2d\sum_{k\in\mathbb{Z}}\rho(kd)\int\limits_{kd}^{(k+1)d}\int\limits_{kd}^{x}|u'(x_1)|^2dx_1dx \\
\leq \;& 2C\sum_{k\in\mathbb{Z}}\int\limits_{kd}^{(k+1)d}\rho(x)|u(x)|^2dx \\
& +2Cd\sum_{k\in\mathbb{Z}}\int\limits_{kd}^{(k+1)d}\rho(x_1)((k+1)d-x_1)|u'(x_1)|^2dx_1 \\
\leq \;& 2C(\|u\|_{0,\gamma}^2+d^2\|u'\|_{0,\gamma}^2)<+\infty.
\end{aligned}
$$

Thus, $\{u(kd)\}_{k\in\mathbb{Z}}\in H_{0,d,\gamma}$.

□

Lemma 4.3 ***(properties of the projector*** Q_d***)***
There exists a constant $C>0$ *independent of* d *such that the following statements hold.*

(a) For any $u\in\mathcal{H}_{1,\gamma}(\mathbb{R})$, *the inequalities*

$$\|(I-Q_d)u\|_{0,\gamma}\leq Cd\|u\|_{1,\gamma} \tag{12}$$

and

$$\|Q_du\|_{1,\gamma}\leq C\|u\|_{1,\gamma} \tag{13}$$

hold.

(b) For any $u\in\mathcal{H}_{2,\gamma}(\mathbb{R})$, *the inequalities*

$$\|(I-Q_d)u\|_{0,\gamma}\leq Cd^2\|u\|_{2,\gamma} \tag{14}$$

and

$$\|(I - Q_d)u\|_{1,\gamma} \le Cd\|u\|_{2,\gamma} \tag{15}$$

hold.

Proof. Let u be a function from the space $\mathcal{H}_{1,\gamma}(\mathbb{R})$ and let $w := Q_d u$. Fix $x = (k+\theta)d$, where $k \in \mathbb{Z}$ and $0 \le \theta \le 1$. The function w can be written as

$$w(x) = u(kd) + \theta(u((k+1)d) - u(kd)) = u(kd) + \theta d \int_0^1 u'((k+\theta_1)d)d\theta_1.$$

Hence,

$$\begin{aligned} u(x) - w(x) &= u((k+\theta)d) - u(kd) - \theta d \int_0^1 u'((k+\theta_1)d)d\theta_1 \\ &= d\int_0^\theta \Big(u'((k+\theta_2)d) - \int_0^1 u'((k+\theta_1)d)d\theta_1\Big)d\theta_2, \end{aligned} \tag{16}$$

and we deduce the following inequality:

$$\begin{aligned} &\int_{kd}^{(k+1)d} \rho(x)|u(x) - w(x)|^2 dx \\ = \;& d\int_0^1 \rho((k+\theta)d)|u((k+\theta)d) - w((k+\theta)d)|^2 d\theta \\ = \;& d^3 \int_0^1 \rho((k+\theta)d)\left|\int_0^\theta \Big(u'((k+\theta_2)d) - \int_0^1 u'((k+\theta_1)d)d\theta_1\Big)d\theta_2\right|^2 d\theta \\ \le \;& d^3 \int_0^1 \theta\rho((k+\theta)d)\int_0^\theta \left|u'((k+\theta_2)d) - \int_0^1 u'((k+\theta_1)d)d\theta_1\right|^2 d\theta_2 d\theta \\ \le \;& 2d^3 \int_0^1 \theta\rho((k+\theta)d)\int_0^\theta |u'((k+\theta_2)d)|^2 d\theta_2 d\theta \\ & +2d^3 \int_0^1 \theta\rho((k+\theta)d)\int_0^\theta \left|\int_0^1 u'((k+\theta_1)d)d\theta_1\right|^2 d\theta_2 d\theta \end{aligned}$$

$$\leq \ 2d^3 \int_0^1 \int_0^\theta \theta\rho((k+\theta)d)|u'((k+\theta_2)d)|^2 d\theta_2 d\theta$$

$$+2d^3 \int_0^1 \int_0^\theta \int_0^1 \theta\rho((k+\theta)d)|u'((k+\theta_1)d)|^2 d\theta_1 d\theta_2 d\theta.$$

Therefore, using the Fubini Theorem we obtain

$$\int_{kd}^{(k+1)d} \rho(x)|u(x)-w(x)|^2 dx$$

$$\leq \ 2d^3 \int_0^1 |u'((k+\theta_2)d)|^2 \int_{\theta_2}^1 \theta\rho((k+\theta)d) d\theta d\theta_2$$

$$+2d^3 \int_0^1 |u'((k+\theta_1)d)|^2 \int_0^1 \int_{\theta_2}^1 \theta\rho((k+\theta)d) d\theta d\theta_2 d\theta_1$$

$$\leq \ Cd^3 \int_0^1 \rho((k+\theta_1)d)|u'((k+\theta_1)d)|^2 d\theta_1$$

$$= \ Cd^2 \int_{kd}^{(k+1)d} \rho(x)|u'(x)|^2 dx.$$

Consequently, we result in the following estimate:

$$\begin{aligned} \|u - Q_d u\|^2_{0,\gamma} &= \int_{\mathbb{R}} \rho(x)|u(x)-w(x)|^2 dx \\ &= \sum_{k\in\mathbb{Z}} \int_{kd}^{(k+1)d} \rho(x)|u(x)-w(x)|^2 dx \\ &\leq Cd^2 \sum_{k\in\mathbb{Z}} \int_{kd}^{(k+1)d} \rho(x)|u'(x)|^2 dx = Cd^2\|u'\|^2_{0,\gamma}. \end{aligned}$$

Thus, inequality (12) is proved.

Inequality (13) follows from the fact that

$$w'((k+\theta)d) = \int_0^1 u'((k+\theta_1)d) d\theta_1$$

and, consequently,

$$\begin{aligned}
\|w'\|^2_{0,\gamma} &= \sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)|w'(x)|^2 dx \\
&= d\sum_{k\in\mathbb{Z}} \int\limits_0^1 \rho((k+\theta)d) \left| \int\limits_0^1 u'((k+\theta_1)d)d\theta_1 \right|^2 d\theta \\
&\leq Cd\sum_{k\in\mathbb{Z}} \int\limits_0^1 \int\limits_0^1 \rho((k+\theta_1)d)|u'((k+\theta_1)d)|^2 d\theta_1 d\theta \\
&= C\|u'\|^2_{0,\gamma}.
\end{aligned}$$

Now let u be a function from the space $\mathcal{H}_{2,\gamma}(\mathbb{R})$. For a fixed $x = (k+\theta)d$, where $k \in \mathbb{Z}$ and $0 \leq \theta \leq 1$, we may continue equality (16) as follows:

$$\begin{aligned}
u(x) - w(x) &= d\int\limits_0^\theta \Big(u'((k+\theta_2)d) - \int\limits_0^1 u'((k+\theta_1)d)d\theta_1 \Big) d\theta_2 \\
&= d\int\limits_0^\theta \int\limits_0^1 \Big(u'((k+\theta_2)d) - u'((k+\theta_1)d) \Big) d\theta_1 d\theta_2 \\
&= d^2 \int\limits_0^\theta \int\limits_0^1 \int\limits_{\theta_1}^{\theta_2} u''((k+\theta_3)d)d\theta_3 d\theta_1 d\theta_2.
\end{aligned}$$

It follows from this equality that

$$\begin{aligned}
&\int\limits_{kd}^{(k+1)d} \rho(x)|u(x) - w(x)|^2 dx \\
= \;& d\int\limits_0^1 \rho((k+\theta)d)|u((k+\theta)d) - w((k+\theta)d)|^2 d\theta \\
= \;& d^5 \int\limits_0^1 \rho((k+\theta)d) \left| \int\limits_0^\theta \int\limits_0^1 \int\limits_{\theta_1}^{\theta_2} u''((k+\theta_3)d)d\theta_3 d\theta_1 d\theta_2 \right|^2 d\theta \\
\leq \;& d^5 \int\limits_0^1 \int\limits_0^\theta \int\limits_0^1 \int\limits_{\theta_1}^{\theta_2} \rho((k+\theta)d)|u''((k+\theta_3)d)|^2 d\theta_3 d\theta_1 d\theta_2 d\theta
\end{aligned}$$

$$\leq Cd^5 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \rho((k+\theta_3)d)|u''((k+\theta_3)d)|^2 d\theta_3 d\theta_1 d\theta_2 d\theta$$
$$= Cd^5 \int_0^1 \rho((k+\theta_3)d)|u''((k+\theta_3)d)|^2 d\theta_3. \qquad (17)$$

Inequality (14) follows from inequality (17) in the same way as in the proof of inequality (12).

Finally, we prove estimate (15). For a fixed $x = (k+\theta)d$, where $k \in \mathbb{Z}$ and $0 \leq \theta \leq 1$, the following equality holds:

$$\begin{aligned} u'(x) - w'(x) &= u'((k+\theta)d) - \frac{1}{d}(u((k+1)d) - u(kd)) \\ &= u'((k+\theta)d) - \int_0^1 u'((k+\theta_1)d)d\theta_1 \\ &= d \int_0^1 \int_{\theta_1}^{\theta} u''((k+\theta_2)d)d\theta_2 d\theta_1 \end{aligned}$$

Hence, the same reasoning as above proves the following estimates:

$$\begin{aligned} & \int_{kd}^{(k+1)d} \rho(x)|u'(x) - w'(x)|^2 dx \\ = \; & d \int_0^1 \rho((k+\theta)d)|u'((k+\theta)d) - w'((k+\theta)d)|^2 d\theta \\ = \; & d^3 \int_0^1 \rho((k+\theta)d) \left| \int_0^1 \int_{\theta_1}^{\theta} u''((k+\theta_2)d)d\theta_2 d\theta_1 \right|^2 d\theta \\ \leq \; & Cd^3 \int_0^1 \rho((k+\theta_2)d)|u''((k+\theta_2)d)|^2 d\theta_2 \\ = \; & Cd^2 \int_{kd}^{(k+1)d} \rho(x)|u''(x)|^2 dx \end{aligned}$$

and

$$\begin{aligned}
\|u-w\|_{1,\gamma}^2 &= \|u-w\|_{0,\gamma}^2 + \|u'-w'\|_{0,\gamma}^2 \\
&= \|u-w\|_{0,\gamma}^2 + \sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)|u'(x)-w'(x)|^2 dx \\
&\le Cd^2\|u\|_{1,\gamma}^2 + Cd^2 \sum_{k\in\mathbb{Z}} \int\limits_{kd}^{(k+1)d} \rho(x)|u''(x)|^2 dx \\
&= Cd^2\|u\|_{2,\gamma}^2.
\end{aligned}$$

□

Lemma 4.4 ***(properties of the projector*** P_d***)***
There exists a constant $C > 0$ *independent of* d *satisfying the following statements.*

(a) For any $u \in \mathcal{H}_{1,\gamma}(\mathbb{R})$*, the inequalities*

$$\|(I-P_d)u\|_{0,\gamma} \le Cd\|u\|_{1,\gamma} \tag{18}$$

and

$$\|(I-P_d)u\|_{1,\gamma} \le C\|u\|_{1,\gamma} \tag{19}$$

hold.

(b) For any $u \in \mathcal{H}_{2,\gamma}(\mathbb{R})$*, the inequalities*

$$\|(I-P_d)u\|_{0,\gamma} \le Cd^2\|u\|_{2,\gamma} \tag{20}$$

and

$$\|(I-P_d)u\|_{1,\gamma} \le Cd\|u\|_{2,\gamma} \tag{21}$$

hold.

Proof. Since P_d is an orthogonal projector, the following inequality holds for all $u \in \mathcal{H}_{1,\gamma}(\mathbb{R})$:

$$\|(I-P_d)u\|_{0,\gamma} \le \|(I-Q_d)u\|_{0,\gamma}.$$

Hence, inequalities (18) and (20) follow immediately from inequalities (12) and (14).

Finally, we prove inequalities (19) and (21). Since the projector P_d is orthogonal, the following estimates hold:

$$\|P_d u - Q_d u\|_{0,\gamma} \le \|(I - Q_d)u\|_{0,\gamma} \le Cd\|u\|_{1,\gamma}\,, \quad \text{if } u \in \mathcal{H}_{1,\gamma}(\mathbb{R})\,,$$

and

$$\|P_d u - Q_d u\|_{0,\gamma} \le Cd^2\|u\|_{2,\gamma}\,, \quad \text{if } u \in \mathcal{H}_{2,\gamma}(\mathbb{R})\,.$$

Next, Lemma 4.1 implies

$$\|P_d u - Q_d u\|_{1,\gamma} \le C\|u\|_{1,\gamma}$$

and

$$\|P_d u - Q_d u\|_{1,\gamma} \le Cd\|u\|_{2,\gamma}.$$

Now inequalities (19) and (21) follow from the last inequalities and inequalities (13) and (15).

□

Lemma 4.5 *Let K be a natural number. There exists a constant $C(K) > 0$ independent of d such that, for all numbers $k \in \mathbb{Z}$, $|k| \le K$, the following inequalities hold:*

$$\|P_d T_{kd}(I - P_d)u\|_{0,\gamma} \le Cd\|u\|_{0,\gamma}, \tag{22}$$

$$\|P_d T_{kd}(I - P_d)u\|_{0,\gamma} \le Cd^2\|u\|_{1,\gamma}, \tag{23}$$

$$\|P_d T_{kd}(I - P_d)u\|_{0,\gamma} \le Cd^3\|u\|_{2,\gamma}. \tag{24}$$

Proof. Since $P_d T_{kd}(I - P_d)u \in \mathcal{V}_d$, we may write the following equality:

$$\begin{aligned}
&\|P_d T_{kd}(I - P_d)u\|_{0,\gamma} \\
=\; & \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\le 1}} \int_{\mathbb{R}} \rho(x)\Big(P_d T_{kd}(I - P_d)u\Big)(x)v(x)dx \\
=\; & \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\le 1}} \int_{\mathbb{R}} \rho(x)\Big(T_{kd}(I - P_d)u\Big)(x)v(x)dx \\
=\; & \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\le 1}} \int_{\mathbb{R}} \rho(x - kd)(I - P_d)u(x)v(x - kd)dx \\
=\; & \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\le 1}} \int_{\mathbb{R}} (\rho(x - kd) - \rho(x))(I - P_d)u(x)v(x - kd)dx \\
& + \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\le 1}} \int_{\mathbb{R}} \rho(x)(I - P_d)u(x)v(x - kd)dx.
\end{aligned} \tag{25}$$

The second integral equals zero because $v(x-kd)$ still belongs to the space $\mathcal{V}_d$. Hence, it follows from equality (25) that

$$\begin{aligned} & \|P_d T_{kd}(I-P_d)u\|_{0,\gamma} \\ \leq & \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\leq 1}} \int_{\mathbb{R}} |\rho(x-kd)-\rho(x)||(I-P_d)u(x)||v(x-kd)|dx \\ \leq & \; Cd \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\leq 1}} \int_{\mathbb{R}} \rho(x)|(I-P_d)u(x)||v(x-kd)|dx \\ \leq & \; Cd \sup_{\substack{v\in\mathcal{V}_d \\ \|v\|_{0,\gamma}\leq 1}} \|(I-P_d)u\|_{0,\gamma}\|T_{-kd}v\|_{0,\gamma}. \end{aligned}$$

The operators T_{-kd} are uniformly bounded in k if $|k|\leq K$. Thus, we deduce the following estimate:

$$\|P_d T_{kd}(I-P_d)u\|_{0,\gamma} \leq Cd\|(I-P_d)u\|_{0,\gamma}. \tag{26}$$

Now estimate (22) follows since the projector P_d is orthogonal. Estimates (23)-(24) follow from estimates (18),(20), and (26).

□

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