Stability and paracontractivity of discrete linear inclusions

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October 22, 1999

Abstract

We study stability properties of a finite set Σ of $n\times n$-matrices such as paracontractivity, BV- and LCP-stability, and their relations to each other. The conjecture on equivalence of paracontractivity and LCP-stability is proved. Moreover, we prove the equivalence of the uniform BV-stability and the property of vanishing length of steps of any trajectory of Σ.

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*This paper was partially written during a visit of A. Vladimirov to the University of Bielefeld in September 1999.

1 Introduction

The interest in *discrete linear inclusions* (DLI) and their stability [1, 3, 6, 10, 9] is caused by their natural occurence in constructing self-similar objects, interpolation schemes, in constructing wavelets of compact support, in studying nonhomogeneous Markov chains, etc. One of the authors became particulary interested in this subject during his work concerned with hysteresis nonlinearities and, in particular, Skorokhod problems and sweeping processes [13, 14, 15, 11].

Briefly, a polyhedral sweeping process is given by a time-dependent polyhedral set

$$Z(t) = \{x \in \mathbb{R}^n : \langle x, p_i \rangle \geq c_i(t), 1 \leq i \leq k\}, \quad 0 \leq t \leq T, \tag{1.1}$$

and an associated set of *projection directions* d_i such that $\langle d_i, p_i \rangle \geq 0$. Here $p_i \in \mathbb{R}^n$, $\|p_i\| = 1$, and $c_i(t)$ are continuous scalar functions such that $Z(t) \neq \emptyset$ for each $t \in [0, T]$.

For each initial value $x_0 \in Z(0)$, by a solution of (1.1) we understand an absolutely continuous function $x(t) : [0, T] \to \mathbb{R}^n$ such that $x(0) = x_0$, $x(t) \in Z(t)$ for all $0 \leq t \leq T$, and

$$\dot{x}(t) \in \{ \sum_{i \in I(x,t)} \alpha_i d_i : \alpha_i \geq 0\} \quad \text{a.e. on } [0, T],$$

where

$$I(x, t) = \{i : \langle p_i, x \rangle = c_i(t)\}.$$

As is known ([4, 5, 12]), sufficient conditions for different types of regularity of sweeping processes (unique solvability, continuity, Lipschitz continuity) can be formulated in terms of different kinds of stability of so called *associated projection systems* (APS), that is, of systems of projections P_i onto hyperplanes $L_i = \{x \in \mathbb{R}^n : \langle x, p_i \rangle = 0\}$ along directions d_i. In particular, notions of *product boundedness* and of finite length of any trajectory of the APS are used. The latter property ensures convergence of a class of discrete-time approximation methods to continuous solutions of the sweeping process.

Another stability property which is widely used in applications is that of left convergent products (LCP); it is known to be equivalent to convergence of each infinite trajectory $x_{j+1} = P_{i_j} x_j$ to some $x^* \in \mathbb{R}^n$ as $j \to \infty$. We will also consider a weaker property of $\lim_{j\to\infty} \|x_{j+1} - x_j\| = 0$ (vanishing steps property).

It seems to be more convenient to study the relations between different kinds of stability for general finite sets of $n\times n$-matrices, the more so that so far we have no reasons to believe that investigation of sets of oblique projections is essentially simpler than that of sets of general matrices.

A matrix A is said to be *paracontracting* with respect to a given norm $\|\cdot\|$ (see, for instance, [8]) if

$$\|Ax\| < \|x\| \quad \text{whenever} \quad Ax \neq x. \tag{1.2}$$

Finite sets of paracontracting matrices are important, for instance, in studying convergence of iterative algorithms. A stronger property of *ℓ-paracontractivity* requires a positive rate of decrease of the norm in (1.2).

In [2], it was conjectured that the LCP property of a finite set of $n\times n$-matrices is equivalent to the paracontractivity of this set with respect to some norm. We prove that this conjecture is true. Moreover, we demonstrate the equivalence of these types of stability to the property of vanishing steps and the property of finite length of all trajectories. The main idea of the proof is induction on the cardinality k of the set Σ.

2 Discrete linear inclusions

Let Σ be a finite set of real $n\times n$-matrices A_i, $i = 1, \ldots, k$. Following [9], by the *discrete linear inclusion* DLI(Σ) we will understand the set of all infinite sequences $\{x_j\}$, $j \geq 0$, of vectors in $\mathbb{R}^n$ such that

$$x_{j+1} = A_{i_j} x_j \tag{2.3}$$

for some $A_{i_j} \in \Sigma$. These sequences and their segments will be called *trajectories* of Σ.

We also introduce (right-infinite) matrix trajectories of Σ. These are sequences $\{M_0, M_1, \ldots\}$ such that $M_0 \in \Sigma$ and

$$M_{i+1} = A_{j_i} M_i, \ 1 \leq j_i \leq k, \ i = 0, 1, \ldots \tag{2.4}$$

A set of matrices $\Sigma = \{A_1, \ldots, A_k\}$ is *product bounded* if there exists a $C > 0$ such that $\|A_{i_1} \ldots A_{i_m}\| < C$ for all finite sequences $\{1 \leq i_j \leq k\}$, $j = 1, \ldots, m$. The following assertion is an easy consequence of well-known results in the theory of DLIs (see, for instance, [1]).

Proposition 2.1 *A finite matrix set Σ is product-bounded iff all its trajectories are bounded.*

A set Σ is *LCP* (left convergent products) if any matrix trajectory of Σ has a limit. This is equivalent to the convergence of any trajectory of Σ (not necessarily to the origin), see [3].

Definition 2.2 A set Σ is called *BV-stable* if all its trajectories

$$x = \{x_0, x_1, \ldots\}$$

have bounded variation, that is,

$$V(x) := \sum_{i=0}^{\infty} |x_{i+1} - x_i| < \infty. \tag{2.5}$$

Proposition 2.3 *A set Σ is BV-stable iff any matrix trajectory*

$$\mathcal{M} = \{M_0, M_1, \ldots\}$$

of Σ has bounded variation, that is, iff

$$\sum_{j=0}^{\infty} \|M_{j+1} - M_j\| < \infty, \tag{2.6}$$

where $\|\cdot\|$ is some matrix norm (they are all equivalent).

Proof. Obviously, (2.6) implies (2.5). Now, if the variation of the sequence M_j of matrices is infinite, then the sum of variations of vector sequences $\{M_0 e_i, M_1 e_i, \ldots\}$ over $i = 1, \ldots, n$ (here e_i are the coordinate vectors) is also infinite and, hence, at least one of these sequences has infinite variation which is a contradiction with the BV-stability of the set Σ. □

We also introduce the formally stronger concept of (uniform) *UBV-stability*. It will turn out to be the same as BV-stability, but this is not obvious.

Definition 2.4 A set Σ is called *UBV-stable* if it is BV-stable and there exists $L > 0$, such that for all trajectories $x = \{x_0, x_1, \ldots\}$

$$V(x) \leq L||x_0|| \tag{2.7}$$

holds.

Now we introduce a new property which is not stronger than LCP but will be proved to be stronger than product boundedness.

Definition 2.5 The set Σ is called VS (vanishing steps) if, for each of its trajectories $\{x_0, \ldots\}$,

$$\lim_{j\to\infty} \|x_{j+1} - x_j\| = 0.$$

For completeness, let us also introduce a property that is stronger than LCP and will be proved to be stronger than BV-stability.

Definition 2.6 The set Σ is *asymptotically stable* (AS) if all its trajectories converge to the origin.

As is easy to see, the notions of PB, VS, LCP, BV, UBV and AS do not depend on the particular norm in $\mathbb{R}^n$ and $\mathbb{R}^{n,n}$. We will now give two definitions that do depend on the norm used.

Definition 2.7 A matrix P is said to be *paracontracting* with respect to the norm $\|\cdot\|$ in $\mathbb{R}^n$ if, for all $x \in \mathbb{R}^n$,

$$Px \neq x \Leftrightarrow \|Px\| < \|x\|.$$

It is *ℓ-paracontracting* with respect to $\|\cdot\|$ if there exists $\gamma > 0$ such that

$$\|Px\| \leq \|x\| - \gamma\|Px - x\|$$

holds for all $x \in \mathbb{R}^n$.

A set of matrices is called paracontracting or ℓ-paracontracting with respect to $\|\cdot\|$ if all its matrices possess the respective property; and it is called just paracontracting or ℓ-paracontracting if there exists a norm in $\mathbb{R}^n$ such that the set possesses the respective property for this norm. We use the abbreviations PC and LPC, respectively. The last two properties, again, do not depend on the particular norm in $\mathbb{R}^n$.

3 Known relations between stability notions

As is known [1], a finite set Σ is AS iff $\rho(\Sigma) < 1$, where $\rho(\Sigma)$ is the generalized spectral radius of Σ, see [3, 1, 6] for definitions and results concerning spectral radii of sets of matrices.

Moreover it follows from results in [1], that Σ is AS iff there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that

$$\max_{i=1,\ldots,k} \|A_i\| < 1,$$

where $\|A\| = \sup_{\{x:\|x\|=1\}} \|Ax\|$. This immediately implies the BV- and UBV-stability of Σ.

In turn, the BV-stability of Σ implies its LCP property because any sequence of bounded variation converges. Further, any convergent sequence has vanishing steps which proves the implication LCP⇒VS.

Proposition 3.1 *For a finite set Σ of $n\times n$-matrices, the properties UBV and LPC are equivalent.*

Proof. Let us demonstrate that the set Σ is UBV-stable iff there exists a seminorm $\|\cdot\|_\Sigma$ in $\mathbb{R}^n$ that decreases at least at rate 1 along any trajectory of the set Σ, that is,

$$\|A_i x\|_\Sigma \le \|x\|_\Sigma - \|A_i x - x\|.$$

Indeed, this seminorm can be chosen as

$$\|x\|_\Sigma = \sup_{x=x_0,x_1,\ldots} \sum_{i=0,1\ldots} \|x_{i+1} - x_i\| < \infty, \tag{3.8}$$

where the supremum is taken over all trajectories of Σ starting from x.

Since the rate of decrease of this seminorm is at least 1 along any trajectory, the rate of decrease of the norm

$$\|x\|_b = \|x\|/2 + \|x\|_\Sigma$$

is at least 1/2. Thus a stronger criterion of UBV-stability can be formulated: A set is UBV-stable iff there exists a norm decreasing at a qualified positive rate along all trajectories of the set. This, however, coincides with the definition of ℓ-paracontractivity with respect to the norm $\|x\|_b$. □

The following result was proved in [8].

Theorem 3.2 *If Σ is a paracontracting set, then it is also an LCP set.*

4 Auxiliary results

It was conjectured in [2] that the LCP property of a finite set Σ implies its paracontractivity. This conjecture was proved for $\Sigma = \{A_1, A_2\}$ and for the case of Σ with continuous limit function, see [2]. We are going to prove it in the general case. First, we will need several auxiliary results. The first one is as follows:

Proposition 4.1 *Suppose a finite set Σ of matrices A_i is a VS set and E is the stationary space of Σ:*

$$E = E(\Sigma) = \{x \in \mathbb{R}^n : A_i x = x, \ i = 1, \ldots, k\}. \tag{4.9}$$

Suppose also that all matrices A_i are reduced by similarity to the form

$$A'_i = \begin{pmatrix} I & B_i \\ 0 & C_i \end{pmatrix}, \quad i = 1, \ldots, k,$$

where I is the identity $m \times m$-matrix corresponding to the subspace E. Then there exist positive constants $\gamma_i > 0$, $i = 1, 2$, such that, for any $x = (p, q) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and any $i = 1, \ldots, k$, the inequalities

$$\gamma_1 \|C_i q - q\| \leq \|A_i x - x\| \leq \gamma_2 \|C_i q - q\| \tag{4.10}$$

hold.

Proof. It suffices to consider the euclidean vector norm. First, note that

$$A'_i x - x = \begin{pmatrix} B_i q \\ C_i q - q \end{pmatrix}, \quad i = 1, \ldots, k;$$

thus, the left inequality in (4.10) with $\gamma_1 = 1$ is obvious.

If there exists a $q \in \mathbb{R}^{n-m}$ and $i \in \{1 \ldots, k\}$ such that $C_i q = q$ and $B_i q \neq 0$, then the sequence $x_j = (jB_i q, q)$, $j = 0, 1, \ldots$ is a trajectory of Σ which contradicts its VS property. Thus $C_i q = q$ implies $B_i q = 0$ for all i and it follows from simple facts of linear algebra that $B_i = D_i(C_i - I)$ for some suitable matrix D_i and hence

$$\|B_i q\| \leq L \|C_i q - q\|$$

for some $L > 0$, all $i = 1, \ldots, k$, and all $q \in \mathbb{R}^{n-m}$. The right-hand inequality with $\gamma_2 = \sqrt{(1 + L^2)}$ follows immediately. $\square$

Now, together with the VS set $\Sigma = \{A_1, \ldots, A_k\}$, let us consider the set $\Sigma' = \{C_1, \ldots, C_k\}$. It follows from Proposition 4.1 that Σ' is also VS and, moreover,

Proposition 4.2 *If Σ is VS then the properties of LCP, BV, and UBV of the set Σ are equivalent to the same properties of Σ'.*

Theorem 4.3 *If $\Sigma = \{A_1, \dots, A_k\}$ is a UBV-stable set then there exists an $\varepsilon > 0$ such that, for any finite trajectory $\{x_0, \dots, x_m\}$ of Σ, the inequality*

$$\max_{i=0,\dots,m-1} \|x_{i+1} - x_i\| \geq \varepsilon \sum_{i=0,\dots,m-1} \|x_{i+1} - x_i\| \tag{4.11}$$

holds.

Proof. Since BV implies VS, by virtue of Proposition 4.2 we can assume $E(\Sigma) = \{0\}$. Let us use induction on the number k of matrices. For a single matrix A without nontrivial invariant vectors, the BV-stability is equivalent to $\rho(A) < 1$ and the required property is obvious.

Next, suppose that the assertion of the theorem holds for all sets of $k-1$ or fewer matrices. Note that (4.11) is equivalent to the following. There exists a $C > 0$ such that the variation of any trajectory of Σ with length of steps equal to or less than 1 is less than C. Suppose the contrary, that is, suppose that there exists a sequence of finite trajectories $\{x_i^j\}$, $j = 1, 2, \dots$, $0 \leq i \leq j$, such that $\|x_{i+1}^j - x_i^j\| \leq 1$ for each admissible i and j but

$$\sum_{i=0,\dots,j-1} \|x_{i+1}^j - x_i^j\| \to \infty \quad \text{as } j \to \infty.$$

The UBV-stability of Σ implies $\lim_{j\to\infty} \|x_0^j\| = \infty$. Let us show that there exists an $M > 0$ such that any piece of any trajectory of variation $V \geq M$ uses all the matrices A_i, $i = 1, \dots, k$. Indeed, this follows from the induction hypothesis.

For each trajectory $\{x^j\}$, let us consider its minimal initial segment of variation $V \geq M$. Obviously $V \leq M + 1$. Since $\|x_0^j\| \to \infty$ as $j \to \infty$ and all matrices A_i participate in each initial segment, we conclude that any limit vector h of the sequence

$$h_j = \frac{x_0^j}{\|x_0^j\|}, \quad j = 1, 2, \dots,$$

is an invariant vector for all A_i and that $\|h\| = 1$. This is a contradiction to $E(\Sigma) = \{0\}$. $\square$

We will also need the following assertion from [7].

Theorem 4.4 *Suppose Σ is an LCP set and $E(\Sigma) = \{0\}$. Then there exists a norm in $\mathbb{R}^n$ and $0 < q < 1$ such that*

$$\|A_i\| \leq 1, \ i = 1, \dots, k,$$

$$\|A_{j_1} \dots A_{j_m}\| \leq q$$

for all products containing each A_i from Σ.

5 Equivalence theorem

Now, let us formulate and prove the main theorem of this paper.

Theorem 5.1 *The following three properties of a set $\Sigma = \{A_1, \dots, A_k\}$ are equivalent:*
(1) The set Σ is UBV;
(2) The set Σ is LCP;
(3) The set Σ is VS;

Proof. The only assertion that needs proof is that (3) implies (1); the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. Again, we will use induction on the number k of matrices in Σ. For a single matrix, this follows from an obvious implication ($\rho(A) < 1$) $\Rightarrow$ UBV. Indeed, if $\rho(A) \geq 1$ and there are no nontrivial invariant vectors, then there exists either a diverging trajectory or a quasiperiodic trajectory $x, Ax, A^2x, \dots$, and in both cases the lengths of steps do not vanish.

Let us now suppose that the statement is true for all sets of $k-1$ or fewer matrices. If all trajectories of Σ have vanishing steps, then so do all trajectories of each proper subset of Σ and, hence, by the induction assumption, each proper subset of Σ is UBV-stable.

Again, we will only consider the case $E(\Sigma) = \{0\}$ because of Proposition 4.2. This implies the existence of a $\delta > 0$ such that, for any $\|x\| > 1$ and some $i = 1, \dots, k$, the inequality

$$\|A_i y - y\| > \delta \tag{5.12}$$

holds for any y such that $\|y - x\| < \delta$. Suppose there exists a trajectory $X = \{x_i\}$ of Σ of infinite variation.

By assumption, $\lim_{i\to\infty}\|x_{i+1}-x_i\|=0$. This implies the convergence of X to zero because of Theorem 4.3. Indeed, suppose X does not converge to zero. Let us consider the set F of all finite segments of X generated by proper subsets of Σ. Theorem 4.3 ensures the existence of a universal constant $L>0$ such that

$$\sum_{i=i_0,\ldots,i_1-1}\|x_{i+1}-x_i\|\leq L\max_{i=i_0,\ldots,i_1-1}\|x_{i+1}-x_i\|$$

for each segment $X_{i_0,i_1}=\{x_{i_0},\ldots,x_{i_1}\}\in F$. Since Σ is VS, the variations of these segments vanish as $i_0\to\infty$. The variation of the whole sequence X is infinite, thus, for each index i_0, there exists a maximal segment X_{i_0,i_1} in F such that $X_{i_0,i_1+1}\notin F$. Since Σ is VS, the variations of extended segments X_{i_0,i_1+1} also vanish as $i_0\to\infty$, and so do their diameters. It remains to notice that, if we choose a sequence of initial indices $i_0^j\to\infty$ as $j\to\infty$ such that $\|x_{i_0^j}\|\not\to 0$ as $j\to\infty$, because of (5.12), for some $\delta>0$ there exist arbitrarily large indices i such that $\|x_{i+1}-x_i\|>\delta$, which is a contradiction with the VS property of Σ.

Thus, all trajectories of Σ converge (because trajectories of bounded variation converge). Thus, we have proved that Σ is an LCP set, and Theorem 4.4 can be now used.

Let us consider a finite trajectory $\{x_0,\ldots,x_m\}$ of Σ, where at least one matrix A_i does not participate. Its total variation is bounded from above by $L\|x_0\|$ where $L>0$ is a universal constant for all i. On the other hand, any finite trajectory generated by all matrices A_i satisfies the inequality $\|x_m\|\leq q\|x_0\|$. Thus, for any infinite trajectory $\{x_0,\ldots\}$ of Σ, its minimal initial segment $\{x_0,\ldots,x_m\}$ for which all the matrices A_i are used satisfies the inequality

$$\sum_{i=0,\ldots,m-1}\|x_{i+1}-x_i\|\leq (L+C)\|x_0\|,$$

where $C=\max_i\|A_i-I\|$.

The second segment $\{x_m\ldots,x_s\}$ satisfies the inequality

$$\sum_{i=m,\ldots,s-1}\|x_{i+1}-x_i\|\leq (L+C)\|x_m\|\leq q(L+C)\|x_0\|,$$

and so on. Since $q<1$, we have $V(x)\leq(1-q)^{-1}(L+C)\|x_0\|$. This shows the UBV-stability of Σ. □

Note that Theorem 5.1 is wrong for general *bounded* sets of matrices. Indeed, the set of all orthogonal projections is VS and paracontracting but not LCP or BV. The set of orthogonal projections in $\mathbb{R}^2$ onto lines $\{y = kx\}$ for all $k = 1, 2, \ldots$ and onto the line $x = 0$ is LCP and paracontracting but not BV.

Finally, let us formulate the most general equivalence theorem which is an obvious consequence of Theorem 5.1 and results collected in Section 3.

Theorem 5.2 *For any finite set Σ of $n\times n$-matrices, the properties UBV, BV, LCP, PC, LPC, and VS are equivalent to each other.*

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