

Dynamics of Patterns in Nonlinear Equivariant PDEs

Wolf-Jürgen Beyn^{*1} and Vera Thümmler¹

¹ Department of Mathematics, Bielefeld University,
PO Box 100131, 33501 Bielefeld, Germany

Received 25 March 2009

Published online XX May 2009

Key words moving patterns, equivariant PDEs, numerical methods, asymptotic stability, partial differential algebraic equations.

MSC (2000) 35K45, 65M12, 65P40

Many solutions of nonlinear time dependent partial differential equations show particular spatio-temporal patterns, such as traveling waves in one space dimension or spiral and scroll waves in higher space dimensions. The purpose of this paper is to review some recent progress on the analytical and the numerical treatment of such patterns. Particular emphasis is put on symmetries and on the dynamical systems viewpoint that goes beyond existence, uniqueness and numerical simulation of solutions for single initial value problems. The nonlinear asymptotic stability of dynamic patterns is discussed and a numerical approach (the *freezing method*) is presented that allows to compute co-moving frames in which solutions converging to the patterns become stationary. The results are related to the theory of relative equilibria for equivariant evolution equations. We discuss several applications to parabolic systems with nonlinearities of FitzHugh-Nagumo and Ginzburg-Landau type.

© 2009 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 Introduction

Nonlinear wave phenomena are ubiquitous in mathematical models for the dynamics of biological, chemical, physical or technical systems. Among the most prominent ones are traveling waves that occur in models for the spread of populations, for neuronal firing or for excitable chemical reactions (see [30],[44],[42], [5] as general references). Further dynamical patterns occurring in two and three space dimensions are rotating and spiral waves (also called dissipative solitons in the physics literature [12]) and scroll waves [23], but also examples in technical systems such as buckling modes of long structures [22] belong to this category.

For the sake of a common framework we restrict in this paper to semilinear parabolic systems of the type

$$u_t = A\Delta u + f(u), \quad x \in \mathbb{R}^d, t \geq 0, \quad (1)$$

where $u(x, t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$ is a positive definite (not necessarily symmetric) matrix and the nonlinearity $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is sufficiently smooth. Several extensions to more different

* Corresponding author: e-mail: beyn@math.uni-bielefeld.de

types of equations (e.g. hyperbolic and coupled hyperbolic-parabolic systems) and to more refined phenomena (e.g. multipulses and multifronts) will be indicated in Section 5.

In dimension $d = 1$ traveling wave solutions of (1) are of the form

$$u(x, t) = v(x - ct), \quad x \in \mathbb{R}, t \in \mathbb{R}, \quad (2)$$

where $v : \mathbb{R} \rightarrow \mathbb{R}^m$ denotes the profile of the wave and $c \in \mathbb{R}$ denotes its speed. For dimension $d = 2$ a model case for dynamic patterns is given by rotating waves of the form

$$u(x, t) = v(R_{-ct}x), \quad x \in \mathbb{R}^2, t \in \mathbb{R}, \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (3)$$

where again $v : \mathbb{R}^2 \rightarrow \mathbb{R}^m$ denotes a fixed profile that rigidly rotates about the origin with angular speed c .

Traveling and rotating waves are special cases of so called *relative equilibria* of the system (1). This notion originates from the theory of *equivariant dynamical systems* in which one studies the influence of symmetries on the longtime behavior of dynamical systems. This theory has been developed to a considerable extent over the last years. Some monographs containing much of the current state are [14],[19],[17] and a few ingredients of the theory needed for this paper will be mentioned in Section 3.

The parabolic system (1) is equivariant with respect to the action of the Euclidean group $SE(d)$ on functions $v : \mathbb{R}^d \rightarrow \mathbb{R}^m$ given by

$$[a(\gamma)v](x) = v(\gamma^{-1}x), \quad x \in \mathbb{R}^d, \gamma \in SE(d). \quad (4)$$

Note that we use γx to denote the action of some group element on a vector $x \in \mathbb{R}^d$ (i.e. $\gamma x = Qx + b$ for some orthogonal matrix Q and some vector $b \in \mathbb{R}^d$) but that we use $a(\gamma)v$ (rather than γv) to denote the action on functions defined by (4), cf. the notions in [14],[17].

Relative equilibria with respect to the action (4) may then be written as

$$u(x, t) = v(\gamma(t)^{-1}x), \quad x \in \mathbb{R}^d, t \in \mathbb{R}, \quad (5)$$

where $\gamma(t) \in SE(d), t \in \mathbb{R}$ denotes the motion of the pattern parametrized by the Euclidean group. Obviously, the traveling and rotating waves from (2), (3) are special cases of (5).

This paper is devoted to various analytical and numerical aspects of relative equilibria for systems that have continuous symmetries such as equation (1). We are particularly interested in numerical methods that make systematic use of the underlying symmetries and in the analysis of these methods and its relation to the stability theory for relative equilibria.

In Section 2 we provide several characteristic examples of relative equilibria and solutions of initial value problems that converge to such equilibria. In addition to the cases (2), (3) above we show waves that rotate and travel simultaneously as well as scroll waves in three space dimensions. The numerical results are based on direct simulations of Cauchy problems for (1) ($u(x, 0) = u_0(x), x \in \mathbb{R}^d$), obtained by truncating to a bounded domain in \mathbb{R}^d , using appropriate boundary conditions and then applying standard space-time discretization methods.

The central topic of Section 3 are stability problems for relative equilibria of (1). We introduce the abstract setting of equivariant dynamical systems and discuss the notion of nonlinear

stability with asymptotic phase for relative equilibria. The main issue here is to infer nonlinear stability from linear stability, which means that the whole spectrum (or an important part of it) of a linearized operator lies in the open left half plane. While such results are well known for equivariant ODEs and for traveling waves in PDEs in one space dimension we report on some recent progress for rotating patterns in two space dimensions.

Section 4 is concerned with the *freezing method* for solving initial value problems for equivariant systems as developed in [10], [11]. In this approach we aim at separating shape dynamics from group dynamics not only for relative equilibria but also for solutions of the Cauchy problem. The form (5) is generalized to

$$u(x, t) = v(\gamma(t)^{-1}x, t), \quad x \in \mathbb{R}^d, t \geq 0, \gamma(t) \in SE(d) \quad (6)$$

and extra phase conditions are imposed in order to compensate for the additional unknowns $\gamma(t)$ in the group. With this ansatz the PDE (1) transforms into a partial differential algebraic equation (PDAE) which is then solved numerically. While the PDE and the PDAE formulation are essentially equivalent on the unbounded domain, the longtime behavior of solutions differs dramatically when restricting to a bounded domain. While the patterns in the original PDE may rotate for ever or may die out when they reach the boundary, the freezing method allows to compute co-moving frames in which solutions converging to a relative equilibrium become stationary at the target pattern. In Section 4 we discuss the feasibility and the implications of this approach and we relate it to the stability problems in Section 3.

The final Section 5 indicates several solved and open problems in the subject area. Numerical experiments indicate that the method works for much more general evolution equations than (1) for which the theoretical background is still under development. It even works in some cases where the underlying symmetry is destroyed (such as stochastic evolution equations with additive noise) and it can be generalized to certain multiple patterns that travel at different speeds and that have weak or strong interactions.

2 Examples

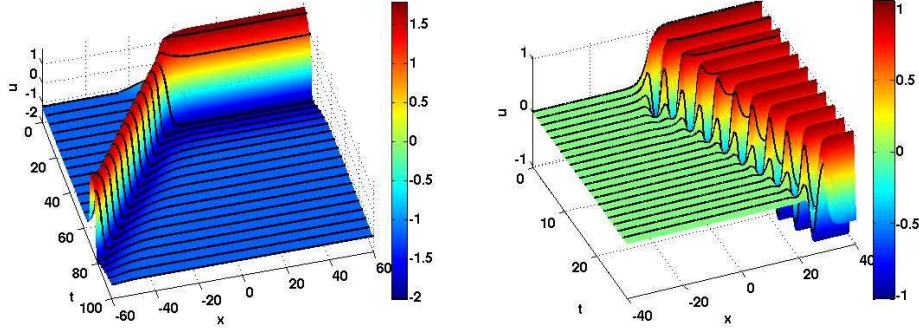
2.1 Traveling and phase-rotating waves in one space dimension

Our first example is the well known FitzHugh Nagumo equation [18] which is a two-dimensional system for $u = (V, R)$ where V is voltage across a nerve membrane and R is a phenomenological parameter

$$\begin{aligned} V_t &= V_{xx} + V - \frac{1}{3}V^3 - R \\ R_t &= \phi(V + a - bR) \end{aligned} \quad x \in \mathbb{R}, \quad t \geq 0. \quad (7)$$

The second equation lacks a diffusion term so that the system is not parabolic and does not satisfy all our assumptions. However, the following simulations can be repeated with a small diffusion term εR_{xx} , $\varepsilon > 0$ without substantial changes in the results.

We use the parameters: $a = 0.7$, $b = 0.8$, $\phi = 0.08$ from [29] and solve the initial value problem on the interval $\Omega = [-60, 60]$ subject to Neumann boundary conditions. In the following, unless stated otherwise, we use the finite element package Comsol MultiphysicsTM[15] with second order elements in space and a BDF method in time. Figure 1 (a) shows that an



(a) FitzHugh-Nagumo: V component of traveling wave (b) Quintic Ginzburg-Landau: real part of traveling and phase-rotating wave

Fig. 1 (online colour at: www.gamm-mitteilungen.org) Moving patterns in one space dimension.

initial ramp function u_0 generates a pulse that travels to the left (with approximate speed $c = -0.812$). When the pulse reaches the left boundary it dies out due to the Neumann boundary conditions. Of course, on the whole real line the pulse will continue to travel to $-\infty$. Considered as dynamical systems we see that the longtime behavior of both systems on the bounded and the unbounded domain will differ substantially, in general. While traveling waves are part of the global attractor on the unbounded domain they occur as transient phenomena on any bounded domain.

Our second example is the complex Ginzburg-Landau equation that occurs as a modulation equation in the study of hydrodynamic instabilities [28]. The quintic Ginzburg-Landau equation reads

$$u_t = \alpha u_{xx} + (\delta + \beta|u|^2 + \rho|u|^4)u, \quad u(x, t) \in \mathbb{C}, x \in \mathbb{R}, t \geq 0, \quad (8)$$

with complex parameters $\alpha, \beta, \delta, \rho$. Assuming $\text{Re}(\alpha) > 0$ one can rewrite (8) as a real two-dimensional parabolic system. Equation (8) has a two dimensional symmetry group $G = S^1 \times \mathbb{R}$, $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ with the action given by

$$[a(\gamma_1, \gamma_2)v](x) = \exp(-i\gamma_1)v(x - \gamma_2), \quad x \in \mathbb{R}, (\gamma_1, \gamma_2) \in S^1 \times \mathbb{R}. \quad (9)$$

Following [36],[40] we take parameters $\alpha = 1, \delta = -0.1, \beta = 3 + i, \rho = -2.75 + i$. Starting at a ramp function the solution converges to a traveling front with simultaneous phase rotation, see Figure 1 (b). These relative equilibria may be written as

$$u(x, t) = \exp(-i\mu_1 t)v(x - \mu_2 t), \quad \text{where } \mu_1, \mu_2 \in \mathbb{R}. \quad (10)$$

At the same parameter values the system also shows pulses that have a phase rotation only, i.e. for which (10) holds with $\mu_1 \neq 0, \mu_2 = 0$, cf. [36],[40].

2.2 Planar spinning solitons

Consider the two-dimensional quintic Ginzburg-Landau system

$$u_t = \alpha \Delta u + (\delta + \beta|u|^2 + \rho|u|^4)u, \quad x \in \mathbb{R}^2, t \geq 0 \quad (11)$$

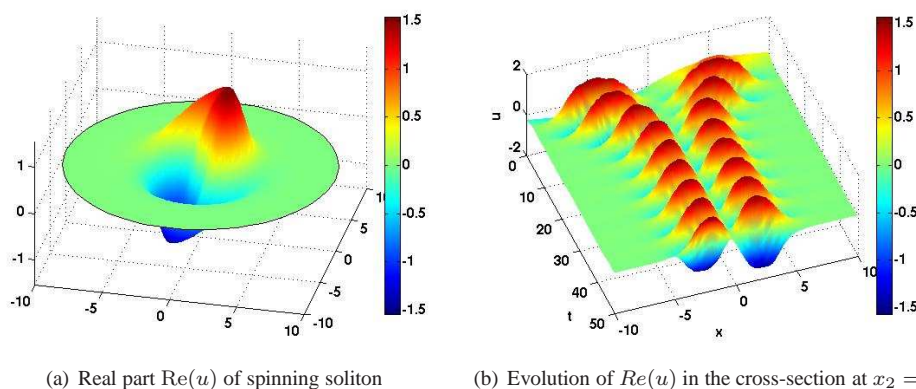


Fig. 2 (online colour at: www.gamm-mitteilungen.org) Quintic Ginzburg-Landau: Real part of spinning soliton

with parameters $\alpha, \beta, \delta, \rho \in \mathbb{C}$ as above. The symmetry group $G = S^1 \times SE(2)$ is now 4-dimensional and acts on functions via

$$[a(\gamma)v](x) = \exp(-i\gamma_1)v(\gamma_2^{-1}x), \quad \gamma = (\gamma_1, \gamma_2) \in S^1 \times SE(2). \quad (12)$$

According to [16] the system (12) shows so called spinning solitons, i.e. strongly localized solutions that are rotating patterns in the sense of (3). These occur at parameter values $\alpha = (1+i)/2$, $\delta = 1/2$, $\beta = 2.5+i$, $\rho = -1-0.1i$, see [16]. Figure 3 (a) shows the real part of a spinning soliton obtained from a simulation on a disc of radius $r = 20$ (the plot is restricted to $r = 10$) with Neumann boundary conditions. In Figure 3 (b) we display the approach toward the soliton within the cross-section $x_2 = 0$ with two Gaussian humps as initial conditions. The center of the soliton turns out to be the origin due to the symmetry of initial conditions. Spinning solitons are rotating patterns as in (3) with a complex valued profile $v : \mathbb{R}^2 \rightarrow \mathbb{C}$. In this special case one finds that the profile v has an extra symmetry given by

$$v(R_{-\theta}x) = \exp(-i\theta)v(x), \quad \text{for } x \in \mathbb{R}^2, \theta \in S^1, \quad (13)$$

so that (3) may equivalently be written as

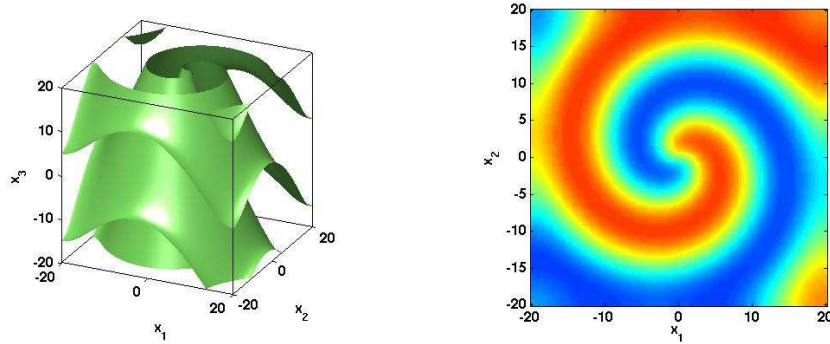
$$u(x, t) = \exp(-ict)v(x). \quad (14)$$

In abstract terms this relative equilibrium has a nontrivial isotropy subgroup which has implications for the stability theory as well as for numerical computations, see Sections 3 and 4.

Another class of rotating patterns are spiral waves for which there is an extensive literature, see e.g. [44], [42], [2]. Usually, spiral waves are not localized (i.e. they do not decay at infinity) as opposed to spinning solitons. While this property does not lead to serious problems with numerical simulations their stability theory turns out to be rather difficult and is not yet complete (see [34] and Section 3). In this paper we will not discuss spiral waves in detail.

2.3 A three-dimensional scroll wave

Scroll waves are special patterns that occur in three-dimensional systems of type (1), cf. [44], [23]. Usually, scroll waves wind around a flow invariant curve, called the filament, and they



(a) Isosurface $Re(u) = 0$ of scroll wave in CGL (b) Spiral pattern of $Re(u) = 0$ in the slice $x_3 = 0$

Fig. 3 (online colour at: www.gamm-mitteilungen.org) Scroll waves in three space dimensions

show spiral patterns in surfaces transversal to the filament. A scroll ring is obtained when the filament is a closed curve; then the number and orientation of rotations of the transverse structure leads to twisted or untwisted scroll rings, cf. [44].

In Figure 3 we show a simple scroll wave that has a straight line on the x_3 axis as its filament. The underlying system is the three-dimensional cubic Ginzburg-Landau equation (11) (also called the $\lambda - \omega$ system in [30]) with parameters $\alpha = 1, \beta = 1 + i, \delta = 1, \rho = 0$. The computational domain is $\Omega = [-20, 20]^3$ with Neumann boundary conditions on the surfaces $x_1 = \pm 20$ and $x_2 = \pm 20$ but periodic boundary conditions in the x_3 direction. This choice of boundary conditions favors the vertical filament. Figure 3 (a) shows a view of the isosurface $Re(u(x)) = 0$ and Figure 3 (b) displays the spiral pattern in the slice $x_3 = 0$. The results are obtained by a modification of Barkley's finite difference code `ezscroll` [3] on a mesh of 125^3 grid points. We refer to Section 4 for some time-dependent simulations in this case.

The symmetry group $G = S^1 \times SE(3)$ now has dimension 7 and acts as in (12) with $SE(2)$ replaced by $SE(3)$. Similar to the two-dimensional case the scroll wave is a relative equilibrium of the type (5) where $\gamma(t)$ denotes rotation about the x_3 -axis. Again the solution has an extra symmetry of the form

$$v(R_{3,\theta}x) = \exp(-i\theta)v(x), \quad \theta \in S^1, \quad R_{3,\theta} = \begin{pmatrix} R_\theta & 0 \\ 0 & 1 \end{pmatrix} \quad (15)$$

which allows to write the scroll wave as a phase-rotating wave.

3 Relative equilibria and their dynamic stability

In this section we first summarize some basic theory on evolution equations that are equivariant with respect to a (not necessarily compact) Lie group G . Then we discuss dynamic stability of relative equilibria for some special cases of the general PDE (1).

3.1 Equivariance

Consider, more generally than (1), an abstract evolution equation

$$u_t = F(u), \tag{16}$$

where the vector field F is defined on a dense subspace Y of some Banach space X and maps into X . Further assume that the group G acts on X via a homomorphism

$$a : G \rightarrow GL(X), \gamma \rightarrow a(\gamma), \tag{17}$$

such that $\gamma \rightarrow a(\gamma)v$ is continuous for every $v \in X$. We use $\mathcal{O}_G(v) = \{a(\gamma)v : \gamma \in G\}$ to denote the group orbit of some element $v \in X$.

Definition 3.1 The system (16) is called *equivariant* with respect to the group action (17) if the following properties hold

- (i) $a(\gamma)(Y) = Y$ for all $\gamma \in G$,
- (ii) $F(a(\gamma)v) = a(\gamma)F(v)$ for all $v \in Y, \gamma \in G$.

The system (1) is equivariant with respect to the group action (4) in suitable Sobolev spaces, e.g.

$$Y = \mathcal{H}^2(\mathbb{R}^d, \mathbb{R}^m), \quad X = \mathcal{L}^2(\mathbb{R}^d, \mathbb{R}^m). \tag{18}$$

This assertion holds if $f(0) = 0$ and if f satisfies appropriate growth conditions. The case $f(u_\infty) = 0$ for some $u_\infty \neq 0$ (as in the FitzHugh Nagumo system (7)) can be reduced to this case by writing the equation in terms of $\tilde{u} = u - u_\infty$. The trick does not work for traveling fronts which typically lie in some affine space $w + \mathcal{H}^2(\mathbb{R}^d, \mathbb{R}^m)$ with a bounded function w that prescribes the behavior at infinity. For this case it is useful to generalize the whole approach from Banach spaces to Banach manifolds using an appropriate formulation of equivariance (cf. [26], [11]). In this paper we avoid the technicalities involved in this generalization.

Let $\mathbb{1}$ denote the unit element in G and let $\mathcal{A} = T_{\mathbb{1}}G$ be the associated Lie algebra. Further, let $L_\gamma : G \rightarrow G, g \mapsto \gamma g$ be the operation of left multiplication with derivative denoted by $dL_\gamma(g) : T_gG \rightarrow T_{\gamma g}G$. The exponential $\gamma(t) = \exp(t\mu) \in G, t \in \mathbb{R}$ for some element $\mu \in \mathcal{A}$ can be defined as the unique solution of the initial value problem

$$\dot{\gamma} = dL_\gamma(\mathbb{1})\mu, \quad \gamma(0) = \mathbb{1}. \tag{19}$$

In general, the map $a(\cdot)v : \gamma \rightarrow a(\gamma)v$ will not be smooth for all $v \in X$, but we assume that this is the case for $v \in Y$ (i.e. Y is in the domain of the infinitesimal generator of the group action) with derivative denoted by $d[a(\gamma)v] : T_\gamma G \rightarrow X$. For example, in the simple shift case $[a(\gamma)v](x) = v(x - \gamma), x \in \mathbb{R}, \gamma \in G = \mathbb{R}$ this holds with

$$X = \mathcal{L}^2(\mathbb{R}, \mathbb{R}^m), v \in Y \subset \mathcal{H}^1(\mathbb{R}, \mathbb{R}^m), \quad d[a(\gamma)v]\mu = \mu v_x \quad \text{for } \mu \in \mathbb{R}. \tag{20}$$

3.2 Relative equilibria

Definition 3.2 A solution $\bar{u}(t) \in Y, t \in \mathbb{R}$ of (16) is called a relative equilibrium if it is of the form

$$\bar{u}(t) = a(\bar{\gamma}(t))\bar{v} \quad \text{for some } \bar{v} \in Y, \bar{\gamma} \in C^1(\mathbb{R}, G). \quad (21)$$

There is no loss of generality in assuming $\bar{\gamma}(0) = \mathbb{1}$. Usually, the whole group orbit $\mathcal{O}_G(\bar{v})$ is called a relative equilibrium [14], [17], [26], but we prefer to include the one-parameter group $\bar{\gamma}(t)$ in the definition because it will be part of the numerical approach in Section 4.

Examples of relative equilibria with respect to the group action (4) are the traveling waves from (2) ($\bar{\gamma}(t) = ct$) and the rotating waves from (3) ($\bar{\gamma}(t) = R_{ct}$). For the complex Ginzburg Landau equation (8) we found a relative equilibrium w.r.t. the group action of $G = S^1 \times \mathbb{R}$ in (9).

It can be shown (see [14, Th.7.2.4]) that any relative equilibrium of (16) can be written as $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ for a suitable $\bar{\mu} \in \mathcal{A}$ such that the following holds

$$0 = F(\bar{v}) - d[a(\mathbb{1})\bar{v}]\bar{\mu}, \quad (22)$$

$$\bar{\gamma}(t) = \exp(t\bar{\mu}), \quad t \in \mathbb{R}. \quad (23)$$

This is proved by inserting $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ into (16). If \bar{v} has a trivial isotropy group

$$H_{\bar{v}} = \{\gamma \in G : a(\gamma)\bar{v} = \bar{v}\}, \quad (24)$$

then $d[a(\mathbb{1})\bar{v}]$ is a one-to-one mapping and hence $\bar{\mu}$ is unique. In the general case, it is only unique up to elements from the Lie algebra of $H_{\bar{v}}$, cf. [14, Ch.7.2]. Note that we found nontrivial isotropy groups for the relative equilibria of the complex Ginzburg Landau equation in dimensions $d \geq 2$ (cf. (13), (15)).

Conversely, if $\bar{v} \in Y, \bar{\mu} \in \mathcal{A}$ satisfies (22) then $a(\exp(t\bar{\mu}))\bar{v}$ is a relative equilibrium of (16).

For traveling waves in the case $d = 1, G = \mathbb{R}$ we obtain from equation (22) the system (cf.(20))

$$0 = Av_{xx} + f(v) + \mu v_x, \quad x \in \mathbb{R}, \quad (25)$$

and in the two-dimensional case (4) we find

$$0 = A\Delta v + f(v) + \mu_3 D_\theta v + \mu_1 D_1 v + \mu_2 D_2 v. \quad (26)$$

For the last equation we used the representation $SE(2) = \mathbb{R}^2 \times S^1$ with the action given by

$$a(\eta, \theta)v(x) = v(R_{-\theta}(x - \eta)), (\eta, \theta) \in SE(2). \quad (27)$$

The partial derivatives in (26) are $D_1 = \frac{\partial}{\partial x_1}, D_2 = \frac{\partial}{\partial x_2}, D_\theta = x_2 D_1 - x_1 D_2$, the constants μ_1, μ_2 denote the translational velocity and μ_3 denotes the rotational velocity. Note that the pattern (3) rotating about the origin satisfies this system with $\mu_1 = \mu_2 = 0, \mu_3 = c$.

In general, the system (22) does not determine \bar{v} and $\bar{\mu}$ uniquely. Since $a(g\bar{\gamma}(t))\bar{v}$ is a relative equilibrium for every $g \in G$ one finds that (22) has a family of solutions $\{(a(g)\bar{v}, Ad_g\bar{\mu}) : g \in G\}$ where $Ad_g\bar{\mu} = \frac{d}{dt}g \exp(t\bar{\mu})g^{-1}|_{t=0}$ is the adjoint action of G on \mathcal{A} . Therefore one needs at least $\dim(G)$ additional constraints (called phase conditions) in order to turn (22) into a well-posed problem for $(\bar{v}, \bar{\mu})$. This aspect is essential for the numerical computation of relative equilibria (for a detailed discussion see [13]) and it will also play a major role for the freezing method in the next section.

3.3 Nonlinear stability

For equivariant evolution equations classical Liapunov stability has to be weakened to stability with asymptotic phase which may be regarded as stability in the orbit space X/G (see [14], [17]). We consider the Cauchy problem

$$u_t = F(u), \quad u(0) = u_0 \in Y \quad (28)$$

and assume that Y is a Banach space with respect to some norm $\|\cdot\|_Y$ that is stronger than the norm in X and with respect to which (28) is well posed. As with relative equilibria we make the dependence on appropriate group orbits explicit in the definition.

Definition 3.3 A relative equilibrium $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ is called *asymptotically stable* if there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ there exists a $\delta > 0$ with the following property. For any $\|u_0 - \bar{v}\|_Y \leq \delta$ equation (28) has a unique solution $u(t) \in Y, t \geq 0$ and there exists an orbit $\gamma(t) \in G, t \geq 0$ such that

$$\|u(t) - a(\gamma(t)\bar{\gamma}(t))\bar{v}\|_Y \begin{cases} \leq \varepsilon & \text{for all } t \geq 0, \\ \rightarrow 0 & \text{as } t \rightarrow \infty. \end{cases} \quad (29)$$

If, in addition, $\gamma(t)$ converges as $t \rightarrow \infty$ to an element γ_∞ in the ε -neighborhood of $\mathbb{1}$ then γ_∞ is called the *asymptotic phase*. The relative equilibrium is then called *stable with asymptotic phase*.

For applications to PDEs the choice of norms is crucial. In some applications one needs $u_0 - \bar{v}$ to be small in a norm stronger than the one for which (29) holds. Moreover, for certain patterns it may be useful to measure the distance of profiles only on compact subsets rather than on the whole space (see [1] for such a discussion). In the finite dimensional ODE case it is well known how to prove asymptotic stability from linearized stability by invoking the classical Liapunov stability theorem in a transversal direction (see [14, Th.7.4.2.]). One of the difficulties in the PDE case is that essential spectra appear because linearized PDE operators on unbounded domains usually lack compactness properties. In the following we will discuss two cases where such a result holds for the general equation (1).

3.4 Traveling waves

This is one of the best studied cases for stability with asymptotic phase (see the monographs [21],[41] and the review [33]). We briefly recall a typical result. Let $u(x, t) = \bar{v}(x - ct)$ be a traveling wave solution of (1) for $d = 1$ where \bar{v} is assumed to be C^2 -bounded on \mathbb{R} such that the limits

$$v_- = \lim_{x \rightarrow -\infty} \bar{v}(x), \quad v_+ = \lim_{x \rightarrow \infty} \bar{v}(x) \quad (30)$$

exist. We impose two conditions that allow to control the essential and the discrete spectrum. **Spectral condition (SC):** There exist constants $\rho, \beta > 0$ such that all solutions $\lambda \in \mathbb{C}$ of the quadratic eigenvalue problems

$$\det(A\lambda^2 + c\lambda I + f'(v_\pm) - sI) = 0 \quad \text{with} \quad \operatorname{Re}(s) \geq -\beta \quad (31)$$

satisfy $|\operatorname{Re}\lambda| \geq \rho$.

This condition ensures that the essential spectrum of the linear operator

$$Lv = Av_{xx} + cv_x + f'(\bar{v}(\cdot))v \quad (32)$$

lies in the half-plane $\operatorname{Re}(s) \leq -\beta$ (say with respect to $\mathcal{H}^1(\mathbb{R}, \mathbb{R}^m)$), cf. [21].

By differentiating (25) with respect to x one finds that 0 is an eigenvalue of L with eigenfunction $\bar{v}_x \in \mathcal{H}^1(\mathbb{R}, \mathbb{R}^m)$. This is the eigenvalue caused by equivariance.

The next condition excludes further eigenvalues on or to the right of the imaginary axis.

Eigenvalue condition (EC): The eigenvalue 0 of the operator L in (32) is simple and L has no further isolated eigenvalues with $\operatorname{Re}(s) \geq -\beta$.

Then the traveling wave is stable with asymptotic phase with respect to the norm $\|\cdot\|_{\mathcal{H}^1}$. More precisely, for $\|u_0 - \bar{v}\|_{\mathcal{H}^1}$ sufficiently small one obtains exponential convergence

$$\|u(\cdot, t) - \bar{v}(\cdot - ct - \gamma(t))\|_{\mathcal{H}^1} \leq C \exp\left(-\frac{\beta}{2}t\right) \|u_0 - \bar{v}\|_{\mathcal{H}^1} \quad (33)$$

for a suitable phase shift $\gamma(t)$ with $\lim_{t \rightarrow \infty} \gamma(t) = \gamma_\infty$, $|\gamma_\infty| \leq C \|u_0 - \bar{v}\|_{\mathcal{H}^1}$. Note that in this case the profile \bar{v} itself need not be an element of \mathcal{H}^1 , rather it is sufficient to have initial values that are small perturbations of \bar{v} in the \mathcal{H}^1 -norm, i.e. we have stability with asymptotic phase in the affine space $\bar{v} + \mathcal{H}^1$.

3.5 Rotating patterns

The stability proof for traveling waves in parabolic systems is greatly simplified by the fact that the linearized operators (32) generate analytic semigroups. This is not the case for the FitzHugh-Nagumo system (7) where (32) is of coupled parabolic hyperbolic type (if $c \neq 0$) and generates a C^0 -semigroup only. For this case Bates and Jones [4] have developed an invariant manifold approach that allows to conclude stability for traveling waves in the FitzHugh-Nagumo system. More generally, in [35] an abstract approach is set up that allows to reduce the dynamics near a relative equilibrium of (16) to a center manifold. Also exponential attraction of the center manifold is proved under spectral assumptions. However, stability with asymptotic phase is not considered in [35].

We review here a recent result [8] that provides nonlinear stability with asymptotic phase for rotating pattern as in (3).

Assume for the nonlinearity $f \in C^4(\mathbb{R}^m, \mathbb{R}^m)$ and let $u(x, t) = \bar{v}(R_{-ct}x)$ be a rotating pattern with $c \neq 0$ that is localized in the sense

$$\sup_{|x| \geq r, 0 \leq |\alpha| \leq 2} |D^\alpha \bar{v}(x)| \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (34)$$

This condition implies that $f(0) = 0$. The following spectral condition requires stability in the far field.

Spectral condition (SC): The matrix $f'(0) \in \mathbb{R}^{m,m}$ is negative definite (not necessarily symmetric).

Since \bar{v} and $\bar{\mu} = (0, 0, c)$ solve equation (26) the linear operator analogous to (32) is

$$Lv = A\Delta v + cD_\theta v + f'(\bar{v}(\cdot))v, \quad (35)$$

defined in the domain $\mathcal{H}_{Eucl}^2 = \{v \in \mathcal{H}^2(\mathbb{R}^2, \mathbb{R}^m) : D_\theta v \in \mathcal{L}^2(\mathbb{R}^2, \mathbb{R}^m)\}$. One can show that this operator generates a C^0 -semigroup on $\mathcal{H}^2(\mathbb{R}^2, \mathbb{R}^m)$ and that it has essential spectrum in

the open left half-plane due to (SC). Moreover, the essential spectrum contains the algebraic set

$$S = \{s \in \mathbb{C} : \det(-\kappa^2 A - incI + f'(0) - sI) = 0 \text{ for some } n \in \mathbb{Z}, \kappa \in \mathbb{R}\}.$$

Note that S consists of a countable number of algebraic curves that are copies of a single parabola-shaped curve shifted by $inc, n \in \mathbb{Z}$ in the imaginary direction. There is a fixed spectral gap between S and the imaginary axis. We refer to [8] for a proof and some illustrative pictures of (numerical) spectra for the case of spinning solitons.

On the contrary, spiral waves that are asymptotically periodic in the radial direction do not satisfy our assumptions and their essential spectrum contains infinitely many parabola-shaped curves that touch the imaginary axis (see [34] for some illuminating results on the spectral behavior of Archimedian spirals and its relation to stability questions).

Applying D_1, D_2, D_θ to the stationary equation (26) immediately shows that the operator L has three eigenvalues $0, \pm ic$ on the imaginary axis with corresponding eigenfunctions $D_\theta \bar{v}, D_1 \bar{v} \pm iD_2 \bar{v}$. These are the eigenvalues caused by $SE(2)$ -equivariance. Therefore we impose the following condition.

Eigenvalue condition (EC): The eigenvalues $0, \pm ic$ of the operator L in (35) are simple with eigenfunctions in \mathcal{H}_{Eucl}^2 and L has no further isolated eigenvalues with $\text{Re}(s) \geq -\beta$ for some $\beta > 0$.

Under these assumptions the rotating pattern is stable with asymptotic phase (in the sense of Definition 3.3) in the function space $\mathcal{H}^2(\mathbb{R}^2, \mathbb{R}^m)$ with the action given by (27). Moreover, we have exponential convergence as in (33)

$$|\eta(t) - \eta_\infty| + |\theta(t) - \theta_\infty| + \|u(\cdot, t) - \bar{v}(R_{-ct-\theta(t)}(\cdot - \eta(t)))\|_{\mathcal{H}^2} \leq C \exp(-\frac{\beta}{2}t) \|u_0 - \bar{v}\|_{\mathcal{H}^2}.$$

The interpretation is that perturbing the initial function leads to a solution which converges to a pattern rotating about a slightly perturbed center at η_∞ with an angular velocity c and with a phase shift θ_∞ . In the proof one has to carefully split the dynamics near \bar{v} into the dynamics within the three-dimensional group orbit $\mathcal{O}_G(\bar{v})$ and in a transversal direction. The nonlinear remainders can be handled in \mathcal{H}^2 by using Sobolev embedding and Gagliardo Nirenberg estimates. Furthermore, there is an abstract perturbation theorem on C^0 -semigroups (see [8, Appendix]) that allows to conclude exponential decay of the C^0 -semigroup generated by the operator L in a suitable subspace.

This theorem applies to the spinning solitons of Section 2.2 when we restrict the action (12) to the subgroup $SE(2)$. In this way we avoid the nontrivial isotropy subgroup caused by (13). Note that the spectral condition (SC) follows from $\delta < 0$. The eigenvalue condition (EC) is hard to prove analytically. Numerical computations reveal that there is a total of 8 additional complex conjugate pairs of simple eigenvalues with real part strictly between the algebraic set S and the imaginary axis (cf. [8]). This indicates that (EC) is satisfied as well.

Because of the symmetry (13) one can also restrict the action (12) to the abelian subgroup $S^1 \times \mathbb{R}^2$ and then study the simpler linearization $Lv = A\Delta v + icv + f'(\bar{v}(\cdot))v$. Note however, that it is easy to destroy the S^1 -equivariance from (12) in the real version of (11) by perturbing the factor of the quintic term. Numerical experiments show that the spinning solitons persist under such perturbations and that the above stability result becomes essential in this case.

4 The freezing method

The theoretical results for relative equilibria in the previous section suggest to realize a splitting into group dynamics and shape dynamics for the general Cauchy problem (28). Such an approach was developed in [10],[11] and called the freezing method. The same idea was used independently and earlier in the paper [32] where it was applied to compute self-similar solutions of Burgers equation. Another precursor of this approach is the work by Marsden and Scheurle on *pattern evocation* for Hamiltonian systems in [27].

4.1 The general principle

We write the solution $u(t)$ of the equivariant Cauchy problem (28) in the form

$$u(t) = a(\gamma(t))v(t), \quad t \geq 0, \quad v(0) = u_0, \gamma(0) = \mathbb{1}, \quad (36)$$

where both $v(t) \in Y$ and $\gamma(t) \in G$ are considered to be unknown and to be determined by a numerical process. Clearly, there is some arbitrariness in the representation (36). We will use this arbitrariness to impose extra conditions (phase conditions) that try to minimize the temporal change of v . In this way we want $v(t)$ to converge to a relative equilibrium (if possible) or at least want to minimize the efforts for mesh adaptation when solving the PDE numerically. Inserting (36) into (28) and using equivariance leads to

$$v_t = F(v) - a(\gamma^{-1})d[a(\gamma)v]\gamma_t. \quad (37)$$

In order to simplify the extra term on the right hand side of (37) it is convenient to introduce $\mu(t) \in \mathcal{A}$ via $\gamma_t = dL_{\gamma}(\mathbb{1})\mu$. Then (28) may be rewritten as a system for $v(t) \in Y$, $\gamma(t) \in G$, $\mu(t) \in \mathcal{A}$

$$v_t = F(v) - d[a(\mathbb{1})v]\mu, \quad v(0) = u_0 \quad (38)$$

$$\gamma_t = dL_{\gamma}(\mathbb{1})\mu, \quad \gamma(0) = \mathbb{1}. \quad (39)$$

We refer to [11, Lemma 3.3] for a precise statement about the equivalence of (28) and the system (38),(39). We also note the important fact that relative equilibria $u(t) = \exp(t\bar{\mu})\bar{v}$ for (28) are in one-to-one correspondence with steady states $(\bar{v}, \bar{\mu})$ of equation (38).

Equation (39) determines the motion on the group and is decoupled from (38). Therefore, it can be solved in an a-posteriori process provided $\mu(t) \in \mathcal{A}$ is known. Following [32] we call (39) the *reconstruction equation*.

In the next step the remaining $p = \dim G$ degrees of freedom in (38) are fixed by adding a set of p phase conditions

$$\psi(v, \mu) = 0. \quad (40)$$

Here we assume that ψ is a given map from $Y \times \mathcal{A}$ into \mathcal{A}^* , the dual of \mathcal{A} which is isomorphic to \mathbb{R}^p . Two different choices for ψ will be discussed below.

Equations (38),(40) comprise a system of differential algebraic equations (DAEs) for the unknowns $v(t) \in Y$, $\mu(t) \in \mathcal{A}$ which, in the applications, will lead to a partial differential algebraic equation (PDAE) that is to be solved numerically.

It is useful to derive expressions for ψ from minimality conditions in terms of an inner product $\langle \cdot, \cdot \rangle_2$ that is continuous with respect to the given norm on X . The first expression

uses a template function $\hat{v} \in Y$ (e.g. $\hat{v} = u_0$) and requires \hat{v} to be the closest point to $v(t)$ on the orbit $\mathcal{O}(\hat{v})$. That is, we require $\|a(\gamma)\hat{v} - v\|_2$ to have a local minimum at $\gamma = \mathbb{1}$ which leads to the following *fixed phase condition*

$$\psi_{\text{fix}}(v)\omega = \langle d[a(\mathbb{1})\hat{v}]\omega, v - \hat{v} \rangle_2 = 0 \quad \text{for all } \omega \in \mathcal{A}. \quad (41)$$

Another possibility is to choose μ in (38) such that $\|v_t\|_2$ is minimal at each time instance. A necessary condition for this is

$$\psi_{\text{min}}(v)\omega = \langle d[a(\mathbb{1})v]\omega, v_t \rangle_2 = 0 \quad \text{for all } \omega \in \mathcal{A}. \quad (42)$$

We note that (41) leads to a DAE of (differentiation) index 2. Differentiating (41) with respect to t and using (38) leads to the index 1 condition

$$\Psi_{\text{fix}}(v, \mu)\omega = \langle d[a(\mathbb{1})\hat{v}]\omega, F(v) - d[a(\mathbb{1})v]\mu \rangle_2 = 0 \quad \text{for all } \omega \in \mathcal{A}, \quad (43)$$

which is a linear system of dimension p for the unknowns $\mu \in \mathcal{A}$. On the other hand, inserting v_t from (38) into (42) directly leads to the second index 1 condition

$$\Psi_{\text{min}}(v, \mu)\omega = \langle d[a(\mathbb{1})v]\omega, F(v) - d[a(\mathbb{1})v]\mu \rangle_2 = 0 \quad \text{for all } \omega \in \mathcal{A}. \quad (44)$$

Writing $\mu = \sum_{j=1}^p \mu_j e^j$ in some basis $\{e^1, \dots, e^p\}$ for \mathcal{A} shows that (43), (44) are linear systems for (μ_1, \dots, μ_p) with matrices

$$B_{\text{fix}} = (\langle d[a(\mathbb{1})\hat{v}]e^i, d[a(\mathbb{1})v]e^j \rangle_2)_{i,j=1}^p, \quad B_{\text{min}} = (\langle d[a(\mathbb{1})v]e^i, d[a(\mathbb{1})v]e^j \rangle_2)_{i,j=1}^p.$$

We note that B_{min} is nonsingular if v has a trivial isotropy group and the same applies to B_{fix} if \hat{v} and v are sufficiently close.

4.2 Examples

In this section we apply the freezing method to the examples from Section 2. For each type of equation we will write down the explicit form of the PDAE obtained from the abstract system (38), (43), resp. (44).

4.2.1 FitzHugh-Nagumo system

In this case the abstract approach is of the simple form $u(x, t) = v(x - \gamma(t))$ and using the fixed phase condition (43) with $\hat{v} = u_0 = (V_0, R_0)$ the system (7) leads to the following

$$\begin{aligned} V_t &= V_{xx} + V - \frac{1}{3}V^3 - R + \mu V_x, & V(x, 0) &= V_0(x) \\ R_t &= \phi(V + a - bR) + \mu R_x, & R(x, 0) &= R_0(x) \\ 0 &= \langle V_{0,x}, V - V_0 \rangle_{\mathcal{L}_2} + \langle R_{0,x}, R - R_0 \rangle_{\mathcal{L}_2}. \end{aligned} \quad (45)$$

This is completed by the reconstruction equation $\gamma_t = \mu, \gamma(0) = 0$. In Figure 4 the solution of (45) is shown when discretized on $\Omega = [-60, 60]$ with Neumann boundary conditions. The initial conditions are the same as in Figure 1(a). After a short transient period, both the wave form (see Figure 4(a)) and the wave speed (see Figure 4(b)) become stationary. From the steady states one can directly read off the asymptotic profile as well as the wave speed.

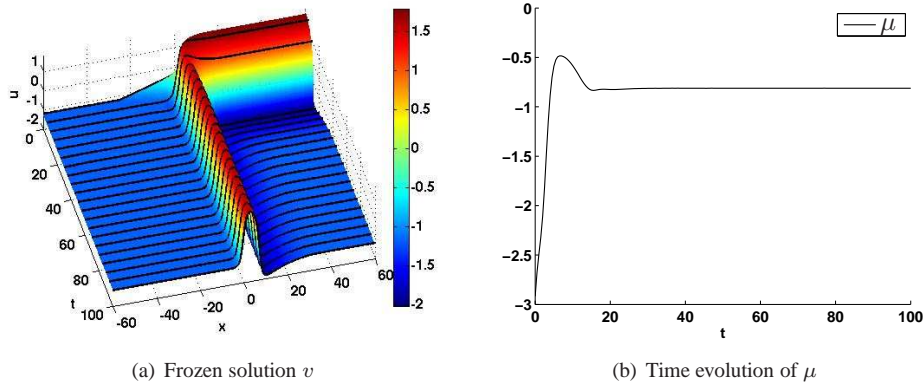


Fig. 4 (online colour at: www.gamm-mitteilungen.org) FitzHugh-Nagumo system.

4.2.2 Quintic complex Ginzburg-Landau equation

The freezing method now uses the ansatz $u(x, t) = \exp(-i\gamma_1(t))v(x - \gamma_2(t))$ in view of the group action (9). Combined with a fixed phase condition and the reconstruction equations we obtain from (8) the system

$$\begin{aligned} v_t &= \alpha v_{xx} + (\delta + \beta|v|^2 + \rho|v|^4)v + \mu_2 v_x + i\mu_1 v, \\ 0 &= \langle u_{0,x}, v - u_0 \rangle_{\mathcal{L}_2} = \langle iu_0, v - u_0 \rangle_{\mathcal{L}_2}. \end{aligned} \quad (46)$$

The motion on the group can be reconstructed by integration from $\gamma_{1,t} = \mu_1$, $\gamma_1(0) = 0$, $\gamma_{2,t} = \mu_2$, $\gamma_2(0) = 0$. Note that in the complex formulation of (46) the inner product should be read as

$$\langle u_1 + iu_2, v_1 + iv_2 \rangle_{\mathcal{L}_2} = \langle u_1, v_1 \rangle_{\mathcal{L}_2} + \langle u_2, v_2 \rangle_{\mathcal{L}_2}.$$

Solving (46) on $\Omega = [-40, 40]$ with Neumann b.c. and starting with the same initial conditions as for the original PDE the solution rapidly stabilizes at the desired profile while the algebraic variables μ_1, μ_2 converge to the values for rotational and translational velocities, see Figure 5 and compare Figure 1(b).

In two space dimensions we use for freezing the subgroup $SE(2)$ rather than $S^1 \times SE(2)$ (see the remarks in Section 3.5). Therefore, we solve the following PDAE on the same computational domain and with the same initial values as in Section 2.2.

$$\begin{aligned} v_t &= \alpha \Delta v + (\delta + \beta|v|^2 + \rho|v|^4)v + \mu_3(x_2 D_1 v - x_1 D_2 v) + \mu_1 D_1 v + \mu_2 D_2 v \\ 0 &= \langle x_2 D_1 u_0 - x_1 D_2 u_0, v - u_0 \rangle_{\mathcal{L}_2} = \langle D_1 u_0, v - u_0 \rangle_{\mathcal{L}_2} = \langle D_2 u_0, v - u_0 \rangle_{\mathcal{L}_2}. \end{aligned} \quad (47)$$

Figure 6 displays the corresponding solution which becomes stationary in contrast to the non-frozen solution shown in Figure 2(b). The parameter μ_3 converges to the angular velocity of the soliton while the translational velocities μ_1 and μ_2 converge to zero (as they should).

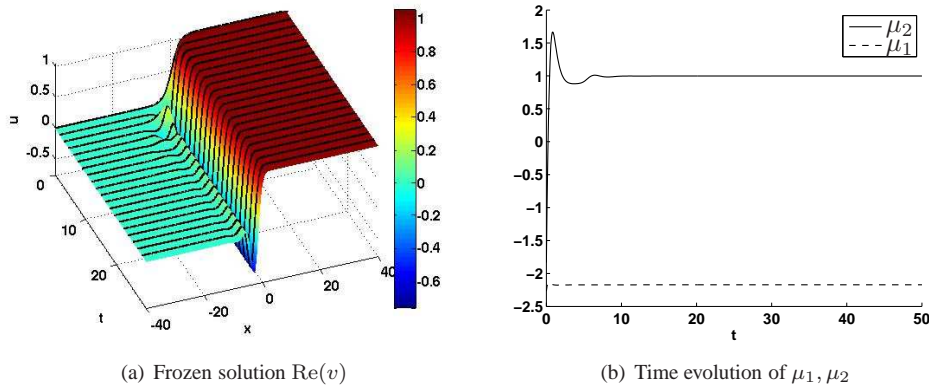


Fig. 5 (online colour at: www.gamm-mitteilungen.org) Quintic Ginzburg-Landau: frozen phase-rotating front in one space dimension.

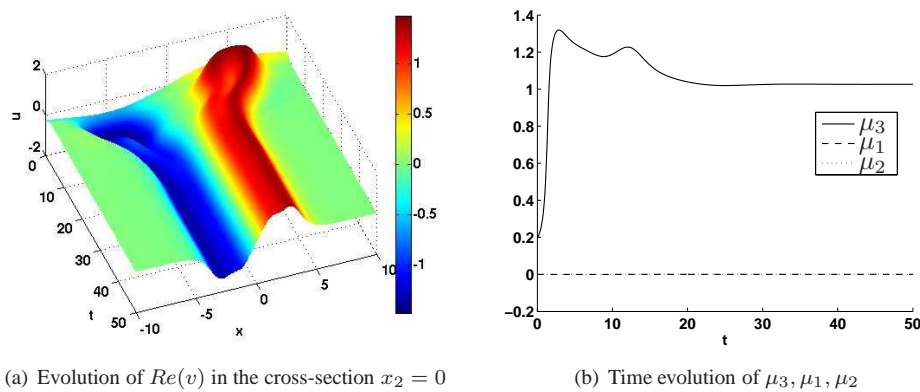


Fig. 6 (online colour at: www.gamm-mitteilungen.org) Quintic Ginzburg-Landau: frozen spinning soliton in two space dimensions.

In three space dimensions we solve a PDAE that has six additional symmetry terms

$$\sum_{j=1}^3 \mu_j D_j v + \mu_4 (x_3 D_2 v - x_2 D_3 v) + \mu_5 (x_1 D_3 v - x_3 D_1 v) + \mu_6 (x_2 D_1 v - x_1 D_2 v)$$

and a corresponding number of phase conditions which we do not write down in detail. In fact, in this case it is convenient to solve the reconstruction equation on $SE(3)$ in terms of quaternions which form a double covering of $SO(3)$ (see [20],[6] for detailed results). Figure 7(a) shows the isosurface of the initial condition that was also used for the direct simulation in Section 2.3. With the same boundary conditions as in Section 2.3 the freezing is successful and stabilizes the isosurface shown in Figure 3(a). The true time dependence of the freezing process can only be seen in a movie, but Figure 7(b) gives an impression of the behavior by looking at the evolution in time of $\text{Re}(v)$ in the cross-section $x_1 = x_3 = 0$.

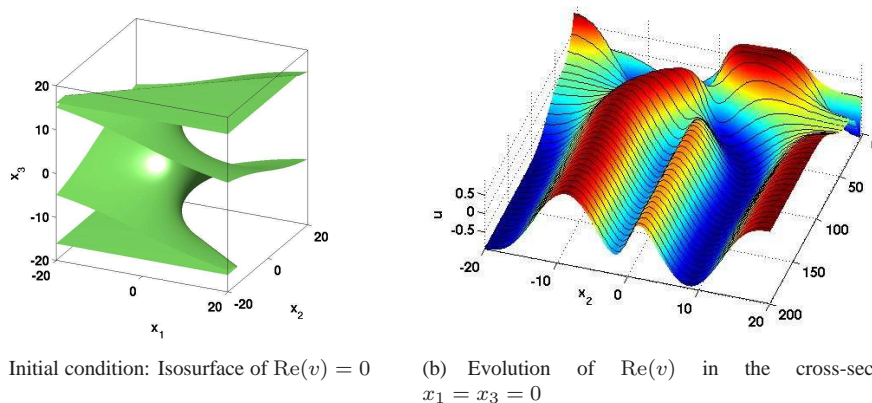


Fig. 7 (online colour at: www.gamm-mitteilungen.org) Quintic Ginzburg-Landau: frozen scroll wave in three space dimensions

5 Conclusions and Perspectives

In this paper we reviewed some recent developments in the numerical and analytical treatment of time dependent PDEs that have continuous symmetries and that are posed on a spatially unbounded domain. Particular emphasis was put on semilinear reaction diffusion equations in \mathbb{R}^d which show a variety of dynamic patterns such as traveling and phase-rotating waves in one, spinning solitons and spiral waves in two and scroll waves in three space dimensions.

These patterns can be identified as relative equilibria when writing the PDE as an abstract evolution equation that is equivariant with respect to the action of a Lie group.

Two closely related issues were discussed for these relative equilibria. First we considered the property of stability with asymptotic phase and the problem of deriving this property from linearized or spectral stability. Usually, the differential operator obtained by linearizing about the relative equilibrium has as many eigenvalues on the imaginary axis as the dimension of the Lie group. The problem then is to prove stability with asymptotic phase by using information on further isolated eigenvalues and on the essential spectrum. This is well known for traveling waves and in the abstract setting there is a general principle of reducing the dynamics to an exponentially attracting center manifold. Further, we provided a stability theorem for localized rotating patterns in \mathbb{R}^2 . An important open problem in this area is the stability of nonlocalized spiral and scroll waves for which the essential spectrum touches the imaginary axis at infinitely many points.

Our second topic was the freezing method which numerically splits shape and group dynamics (as it is done in the stability proof for relative equilibria) for solutions of the general Cauchy problem. The given PDE is transformed into a PDAE where the extra constraints (phase conditions) are derived from minimization principles. Solving the PDAE numerically (on a bounded domain and with space-time discretization) allows to obtain solutions that converge to the unknown pattern and to obtain algebraic values that converge to the velocities on the group. This approach was demonstrated for several applications to systems of

FitzHugh-Nagumo and Ginzburg-Landau type. The method is particularly effective for solutions that converge to a relative equilibrium that is stable with asymptotic phase. Essentially, our approach transforms such a relative equilibrium into a steady state of a PDAE such that it becomes asymptotically stable in the classical Liapunov sense.

The question arises whether this can be proved for the continuous equation or, even more important, for numerical approximations. For traveling waves (and more general relative equilibria in one space dimension) this has been achieved in the papers [37], [39],[38]. In [39] it is shown that a finite difference discretization of (25) on a sufficiently large interval has a unique solution that approximates the relative equilibrium to a certain order. Moreover, if the stability conditions hold for the traveling wave (see (SC) and (EC) in Section 3.5) then the approximate relative equilibrium is asymptotically stable for a full space-time discretization of the PDAE (see (45) for the FitzHugh-Nagumo case) with rates uniform in the discretization parameters (see [38]). These results also hold for more general group actions but are essentially limited to the one-dimensional case. So far, there are no corresponding results for the higher dimensional case $d \geq 2$, such as the two-dimensional rotating patterns in Section 3.5.

It may be no surprise that the freezing method works numerically for other and more general equations than (1).

For example, it can be used for freezing viscous shock waves of conservation laws (see [32])

$$u_t + f(u)_x = Au_{xx}, \quad x \in \mathbb{R}, \quad u(x, 0) = u_0(x).$$

Stability proofs for strong shocks are quite delicate (cf. [24],[43],[7]), since the essential spectrum of the linearized operator touches the imaginary axis. However, some prestudies indicate that stability can be transferred from the PDE to the PDAE formulation in certain situations [31].

The freezing method seems to even retain its favorable properties in situations where the symmetry in (1) is broken. This occurs, for example, with stochastic PDEs. Numerical experiments show that even in this case the freezing ansatz leads to reasonable results (cf.[25]).

Finally, we mention multiple pulses and multiple fronts that frequently occur in systems of type (1). If these travel at different speeds and interact strongly or repel each other then it is clearly impossible to set up a common moving frame in which all waves become stationary (see [10] for such a case in the FitzHugh-Nagumo equation). In the recent paper [9] we managed to extend the freezing method to cope with such multifronts and multipulses. Essentially we write the multipulse as a superposition of single pulses each of which has its own coordinate system and requires its own phase condition. The vector field is decomposed by a dynamic partition of unity and the nonlinear interactions are fully retained. The stability analysis of this procedure is work in progress. However, the generalization of this 'decompose and freeze' approach to higher space dimensions is wide open.

Acknowledgements The authors would like to thank Oliver Gronau for providing the numerical results on scroll waves.

Also the support of the CRC 701 'Spectral Structures and Topological Methods in Mathematics' at Bielefeld University is gratefully acknowledged.

References

- [1] P. Ashwin. Patchwork patterns: dynamics on unbounded domains. In J. Buescu, S. Castro, A. Dias, and I. Labouriau, editors, *Bifurcation, Symmetry, and Patterns*, Trends in Mathematics, pages 67–74. Birkhäuser, 2003.
- [2] D. Barkley. Euclidean symmetry and the dynamics of rotating spiral waves. *Phys. Rev. Lett.*, 72:164–167, 1994.
- [3] D. Barkley. Ez software (ezscroll and ezspiral). http://www.warwick.ac.uk/~masax/Software/ez_software.html, 2008.
- [4] P. W. Bates and C. Jones. Invariant manifolds for semilinear partial differential equations. In H.-O. Walther and U. Kirchgraber, editors, *Dynamics Reported*, volume 2 of *Dynam. Systems Appl.*, pages 1–38. Teubner, 1989.
- [5] W.-J. Beyn, B. Fiedler, and B. Sandstede. Dynamics of patterns, report no. 57/2008. In *Oberwolfach Reports*. European Mathematical Society, 2009.
- [6] W.-J. Beyn, O. Gronau, and V. Thümmeler. Freezing spatio-temporal patterns in three space dimensions by using quaternions. Technical report, Bielefeld University, 2009. (in preparation).
- [7] W.-J. Beyn and J. Lorenz. Stability of viscous profiles: proofs via dichotomies. *J. Dynam. Differential Equations*, 18:141–195, 2006.
- [8] W.-J. Beyn and J. Lorenz. Nonlinear stability of rotating patterns. *Dynamics of PDEs*, 5:349–400, 2009.
- [9] W.-J. Beyn, S. Selle, and V. Thümmeler. Freezing multipulses and multifronts. *SIAM Journal on Applied Dynamical Systems*, 7:577–608, 2008.
- [10] W.-J. Beyn and V. Thümmeler. Freezing solutions of equivariant evolution equations. *SIAM Journal on Applied Dynamical Systems*, 3(2):85–116, 2004.
- [11] W.-J. Beyn and V. Thümmeler. Phase conditions, Symmetries, and PDE Continuation. In B. Krauskopf, H. Osinga, and J. Galan-Vioque, editors, *Numerical Continuation Methods for Dynamical Systems*, Series in Complexity, pages 301–330. Springer, 2007.
- [12] M. Bode, A. W. Liehr, C. P. Schenk, and H.-G. Purwins. Interaction of dissipative solitons: particle-like behaviour of localized structures in a three-component reaction-diffusion system. *Phys. D*, 161:45–66, 2002.
- [13] A. Champneys and B. Sandstede. Numerical computation of coherent structures. In B. Krauskopf, H. Osinga, and J. Galan-Vioque, editors, *Numerical Continuation Methods for Dynamical Systems*, pages 331–358. Springer, 2007.
- [14] P. Chossat and R. Lauterbach. *Methods in equivariant bifurcations and dynamical systems*, volume 15 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
- [15] Comsol Multiphysics 3.3, 2007. Comsol Inc., www.comsol.com.
- [16] L.-C. Crasovan, B. A. Malomed, and D. Mihalache. Spinning solitons in cubic-quintic nonlinear media. *Pramana Journal of Physics*, 57(5/6):1041–1059, 2001.
- [17] M. J. Field. *Dynamics and symmetry*, volume 3 of *ICP Advanced Texts in Mathematics*. Imperial College Press, London, 2007.
- [18] R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1:445–466, 1961.
- [19] M. Golubitsky and I. Stewart. *The symmetry perspective*, volume 200 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002.
- [20] O. Gronau. Einfrieren von raumzeitlichen Mustern in dreidimensionalen anregbaren Systemen mit Quaternionen. Master’s thesis, Universität Bielefeld, 2008.
- [21] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer, 1981.
- [22] G. W. Hunt, M. A. Peletier, A. R. Champneys, P. D. Woods, M. A. Wadee, C. J. Budd, and G. J. Lord. Cellular buckling in long structures. *Nonlinear Dynam.*, 21(1):3–29, 2000.

- [23] J. P. Keener and J. J. Tyson. The dynamics of scroll waves in excitable media. *SIAM Rev.*, 34:1–39, 1992.
- [24] H.-O. Kreiss and G. Kreiss. Stability of systems of viscous conservation laws. *Commun. Pure Appl. Math.*, 51:1397–1424, 1998.
- [25] G. Lord and V. Thümmeler. Freezing stochastic traveling waves. Technical Report 08-010, CRC 701, Bielefeld University, 2008.
- [26] J. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry (2nd ed.)*, volume 17 of *Texts in Applied Mathematics*. Springer, 1999.
- [27] J. E. Marsden and J. Scheurle. Pattern evocation and geometric phases in mechanical systems with symmetry. *Dynam. Stability Systems*, 10(4):315–338, 1995.
- [28] A. Mielke. The Ginzburg-Landau equation in its role as a modulation equation. In B. Fiedler, editor, *Handbook of dynamical systems, Vol. 2*, pages 759–834. North-Holland, Amsterdam, 2002.
- [29] R. M. Miura. Accurate computation of the stable solitary waves for the FitzHugh-Nagumo equations. *Journal of Mathematical Biology*, 13:247–269, 1982.
- [30] J. D. Murray. *Mathematical Biology*, volume 19 of *Biomathematics*. Springer-Verlag, Berlin, second edition, 1993.
- [31] J. Rottmann-Matthes. Spectral analysis of coupled hyperbolic-parabolic systems on finite and infinite intervals. In *Hyperbolic Problems: Theory, Numerics, Applications (Conference Lyon 2006)*, pages 901–909, 2008.
- [32] C. W. Rowley, I. G. Kevrekidis, J. E. Marsden, and K. Lust. Reduction and reconstruction for self-similar dynamical systems. *Nonlinearity*, 16(4):1257–1275, 2003.
- [33] B. Sandstede. Stability of traveling waves. In B. Fiedler, editor, *Handbook of Dynamical Systems*, volume 2, pages 983–1055. North Holland, 2002.
- [34] B. Sandstede and A. Scheel. Absolute versus convective instability of spiral waves. *Phys. Rev. E*, 7708-7714, 2000.
- [35] B. Sandstede, A. Scheel, and C. Wulff. Dynamics of spiral waves on unbounded domains using center-manifold reductions. *J. Diff. Eqns.*, 141:122–149, 1997.
- [36] O. Thual and S. Fauve. Localized structures generated by subcritical instabilities. *J. Phys. France*, 49:1829–1833, 1988.
- [37] V. Thümmeler. *Numerical Analysis of the method of freezing traveling waves*. PhD thesis, Bielefeld University, 2005.
- [38] V. Thümmeler. The effect of freezing and discretization to the asymptotic stability of relative equilibria. *J. Dynam. Differential Equations*, 20:425–477, 2008.
- [39] V. Thümmeler. Numerical approximation of relative equilibria for equivariant PDEs. *SIAM J. Numer. Anal.*, 46:2978–3005, 2008.
- [40] W. van Saarloos and P. C. Hohenberg. Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. *Phys. D*, 56(4):303–367, 1992.
- [41] A. Volpert, V. A. Volpert, and V. A. Volpert. *Traveling wave solutions of parabolic systems*, volume 140 of *Translations of Mathematical Monographs*. AMS, 1994.
- [42] A. T. Winfree. *The Geometry of Biological Time*. Springer, 1990.
- [43] K. Zumbrun and P. Howard. Pointwise semigroup methods and stability of viscous conservation laws. *Indiana Univ. Math. J.*, 47:741–871, 1998.
- [44] V. Zykov. *Simulaton of wave processes in excitable media*. Manchester University Press, 1987.