Local and Global Stability Analysis of an Unsupervised Competitive Neural Network

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Abstract

Unsupervised competitive neural networks (UCNN) are an established technique in pattern recognition for feature extraction and cluster analysis. A novel model of an unsupervised competitive neural network implementing a multi–time scale dynamics is proposed in this paper. The local and global asymptotic stability of the equilibrium points of this continuous–time recurrent system whose weights are adapted based on a competitive learning law is mathematically analyzed. The proposed neural network and the derived results are compared with those obtained from other multi–time scale architectures.

1 Introduction

Unsupervised competitive neural networks (UCNN) have emerged over the past years as an important technique in signal processing and pattern recognition [5, 4]. These networks implement the winner–take–all (WTA) paradigm which enforces based on lateral inhibition a localized representation of a single active neuron. When used for unsupervised learning, they yield data representations similar to those obtained based on vector quantization. They perform for each input pattern a global search for the "winner neuron".

The proposed UCNN represents a nonlinear dynamical system which includes the mutual interference between neuron and learning dynamics. It is based on the standard competitive learning law [5] to determine the best–matching representant among all neurons for a given input.

Biological systems are characterized by continuous changes in the neural activity and synapses as they sample new stimuli from the environment. The different time-scales in our feedback system should capture the fact that neurons fluctuate faster than synapses. This dynamical asymmetry in this neural feedback system produces the famous stability convergence dilemma.

Recently, some articles have discussed neural systems with time–varying weights based on the competitive learning law In der ersten Spalte unten [9]. The winner–take–all competition between group of neurons was studied in [18]. The inhibitory connectivity is determined by an online learning rule. A stability analysis identifies the winning groups as the steady states of the system. In [2], the unsupervised model of neural unit has the ability to self–tune to a single centroid of a cluster without employing the winner–take–all paradigm. This is performed by employing a varying threshold strategy during the learning stage. A different technique forms the basis of predicting the active neuron in [7]. A simple algorithm decides on the neurons' activity without requiring the numerical integration of the network's differential equations. Both algorithms facilitate hybrid implementations of the UCNNs in VLSI techniques. Hardware implementations can make full use of the inherent property of parallelism

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found in biological neural networks. Therefore, hardware elements such as very large-scale integration (VLSI) are very beneficial. However, the realization of the learning mechanism in hardware becomes a difficult and important issue. The backpropagation method is a well-established learning method but has some major drawbacks when implemented as a hardware, such as wiring for weight modifications or calculations of the derivative of the sigmoid function. Thus, learning methods based on the gradient methods are very complex and inefficient for implementing large-scale neural networks [1]. Our proposed learning mechanism is based on computing an inner product and is usually easier than the Euclidean distance used in the classical SOM or the backpropagation algorithm to compute in VLSI hardware. A modular analog circuit implementation based on small transductance multipliers of a coupled neural network with Hebbian learning and STM was described in [8].

Our proposed UCNN models the dynamics of both the neural activity levels, the short–term memory (STM), and the dynamics of unsupervised synaptic modifications, the long–term memory (LTM). It represents a coupled system of nonlinear differential equations evolving at two different time-scales.

The UCNN model emulates realistically the Willshaw–Malsburg model [17] of topographic formation, solving the equations of synaptic self–organization coupled with the field equation of neural excitations. In other words, we study the dynamics of cortical cognitive maps developed by self–organization which can be found in the nervous system. The network combines an additive activation dynamics of Hopfield–type [3] with a competitive learning law for the synaptic modifications. The model reduces to the Hopfield neural network if no learning occurs and to a simple competitive learning if the neural activity is assumed time–constant. The competitive learning law modulates the signal–synaptic difference with the neural output signal, which has to be a nonlinear monotonic increasing and sufficiently steep to approximate a binary threshold function. Differently from the competitive learning employed in the self–organizing maps, it does not employ an Euclidean distance as a parameter of a neighborhood function.

The general neural network equations describing the temporal evolution of the STM and LTM states for the ith neuron of a n–neuron network are

$$\begin{aligned} \text{STM:} \quad & \epsilon \dot{x}_i = -a_i x_i + \sum_{j=1}^{n} d_{ji} f(x_j) + b_i \sum_{j=1}^{p} m_{ji} y_j, \\ \text{LTM:} \quad & \dot{m}_{ji} = f(x_i)(y_j - m_{ji}), \end{aligned} \tag{1}$$

where x_i is the current activity level, $a_i > 0$ is the time constant of the neuron, $b_i > 0$ is the contribution of the external stimulus term, $f(x_i)$ is the neuron's output, y_j, $j = 1, \ldots, p$ is the time–constant external stimulus, and m_{ji} is the synaptic efficiency and ϵ is the fast time–scale associated with the STM state. d_{ji} represents a synaptic connection parameter between the ith neuron and the jth neuron. We assume here, that the recurrent neural network consists of both feedforward and feedback connections between the layers and neurons forming complicated dynamics. The neural network is modelled by a system of deterministic equations with a time–dependent input vector rather than a source emitting input signals with a prescribed probability distribution.

Defining the matrices $A = \mathrm{diag}(a_j,\ j = 1, ..., n)$, $B = \mathrm{diag}(b_j,\ j = 1, ..., n)$, $D = (d_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} \in \mathbb{R}^{n,n}$, $M = (m_{ij})_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}} \in \mathbb{R}^{p,n}$ and defining $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ by $\bar{f}(x)_i = f(x_i)$ we can write (1) in the compact form

$$\begin{aligned} \epsilon \dot{x} &= -Ax + D^T \bar{f}(x) + BM^T y, \\ \dot{M} &= -M \mathrm{diag}(\bar{f}(x)) + y \bar{f}(x)^T. \end{aligned} \tag{2}$$

2 Reduction

Since the nonlinearity in the second equation is a rank-one matrix only, we can use the vec notation and the Kronecker product for matrices $A \in \mathbb{R}^{p,q}$, $B \in \mathbb{R}^{k,l}$

$$\mathrm{vec}(A) = \mathrm{vec}([A_1, ..., A_q]) = \begin{pmatrix} A_1^T, & \dots & A_q^T \end{pmatrix} \in \mathbb{R}^{pq},$$
$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \in \mathbb{R}^{pk,ql}$$

and the relations [6]

$$\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$$
$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

to rewrite (2) by introducing the variable $z = \mathrm{vec}(M^T) \in \mathbb{R}^{pn}$ as

$$\begin{aligned} \dot{z} &= -(I_p \otimes \mathrm{diag}(\bar{f}(x)))z + y \otimes \bar{f}(x), \\ \epsilon\dot{x} &= -Ax + D^T\bar{f}(x) + (B \otimes I)(y^T \otimes I_n)z. \end{aligned} \tag{3}$$

By introducing the dynamic variable $s = M^T y \in \mathbb{R}^n$ and assuming that the input stimuli are time–constant normalized vectors of unit magnitude $|y|^2 = 1$ we obtain the projected system

$$\dot{s} = -\mathrm{diag}(\bar{f}(x))s + \bar{f}(x), \tag{4a}$$
$$\epsilon\dot{x} = -Ax + D^T\bar{f}(x) + Bs, \tag{4b}$$

which we call the state space representation of the LTM and STM equations. The following lemma shows that the solutions of (3) and (4) can be transformed into each other.

Lemma 2.1. *If (x, z) is a solution of (3) then (x, s) with $s = (y^T \otimes I_n)z$ is a solution of (4). Conversely, if (x, s) is a solution of (4), then (x, z) with $z = y \otimes s$ is a solution of (3).*

Proof. With
$$s = \mathrm{vec}(M^T y) = (y^T \otimes I_n)\mathrm{vec}(M^T) = (y^T \otimes I_n)z$$
we obtain immediately

$$\epsilon\dot{x} = -Ax + D^T\bar{f}(x) + (B \otimes I)s$$

as well as

$$\begin{aligned} \dot{s} &= (y^T \otimes I_n)\dot{z} \\ &= (y^T \otimes I_n)(-(I_p \otimes \mathrm{diag}(\bar{f}(x)))z + y \otimes \bar{f}(x)) \\ &= -\mathrm{diag}(\bar{f}(x))(y^T \otimes I_n)z + \bar{f}(x) = -\mathrm{diag}(\bar{f}(x))s + \bar{f}(x). \end{aligned}$$

Conversely, if (x, s) solves (4), then

$$\begin{aligned} \dot{z} &= y \otimes \dot{s} = y \otimes (-\mathrm{diag}(\bar{f}(x))s + \bar{f}(x)) \\ &= -y \otimes \mathrm{diag}(\bar{f}(x))s + y \otimes \bar{f}(x) \\ &= -(I_p \otimes \mathrm{diag}(\bar{f}(x)))(y \otimes s) + y \otimes \bar{f}(x) \\ &= -(I_p \otimes \mathrm{diag}(\bar{f}(x)))z + y \otimes \bar{f}(x) \end{aligned}$$

and

$$\begin{aligned}\dot{x} &= -Ax + D^T \bar{f}(x) + Bs \\ &= -Ax + D^T \bar{f}(x) + B(y^T \otimes I_n)(y \otimes s) \\ &= -Ax + D^T \bar{f}(x) + (B \otimes I)(y^T \otimes I_n)z\end{aligned}$$

hold. □

Thus the solutions of the system of lower dimension give all information about stability. We will analyze this system in detail.

3 Equilibrium

The choice of the nonlinearity is crucial for the overall dynamics: For the UCNN different f have been used [7], [10]:

1. a positive sigmoid function,
2. a sigmoid function with $f(0) = 0$,
3. the Hill function

$$f_\theta(x) = \begin{cases} \frac{x}{x+\theta}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \qquad \theta \geq 0.$$

If we assume $f \in \mathcal{C}^1$ with $|f'(x)| \leq L_f$ for all $x \in \mathbb{R}$ and

$$a_i > L_f d_{ii} + \sum_{j=1}^{n} |L_f d_{ji}|, \tag{5}$$

then we obtain with Banach's fixed-point theorem an unique equilibrium of (4) in the same way as in [12]. It is given by $e = (\tilde{x}, \tilde{s})$ where $\tilde{s} = (1, ..., 1)$ and $\tilde{x}$ is the solution of

$$x = A^{-1}(D^T \bar{f}(x) + b),$$

or equivalently, the x_i solve

$$x_i = \frac{1}{a_i}(b_i + \sum_{j=1}^{n} d_{ji} f(x_j)).$$

If f has any zeros, then additional equilibria are given by $(\tilde{x}, \tilde{s})$, where $f(\tilde{x}) = 0$ and $\tilde{s} = B^{-1}A\tilde{x}$.

In this paper we consider the first case of a positive function only and employ the following hypothesis.

Hypothesis 3.1. *Let $f \in \mathcal{C}^1$ with $|f'(x)| \leq L_f$ for all $x \in \mathbb{R}$ be positive and bounded, i.e. $0 < f(x) < C_f$ for all $x \in \mathbb{R}$ for some bound $C_f > 0$.*

Note that these bounds on f imply that the x-components of the equilibrium lie in the intervals

$$\tilde{x}_i \in \Big[\frac{1}{a_i}(b_i - C_f \sum_{j=1}^{n} |d_{ji}|), \frac{1}{a_i}(b_i + C_f \sum_{j=1}^{n} |d_{ji}|)\Big].$$

In the following we apply the theory of flow–invariance to give the mathematical conditions for showing when the STM and LTM trajectories are locally and globally bounded. Our method is more general than that given in [7] since it is not necessary to assume a high gain approximation and it doesn't treat the two dynamics separately. In addition, it doesn't require the excitatory region to comprise only one neuron. We also give a strict Lyapunov function for the neural multi–time scale system, show the existence and uniqueness of the equilibrium, and prove local and global asymptotic stability for the equilibrium.

4 Global Attractor

The existence and uniqueness of the equilibrium is given based on flow–invariance (cf. [14]) while the global asymptotic stability of the equilibrium is shown by a strict Lyapunov function (cf. [11]) on the global attractor.
The theory of flow–invariance gives a qualitative interpretation of the dynamics of a system, taking into account the invariance of the flow of the system.
Before we state the stability results based on this concept we will first give some useful definitions used in nonlinear analysis.

4.1 Definitions

Definition 4.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous map and let $\mathcal{S}$ be a subset of $\mathbb{R}^n$. We say that $\mathcal{S}$ is flow-invariant with respect to the system of differential equations*

$$\dot{x}(t) = F(x(t)), \tag{6}$$

if any solution $x(t)$ starting in S at $t = 0$ remains in $\mathcal{S}$ for all $t \geq 0$ as long as $x(t)$ is defined.

In dynamical systems terminology, such sets are called positively invariant under the flow generated by (6).

Definition 4.2. *We say that the system (6) is dissipative in $\mathbb{R}^n$ if there exists a bounded set $\mathcal{U} \subset \mathbb{R}^n$ such that for any solution $x(t)$ of (6) there exists $T \geq 0$ such that $x(t) \in \mathcal{U}$ for all $t \geq T$. In other words, all solutions of (6) enter this bounded set $\mathcal{U}$ in finite time.*

If (6) is dissipative then all solutions of (6) are defined for $t \geq 0$, and there exists a compact set $\mathcal{A} \subset \mathcal{U}$ which attracts all solutions of (6). The set $\mathcal{A}$ is invariant under the flow of (6) and it is called the *global attractor* of (6) in $\mathbb{R}^n$.

4.2 Results

Theorem 4.3. *Consider the system of differential equations (4) and suppose that f satisfies Hypothesis 3.1. Then (4) is dissipative in $\mathbb{R}^{2n}$ and has a compact global attractor*

$$\mathcal{A} \subseteq \mathcal{D} = \prod_{i=1}^{n} [-l_i, l_i] \times \prod_{i=1}^{n} 1,$$

where

$$l_i = \frac{1}{a_i}\Big(b_i + C_f \sum_{j=1}^{n} |d_{ji}|\Big) > 0, \quad i = 1, \ldots, n. \tag{7}$$

Proof. Since f is locally Lipschitz, system (4) enjoys local existence and uniqueness of solutions. Moreover, since f is uniformly bounded, all solutions are defined for all $t \geq 0$.
Given $\rho > 0$, we define

$$\delta_i = \min(\frac{\rho}{2} a_i b_i,\ \rho) \tag{8}$$

for $i = 1, \ldots, n$. It follows that $\delta_i > 0$ and $-b_i\delta_i + a_i\rho \geq a_i\frac{\rho}{2}$ for all $i = 1, \ldots, n$.
Then for $t \geq 0$ and for $s_i(t) \leq 1 - \delta_i$ the following inequality holds:

$$\dot{s}_i(t) \geq f(x_i(t))\delta_i > 0.$$

Similarly, for $t \geq 0$ and for $s_i(t) \geq 1 + \delta_i$ we have that

$$\dot{s}_i(t) \leq f(x_i(t))(-\delta_i) < 0.$$

If $s_i(t) = 1$ then we have $\dot{s}(t) = 0$. Therefore, for any $i \in \{1, \dots, n\}$ there exists a $T_i^s \geq 0$ such that

$$s_i(t) \in [1 - \delta_i, 1 + \delta_i] \subseteq [1 - \rho, 1 + \rho] \tag{9}$$

for all $t \geq T_i^s$. Let $T^s = \max_i T_i^s$, then (4a) holds for all $t \geq T^s$.

Now we consider $t \geq T^s$. For $x_i(t) \leq -l_i - \rho$, we have

$$\epsilon \dot{x}_i(t) \geq a_i(l_i + \rho) + \sum_{j=1}^{n} d_{ji} f(x_j) + b_i s_i.$$

Then (8) and (9) and using the definition of l_i given by (7), we find that for $t \geq T^s$ and $x_i(t) \leq -l_i - \rho$,

$$\begin{aligned} \epsilon \dot{x}_i(t) &\geq a_i l_i + a_i \rho - C_f \sum_{j=1}^{n} |d_{ji}| - b_i(1 + \delta_i) \\ &= -b_i \delta_i + a_i \rho \geq a_i \frac{\rho}{2} > 0. \end{aligned}$$

Similarly, for $t \geq T^s$ and for $x_i(t) \geq l_i + \rho$, (8) and (4a) imply that

$$\epsilon \dot{x}_i(t) \leq -a_i(l_i + \rho) + \sum_{j=1}^{n} d_{ji} f(x_j) + b_i s_i.$$

Using (7) again, we find that for $t \geq T^s$ and $x_i(t) \geq l_i + \rho$,

$$\begin{aligned} \epsilon \dot{x}_i(t) &\leq -a_i l_i - a_i \rho + C_f \sum_{j=1}^{n} |d_{ji}| + b_i(1 + \delta_i) \\ &= -a_i \rho + b_i \delta_i \leq -a_i \frac{\rho}{2} < 0. \end{aligned}$$

Consequently, for any $i \in \{1, \dots, n\}$ there exist $T_i^x \geq T^s \geq 0$ and $\epsilon^* > 0$ such that

$$x_i(t) \in [-l_i - \rho, l_i + \rho] \tag{10}$$

for all $t \geq T_i^x$ and all $\epsilon \in (0, \epsilon^*)$. Let $T = \max_i T_i^x$, then both (9) and (10) hold for all $i \in \{1, \dots, n\}$ and all $t \geq T$.
Then for any $\rho > 0$ and for any $\epsilon \in (0, \epsilon^*)$ and initial condition $\{x_i(0), s_i(0)\} \in \mathbb{R}^{2n}$ there exists a $T \geq 0$ such that

$$s_i(t) \in [1 - \rho, 1 + \rho], \quad x_i(t) \in [-l_i - \rho, l_i + \rho]$$

for all $i = 1, \dots, n$ and all $t \geq T$. Therefore the set $\mathcal{D}$ is a positively invariant set of (4), that is, any solution starting in $\mathcal{D}$ at $t = 0$ remains in $\mathcal{D}$ for all $t \geq 0$. □

Note that even without employing condition (5) which ensures the existence and uniqueness of the equilibrium e by Banach's fixed-point theorem, one can conclude that the existence of e by using Brower's fixed-point theorem as the following corollary shows. The proof of the uniqueness of e is not necessary since we prove in Theorem 5.3 that any solution converges to the equilibrium.

Corollary 4.4. *Since the set $\mathcal{D}$ is homotopically equivalent to a point and $\mathcal{D}$ is flow-invariant with respect to (4), the Brower fixed point theorem implies that there exists a point $e \in \mathcal{D}$ which is fixed under the flow of (4), that is, $e \in \mathcal{D}$ is an equilibrium of (4).*

We introduce the change of variables $\phi = x - \tilde{x}$, $\psi = s - \tilde{s} = s - 1$ which shifts e to the origin. Specifically, if we denote $\bar{h}(\phi) = \bar{f}(\phi + \tilde{x}) - \bar{f}(\tilde{x})$ and $\bar{g}(\phi) = \bar{f}(\phi + \tilde{x})$, then $\bar{h}(0) = 0$ and (4) may be rewritten as

$$\dot{\psi} = -\mathrm{diag}(\bar{g}(\phi))\psi, \tag{11a}$$

$$\epsilon\dot{\phi} = -A\phi + B\psi + D^T\bar{h}(\phi). \tag{11b}$$

Before presenting conditions for global stability we give conditions for the local stability of the equilibrium e.

5 Local Asymptotic Stability Analysis

We study the local dynamic behavior of the UCNN by employing Lyapunov's linearization method [16]. This will allow us to draw conclusions about a nonlinear system by studying the behavior of a linear system in the same way as in [11]. For that we have to restrict further to functions, which have positive derivative at the equilibrium:

Hypothesis 5.1. *In addition to Hypothesis 3.1, assume that $f'(\tilde{x}_i) > 0$, for all $i = 1, ..., n$.*

Linearizing system (11) about the equilibrium zero we obtain with $\bar{g}(0) = \bar{f}(\tilde{x})$ and $\mathrm{diag}(\bar{h}'(0)) = \mathrm{diag}(\bar{g}'(0)) = \mathrm{diag}(\bar{f}'(\tilde{x}))$ for the total derivative DF of the r.h.s. the following:

$$DF = \begin{pmatrix} -\mathrm{diag}(\bar{f}(\tilde{x})) & 0 \\ B & D^T\mathrm{diag}(\bar{f}'(\tilde{x})) - A \end{pmatrix}.$$

In this case, the eigenvalues of DF are directly given by the union of the eigenvalues of the diagonal blocks. Thus, the zero solution is locally asymptotically stable for all $\epsilon > 0$, provided both blocks are of Hurwitz type, i.e. all of their eigenvalues have negative real parts. To show if a matrix M is Hurwitz, we apply a well–known eigenvalues localization theorem, the so called Gersgorin's Theorem [15]. This theorem states that the eigenvalues of a real $n \times n$ matrix M are contained in the union of the n disks of the complex λ-plane

$$|\lambda - M_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |M_{ij}|, \tag{12}$$

i.e. $-M_{ii} > \sum_{i \neq j} |M_{ij}|$ to guarantee stability.
Although stability itself is not affected by the choice of the singular perturbation parameter ϵ, the time needed to reach the equilibrium increases as ϵ is decreased. The reason for that is that for small ϵ the approach to the equilibium will be mostly along the slow manifold.
The following theorem gives conditions for the equilibrium of the competitive neural network (1) to be asymptotically stable.

Theorem 5.2. *Let f satisfy Hypothesis 5.1 and assume*

$$a_i > d_{ii}f'(\tilde{x}_i) + \sum_{j \neq i} |d_{ji}f'(\tilde{x}_j)|, \tag{13}$$

for all $i = 1, \ldots, n$. Then the equilibrium e of (4) is asymptotically stable for all $\epsilon > 0$.

Proof. The conditions on f ensure that $-\mathrm{diag}(\bar{f}(\tilde{x}))$ is Hurwitz and from (13) we obtain using Gershgorin's Theorem (12) that $D^T\mathrm{diag}(\bar{f}'(\tilde{x})) - A$ is Hurwitz. □

Dynamics on the global attractor

In the following we show that the equilibrium e is attractive by constructing a local Lyapunov function. Here we use the same method as in [12] and [13].

Theorem 5.3. *Let f satisfy Hypothesis 5.1 and assume*

$$a_i > b_i + \frac{L_f}{2}\sum_{j=1}^{n}(|d_{ij}| + |d_{ji}|) \tag{14}$$

for all $i = 1, \dots, n$.
Then e is a global attractor for system (4). Moreover, any solution of (4) converges to e asymptotically as $t \to \infty$.

Proof. We prove global convergence by presenting a Lyapunov function for (11) which is locally defined on the shifted invariant set $\mathcal{D} - e$. For

$$V(\phi, \psi) = \frac{1}{2}(\psi^T\psi + \epsilon\phi^T\phi).$$

we obtain

$$\frac{d}{dt}V = -\psi^T \operatorname{diag}(\bar{g}(\phi))\psi + \phi^T(B\psi - A\phi - D^T\bar{h}(\phi)).$$

Assume that $\phi_i \in [-l_i - \tilde{x}_i + \rho, l_i - \tilde{x}_i + \rho]$ and $|\psi_i| < \rho$ for all $i = 1, ..., n$, and some $\rho > 0$, where l_i has been given in Theorem 4.3. Then there exists $c_f > 0$ with $f(\phi) > c_f$. Together with the estimate $|f'(\tilde{x})| < L_f$ and the definitions of $\bar{g}$ and $\bar{h}$ this leads to

$$\begin{aligned}
\frac{d}{dt}V &= -\sum_{i=1}^{n}(f(\phi_i + \tilde{x}_i)\psi_i^2 + a_i\phi_i^2) + \sum_{i=1}^{n} b_i\phi_i\psi_i \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}\ \phi_i(f(\phi_i + \tilde{x}_i) - f(\phi_i)) \\
&< -\sum_{i=1}^{n}(c_f\psi_i^2 + a_i\phi_i^2) + \sum_{i=1}^{n} b_i|\phi_i||\psi_i| \\
&\quad + L_f\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ji}||\phi_i||\phi_j|.
\end{aligned}$$

The r.h.s can then be written as a quadratic form vQv^T, with $v = (|\psi_1|, |\phi_1|, ..., |\psi_n|, |\phi_n|)$ and Q a matrix with the block structure

$$Q_{ij} = \begin{cases} \begin{pmatrix} -c_f & 0 \\ b_i & -a_i + \frac{1}{2}\kappa_{ii} \end{pmatrix}, & \text{if } i = j, \\ \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\kappa_{ij} \end{pmatrix}, & \text{if } i \neq j, \end{cases} \qquad i, j = 1, ..., n$$

and $\kappa_{ij} = L_f(|d_{ij}| + |d_{ji}|)$. Condition (14) implies via Gershgorin's Theorem that Q is negative definite. Thus we have

$$\frac{d}{dt}V < -\alpha(\|\psi\|^2 + \|\phi\|^2) < 0,$$

for some $\alpha > 0$. This implies that the solutions $(\phi(t), \psi(t))$ of (11) which start in the set $\mathcal{D} - e$ converge to the origin asymptotically. In terms of the system (4), its solutions $(x(t), s(t))$ which start near the invariant set $\mathcal{D}$, converge to e asymptotically. From Theorem 4.3 we obtain that all solutions of (11) converge to $\mathcal{D} - e$, thus we obtain global convergence to e. □

Note that although we have constructed a local Lyapunov function only it is known [19] that a globally defined Lyapunov function exists.

6 Examples

The simplest example consists of only one neuron.

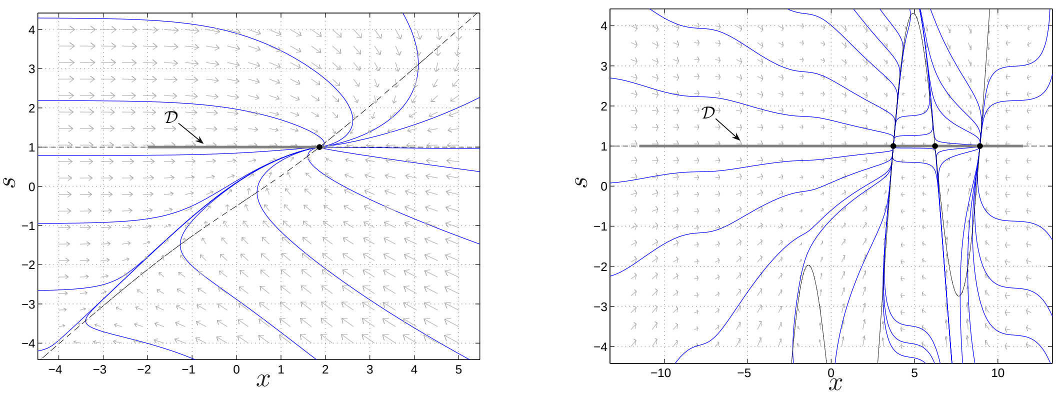
Example 6.1. Let $n = 1$, $a_i = 1$, $b_i = b$, $d_{ji} = d$ and the nonlinearity be either a sigmoid function $f(x) = 1 - \frac{1}{1+e^x}$ with the bounds $f'(x) \leq \frac{1}{2}$ and $0 < f(x) < 1$ or $f(x) = 1.1 + \sin(x)$ with $f'(x) \leq 1$ and $0 < f(x) < 2.1$.
Then system (4) reads

$$\dot{s} = f(x)(1-s),$$
$$\epsilon\dot{x} = -x + d \cdot f(x) + bs$$

and the linearization about $e = (\tilde{x}, 1)$ is given by

$$M = \begin{pmatrix} -f(\tilde{x}) & 0 \\ b & -1 + d \cdot f'(\tilde{x}) \end{pmatrix}.$$

Thus the local stability condition (13) is given by $d \cdot f'(\tilde{x}) < 1$ and the global stability condition (14) reads $dL_f < 1$. The invariant set $\mathcal{D}$ is given by $[-l, l] \times 1$ with $l = L_f d + b$. Note, that the time to reach the set $\{(s, x) \in \mathbb{R}^2 : |s| < 1 + \rho\}$ for some $\rho > 0$ can be quite large and increases as ϵ decreases.



(a) $f(x) = 1 - \frac{1}{1+e^x}$, $d = 1$, $b = 1$, $\epsilon = 1$ (b) $f(x) = 1.1 + \sin(x)$, $d = 5$, $b = 1$, $\epsilon = 1$

Figure 1: Dynamics of solutions different f.

In the first case shown in Figure 1 (a) the attractor $\mathcal{A}$ is the unique asymptotically stable equilibrium e whereas it consists of the three equilibria and the connecting orbits between them in the second case shown in Figure 1 (b). In this case Theorem 4.3 holds, but not Theorem 5.3 since condition (14) is not satisfied.

Example 6.2. Let $n = 20$, $a_i = a$, $b_i = b$, $D_{ii} = \alpha \geq 0$, $D_{ji} = -\beta \leq 0$, $i > j$ and $D_{ji} = \beta \geq 0$, $i < j$. and the nonlinearity again the sigmoid function. In Figure 2 a simulation for 30 randomly choosen initial points is shown. After a transient time all points converge to the equilibrium e.The bounds l_i which define invariant set $\mathcal{D}$ are $l_i = 4$ for $i = 1, 20$ and $l_i = 4.5$ for $i = 2, \ldots, 19$ and condition (14) which ensures global stability of e is satisfied.
Figure 3 shows how the time needed to reach the equilibrium increases as the singular perturbation parameter ϵ is decreased.

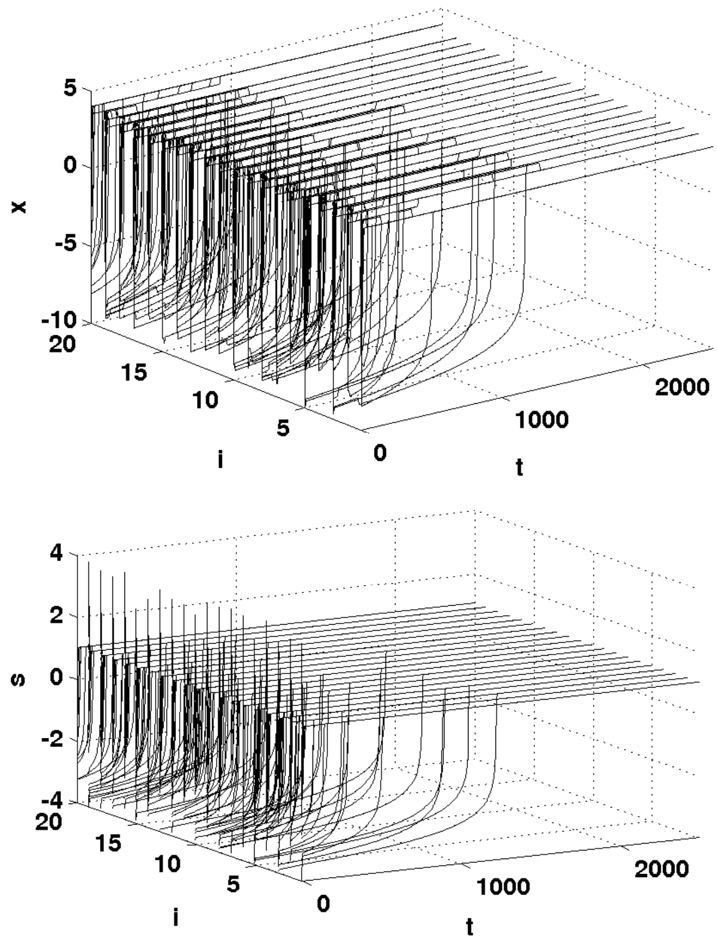


Figure 2: STM and LTM states for $a = 2$, $b = 5$, $\alpha = 2$, $\beta = 1$, $\epsilon = 1$.

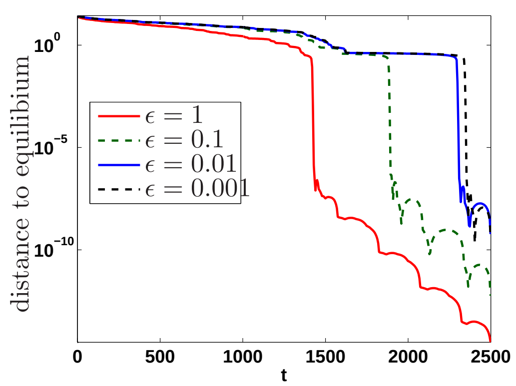


Figure 3: Mean distance (30 samples) to the equilibrium for varying ϵ.

7 Conclusions and future work

In this paper which is an elaborated and detailed version of the IJCNN2006 conference paper submission, we proved local and global asymptotic stability of a unsupervised competitive neural network with fast and slow dynamics, the proposed architecture of which can lead to hybrid implementations in VLSI techniques. We justify the projection ansatz in [13], [14] and based on the singular perturbation and flow invariance techniques we give conditions for the LTM and STM trajectories to be bounded which are less restrictive than with the K–monotone theory. We also presented a strict Lyapunov function and based on it we have shown global asymptotic stability of the equilibrium point.
We plan to expand our investigation to apply the proposed unsupervised competitive neural network to the segmentation and classification of dynamic breast MR data sets. Each lesion tissue type is modeled by a multivariate Gaussian distribution. This technique will enable the extraction of spatial and temporal features of dynamic MRI data stemming from patients with confirmed lesion diagnosis.

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