Spectral properties of mixed hyperbolic-parabolic systems

Diplomarbeit

vorgelegt von: Jens Rottmann
Betreuer: Prof. Dr. W.-J. Beyn

Universität Bielefeld
Fakultät für Mathematik
im Mai 2005

Contents

Notations

$L(E,F)$	Vector-space of bounded linear operators from the Banach space E to the Banach space F.
$\mathcal{C}^k$	Vector-space of k-times differentiable mappings.
$\mathcal{R}(P)$	Range of the operator P.
$\mathcal{N}(P)$	The null space of the operator P.
v^*	Complex conjugate and transposed of a vector $v \in \mathbb{C}^l$.
$\langle u, v\rangle$	Euclidean inner product of vectors u and $v \in \mathbb{C}^l$.
$\lvert v\rvert$	Euclidean norm of a vector $v \in \mathbb{C}^l$.
$\lvert v\rvert_\infty$	Maximum norm of a vector $v \in \mathbb{C}^l$.
$\lvert M\rvert$	Matrix norm to the Euclidean vector norm.
$\Vert M\Vert_\infty$	Sup norm for a matrix-valued function $M \in \mathcal{C}(J)$.
dist	Hausdorff semi-distance.
$\lvert u\rvert_\Gamma^2$	Boundary norm of $u \in \mathcal{C}([x_-, x_+])$ given by $\lvert u\rvert_\Gamma^2 := \lvert u(x_-)\rvert^2 + \lvert u(x_+)\rvert^2$.
$\Vert u\Vert = \Vert u\Vert_{L_2}$	L_2-norm of a function u.
$(u,v)_{L_2}$	L_2-inner product for functions u and v.
$\lvert J\rvert$	Length of the compact interval J.
E^*	Dual space of the Banach space E.
$\langle f, v\rangle$	Evaluation of $f \in E^*$ at $v \in E$.
codim(U)	If U is a subspace of V the codimension of U in V is denoted by codim(U).
ind(P)	Fredholm index of the operator P.
$\rho(P)$	Resolvent set of the operator P.
$\sigma_{ess}(P)$	Essential spectrum of the operator P.
$\sigma_{eig}(P)$	Eigenspectrum of the operator P.
$\sigma_p(P)$	Point-spectrum of the operator P.
$\sigma_\Delta(P)$	$\sigma_\Delta(P) := \sigma(P) \setminus \sigma_p(P)$.

1 Introduction

The aim of this thesis is to analyze the spectral properties of linear partial differential operators on the whole real line and how the properties are related to the same operator 'restricted' to a finite interval.

In many applied problems in biology, physics or chemistry there arise travelling waves as solutions of systems of partial differential equations PDE of the form

$$U_t = f(U, U_x, U_{xx}) \quad \text{in } [0, \infty) \times \mathbb{R}, \tag{1.1}$$

for example see [Mur93] or [KS94]. A solution U of (1.1) is called a **travelling wave solution** if it satisfies

$$U(t, x) = W(x - ct) \tag{1.2}$$

for some $c \in \mathbb{R}$ and some function W. One calls c the **speed** and W the **profile** of the wave. The travelling wave is called a **pulse** if the limits

$$\lim_{x \to \infty} W(x) \quad \text{and} \quad \lim_{x \to -\infty} W(x)$$

exist and are equal.

A travelling wave has the special property that it is constant if one looks at it in a moving frame. More precisely this means that the function

$$\tilde{U}(t, x) := U(t, x + ct) = W(x)$$

is constant in time and therefore the function $\tilde{U}$ is a steady state of the transformed PDE

$$\begin{aligned} 0 = &\tilde{U}_t(t, x) = \frac{d}{dt} U(t, x + ct) = U_t(t, x + ct) + cU_x(t, x + ct) \\ = &f(\tilde{U}(t, x), \tilde{U}_x(t, x), \tilde{U}_{xx}(t, x)) + c\tilde{U}_x(t, x) \quad \text{in } [0, \infty) \times \mathbb{R}. \end{aligned} \tag{1.3}$$

In [KKP94] the asymptotic stability of a steady state of equation (1.3) is deduced from spectral properties of the linearization* of that equation at the steady state. The linearization of (1.3) at a steady state $\tilde{U}$ reads

$$\tilde{V}_t = f_U(\tilde{U}, \tilde{U}_x, \tilde{U}_{xx})\tilde{V} + f_{U_x}(\tilde{U}, \tilde{U}_x, \tilde{U}_{xx})\tilde{V}_x + f_{U_{xx}}(\tilde{U}, \tilde{U}_x, \tilde{U}_{xx})\tilde{V}_{xx} + c\tilde{V}_x =: P\tilde{V}, \tag{1.4}$$

where the indices denote the partial derivatives of f with respect to the corresponding component.

*See [KL89, Chapter 1] for linearization of partial differential equations.

More precisely in [KKP94, Chapter 6] the authors show with the help of resolvent estimates and under some assumptions on the structure of P the following "linearized stability implies nonlinear stability" result.

If zero is a simple eigenvalue of P *and the rest of the spectrum is strictly to the left of the imaginary axis, solutions* $\tilde{V} = \tilde{U} + \delta U$ *of* (1.3)*, with* δU *a small perturbation, converge to some shift* $\tilde{W}(x) = W(x + x_0)$ *of the steady state* $\tilde{U}(t,x) = W(x)$ *as* $t \to \infty$*.*

For the original system this means that initial values close to the travelling wave solution U converge to some shifted version of the travelling wave. This is known as 'stability with asymptotic phase', for example see [VVV94, Chapter 5].

Remark. *In the case that* (1.4) *is obtained by linearization about a travelling wave solution there always is a zero eigenvalue. This is caused by the shift invariance of equation* (1.3)*.*

If $\tilde{U}(t,x) = W(x)$ *is a solution of* (1.3) *all its shifts* $\tilde{U}_\alpha(t,x) = W(x+\alpha)$ *are also solutions of* (1.3)*. This shows that for every* $c \in \mathbb{R}$ *the equality*

$$0 = f\big(W(x+c), W_x(x+c), W_{xx}(x+c)\big) + cW_x(x+c)$$

holds such that differentiation with respect to c *and evaluation at* $c = 0$ *leads to*

$$0 = f_U(W,W_x,W_{xx})(W_x) + f_{U_x}(W,W_x,W_{xx})(W_x)_x + f_{U_{xx}}(W,W_x,W_{xx})(W_x)_{xx} + c(W_x)_x.$$

This shows that for a non constant wave profile W *the derivative* $W_x \neq 0$ *is an eigenfunction of the operator* P*.*

As indicated above the 'stability with asymptotic phase' of a travelling wave solution is closely related to spectral properties of the operator linearized at the wave-profile.

In [BL99] the case that P from (1.4) is a parabolic operator is analyzed. The authors show that if one restricts the operator P to a finite interval and uses 'suitable' boundary conditions, the spectral properties of the all line operator P are preserved by this finite interval approximation. This is important if one for example wants to decide numerically for the 'stability with asymptotic phase' of a travelling wave.

We extend the results from [BL99] to strictly hyperbolic and mixed hyperbolic-parabolic systems in this thesis and we improve some of the resolvent estimates from [KKP94]. We also give an algebraic criterion whether the considered artificial boundary conditions are 'suitable'.

A main starting point besides the article [BL99] were personal notes by J. Lorenz [Lor99] which mainly consist of a guide of what to do and were very helpful for the analysis in Section 3.2.

We now give a brief overview of the contents of this thesis.

In Chapter 3 we assume that the linearized equation is strictly hyperbolic and in Chapter 4 we consider mixed hyperbolic-parabolic systems.

The structure of Chapters 3 and 4 is quite similar:

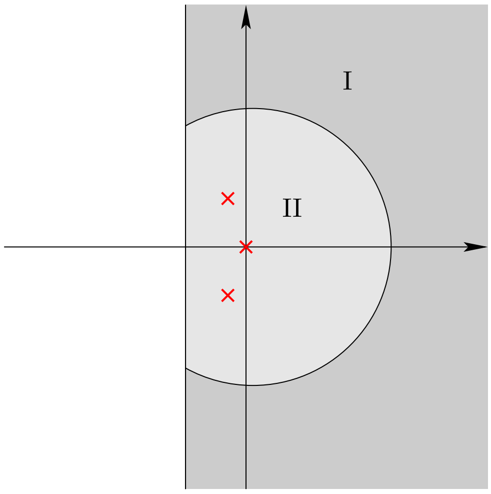


Figure 1.1: Regions in the complex plane. We will see that for the operators we consider there is no spectrum of the all-line operator and its restriction to finite intervals in Region I. Furthermore there are only isolated eigenvalues of the all-line operator in Region II.

After some prelimary considerations we first analyze in Sections 3.2 and 4.4 the behavior of the spectrum in region I of Figure 1.1. In Section 4.4 we combine the results we obtain in the analysis of the hyperbolic case (Section 3.2) for this region with the results from [BL99, Chapter 2] in order to obtain similar results in the mixed case.

In the second part we then analyze in Sections 3.3 and 4.5 the spectral behavior in compact regions of the complex plane.

In the last part of the spectral analysis we show in Sections 3.4 and 4.6 the approximation of isolated eigenvalues of the operator P in the right half-plane.

For the analysis in the second and third part we make use of a generalization of the theory of discrete approximations to directed sets. This generalization is presented in Chapter 2 and might be useful also in other applications.

We use the rather abstract theory since it appeared during retracing the proofs in [BL99, Chapter 4] that the main problem is to show that for every sequence of solutions of the approximative problems a subsequence converges to a solution of the all line problem which lead to 'regular convergence'. We use the ideas of the proofs in [BL99, Chapter 4] to show that the finite interval approximations of the all line boundary value problem 'regularly converge' to the all line problem. The resulting, and for the analysis in Chapters 3 and 4 basic Theorem 2.29 is given in Section 2.5.

This abstraction simplifies and harmonizes the analysis of the spectral properties in compact regions of the complex plane and of the eigenvalue approximations of isolated eigenvalues in the hyperbolic and the mixed case.

By adaption of a result from [Bey80] we are also able to allow for so called 's-dependent' boundary conditions for the approximation of simple eigenvalues. These boundary conditions should lead to better approximations of the eigenvalues and allow for shorter intervals.

We finish the thesis with an analysis of the sufficient conditions obtained for the artificial boundary operators. We show that certain natural choices namely periodic boundary conditions and characteristic boundary conditions satisfy the requirements for the resolvent estimates from Chapter 3 and Chapter 4. This is done in Section 5.1 for the hyperbolic case and in Section 5.2 for the mixed hyperbolic-parabolic case. Finally we numerically test our theoretical results at the FitzHugh-Nagumo system in Section 5.3 which is of the form analyzed in Chapter 4.

In the Appendices we collect and show basic results which are used throughout the thesis. In particular in Appendix A we show a perturbation Lemma (Lemma A.7) which is essential for the analysis in Section 3.2. In Appendix B we introduce and review some results from the theory of exponential dichotomies which basic for most of the analysis in Chapters 3 and 4.

At this place I would like to thank W.-J. Beyn for this interesting task that lead into a fascinating area of applied mathematics. I am very grateful for his very good support and his interest in my advancements. I also would like to thank V. Thümmler for her providence of numerical data.

2 Discrete approximations

In this chapter we introduce the concept of discrete approximations. The concept was developed in the 1970s for example in [Stu71a, Stu71b, Stu72], [Gri75a, Gri75b, Gri76], and [Vai77a, Vai77b]. We will need a generalization of the theory to the case when the index set is not the set of integers but more generally a directed set. The aim is to treat the approximation of boundary value problems on the infinite line by finite boundary value problems.

Therefore we also have to consider nets and subnets instead of sequences and subsequences. In Section 2.1 we give the exact definitions and basic results which will be used in this thesis. The next Section 2.2 is concerned with the review and reformulation of some results to our setting. The most important results in this chapter which are essential for the application in Chapter 3 and Chapter 4 are Theorem 2.26 presented in Section 2.3, Theorem 2.28 from Section 2.4, and Theorem 2.29 in Section 2.5.

2.1 The language of discrete approximations

As already mentioned in the introduction we will present in this section some basics of the theory of directed sets, nets, and discrete approximations. Especially the theory of nets is streamlined for our demands and differs in some points from the usual theory. More general definitions and further results on nets and subnets as well as some enlighting basic examples can be found in texts on general topology for example see [Wil70, §11] and [Kel75, Chapter 2]. Our presentation of the theory of discrete approximations follows the book of G. Vainikko [Vai76].

Definition 2.1. A **directed set** $(H, \succ)$ is a nonempty set H together with a relation $\succ$, called **direction**, that has the properties

(D1) for all $J \in H$ holds $J \succ J$,

(D2) for all $J_1, J_2, J_3 \in H$ with $J_1 \succ J_2,\ J_2 \succ J_3$ holds $J_1 \succ J_3$,

(D3) for all $J_1, J_2 \in H$ exists $J_3 \in H$ with $J_3 \succ J_1, J_2$.

Notice the lack of antisymmetry so that a directed set is not necessarily a partially ordered set. We will simply write H for the directed set $(H, \succ)$ if the direction $\succ$ is clear from the context.

Example 1. *1. The integers with the usual order forms a directed set* $(\mathbb{N}, \geq)$*.*

2. The pair $(\{1\}, \succ)$ *with* $1 \succ 1$ *is a directed set.*

3. *The set $H := \{J : J = [a,b]; a \leq b \in \mathbb{R}\}$ with the direction $J_1 \succ J_2 :\Leftrightarrow J_1 \supset J_2$ is a directed set. This is the archetype of a directed set we will always have in mind since it will be the directed set we consider in Chapters 3 and 4.*

4. *The set $H_\zeta := \mathbb{R} \setminus \{\zeta\}$ together with the direction $x \succ y :\Leftrightarrow |\zeta - x| \leq |\zeta - y|$ is a directed set. This is an example of a directed set which is not partially ordered.*

Definition 2.2. A nonempty subset $H' \subset H$ is called a **cofinal subset** of H if and only if for every $J \in H$ there is an element $J' \in H'$ with $J' \succ J$.

It is clear that H' together with the relation $\succ$ restricted to H' is a directed set again. By subsets $H', H'', \ldots$ of a directed set H we will always mean cofinal subsets of H together with the direction of H restricted to the subset.

Definition 2.3. Let $(H, \succ_H)$ and $(I, \succ_I)$ be two directed sets. We call a mapping $\phi : I \to H$

- **cofinal** if and only if for every $J \in H$ there is an $i_0 \in I$ with $\phi(i) \succ_H J$ for all $i \in I$ with $i \succ_I i_0$,
- **monotone** if and only if $i_1, i_2 \in I$ with $i_1 \succ_I i_2$ implies $\phi(i_1) \succ_H \phi(i_2)$,
- **strictly monotone** if and only if for all $i_1, i_2 \in I$ ($i_1 \succ_I i_2$ and $i_2 \not\succ_I i_1$) implies ($\phi(i_1) \succ_H \phi(i_2)$ and $\phi(i_2) \not\succ_H \phi(i_1)$).

Now we can define the notion of a cofinal sequence in a directed set. Most of the results from [Vai76] will be transferred to the setting of directed sets considered here by using cofinal or strictly monotone cofinal sequences which are defined next.

Definition 2.4. Let $(H, \succ)$ be a directed set and consider $\mathbb{N}$ as a directed set as in example 1. A **cofinal sequence** in H is a cofinal map $\phi : (\mathbb{N}, \geq) \to (H, \succ)$. The cofinal sequence is called **monotone** respectively **strictly monotone** if the map ϕ is monotone respectively strictly monotone. By setting $J_n := \phi(n)$ we will simply write $(J_n)_{n \in \mathbb{N}}$ for the cofinal sequence ϕ.

The most important property of the natural numbers we need to imitate for a directed set in order to adapt the proofs of [Vai76] will be formulated in the next definition.

Definition 2.5. A directed set H is called (sequentially) **unbounded** if there is a strictly monotone cofinal sequence in H.

Example (Example 1 continued). *One easily sees that $(\{1\}, \succ)$ is not unbounded but the other examples are. A strictly monotone cofinal sequence for the third example is given by $J_n = [-n, n]$, $n \in \mathbb{N}$.*

Lemma 2.6. *Let H be an unbounded directed set. Then every cofinal subset H' of H is also an unbounded directed set.*

Proof. The property of H' being directed simply follows from the cofinality and the properties of the direction. To see the unboundedness of H' let $(J_n)_{n\in\mathbb{N}}$ be any strictly monotone cofinal sequence in H. Take any $J'_0 \in H'$. Assume that elements $J'_0, \ldots, J'_n$ in H' with $J'_i \succ J'_{i-1}$ and $J_{i-1} \not\succ J_i$ for $i = 1, \ldots, n$ are constructed. Then there is $n_0 \in \mathbb{N}$ with $J_m \succ J'_n$ for all $m \geq n_0$. Since H' is cofinal there is an element $J'_{n+1} \in H'$ with $J'_{n+1} \succ J_{n_0+1}$. It follows from (D2) $J'_{n+1} \succ J'_n$ and $J'_n \not\succ J'_{n+1}$. Therefore the sequence $(J'_n)_{n\in\mathbb{N}}$ in H' constructed in this way is strictly monotone and cofinal and so H' is unbounded. □

After these basic definitions about directed sets we are able to introduce our notion of nets and convergence of nets.

Definition 2.7. Let H be a directed set and assume that for every $J \in H$ a complex Banach space X_J is given. A family of elements $(z_J)_{J\in H}$ with $z_J \in X_J$ for all $J \in H$ is called a **net** in (X_J).

If $H' \subset H$ is a cofinal subset of H then we call the net $(z_J)_{J\in H'}$ a **subnet** of $(z_J)_{J\in H}$.

REMARK 2.8. *If $(J_n)_{n\in\mathbb{N}}$ is a cofinal sequence in an unbounded directed set H we identify the sequence $(z_{J_n})_{n\in\mathbb{N}}$ with the subnet $(z_J)_{J\in H'}$ where $H' \subset H$ is the image of the sequence $(J_n)_{n\in\mathbb{N}}$. By subsets $\mathbb{N}', \mathbb{N}'', \ldots$ of $\mathbb{N}$ we always mean infinite subsets of $\mathbb{N}$ and then $(z_{J_n})_{n\in\mathbb{N}'}$ is a cofinal subsequence of the cofinal sequence $(z_{J_n})_{n\in\mathbb{N}}$.*

Definition 2.9. A net $(z_J)_{J\in H}$ in (X_J) is called bounded if and only if there is an index $J_0 \in H$ and a $C > 0$ so that $\|z_J\|_{X_J} \leq C$ for all $J \succ J_0$.

Definition 2.10. If $(z_J)_{J\in H}$ is a net in a (constant) Banach space X, i.e. $z_J \in X \ \forall J \in H$, we say z_J **converges** to $z \in X$ if and only if for every $\varepsilon > 0$ there is a $J_0 \in H$ with $\|z_J - z\| \leq \varepsilon \ \forall J \succ J_0$. We will simply write this as $z_J \to z \ (J \in H)$.

We say $z \in X$ is a **cluster point** of the net $(z_J)_{J\in H}$ iff some subnet $(z_J)_{J\in H'}$ of (z_J) converges to z.

When we consider subnets we will write $z_J \to z \ (J \in H')$ as a shorthand for the convergence of the subnet $(z_J)_{J\in H'}$ of $(z_J)_{J\in H}$ to the element z.

From now on we always assume that the index set H is an arbitrary unbounded directed set. Most of the definitions will also make sense if H is not unbounded but some of the Theorems and Lemmata will not hold if we drop the condition of unboundedness. Furthermore we assume that E, F, and for every index $J \in H$ also E_J and F_J will denote separable complex Banach spaces with norms $\|\cdot\|_*$, $* \in \{E, F, E_J, F_J\}$. Sometimes we simply write $\|\cdot\|$ or $\|\cdot\|_J$ if it is clear from the context which norm is meant. The lower index J then indicates the corresponding index of the spaces E_J and F_J.

By $\mathcal{P}$ and $\mathcal{Q}$ we denote nets of linear continuous operators $\mathcal{P} := (p_J)_{J\in H}$ and $\mathcal{Q} := (q_J)_{J\in H}$ with $p_J \in L(E, E_J)$ and $q_J \in L(F, F_J)$. Here and in the sequel $L(X, Y)$ always denotes the set of linear bounded operators from the Banach space X to the Banach space Y. We require

$$\|p_J z\|_{E_J} \to \|z\|_E \ (J \in H) \quad \text{for every } z \in E, \tag{2.1}$$

$$\|q_J r\|_{F_J} \to \|r\|_F \ (J \in H) \quad \text{for every } r \in F. \tag{2.2}$$

Lemma 2.11. *The nets $\mathcal{P}$ and $\mathcal{Q}$ are bounded.*

Proof. Assume $\mathcal{P}$ is unbounded. Then there is a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H with $\|p_{J_n}\|_{L(E,E_{J_n})} \geq n$. Now $(p_{J_n})_{n\in\mathbb{N}}$ satisfies

$$\sup_{n\in\mathbb{N}} \|p_{J_n} z\|_{E_{J_n}} < \infty \text{ for all } z \in E$$

because of the convergence $\lim_{n\to\infty} \|p_{J_n} z\|_{E_{J_n}} = \|z\|_E$. The Theorem of Banach-Steinhaus (see [Alt99, 5.3] or [Yos78, p. 73]) therefore implies

$$\sup_{n\in\mathbb{N}} \|p_{J_n}\|_{L(E,E_{J_n})} < \infty$$

what contradicts the assumption. □

Notice that the proof of the Theorem of Banach-Steinhaus presented in [Alt99, 5.3] does not use the image space of the operators but uses that the family of functions $f_{p_{J_n}} \in \mathcal{C}^0(E,\mathbb{R})$ defined by $f_{p_{J_n}}(z) = \|p_{J_n}(z)\|_{E_{J_n}}$ is point-wise bounded and so the theorem can be applied in the setting considered here.

Definition 2.12 (cf. [Vai76, §1]). Let $\mathcal{P}$ be as above. A net $(z_J)_{J\in H}$, $z_J \in E_J$, is called

$\mathcal{P}$-convergent to $z \in E$ if and only if $\|z_J - p_J z\|_{E_J} \to 0$ $(J \in H)$. We abbreviate this by writing $z_J \xrightarrow{\mathcal{P}} z$ $(J \in H)$ and call z the $\mathcal{P}$-limit (or simply the limit) of the net $(z_J)_{J\in H}$.

$\mathcal{P}$-compact if and only if for every subnet $(z_J)_{J\in H'}$, $H' \subset H$, there is a subnet $H'' \subset H'$ and a $z \in E$ with $z_J \xrightarrow{\mathcal{P}} z$ $(J \in H'')$.

Moreover every point $z \in E$ so that there is a subnet $(z_J)_{J\in H'}$ of $(z_J)_{J\in H}$ with $z_J \xrightarrow{\mathcal{P}} z$ $(J \in H')$ is called a **cluster-point** of $(z_J)_{J\in H}$.

The definition directly implies that for every $z \in E$ the net $(p_J z)_{J\in H}$ in (E_J) $\mathcal{P}$-converges to z.

Remark (Remark 2.8 continued). *The unboundedness condition in Remark 2.8 cannot be dropped since otherwise the notion of a $\mathcal{P}$-convergent subnet is not well-defined. For example consider $(H,\succ)$ with $H = \{a,b\}$ and $a \succ b$, $b \succ a$, $a \succ a$, $b \succ b$, and the net $z_a = 0 \in \mathbb{R}$ and $z_b = 1 \in \mathbb{R}$. Then the net is not convergent. Consider the cofinal sequence*

$$J_0 = a, \text{ and } J_n = b \;\forall n \geq 1.$$

Then the sequence $(z_{J_n})_{n\in\mathbb{N}}$ is convergent but the subnet under the identification from above is not. The next Lemma shows that this cannot happen if we have an unbounded directed set H.

Lemma 2.13. *Let H be an unbounded directed set and let $(z_J)_{J\in H}$ be a net in (E_J). Then for every cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H the following equivalence holds.*
The sequence $(z_{J_n})_{n\in\mathbb{N}}$ $\mathcal{P}$-converges to $z \in E$ if and only if
the subnet $(z_J)_{J\in H'}$ where $H' = \{J \in H : J = J_n \text{ for some } n \in \mathbb{N}\}$, $\mathcal{P}$-converges to $z \in E$.

Proof. "⇒" Let $(J'_n)_{n\in\mathbb{N}}$ be a strictly monotone cofinal sequence in H. Let $\varepsilon > 0$ be given. Then by assumption there is $n_0 \in \mathbb{N}$ with $\|z_{J_n} - p_{J_n}(z)\|_J \le \varepsilon$ for all $n \ge n_0$. By cofinality of the sequence $(J'_n)_{n\in\mathbb{N}}$ there is $m \in \mathbb{N}$ with $J'_m \succ J_n$ for all $n < n_0$ and by cofinality of the sequence $(J_n)_{n\in\mathbb{N}}$ there is $N_0 \in \mathbb{N}$ with $J_n \succ J'_{m+1}$ for all $n \ge N_0$. Because $J'_m \not\succ J'_{m+1}$ it follows

$$\|z_{J_n} - p_{J_n} z\|_{J_n} \le \varepsilon \; \forall J_n \in H' \text{ with } J_n \succ J_{N_0}.$$

"⇐" Let $\varepsilon > 0$ be given. By definition there is $J^0 \in H'$ with $\|z_J - p_J z\|_J \le \varepsilon$ for all $J \in H'$ with $J \succ J^0$. Because of the cofinality of $(J_n)_{n\in\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $J_n \succ J^0$ for all $n \ge n_0$ and so

$$\|z_{J_n} - p_{J_n} z\|_{J_n} \le \varepsilon \; \forall n \ge n_0.$$

□

The proof shows that the backward implication holds for every cofinal sequence, but the forward implication needs the unboundedness of the underlying directed set H.

Lemma 2.14. *Every $\mathcal{P}$-compact net $(z_J)_{J\in H}$ is bounded.*

Note that this in particular states that every $\mathcal{P}$-convergent net is bounded.

Proof. Assume that $(z_J)_{J\in H}$ is an unbounded $\mathcal{P}$-compact net. Then there must be a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ with $\|z_{J_n}\|_{E_{J_n}} \ge n$. Since $(z_{J_n})_{n\in\mathbb{N}}$ is a subnet of $(z_J)_{J\in H}$ (recall that we assume H is unbounded) there is a subnet $\mathbb{N}' \subset \mathbb{N}$ and an element $z \in E$ with $z_{J_n} \xrightarrow{\mathcal{P}} z$ $(n \in \mathbb{N}')$. Hence the sequence $(z_{J_n})_{n\in\mathbb{N}'}$ must be bounded because of property (2.1). This contradicts the assumption. □

Lemma 2.15. *The $\mathcal{P}$-limit of a $\mathcal{P}$-convergent net $(z_J)_{J\in H}$ in E_J is unique.*

Proof. Let $z_1, z_2 \in E$ be $\mathcal{P}$-limits of the net (z_J). Then

$$\|p_J(z_1 - z_2)\|_J = \|(z_J - p_J z_1) - (z_J - p_J z_2)\|_J \le \|z_J - p_J z_1\|_J + \|z_J - p_J z_2\|_J.$$

By property (2.1) this implies $\|p_J(z_1 - z_2)\|_J \to 0$ $(J \in H)$ and so

$$0 = \|z_1 - z_2\|_E.$$

□

Example 2. *In this example we take* $(\mathbb{N}, \geq)$ *as* $(H, \succ)$ *and denote by* E *the Banach-space* l_1 *of summable sequences* $x = (x_0, x_1, \dots) \in \mathbb{R}^{\mathbb{N}}$ *with the norm* $\|x\|_E = \sum_{n=0}^{\infty} |x_n|$*. Denote by* E_n *the space of* $n+1$*-vectors* $x = (x_0, \dots, x_n)$ *with the norm* $\|x\|_{E_n} = \sum_{i=0}^{n} |x_n|$*. Furthermore denote by* $p_n \in L(E, E_n)$ *the operator* $p_n : E \ni x = (x_0, x_1, \dots) \mapsto (x_0, \dots, x_n) \in E_n$*. Then* (2.1) *is obviously satisfied.*

Let $\mathcal{A} \in L(E, F)$ and for every $J \in H$ let $\mathcal{A}_J \in L(E_J, F_J)$, be given. The situation is illustrated in the diagram to the right which in general does not commute.

$$\begin{array}{ccc} z \in E & \xrightarrow{\mathcal{A}} & F \ni r \\ p_J \downarrow & & \downarrow q_J \\ z_J \in E_J & \xrightarrow[\mathcal{A}_J]{} & F_J \ni r_J \end{array}$$

Definition 2.16 (cf. [Vai76, §2])**.** A net $(\mathcal{A}_J)_{J \in H}$ of linear continuous operators $\mathcal{A}_J \in L(E_J, F_J)$ is called

- $\mathcal{PQ}$**-convergent** to an operator $\mathcal{A} \in L(E, F)$ if and only if for every $\mathcal{P}$-convergent net $(z_J)_{J \in H}$, $z_J \xrightarrow{\mathcal{P}} z$ $(J \in H)$, holds $\mathcal{A}_J z_J \xrightarrow{\mathcal{Q}} \mathcal{A}z$ $(J \in H)$. We write this as $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$.
- $\mathcal{PQ}$**-regularly convergent** to an operator $\mathcal{A} \in L(E, F)$ if and only if $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ and in addition the following regularity condition holds:

  Every bounded net $(z_J)_{J \in H}$ for which the net $(\mathcal{A}_J z_J)_{J \in H}$ is $\mathcal{Q}$-compact, is $\mathcal{P}$-compact. We simply write this as $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ regularly.
- $\mathcal{PQ}$**-stably convergent** to an operator $\mathcal{A} \in L(E, F)$ if and only if there is an index $J_0 \in H$ so that for all $J \succ J_0$ the inverse operator $\mathcal{A}_J^{-1} \in L(F_J, E_J)$ exists and satisfies $\|\mathcal{A}_J^{-1}\|_{L(F_J, E_J)} \leq \text{const}$ with a constant independent of J. This is again abbreviated to $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ stably.
- In the special case where $F_J = F = \mathbb{C}$ and $q_J = id_{\mathbb{C}}$ $\forall J \in H$ we say that $f_J \in E_J^*$ **weakly** $\mathcal{P}$**-converges** to $f \in E^*$ and write $f_J \overset{\mathcal{P}}{\rightharpoonup} f$ iff $f_J \xrightarrow{\mathcal{PQ}} f$ $(J \in H)$. Here E^* denotes the dual space of E.

The following lemma is obvious and we omit the proof.

Lemma 2.17. *If a net of linear continuous operators* $(\mathcal{A}_J)_{J \in H}$ *is (regularly/stably)-*$\mathcal{PQ}$*-convergent then every subnet* $(\mathcal{A}_J)_{J \in H'}$*,* $H' \subset H$*, has the same property.*

Lemma 2.18 (cf. [Vai76, §2 (8)])**.** *With the notation from above* $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ *if and only if the net* $(\mathcal{A}_J)_{J \in H}$ *is bounded and for all* $z \in E$ *holds* $\mathcal{A}_J p_J z \xrightarrow{\mathcal{Q}} \mathcal{A}z$*.*

Proof. First we show the necessity. The property $\mathcal{A}_J p_J z \xrightarrow{\mathcal{Q}} \mathcal{A}z$ $\forall z \in E$ obviously follows from the $\mathcal{PQ}$-convergence and therefore it remains to show the boundedness of $(\mathcal{A}_J)_{J \in H}$. Assume the net $(\mathcal{A}_J)_{J \in H}$ is not bounded and let $(J_n)_{n \in \mathbb{N}}$ be a cofinal sequence in H. Then for every $n \in \mathbb{N}$ there is J'_n in H with $J'_n \succ J_n$ so that there exists a $z_n \in E_{J'_n}$ with $\|z_n\|_{E_{J'_n}} = 1$ and $\|\mathcal{A}_{J'_n} z_n\|_{F_{J'_n}} \geq n + 1$. For every $n \in \mathbb{N}$

define $z'_n := \frac{z_n}{\|\mathcal{A}_{J'_n} z_n\|_{F_{J'_n}}}$. Then $z'_n \xrightarrow{\mathcal{P}} 0$ ($n \in \mathbb{N}$). From the assumption of $\mathcal{PQ}$-convergence and Lemma 2.17 follows $\mathcal{A}_{J'_n} z'_n \xrightarrow{\mathcal{Q}} 0$ ($n \in \mathbb{N}$) but $\|\mathcal{A}_{J'_n} z'_n\|_{F_{J'_n}} = 1$ for all $n \in \mathbb{N}$ which contradicts (2.2).

Second we show the sufficiency. Assume $z_J \xrightarrow{\mathcal{P}} z$ ($J \in H$). Then by the triangle inequality we obtain

$$\begin{aligned}\|\mathcal{A}_J z_J - q_J \mathcal{A} z\| &\leq \|\mathcal{A}_J(z_J - p_J z)\| + \|\mathcal{A}_J p_J z - q_J \mathcal{A} z\| \\ &\leq \|\mathcal{A}_J\|_{L(E_J,F_J)} \|z_J - p_J z\|_{E_J} + \|\mathcal{A}_J p_J z - q_J \mathcal{A} z\|.\end{aligned}$$

The second summand converges to zero by assumption and the first summand converges to zero because the net $(\mathcal{A}_J)_{J\in H}$ is bounded and $z_J \xrightarrow{\mathcal{P}} z$ ($J \in H$). □

Example (Example 2 continued). *Let $F_n = E_n$, $F = E$, and $\mathcal{Q} = \mathcal{P}$. Denote by I_n the identity map on E_n and by I the identity map on E. It is easy to see $I_n \xrightarrow{\mathcal{PQ}} I$ regularly and stably.*

Consider the operators

$$\mathcal{A}_n(x_0, \ldots, x_n) = (\frac{x_0}{n+1}, \ldots, \frac{x_n}{n+1}),\ \mathcal{A}(x_0, x_1, \ldots) = (0, 0, \ldots).$$

Then $\mathcal{A}_n \xrightarrow{\mathcal{PQ}} \mathcal{A}$, but the convergence is neither regular nor stable.

Let $n_0 \in \mathbb{N}$ be fixed and consider the operators $\mathcal{A}_n$ which are arbitrarily defined for $0 \leq n \leq n_0 - 1$ and for $n \geq n_0$ they shall be defined by

$$\mathcal{A}_n(x_0, \ldots, x_n) = (0, \ldots, 0, x_{n_0}, \ldots, x_n) \text{ and } \mathcal{A} = (x_0, x_1, \ldots) = (0, \ldots, 0, x_{n_0}, \ldots),$$

i.e. the first n_0 elements are mapped to zero. Then $\mathcal{A}_n \xrightarrow{\mathcal{PQ}} \mathcal{A}$ regularly, but not stably. The regularity follows from the compactness of the closed unit ball in $\mathbb{R}^{n_0}$.

Finally consider the operators

$$\mathcal{A}_n = (x_0, \ldots, x_n) = (x_n, x_0, \ldots, x_{n-1}) \text{ and } \mathcal{A} = (x_0, x_1, \ldots) = (0, x_0, x_1, \ldots).$$

Then $\mathcal{A}_n \xrightarrow{\mathcal{PQ}} \mathcal{A}$ stably, but not regularly, as one observes by looking at the sequence $z_n = (0, \ldots, 0, 1)$ whose image under $\mathcal{A}_n$ is $\mathcal{Q}$-compact, but the sequence $(z_n)_{n\in\mathbb{N}}$ is not $\mathcal{P}$-compact.

2.2 Review of basic results

In this section we state some results presented in [Vai76] which we adapt to the setting here and which will be used in the sequel.

Lemma 2.19 (cf. [Vai76, §1 (12)]). *Let $(u_1, \ldots, u_k)$ be a basis of a subspace $U \subset E$ and assume for $j = 1, \ldots, k$ there are given nets $(u_{j,J})_{J\in H}$ with*

$$u_{j,J} \xrightarrow{\mathcal{P}} u_j\ (J \in H).$$

Moreover let $(u_J)_{J\in H}$ *be a bounded net in* (E_J) *of the form* $u_J = \sum_j \alpha_{j,J} u_{j,J}$ *with* $\alpha_{j,J} \in \mathbb{C}$ *for all* $j \in \{1,\dots,k\}$ *and for all* $J \in H$. *Then there is* $J_0 \in H$ *and* $C \geq 0$ *with*

$$\sum_j |\alpha_{j,J}| \leq C \;\forall J \in H,\; J \succ J_0.$$

Proof. Assume the assertion is false. Then as in the proof of Lemma 2.18 there is a cofinal sequence $(J_n)_{n\in H}$ with $\sum_j |\alpha_{j,J_n}| \geq n+1$. Define a sequence $(\tilde{u}_{J_n})_{n\in\mathbb{N}}$ in (E_{J_n}) by $\tilde{u}_{J_n} = \sum_j \beta_{j,n} u_{j,J_n}$ where $\beta_{j,n} := \frac{\alpha_{j,J_n}}{\sum_i |\alpha_{i,J_n}|}$. Then there is a subsequence $\mathbb{N}' \subset \mathbb{N}$ and scalars $\beta_1,\dots,\beta_k \in \mathbb{C}$ with $\beta_{j,n} \to \beta_j$ $(n \in \mathbb{N}')$. Define $\tilde{u} := \sum_j \beta_j u_j$. By construction holds $\tilde{u}_{J_n} \xrightarrow{\mathcal{P}} \tilde{u}$ $(n \in \mathbb{N}')$ as well as $\|\tilde{u}_{J_n}\| \to 0$ $(n \in \mathbb{N}')$ which implies $\|\tilde{u}\| = 0$ by property (2.1). But the construction of the $\beta_1,\dots,\beta_k$ also shows $\sum_{j=1}^k |\beta_j| = 1$ and therefore the linear independency of $u_1,\dots u_k$ in E yields $\tilde{u} := \sum_j \beta_j u_j \neq 0$. This contradicts $\|\tilde{u}\| = 0$. □

We will use the previous lemma to show that the images of linearly independent elements in E under p_J are still linearly independent for sufficiently 'large' indices J.

Let $u_1,\dots,u_k$ and $u_{1,J},\dots,u_{k,J}$ be given as in Lemma 2.19 and define for all $j = 1,\dots,k$ and all $J \in H$ the subspaces

$$\begin{aligned} G_{j,J} &:= \operatorname{span}\{u_{1,J},\dots,u_{j-1,J},u_{j+1,J},\dots,u_{k,J}\},\\ G_J &:= \operatorname{span}\{u_{1,J},\dots,u_{k,J}\},\\ G_j &:= \operatorname{span}\{u_1,\dots,u_{j-1},u_{j+1},\dots,u_k\},\\ G &:= \operatorname{span}\{u_1,\dots,u_k\}. \end{aligned}$$

Lemma 2.20 (cf. [Vai76, §1 (13)]). *Under the assumptions and notations from above there is* $J_0 \in H$ *so that*

$$d_{j,J} := \operatorname{dist}(u_{j,J}, G_{j,J}) \geq \frac{1}{2}\operatorname{dist}(u_j, G_j) =: \frac{1}{2} d_j \quad \forall J \succ J_0,\; j = 1,\dots,k.$$

Proof. Assume the lemma does not hold. Then there is a $j \in \{1,\dots,k\}$ and a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H such that

$$\|u_{j,J_n} - \sum_{i\neq j} \alpha_{i,J_n} u_{i,J_n}\| = d_{j,J_n} < \frac{1}{2} d_j \;\forall n \in \mathbb{N}.$$

From Lemma 2.19 follows that the sequence $\sum_{i\neq j} |\alpha_{i,J_n}|$ is bounded. Hence there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and scalars $\alpha_i \in \mathbb{C}$, $i \neq j$, with $\alpha_{i,J_n} \to \alpha_i$ $(n \in$

$\mathbb{N}'), i \neq j$. This implies

$$\begin{aligned}\frac{d_j}{2} \geq d_{j,n} &= \|u_{j,J_n} - \sum_{i\neq j} \alpha_{i,J_n} u_{i,J_n}\| \\ &\geq \|p_{J_n}(u_j - \sum_{i\neq j} \alpha_i u_i)\| - \|p_{J_n}(\sum_{i\neq j}(\alpha_{i,J_n} - \alpha_i) u_i)\| \\ &\quad - \|u_{j,J_n} - \sum_{i\neq j} \alpha_{i,J_n} u_{i,J_n} - p_{J_n}(u_j - \sum_{i\neq j} \alpha_{i,J_n} u_i)\| \\ &\xrightarrow[\substack{n\to\infty \\ n\in\mathbb{N}'}]{} \|u_j - \sum_{i\neq j} \alpha_i u_i\| \geq d_j,\end{aligned}$$

where we used the $\mathcal{P}$-convergence $u_{l,J_n} \xrightarrow{\mathcal{P}} u_l$ $(n \in \mathbb{N}')$ for all $l = 1,\ldots,k$, and the boundedness of the net $\mathcal{P}$ (see Lemma 2.11). This convergence contradicts the assumption. ☐

In Sections 2.3 and 2.4 we will considerably make use of the following Lemma.

Lemma 2.21 (cf. [Vai76, §2 (60)])**.** *Assume $\mathcal{A}_J \in L(E_J, F_J)$ for all $J \in H$, $\mathcal{A} \in L(E,F)$, $\mathcal{N}(\mathcal{A}) = \{0\}$, and $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ regularly. Furthermore assume there is an index $J_0 \in H$ so that $\mathcal{A}_J$ is Fredholm of index zero for all $J \in H$ with $J \succ J_0$. Then $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ stably.*

Proof. Contrary to the assertion assume there is a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H so that for every $n \in \mathbb{N}$ there is an element $v_n \in E_{J_n}$ with $\|v_n\|_{E_{J_n}} = 1$ and $\|\mathcal{A}_{J_n} v_n\|_{F_{J_n}} \leq \frac{1}{n+1}$. Then the sequence $(\mathcal{A}_{J_n} v_n)_{n\in\mathbb{N}}$ is $\mathcal{Q}$-compact and thus by regularity (cf. Lemma 2.17) there exist $\mathbb{N}' \subset \mathbb{N}$ and $v \in E$ with $v_n \xrightarrow{\mathcal{P}} v$ $(n \in \mathbb{N}')$. Then $\|v\|_E = 1$ and from the assumption follows $\mathcal{A}v \neq 0$.
This contradicts $\mathcal{A}_{J_n} v_n \xrightarrow{\mathcal{Q}} 0$ $(n \in \mathbb{N}')$ and $\mathcal{A}_{J_n} v_n \xrightarrow{\mathcal{Q}} \mathcal{A}v$ $(n \in \mathbb{N}')$. ☐

Similarly we can adapt the proof of [Vai76, §1 (37)] and show the following lemma. Note that in the proof the unboundedness of H is indispensable since it uses a kind of diagonal sequence argument, which is not possible if we drop the unboundedness assumption. Recall that we assume that all spaces are separable.

Lemma 2.22. *Let $f \in E^*$ then there is a net of linear functionals $(f_J)_{J\in H}$ in (E_J^*), with the properties*

$$f_J \stackrel{\mathcal{P}}{\rightharpoonup} f \ (J \in H) \text{ and } \|f_J\|_{E_J^*} \to \|f\|_{E^*}.$$

Proof. Let $f \in E^*$ be arbitrary. Without loss of generality $f \neq 0$. Let $\{x^0, x^1, \ldots\}$ be linearly independent elements in E with $\operatorname{span}\{x^0, x^1, \ldots\}$ being dense in E. The set $\{x^0, \ldots\}$ may also be finite. Let $X^k := \operatorname{span}\{x^0, \ldots, x^k\}$ and set $x_J^k := p_J x^k$. Then $x_J^k \xrightarrow{\mathcal{P}} x^k$ and by Lemma 2.20 for every $k \in \mathbb{N}$ there exists $J_k \in H$ so that $\{x_J^0, \ldots, x_J^k\}$ are linearly independent in E_J for all $J \succ J_k$.

For $J \succ J_k$ define f_J^k on $X_J^k := \operatorname{span}\{x_J^0, \ldots, x_J^k\}$ by

$$\langle f_J^k, x_J^i \rangle := \langle f, x^i \rangle, \ i = 0, \ldots, k.$$

Then it holds

$$\sup_{\substack{x_J \in X_J^k \\ \|x_J\|=1}} |\langle f_J^k, x_J\rangle| \to \sup_{\substack{x \in X^k \\ \|x\|=1}} |\langle f, x\rangle| \le \sup_{\substack{x \in E \\ \|x\|=1}} |\langle f, x\rangle| \ (J \in H).$$

For every $J \in H$ we continue f_J^k to the whole space E_J with preserving its norm by applying the Hahn-Banach Theorem (cf. [Alt99, 4.15]). So for every $\varepsilon > 0$ and every $k \in \mathbb{N}$ we find a $J_\varepsilon^k \in H$ with $J_\varepsilon^k \succ J_k$ and with

$$\|f_J^k\| \le \|f\| + \varepsilon \ \forall J \succ J_\varepsilon^k, J \in H.$$

By the unboundedness of H we can find a strictly monotone cofinal sequence $(J'_n)_{n\in\mathbb{N}}$ with $J'_n \succ J^n_{\frac{1}{n+1}}$ $\forall n \in \mathbb{N}$, i.e. with

$$\|f_J^n\| \le \|f\| + \frac{1}{n+1} \ \forall J \succ J'_n, \ J \in H.$$

Now define f_J by

$$f_J = \begin{cases} 0, & \text{if } J \not\succ J'_0, \\ f_J^k, & \text{if } J \succ J'_k \text{ and } J \not\succ J'_{k+1}, \ k = 1, 2, \dots \end{cases}$$

then for every $\varepsilon_0 > 0$ there is a $\bar{J}_{\varepsilon_0} \in H$ with

$$\|f_J\| \le \|f\| + \varepsilon_0 \ \forall J \succ \bar{J}_{\varepsilon_0}.$$

Assume $x_J \xrightarrow{\mathcal{P}} x$ and let $\varepsilon > 0$ be given. Let $k > 0$ be so large so that there exist $\alpha_0, \dots, \alpha_k \in \mathbb{C}$ with $\|x - \tilde{x}\| \le \varepsilon$, where $\tilde{x} = \sum_{j=0}^k \alpha_j x^j$, which is possible by the separability assumption. Choose $\bar{J} \succ J'_k, \bar{J}_\varepsilon$ with $\|x_J - p_J x\| \le \varepsilon$ and $\|p_J(x - \tilde{x})\| \le 2\varepsilon$ for all $J \succ \bar{J}$. For all $J \succ \bar{J}$ one obtains because of

$$\langle f_J, p_J \tilde{x}\rangle = \langle f, \tilde{x}\rangle \ \forall J \succ J'_k$$

the estimates

$$\begin{aligned} |\langle f_J, x_J\rangle - \langle f, x\rangle| &\le |\langle f_J, x_J - p_J x\rangle| + |\langle f_J, p_J(x - \tilde{x})\rangle| + |\langle f_J, p_J \tilde{x}\rangle - \langle f, x\rangle| \\ &\le \|f_J\|\varepsilon + \|f_J\| 2\varepsilon + \|f\|\|\tilde{x} - x\| \\ &\le (\|f\| + \varepsilon)\varepsilon + (\|f\| + \varepsilon)2\varepsilon + \|f\|\varepsilon \le (4\|f\| + \varepsilon)\varepsilon. \end{aligned}$$

This shows $f_J \overset{\mathcal{P}}{\rightharpoonup} f$ since $\varepsilon > 0$ was arbitrary.

We have already seen that for every $\varepsilon > 0$ there is $J^0 \in H$ with

$$\|f_J\|_{E_J^*} \le \|f\|_{E^*} + \varepsilon \ \forall J \succ J^0.$$

Let $\varepsilon > 0$ be given. It exists $x \in E$ with $\|x\| = 1$ and

$$|\langle f, x\rangle| \ge \|f\| - \frac{\varepsilon}{3}.$$

By the computations from above there is $J'_\varepsilon \in H$ with

$$|\langle f_J, p_J x\rangle - \langle f, x\rangle| \le \frac{\varepsilon}{3} \;\forall J \succ J'_\varepsilon.$$

Thus $|\langle f_J, p_J x\rangle| \ge \|f\| - 2\frac{\varepsilon}{3}$ for all $J \succ J'_\varepsilon$. Choose $J^1 \in H$ with $J^1 \succ J'_\varepsilon$ so that

$$\|p_J x\| \le 1 + \frac{\varepsilon}{3\|f\| - 3\varepsilon} \;\forall J \in H \text{ with } J \succ J^1$$

which is possible since $\|p_J x\| \to 1$ $(J \in H)$. Now for all $J \succ J^1$ holds

$$\|f_J\|_{E_J^*} \ge \|f\|_{E^*} - \varepsilon.$$

This finishes the proof. □

The next result will be needed in the next section and also in Chapter 3. We do not give the proof.

Lemma 2.23. *Assume families of linear operators $\mathcal{P}$ and $\mathcal{Q}$ as above and furthermore $\mathcal{O}$ with $o_J \in L(G, G_J)$ with the same properties as $\mathcal{P}$ and $\mathcal{Q}$.*
Then $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ (regularly/stably), $\mathcal{B}_J \xrightarrow{\mathcal{QO}} \mathcal{B}$ (regularly/stably) implies

$$\mathcal{B}_J\mathcal{A}_J \xrightarrow{\mathcal{PO}} \mathcal{BA} \text{ (regularly/stably).}$$

The benefit of the previous lemma for the next section will lie in the combination with the next lemma.

Lemma 2.24. *If $\mathcal{A}_J \xrightarrow{\mathcal{PQ}} \mathcal{A}$ regularly and stably and if exists $\mathcal{A}^{-1} \in L(F, E)$ then there is $J_0 \in H$ such that*

$$\mathcal{A}_J^{-1} \xrightarrow{\mathcal{QP}} \mathcal{A}^{-1} \; (J \in H') \;\text{ regularly and stably,}$$

where $H' := \{J \in H : J \succ J_0\}$.

Proof. Let $(u_J)_{J\in H}$ be a net in F_J with $u_J \xrightarrow{\mathcal{Q}} u$ $(J \in H')$. Then the stability assumption and Lemma 2.14 imply that $(\mathcal{A}_J^{-1}u_J)_{J\in H'}$ is a bounded net. Obviously $(\mathcal{A}_J(\mathcal{A}_J^{-1}u_J))_{J\in H'}$ is $\mathcal{Q}$-compact. Thus by the regularity assumption $(\mathcal{A}_J^{-1}u_J)_{J\in H'}$ is $\mathcal{P}$-compact. Let $v := \mathcal{A}^{-1}u$ and assume $(\mathcal{A}_J^{-1}u_J)_{J\in H'}$ does not converge to v. Then there is a cofinal subset $H'' \subset H'$ and $\eta > 0$ with $\|\mathcal{A}_J^{-1}u_J - p_J v\| \ge \eta$ for all $J \in H''$. By compactness there is a convergent subnet $\mathcal{A}_J^{-1}u_J \xrightarrow{\mathcal{P}} \tilde{v}$ $(J \in H''')$ with $H''' \subset H''$. But then

$$\mathcal{A}_J(\mathcal{A}_J^{-1}u_J) \to \mathcal{A}\tilde{v} \; (J \in H''')$$

and

$$\mathcal{A}_J(\mathcal{A}_J^{-1}u_J) \to u \; (J \in H'''),$$

thus $\tilde{v} = \mathcal{A}^{-1}u$ and this shows the $\mathcal{QP}$-convergence of $\mathcal{A}_J^{-1}$ to $\mathcal{A}^{-1}$.

The stable convergence follows directly from the boundedness of the net $(\mathcal{A}_J)_{J\in H}$.

The regularity is shown as follows: Let $(u_J)_{J\in H'}$ be a bounded net in (F_J) and assume that $(\mathcal{A}_J^{-1}u_J)_{J\in H'}$ is $\mathcal{P}$-compact. Then $(u_J)_{J\in H'}$ is $\mathcal{Q}$-compact since $u_J = \mathcal{A}_J(\mathcal{A}_J^{-1}u_J)$. □

2.3 Generalized eigenvalue problems

The main result in this section is Theorem 2.26 which basically is a reformulation of [Vai76, §4 Konvergenzsatz(62)]. It is already presented in a similar version in [Vai77a, §7], but the proof is only indicated there.

We assume that H is an unbounded directed set and we denote its elements by J. We use the same notations and assumptions on $\mathcal{P}$, $\mathcal{Q}$, E_J, F_J, E, and F as in sections 2.1 and 2.2. To simplify notation we do not distinguish in notation between the different norms and write $\|\cdot\|$ since it is obvious from the context which norm is used. We also write I_J for the identity map on E_J and I for the identity map on E. Finally dist stands for the usual Hausdorff semi distance of sets in a Banach space, defined by $\operatorname{dist}(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$ and in the case that $a \in A$ one defines $\operatorname{dist}(a,B) = \inf_{b\in B}\|a-b\|$.

Theorem 2.25 (cf. [Vai76, §4 (62)])**.** *Let $T \in L(E,E)$ and for every $J \in H$ let $T_J \in L(E_J,E_J)$. Consider the operator-valued functions $\mathcal{A}_J(\mu) := \mu I_J - T_J \in L(E_J,E_J)$ for all $J \in H$ and $\mathcal{A}(\mu) := \mu I - T \in L(E,E)$ for all $\mu \in \mathbb{C}$. Let $\Lambda \subset \mathbb{C}$ be an open domain in $\mathbb{C}$ and assume that the operators $\mathcal{A}(\mu)$ and for each $J \in H$ the operators $\mathcal{A}_J(\mu)$ are Fredholm of index zero for all $\mu \in \Lambda$. Furthermore assume $\mathcal{A}_J(\mu) \xrightarrow{\mathcal{PP}} \mathcal{A}(\mu)$ $(J \in H)$ regularly for all $\mu \in \Lambda$ and that $\sigma(T)\cap\Lambda$ only consists of isolated points. Then the following properties hold true.*

(i) For every $\mu_0 \in \sigma(T)\cap\Lambda$ there is a net $(\mu_J)_{J\in H}$ and an element $J_0 \in H$ such that $\mu_J \in \sigma(T_J)$ for all $J \in H$ with $J \succ J_0$ and μ_J converges to μ_0.

(ii) If $\mu_0 \in \Lambda$ is a clusterpoint of a net $(\mu_J)_{J\in H}$ with $\mu_J \in \sigma(T_J)$ $\forall J \in H, J \succ J_0$, for some $J_0 \in H$ then it holds $\mu_0 \in \sigma(T)$.

(iii) If $(\mu_J)_{J\in H}$ is a net in $\mathbb{C}$ and $(v_J)_{J\in H}$ is a net in (E_J) with $\mu_J \to \mu_0$ $(J \in H)$ and $\|v_J\|_{E_J} = 1$, so that there is $J_0 \in H$ with $\mu_J \in \sigma(T_J)$ for all $J \succ J_0$ and with $\mathcal{A}_J(\mu_J)v_J = 0$. Then the net $(v_J)_{J\in H}$ is $\mathcal{P}$-compact and every cluster-point $v_0 \in E$ is a normalized eigenfunction of T to the eigenvalue μ_0.

(iv) For the eigenvalues and eigenfunctions from (iii) the following estimates hold

(a) $|\mu_J - \mu_0| \le C\tilde{\epsilon}_J^{\frac{1}{\kappa}}$,

(b) $\inf_{v_0\in\mathcal{N}(\mathcal{A}(\mu_0))}\|v_J - p_J v_0\|_{E_J} \le C\tilde{\epsilon}_J^{\frac{1}{\kappa}}$,

where κ is the smallest integer so that $\mathcal{N}\left((\mu_0 I - T)^\kappa\right) = \bigcup_{j=0}^\infty \mathcal{N}\left((\mu_0 I - T)^j\right)$ and $\tilde{\epsilon}_J$ is given at the end of the Theorem.

(v) Let $\delta > 0$ be so small that $\overline{K_\delta(\mu_0)} \subset \Lambda$ and $\overline{K_\delta(\mu_0)} \cap \sigma(T) = \{\mu_0\}$. Denote by $W_J := W_J(T_J,\mu_0)$ the linear hull of all generalized eigenspaces of T_J to eigenvalues $\mu_J \in \sigma(T_J)\cap\overline{K_\delta(\mu_0)}$ and denote by $W := W(T,\mu_0)$ the generalized eigenspace of T to the eigenvalue μ_0. Then there is a $J_0 \in H$ so that for all $J \in H$ with $J \succ J_0$ hold

(c) $\dim W(T, \mu_0) = \dim W(T_J, \mu_0) < \infty$,

(d) $\vartheta(W, W_J) := \sup_{\substack{v\in W \\ \|v\|=1}} \operatorname{dist}(p_J(v), W_J) \leq C\tilde{\epsilon}_J$,

(e) $\vartheta(W_J, W) := \sup_{\substack{v_J\in W_J \\ \|v_J\|=1}} \operatorname{dist}(v_J, p_J(W)) \leq C\tilde{\epsilon}_J$.

Moreover constants in (iv) and (v) are independent of J for $J \succ J_0$ with a suitable $J_0 \in H$ and $\tilde{\epsilon}_J$ in (iv) and (v) is given by

$$\tilde{\epsilon}_J := \max_{\substack{v\in W \\ \|v\|=1}} \|T_J p_J v - p_J T v\|.$$

Remark. *Note that κ in (iv) is finite because of the Fredholm properties (cf. [Vai76, §4 Satz(26)]).*

We do not give the proof here but rather indicate what must be done in case one wants to follow the proof in [Vai76].

Indication of a proof. One has to be careful in several points. First one needs the unboundedness of H since in one step of the proof (cf. [Vai76, §4 (18)]) Lemma 2.22 is needed. Second the measure of non-compactness has to be generalized for our setting. It is used for the proof of [Vai76, §4 (55)] which is a crucial part in the proof of (v). We have used the following definition: For every net $(z_J)_{J\in H}$ in (E_J) the measure of non-compactness is defined as

$$\mu(z_J) := \inf\{\varepsilon > 0 : \forall \text{ cofinal } H' \subset H \ \exists \text{ cofinal } H'' \subset H', z' \in E \text{ so that } \|z_J - p_J z'\| \leq \varepsilon \ (J \in H'')\}.$$

It is clear that μ has the properties [Vai76, §2 (72)-(77)], where one uses the unboundedness of H for the proof of [Vai76, §2 (73)] because of a diagonal sequence argument. In the proof one also needs the result of [Vai76, §2 (78)] in the easier setting of [Vai76, §2 (79)] (it also holds in the case of unbounded directed sets, but we omit its proof) and so the proof of [Vai76, §4 (55)] can be adapted to our setting. The remaining steps in the proof of [Vai76, §4 (62)] do not cause any trouble if one tries to prove them in the setting of directed sets. □

The next Theorem is the main result in this section. It will be proved by application of Theorem 2.25.

Theorem 2.26 (cf. [Vai77a, §7 (89)])**.** *Assume an unbounded directed set H and families of operators $\mathcal{P}$ and $\mathcal{Q}$ as above. Let A, $B \in L(E, F)$, and A_J, $B_J \in L(E_J, F_J)$ for all $J \in H$. Denote by $\mathcal{A}$ the operator-valued function $\mathcal{A}(s) = sB - A \in L(E, F)$ and by $\mathcal{A}_J$ the function $\mathcal{A}_J(s) = sB_J - A_J \in L(E_J, F_J)$. Let Σ' be an open and bounded domain in $\mathbb{C}$ and let Σ be an open connected neighborhood of the closure $\overline{\Sigma'}$ of Σ'. Assume that for all $s \in \Sigma$ the operators $\mathcal{A}(s)$ and $\mathcal{A}_J(s)$ are Fredholm of index zero and assume there is $\underline{s} \in \rho(\mathcal{A}) \cap \Sigma \setminus \overline{\Sigma'}$.* Finally assume that for every $s \in \Sigma$ the operators $\mathcal{A}_J(s)$ regularly $\mathcal{PQ}$-converge to $\mathcal{A}(s)$. Then the following properties hold.*

*For the definition of $\rho(\mathcal{A})$ and $\sigma(\mathcal{A})$ see Definition C.6.

(1) *For every* $s_0 \in \sigma(\mathcal{A}) \cap \Sigma'$ *there is a net* $(s_J)_{J\in H}$ *and* $J_0 \in H$ *so that for all* $J \succ J_0$ $s_J \in \sigma(\mathcal{A}_J)$ *and* $s_J \to s_0$ $(J \in H)$.

(2) *If* $(s_J)_{J\in H}$ *is a net so that there is* $J_0 \in H$ *with* $s_J \in \sigma(\mathcal{A}_J)$ *for all* $J \succ J_0$, *then every cluster point of* $(s_J)_{J\in H}$ *which lies in* Σ' *is an eigenvalue of* $\mathcal{A}(\cdot)$.

(3) *Assume* $(s_J)_{J\in H}$ *is a net in* $\mathbb{C}$ *and* $(v_J)_{J\in H}$ *is a net in* (E_J) *so that there is* $J_0 \in H$ *with* s_J *is an eigenvalue of* $\mathcal{A}_J(\cdot)$ *and* v_J *is a normalized eigenelement of* $\mathcal{A}_J(\cdot)$ *to the eigenvalue* s_J *for all* $J \succ J_0$*. *Then the net* $(v_J)_{J\in H}$ *is* $\mathcal{P}$-*compact and every cluster-point* $v_0 \in E$ *is a normalized eigenfunction of* $\mathcal{A}(\cdot)$ *to the eigenvalue* s_0.

(4) *For the nets from (3) the following estimates hold*

(a) $|s_J - s_0| \le C\epsilon_J^{\frac{1}{\kappa}}$,

(b) $\inf_{v_0\in\mathcal{N}(\mathcal{A}(s_0))} \|v_J - p_J v_0\| \le C\epsilon_J^{\frac{1}{\kappa}}$,

where κ *is the largest order of all root-polynomials to the eigenvalue* s_0 *and* ϵ_J *is given at the end of the theorem.*

(5) *Let* $s_0 \in \Sigma' \cap \sigma(\mathcal{A})$ *and* $\varepsilon_0 > 0$ *with* $\overline{K_{\varepsilon_0}(s_0)} \subset \Sigma'$ *and* $\overline{K_{\varepsilon_0}(s_0)} \cap \sigma(\mathcal{A}) = \{s_0\}$. *Let* $\mathcal{W}$ *denote the root-subspace*† *of* $\mathcal{A}$ *to the eigenvalue* s_0 *and let* $\mathcal{W}_J$ *denote the linear hull of all root-subspaces of* $\mathcal{A}_J$ *to eigenvalues* $s_J \in \sigma(\mathcal{A}_J) \cap \overline{K_{\varepsilon_0}(s_0)}$. *Then there is an index* $J_0 \in H$ *so that for all* $J \in H$ *with* $J \succ J_0$ *hold*

(c) $\dim \mathcal{W}_J = \dim \mathcal{W} < \infty$,

(d) $\vartheta(\mathcal{W}, \mathcal{W}_J) := \sup_{\substack{v\in\mathcal{W}\\ \|v\|=1}} \operatorname{dist}(p_J v, \mathcal{W}_J) \le C\epsilon_J$.

(e) $\vartheta(\mathcal{W}_J, \mathcal{W}) := \sup_{\substack{v_J\in\mathcal{W}_J\\ \|v_J\|=1}} \operatorname{dist}(v_J, p_J\mathcal{W}) \le C\epsilon_J$,

The constants in (a), (b), (d), and (e) are independent of J *for* $J \succ J_0$ *with a suitable* $J_0 \in H$. *Finally* ϵ_J *in (4) and (5) is given by*

$$\epsilon_J = \max_{\substack{v,v'\in\mathcal{W}\\ \|v\|=1\\ \mathcal{A}(s_0)v=Bv'}} \|\mathcal{A}_J(s_0)p_J v - B_J p_J v'\|.$$

Before we prove the theorem, we show a characterization of the root-subspaces of $\mathcal{A}(\cdot)$ which will be essential for the application of Theorem 2.25 in the proof of Theorem 2.26.

Lemma 2.27. *Let* $A, B \in L(E, F)$ *and assume* $\mathcal{A}(s) = sB - A \in L(E, F)$ *is Fredholm of index zero for all* $s \in \Sigma$ *with* Σ *an open domain in* $\mathbb{C}$. *Assume there is* $\underline{s} \in \Sigma$ *with* $\mathcal{A}(\underline{s})^{-1} \in L(F, E)$ *and denote by* T *the linear operator*

$$T := \mathcal{A}(\underline{s})^{-1}B \in L(E, E).$$

*See Definition C.6.

†See Definition C.6.

Let $s_0 \neq \underline{s}$ and set $\mu_0 := \frac{1}{\underline{s}-s_0}$. Then the generalized eigenspace of T to the eigenvalue μ_0 given by

$$W(T,\mu_0) := \bigcup_{k\geq 0} \{v \in E : (\mu I - T)^k v = 0\}$$

coincides with the root-subspace of $\mathcal{A}(\cdot)$ to the eigenvalue s_0

$$\mathcal{W}(\mathcal{A}, s_0).$$

Moreover we have the equivalence
$0 \neq v \in E$ with $(\mu_0 I - T)^k v \neq 0$ and $(\mu_0 I - T)^{k+1} v = 0$ for some $k \in \mathbb{N}$
if and only if
there is a sequence $v_0, \dots, v_k$ of nonzero elements in E with $v_k = v$ and

$$\begin{aligned} \mathcal{A}(s_0)v_0 &= 0, \\ \mathcal{A}(s_0)v_{i+1} &= Bv_i,\ i = 0,\dots,k-1. \end{aligned} \tag{2.3}$$

Furthermore for every $v \in W$ there is a unique $v' \in W$ with $\mathcal{A}(s_0)v = Bv'$.

Remark. *Note that [Vai76, §4 (26)] states that the order of the eigenelements of $\mathcal{A}(s_0)$ is bounded.*

Proof. We prove the first part of the lemma by induction. Clearly $0 \in W$ and $0 \in \mathcal{W}$ so without loss of generality assume $0 \neq v \in E$.

It holds $(\mu_0 I - T)v = 0$ if and only if $\mathcal{A}(\underline{s})(I - (\underline{s} - s_0)T)v = \mathcal{A}(s_0)v = 0$. Thus the case $k = 0$ is shown and in particular yields the equality

$$\mathcal{N}(\mu_0 I - T) = \mathcal{N}(\mathcal{A}(s_0)). \tag{2.4}$$

Now show $k-1 \to k$:
Necessity. Assume $(\mu_0 I - T)^k v \neq 0$ and $(\mu_0 I - T)^{k+1} v = 0$. From (2.4) follows $0 \neq (\mu_0 I - T)^k v \in \mathcal{N}(\mathcal{A}(s_0))$. This implies

$$\begin{aligned} 0 = \mathcal{A}(s_0)(\mu_0 I - T)^k v &= \mathcal{A}(s_0) \sum_{i=0}^{k} \binom{k}{i} \mu_0^{k-i} (-T)^i v \\ &= \mu_0^k \mathcal{A}(s_0)v - \mathcal{A}(s_0)T \sum_{i=1}^{k} \binom{k}{i} \mu_0^{k-i} (-T)^{i-1} v. \end{aligned}$$

The equation above is equivalent to

$$\begin{aligned} \mathcal{A}(s_0)v &= \mathcal{A}(s_0)T \sum_{i=1}^{k} \binom{k}{i} \mu_0^{-i} (-T)^{i-1} v \\ &= B(I - (\underline{s} - s_0)T) \sum_{i=1}^{k} \binom{k}{i} \mu_0^{-i} (-T)^{i-1} v \\ &= B(\mu_0 I - T) \sum_{i=1}^{k} \binom{k}{i} \mu_0^{-1-i} (-T)^{i-1} v. \end{aligned}$$

Now define $v_{k-1} := (\mu_0 I - T)\sum_{i=1}^{k}\binom{k}{i}\mu_0^{-1-i}(-T)^{i-1}v$ and note that $(\mu_0 I - T)$ commutes with the sum. Hence v_{k-1} satisfies $(\mu_0 I - T)^k v_{k-1} = 0$ and also $(\mu_0 I - T)^{k-1}v_{k-1} \neq 0$. Since if $(\mu_0 I - T)^{k-1}v_{k-1}$ is zero, then

$$(\mu_0 I - T)^k \sum_{i=1}^{k}\binom{k}{i}\mu_0^{-1-i}(-T)^{i-1}v = 0.$$

But because of $k^2 \geq k+1$ this leads to

$$\begin{aligned}
0 &= (\mu_0 I - T)^k(\mu_0 I - T)^k v \\
&= (\mu_0 I - T)^k \mu_0^k v - T(\mu_0 I - T)^k \sum_{i=1}^{k}\binom{k}{i}\mu_0^{k-i}(-T)^{i-1}v = \mu_0^k(\mu_0 I - T)^k v \\
&\Rightarrow (\mu_0 I - T)^k v = 0
\end{aligned}$$

what contradicts the assumption. This also implies that v_{k-1} cannot be equal to zero. Thus by the induction hypothesis there are nonzero elements $v_0, \ldots, v_{k-2} \in E$ with (2.3) and so by setting $v_k := v$ the necessity follows.

Sufficiency. Let $v_0, \ldots, v_k$ with (2.3) be given. Then from the equality $\mathcal{A}(s_0)v_k = Bv_{k-1}$ we obtain (using $\mathcal{A}(\underline{s})^{-1}\mathcal{A}(s_0) = I - (\underline{s} - s_0)T$)

$$\begin{aligned}
0 = (\underline{s} - s_0)^k B(\mu_0 I - T)^k v_{k-1} &= B(I - (\underline{s} - s_0)T)^k v_{k-1} \\
&= (\mathcal{A}(s_0)\mathcal{A}(\underline{s})^{-1})^k \mathcal{A}(s_0)v_k = \mathcal{A}(S_0)\big(\mathcal{A}(\underline{s})^{-1}\mathcal{A}(s_0)\big)^k v_k \\
&= \mathcal{A}(s_0)\big(I - (\underline{s} - s_0)T\big)^{k-1}v_k, \quad (2.5)
\end{aligned}$$

where we use the induction hypothesis for v_{k-1} and

$$B\big(I - (\underline{s} - s_0)\mathcal{A}(\underline{s})^{-1}\big) = \mathcal{A}(s_0)\mathcal{A}(\underline{s})^{-1}B.$$

By the equality $\mathcal{N}(\mathcal{A}(s_0)) = \mathcal{N}(\mu_0 I - T)$ we thus find

$$(\mu_0 I - T)^{k+1}v_k = 0.$$

Now assume $(\mu_0 I - T)^k v_k = 0$. One shows similar to (2.5) the equalities

$$0 = (\mathcal{A}(s_0)\mathcal{A}(\underline{s}))^{k-1}\mathcal{A}(s_0)v_k = (\underline{s} - s_0)^{k-1}B(\mu_0 I - T)^{k-1}v_{k-1}.$$

By multiplication this equation from the left with $\mathcal{A}(\underline{s})^{-1}$ one obtains

$$0 = (\mu_0 I - T)^k v_k = \mu_0(\mu_0 I - T)^{k-1}v_{k-1} \neq 0$$

where we used the induction hypothesis and $T = \mathcal{A}(\underline{s})^{-1}B$. This finishes the proof of the equality $W(T, \mu_0) = \mathcal{W}(\mathcal{A}, s_0)$. We denote by W this space.

Uniqueness of v'. To show that for every $v \in W$ there is atmost one $v' \in W$ with $\mathcal{A}(s_0)v = Bv'$ assume there is $v'' \neq v'$, $v'' \in W$ with $\mathcal{A}(s_0)v = Bv''$. Then $w := v' - v'' \in W$ satisfies $w \neq 0$ and $Bw = 0$. But then $w \in \mathcal{N}(T)$ and since $\mu_0 \neq 0$ we obtain $w \notin W$ which is a contradiction and proves the uniqueness. □

Remark. *The proof also shows that the lengths of the Jordan chains for T coincide with the orders of the root-polynomials. Therefore the maximal length of all Jordan chains of T to the eigenvalue s_0 is the same as the maximal order of all root-polynomials of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.*

Now we can prove Theorem 2.26 by application of Theorem 2.25.

Proof of Theorem 2.26. Lemma 2.24 shows the existence of $J_1 \in H$ so that by setting $H_1 := \{J \in H : J \succ J_1\}$ one obtains

$$\mathcal{A}_J(\underline{s})^{-1} \xrightarrow{\mathcal{QP}} \mathcal{A}(\underline{s})^{-1} \ (J \in H_1) \text{ regularly and stably.}$$

Therefore Lemma 2.23 yields

$$\mathcal{A}_J(\underline{s})^{-1}\mathcal{A}_J(s) \xrightarrow{\mathcal{PP}} \mathcal{A}(\underline{s})^{-1}\mathcal{A}(s) \ (J \in H_1) \text{ regularly for all } s \in \Sigma.$$

Define $T_J := \mathcal{A}_J(\underline{s})^{-1}B_J \in L(E_J, E_J)$ for all $J \in H_1$ and also $T := \mathcal{A}(\underline{s})^{-1}B \in L(E, E)$. With this choice we have

$$\begin{aligned} \mathcal{A}_J(\underline{s})^{-1}\mathcal{A}_J(s_0) &= (I_J - (\underline{s} - s_0)T_J) \ \forall J \in H_1 \ \text{ and} \\ \mathcal{A}(\underline{s})^{-1}\mathcal{A}(s_0) &= (I - (\underline{s} - s_0)T). \end{aligned} \tag{2.6}$$

Define the map $\eta : \Sigma' \to \mathbb{C}, s \mapsto \frac{1}{\underline{s}-s}$ and set $\Lambda := \eta(\Sigma')$. From the assumptions on $\underline{s}$ and Σ' we obtain

$$0 < C_2 := \frac{1}{\operatorname{dist}(\overline{\Sigma'}, \underline{s})} \le |\mu| \le \frac{1}{\operatorname{dist}(\underline{s}, \overline{\Sigma'})} =: C_1 \ \forall \mu \in \Lambda. \tag{2.7}$$

Furthermore the assumptions on $\mathcal{A}$ and $\mathcal{A}_J$ yield that for all $J \in H_1$ and all $\mu \in \Lambda$ the operators $\mu I_J - T_J \in L(E_J, E_J)$ and $\mu I - T \in L(E, E)$ are Fredholm of index zero and $\sigma(T) \cap \Lambda$ consists of isolated points (cf. [Vai76, §4 (7)]). From (2.6) we conclude

$$\begin{aligned} \mu_J \in \sigma(T_J) \text{ with } \mu_J \neq 0 \text{ if and only if } \underline{s} - \frac{1}{\mu_J} \in \sigma(\mathcal{A}_J) \text{ and} \\ \mu \in \sigma(T) \text{ with } \mu \neq 0 \text{ if and only if } \underline{s} - \frac{1}{\mu} \in \sigma(\mathcal{A}). \end{aligned} \tag{2.8}$$

Now Theorem 2.25 is applicable.

(1) Let $\mu_0 := \frac{1}{\underline{s}-s_0}$ with $s_0 \in \sigma(\mathcal{A}) \cap \Sigma'$. Then (2.8) implies $\mu_0 \in \sigma(T) \cap \Lambda$. By Theorem 2.25 (i) there is a net $(\mu_J)_{J \in H_1}$ with $\mu_J \in \sigma(T_J)$ for all $J \succ J_2$ where J_2 is some element of H_1, and with $\mu_J \to \mu_0$ $(J \in H_1)$. Because of $\mu_0 \in \Lambda$ and (2.7), $J_2 \in H_1$ can be chosen such that $|\mu_J| \ge \frac{C_2}{2}$ for all $J \in H_1$ with $J \succ J_2$. Thus, again by (2.8), it is $s_J := \underline{s} - \frac{1}{\mu_J} \in \sigma(\mathcal{A}_J)$ for all $J \in H_1$ with $J \succ J_2$ and s_J satisfies the estimate

$$|s_J - s_0| = |\frac{1}{\mu_0} - \frac{1}{\mu_J}| = |\frac{\mu_J - \mu_0}{\mu_0 \mu_J}| \le \frac{2}{C_2^2}|\mu_J - \mu_0|.$$

This implies (1).

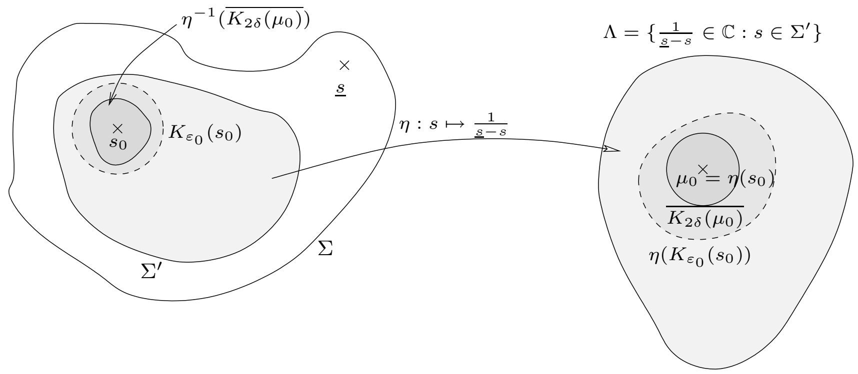


Figure 2.1: Visualization of the sets in the proof of Theorem 2.26.

(2) Let $(s_J)_{J\in H}$ and $s_0 \in \Sigma'$ be given as in Theorem 2.26 (2). Then $\mu_0 := \frac{1}{\underline{s}-s_0}$ is an element of Λ. Furthermore $s_J \neq \underline{s}$ for all $J \succ J_0$ and so $\mu_J := \frac{1}{\underline{s}-s_J}$ is a net in $\mathbb{C}$ with $\mu_J \in \sigma(T_J)\ \forall J \succ J_0$. Now the assumption on $(s_J)_{J\in H}$ shows that the net $(\mu_J)_{J\in H}$ has $\mu_0 \in \Lambda$ as a cluster point. Then from Theorem 2.25 (ii) follows $\mu_0 \in \sigma(T) \cap \Lambda$ and the equivalence (2.8) proves (2).

(3) Assertion (3) follows from (iii) of Theorem 2.25 by application of (1), (2), equivalence (2.8), and the equivalences of $\mathcal{N}(\mathcal{A}_J(s))$ and $\mathcal{N}(\frac{1}{\underline{s}-s}I_J - T_J)$ for all $J \in H_1$ and of $\mathcal{N}(\mathcal{A}(s))$ and $\mathcal{N}(\frac{1}{\underline{s}-s}I - T)$ (cf. Lemma 2.27).

(4,5) For the proof of (5) let $\varepsilon_0 > 0$ be given as in Theorem 2.26. Note that $\eta : \Sigma' \to \Lambda$ is a homeomorphism. Set $\mu_0 := \eta(s_0)$ and choose $\delta > 0$ so that $\overline{K_{2\delta}(\mu_0)} \subset \eta(K_{\varepsilon_0}(s_0))$. For a visualization see Figure 2.1.

The results of (1) and (2) imply that there is $J_3 \in H_1$ with

$$\sigma(\mathcal{A}_J) \cap K_{\varepsilon_0}(s_0) \subset \eta^{-1}(\overline{K_{2\delta}(\mu_0)})\ \forall J \succ J_3$$

since otherwise there would be a cluster point $\tilde{s}$ of $(s_J)_{J\in H}$ in the compact set $\overline{K_{\varepsilon_0}(s_0)} \setminus \eta^{-1}(K_\delta(s_0)) \subset \Sigma'$. But then by property (2), $\tilde{s}$ would be an element of $\sigma(\mathcal{A})$ what contradicts the choice of ε_0 and s_0. Together with the result of Lemma 2.27 this yields that for $J \in H$ with $J \succ J_3$ the space

$$\begin{aligned} W_J := W(T_J, \mu_0) :=& \text{linear hull of all generalized eigenfunctions of } T_J \\ & \text{to eigenvalues } \mu_j \text{ with } |\mu_J - \mu_0| \leq 2\delta \end{aligned}$$

is equal to

$$\begin{aligned} \mathcal{W}_J := \mathcal{W}(\mathcal{A}_J, s_0) :=& \text{linear hull of the root-subspaces to} \\ & \text{eigenvalues } s_J \text{ of } \mathcal{A}_J(\cdot) \text{ with } |s_J - s_0| \leq \varepsilon_0. \end{aligned}$$

Now (v) of Theorem 2.25 shows with the definition

$$\tilde{\epsilon}_J := \max_{\substack{v\in\mathcal{W} \\ \|v\|=1}} \|p_J T v - T_J p_J v\|_{E_J}, \tag{2.9}$$

that there is $J_4 \succ J_3$ so that for all $J \succ J_4$ the estimates

$$\dim \mathcal{W}_J = \dim \mathcal{W} < \infty$$

as well as

$$\vartheta(\mathcal{W}_J, \mathcal{W}) \leq C\tilde{\epsilon}_J$$

and

$$\vartheta(\mathcal{W}, \mathcal{W}_J) \leq C\tilde{\epsilon}_J$$

hold.

Therefore it remains to show that there is $J_5 \in H_1$ and a constant const > 0 such that

$$\max_{\substack{v\in\mathcal{W}\\ \|v\|=1}} \|p_J T v - T_J p_J v\|_{E_J} \leq \text{const} \max_{\substack{v,v'\in\mathcal{W}\\ \|v\|=1\\ \mathcal{A}(s_0)v=Bv'}} \|\mathcal{A}_J(s_0)p_J v - B_J p_J v'\| \;\forall J \succ J_5. \tag{2.10}$$

Recall that by Lemma 2.27 for every $v \in \mathcal{W}$ there is a unique $v' \in \mathcal{W}$ with $\mathcal{A}(s_0)v = Bv'$. Such a pair (v, v') of elements of $\mathcal{W}$ satisfies

$$BTv' = B\mathcal{A}(\underline{s})^{-1}\mathcal{A}(s_0)v = B(I - (\underline{s} - s_0)T)v = \mathcal{A}(s_0)Tv \tag{2.11}$$

and also

$$\begin{aligned}\mathcal{A}_J(\underline{s})^{-1}\left(\mathcal{A}_J(s_0)p_J T v - B_J p_J T v'\right) &= (I - (\underline{s} - s_0)T_J)p_J T v - T_J p_J T v'\\ &= p_J T v - T_J p_J((\underline{s} - s_0)Tv + \mathcal{A}(\underline{s})^{-1}\mathcal{A}(s_0)v) = p_J T v - T_J p_J v.\end{aligned} \tag{2.12}$$

Then by the stable convergence of $\mathcal{A}_J(\underline{s}) \xrightarrow{\mathcal{PQ}} \mathcal{A}(\underline{s})$ $(J \in H_1)$ we find an index $J_5 \in H_1$ so that $\|\mathcal{A}_J(\underline{s})^{-1}\| \leq \text{const}\ \forall J \succ J_5$. Hence (2.12) shows

$$\|p_J T v - T_J p_J v\| \leq \text{const}\|\mathcal{A}_J(s_0)p_J T v - B_J p_J T v'\|.$$

Finally the linearity of the operators p_J, B_J, $\mathcal{A}_J$, and the inclusion

$$\begin{aligned}&\{(Tv, Tv') : (v, v') \in \mathcal{W}^2, \|v\| \leq 1, \mathcal{A}(s_0)v = Bv'\}\\ &\qquad\subset \{(v, v') \in \mathcal{W}^2 : \|v\| \leq \|T\|, \mathcal{A}(s_0)v = Bv'\},\end{aligned}$$

which is a result of (2.11), imply

$$\begin{aligned}&\max_{\substack{v\in\mathcal{W}\\ \|v\|=1}} \|p_J T v - T_J p_J v\|\\ &\quad\leq \text{const} \max_{\substack{v,v'\in\mathcal{W}\\ \|v\|=1\\ \mathcal{A}(s_0)v=Bv'}} \|\mathcal{A}_J(s_0)p_J T v - B_J p_J T v'\|\\ &\quad\quad\leq \text{const}\|T\| \max_{\substack{v,v'\in\mathcal{W}\\ \|v\|=1\\ \mathcal{A}(s_0)v=Bv'}} \|\mathcal{A}_J(s_0)p_J v - B_J p_J v'\|.\end{aligned}$$

So the required estimate (2.10) is shown.

Note that the last estimate does only depend on J_5 and s_0. Therefore it directly implies (4) by using the results

$$|s_J - s_0| \leq \frac{2}{C_2^2}|\mu_J - \mu_0| \leq \tilde{\epsilon}_J^{\frac{1}{\kappa}}$$

and

$$\inf_{v_0 \in \mathcal{N}(\mathcal{A}(s_0))} \|v_J - p_J v_0\| \leq C\tilde{\epsilon}_J^{\frac{1}{\kappa}}$$

of Theorem 2.25 (iv) for the nets from (3).

□

2.4 Simple eigenvalues

In this section we present a result about the convergence of eigenvalue approximations for holomorphic operator-valued functions in the setting of discrete approximations.

The application of the Theorem 2.28 we have in mind is to allow for so called projection boundary conditions in the approximation of eigenvalues of the boundary value problems on the infinite line by finite interval approximations. The advantages of projection-boundary conditions for the computation of connecting orbits are discussed in [Bey90].

Note that the result of Theorem 2.26 does not apply to general holomorphic operator functions since the proof makes substantial use of the polynomial structure of the operators. The result in this section is a reformulation of Lemma 1 of [Bey80].

We use the same notations and assumptions as in the previous sections. Especially, we assume that H is an unbounded directed set and the spaces E, F, E_J, F_J are separable.

Theorem 2.28. *Let Σ be an open subset of $\mathbb{C}$. Let $\mathcal{A} : \Sigma \to L(E, F)$ be a holomorphic and operator-valued function. Let $s_0 \in \Sigma$ be a simple eigenvalue of $\mathcal{A}(\cdot)$ with eigenfunction $v_0 \in E$, $v_0 \neq 0$. Assume that $(\mathcal{A}_J)_{J\in H}$ is a family of holomorphic operator-valued functions $\mathcal{A}_J : \Sigma \to L(E_J, F_J)$ such that there is an index $J_1 \in H$ with $\mathcal{A}_J(s_0)$ is Fredholm of index zero for all $J \in H$, $J \succ J_1$. Furthermore assume*

(i) $\mathcal{A}_J(s_0) \xrightarrow{\mathcal{PQ}} \mathcal{A}(s_0)$ *regularly,*

(ii) $\mathcal{A}'_J(s_0) \xrightarrow{\mathcal{PQ}} \mathcal{A}'(s_0)$,

(iii) for every $\varepsilon > 0$ there is $J_2 \succ J_1$, $J_2 \in H$, and $\delta > 0$ such that

$$\|\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0)\| \leq \varepsilon \; \forall s \in \overline{K_\delta(s_0)} \subset \Sigma, \; J \succ J_2.$$

Then there is an index $J_0 \in H$ and a positive constant δ_0 such that for all $J \succ J_0$ the function $\mathcal{A}_J(\cdot)$ has exactly one simple eigenvalue $s_J \in \overline{K_{\delta_0}(s_0)}$.
Moreover, for each $J \in H$ with $J \succ J_0$ there is a corresponding eigenfunction $v_J \in E_J$ with

$$|s_J - s_0| + \|v_J - p_J v_0\| \leq \text{const}\|\mathcal{A}_J(s_0)p_J v_0\|. \tag{2.13}$$

We follow the proof of Lemma 1 in [Bey80], but rather than referring to [Vai76, §3 (14)] for the existence part, we show this directly by applying the contraction mapping Theorem (Lemma C.4) in order to circumvent the problem that [Vai76, §3 (14)] is not formulated in the setting of directed sets. The idea for the existence part lies in a kind of bordering the operators $\mathcal{A}_J(\cdot)$ so that Lemma C.4 is applicable. The second part, where simplicity of the eigenvalue is shown, is essentially an adaption of the proof in [Bey80].

Proof. By a corollary of the Hahn-Banach Theorem (cf. [Alt99, Folgerung 4.17]) there is a linear and continuous functional $g_0 \in E^*$ with $\langle g_0, v_0 \rangle = 1$. This leads to a splitting $E = \text{span}(v_0) \oplus W$, where $W := \mathcal{N}(g_0)$. By Lemma 2.22 we find a net $(f_J)_{J\in H}$ in $(E_J)^*$ with $f_J \overset{\mathcal{P}}{\rightharpoonup} g_0$ $(J \in H)$.
Thus there is $J_1 \in H$ with $|\langle f_J, p_J v_0\rangle| \geq \frac{1}{2}$ for all $J \in H$ with $J \succ J_1$ and so we can define

$$g_J := \begin{cases} \frac{1}{\langle f_J, p_J v_0\rangle} f_J, & \text{for all } J \succ J_1, \\ f_J, & \text{otherwise.} \end{cases}$$

Since Lemma 2.11 and Lemma 2.18 imply that there is $J_2 \in H$ with $\|p_J\|, \|f_J\| \leq \text{const} < \infty$ for all $J \succ J_2$ it follows that $\|g_J\| \leq \text{const} < \infty$ for all $J \succ J_2$. Thus the inequality

$$|\langle g_J, p_J x\rangle - \langle g, x\rangle| \leq \left|\frac{1}{\langle f_J, p_J v_0\rangle} - 1\right| |\langle f_J, p_J x\rangle| + |\langle f_J, p_J x\rangle - \langle g, x\rangle|$$

which holds for all $x \in E$ and all $J \succ J_1, J_2$ shows

$$g_J \overset{\mathcal{P}}{\rightharpoonup} g_0. \tag{2.14}$$

Define the operators

$$B : \Sigma \times E \to \mathbb{C} \times F,\ (s, v) \mapsto (\langle g, v\rangle - 1, \mathcal{A}(s)v) \tag{2.15}$$

and

$$B_J : \Sigma \times E_J \to \mathbb{C} \times F_J,\ (s, v) \mapsto (\langle g_J, v\rangle - 1, \mathcal{A}_J(s)v). \tag{2.16}$$

For the rest of this proof we denote by an upper index "0" the evaluation at s_0, v_0, $p_J v_0$, (s_0, v_0), or $(s_0, p_J v_0)$, where the exact meaning will be clear from the context.
Furthermore define families of linear continuous operators $\tilde{\mathcal{P}} = (\tilde{p}_J)_{J\in H}$ and $\tilde{\mathcal{Q}} = (\tilde{q}_J)_{J\in H}$ by

$$\tilde{p}_J : \mathbb{C} \times E \to \mathbb{C} \times E_J,\ (s, v) \mapsto (s, p_J v)$$

and

$$\tilde{q}_J : \mathbb{C} \times F \to \mathbb{C} \times F_J,\, (s, v) \mapsto (s, q_J v).$$

(On products of Banach space $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we use the usual product norm given by $\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$.) It is clear that the families $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ satisfy the properties (2.1) and (2.2).

Step 1: It holds $\mathcal{N}(DB^0) = \{0\}$, where DB^0 is the total derivative of B evaluated at (s_0, v_0). (In the following D will always mean the total derivative and $'$ will stand for the derivative with respect to s.)

Let $(\zeta, \phi) \in \mathbb{C} \times E$ with

$$0 = DB^0(\zeta, \phi) = (\langle g_0, \phi\rangle, \zeta\mathcal{A}'(s_0)v_0 + \mathcal{A}(s_0)\phi).$$

The simplicity assumption of the eigenvalue s_0 implies $\zeta = 0$ and so $\phi \in \mathcal{N}(\mathcal{A}(s_0))$. This means $\phi = cv_0$ for some $c \in \mathbb{C}$, but $\langle g_0, \phi\rangle = \langle g_0, cv_0\rangle = c$ and so $\phi = 0$. This finishes step 1.

Step 2: The operators DB^0_J are Fredholm of index zero for $J \succ J_1$.

The (Fréchet-)derivative of B_J at $(s_0, p_J v_0)$ can be written as an operator matrix in the form

$$DB^0_J = \begin{bmatrix} 0 & g_J \\ \mathcal{A}'_J(s_0)p_J v_0 & \mathcal{A}^0_J \end{bmatrix} : \mathbb{C} \times E_J \to \mathbb{C} \times F_J.$$

Now the Bordering Lemma C.9 shows the second step.

Step 3: The operators DB^0_J regularly-$\tilde{\mathcal{P}}\tilde{\mathcal{Q}}$-converge to DB^0.

The convergence $DB^0_J \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} DB^0$ follows directly from assumptions (i) and (ii), (2.14), and the definitions of $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$. Thus it remains to show regularity.

Let $(\zeta_J, \phi_J)_{J\in H}$ be a bounded net in $(\mathbb{C} \times E_J)$, so that the net $(DB^0_J(\zeta_J, \phi_J))_{J\in H}$ is $\tilde{\mathcal{Q}}$ compact.

Let $(\zeta_J, \phi_J)_{J\in H'}$, $H' \subset H$, be any cofinal subnet. From the boundedness follows that there is a cofinal subset $H'' \subset H'$ with $\zeta_J \to \zeta$ $(J \in H'')$. Because of the $\tilde{Q}$ compactness of $(DB^0_J(\zeta_J, \phi_J))_{J\in H}$ there is a cofinal subset $H''' \subset H''$ and $(\lambda, \nu) \in \mathbb{C} \times F$ with

$$DB^0_J(\zeta_J, \phi_J) \xrightarrow{\tilde{\mathcal{Q}}} (\lambda, \nu)\ (J \in H''').$$

This implies $\zeta_J\mathcal{A}'_J(s_0)p_J v_0 + \mathcal{A}^0_J\phi_J \xrightarrow{\mathcal{Q}} \nu$ $(J \in H''')$. Using the triangle inequality and $\zeta_J\mathcal{A}'_J(s_0)p_J v_0 \to \zeta\mathcal{A}'(s_0)v_0$, we obtain from this

$$\mathcal{A}^0_J\phi_J \xrightarrow{\mathcal{Q}} \nu - \zeta\mathcal{A}'(s_0)v_0\ (J \in H''').$$

This shows that the net $(\mathcal{A}^0_J\phi_J)_{J\in H}$ is $\mathcal{Q}$ compact and so by assumption (i) $(\phi_J)_{J\in H}$ is $\mathcal{P}$ compact. So there is $H'''' \subset H'''$ and $\phi \in E$ with

$$(\zeta_J, \phi_J) \xrightarrow{\tilde{\mathcal{P}}} (\zeta, \phi)\ (J \in H'''').$$

Step 3 is proven.

Step 4: There is $J_4 \in H$ and $\delta > 0$ so that B_J has a unique zero (s_J, v_J) in $\overline{K_\delta(s_0, p_J v_0)}$ and this satisfies the estimate (2.13).

The results of steps 1–3 together with Lemma 2.21 imply

$$DB_J^0 \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} DB^0 \text{ regularly and stably.}$$

Thus there are $J_3 \succ J_2$ and $\kappa, \tau > 0$ such that DB_J^0 is a homeomorphism for all $J \succ J_3$ and satisfies

$$\|(DB_J^0)^{-1}\|_{L(\mathbb{C}\times F_J, \mathbb{C}\times E_J)} \leq \kappa \;\forall J \succ J_3 \tag{2.17}$$

as well as

$$\|DB_J^0\|_{L(\mathbb{C}\times E_J, \mathbb{C}\times F_J)} < \tau \;\forall J \succ J_3. \tag{2.18}$$

Now choose $0 < q < 1$ and $\varepsilon > 0$ such that

$$\big(\|\mathcal{A}'_J(s_0)\| + \|p_J v_0\| + \varepsilon\big)\varepsilon \leq \frac{q}{\kappa} \;\forall J \succ J_3.$$

By assumption (iii) there is $\delta_1 > 0$ with

$$\|\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0)\| \leq \varepsilon \text{ for all } s \in \overline{K_{\delta_1}(s_0)} \subset \Sigma \text{ and } J \succ J_3, J_2.$$

Choosing $\delta := \min(\varepsilon, \delta_1)$ one finds for all $(s, v) \in \overline{K_\delta(s_0, p_J v_0)}$

$$\begin{aligned}
&\|(DB_J(s,v) - DB_J^0)(\zeta,\phi)\|_{\mathbb{C}\times F_J} \\
&\quad = \big\|\big(\langle g_J,\phi\rangle - \langle g_J,\phi\rangle, \zeta(\mathcal{A}'_J(s)v - \mathcal{A}'_J(s_0)p_J v_0) + (\mathcal{A}_J(s) - \mathcal{A}_J(s_0))\phi\big)\big\| \\
&\quad \leq |\zeta|\big\|\mathcal{A}'_J(s)v - \mathcal{A}'_J(s_0)p_J v_0\big\| + \big\|\mathcal{A}_J(s) - \mathcal{A}_J(s_0)\big\|\|\phi\| \\
&\quad \leq \big(\|\mathcal{A}'_J(s_0)(v - p_J v_0)\| + \|(\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0))v\|\big)|\zeta| \\
&\qquad\quad + \Big\|\int_0^1 \mathcal{A}'_J(s_0 + t(s - s_0))(s - s_0)dt\Big\|\|\phi\| \\
&\quad \leq \big(\|\mathcal{A}'_J(s_0)\|\|v - p_J v_0\| + \|\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0)\|\|v\|\big)|\zeta| \\
&\qquad\quad + \int_0^1 \|\mathcal{A}'_J(s_0 + t(s - s_0))\|dt|s - s_0|\|\phi\| \\
&\quad \leq \|\mathcal{A}'_J(s_0)\|\varepsilon(|\zeta| + \|\phi\|) + (\|p_J v_0\| + \varepsilon)\varepsilon|\zeta| + \varepsilon^2\|\phi\| \\
&\quad \leq \big(\|\mathcal{A}'_J(s_0)\| + \|p_J v_0\| + \varepsilon\big)\varepsilon(|\zeta| + \|\phi\|) \\
&\quad \leq \frac{q}{\kappa}(|\zeta| + \|\phi\|_{E_J}) = \frac{q}{\kappa}\|(\zeta,\phi)\|_{\mathbb{C}\times E_J},
\end{aligned}$$

which implies

$$\|DB_J(s,v) - DB_J(s_0, p_J v_0)\|_{\mathbb{C}\times E_J \to \mathbb{C}\times F_J} \leq \frac{q}{\kappa} \;\forall (s,v) \in \overline{K_\delta(s_0, p_J v_0)}. \tag{2.19}$$

Finally, B_J^0 $\tilde{\mathcal{Q}}$-converges to B^0 as seen by

$$\begin{aligned}\|B_J^0 - \tilde{q}_J B^0\| &= \big\|\big(\langle g_J, p_J v_0\rangle - \langle g, v_0\rangle, \mathcal{A}_J(s_0)p_J v_0 - q_J\mathcal{A}(s_0)v_0\big)\big\| \\ &= |\langle g_J, p_J v_0\rangle - \langle g, v_0\rangle| + \|\mathcal{A}_J(s_0)p_J v_0 - q_J\mathcal{A}(s_0)v_0\|\end{aligned}$$

and the convergences $g_J \xrightarrow{\mathcal{P}} g$ and $\mathcal{A}_J^0 \xrightarrow{\mathcal{PQ}} \mathcal{A}^0$.

Thus there is $J_4 \in H$ with $J_4 \succ J_3$ so that for all $J \succ J_4$ holds

$$\|B_J^0\|_{\mathbb{C}\times F_J} \le \|B^0\|_{\mathbb{C}\times F} + \delta\frac{1-q}{\kappa} = \delta\frac{1-q}{\kappa}. \tag{2.20}$$

The inequalities (2.17), (2.18), (2.19), (2.20) show that the assumptions of Lemma C.4 with the choice $y = 0$ are fulfilled. Therefore for all $J \succ J_4$ there is a unique zero (s_J, v_J) of B_J in $\overline{K_\delta(s_0, p_J v_0)}$ and moreover the estimate

$$\|(s_J, v_J) - (s_0, p_J v_0)\| \le \frac{\kappa}{1-q}\|B_J^0 - 0\|_{\mathbb{C}\times F_J} = \frac{\kappa}{1-q}\|\mathcal{A}_J(s_0)p_J v_0\|_{F_J}$$

holds. This finishes the proof of step 4.

Now it remains to prove the uniqueness of the eigenvalue in $\overline{K_{\delta_0}(s_0, p_J v_0)}$ for a suitable δ_0 and its simplicity.

Step 5: There is $J_5 \succ J_4$ and $0 < \delta_1 \le \delta$ so that s_J is the only eigenvalue of $\mathcal{A}_J$ in $\overline{K_{\delta_1}(s_0, p_J v_0)}$ for all $J \in H$ with $J \succ J_5$ and $\dim \mathcal{N}(\mathcal{A}_J(s_J)) = 1$.

Assume the assertion is false. Then there is a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H with $(\lambda_n, \phi_n) \in \Sigma \times E_{J_n}$ so that $|\lambda_n - s_0| \le \min(\frac{1}{n+1}, \delta)$, $\|\phi_n\|_{J_n} = 1$, and $\mathcal{A}_{J_n}(\lambda_n)\phi_n = 0$, but $\lambda_n \ne s_{J_n}$ or $\phi_n \notin \operatorname{span}(v_{J_n})$. Then it holds

$$\begin{aligned}\|\mathcal{A}_{J_n}(s_0)\phi_n\| &= \|\big(\mathcal{A}_{J_n}(s_0) - \mathcal{A}_{J_n}(\lambda_n)\big)\phi_n\| \\ &\le \int_0^1 \|\mathcal{A}'_{J_n}(s_0 + t(\lambda_n - s_0))\||\lambda_n - s_0| dt \\ &\le \int_0^1 \big(\|\mathcal{A}'_{J_n}(s_0)\| + \varepsilon\big)dt|\lambda_n - s_0|\end{aligned}$$

and thus $\lim_{n\to\infty}\|\mathcal{A}_{J_n}(s_0)\phi_n\| = 0$ which implies $\mathcal{A}_{J_n}(s_0)\phi_n \xrightarrow{\mathcal{Q}} 0$. Therefore assumption (i) together with Lemma 2.17 shows that $(\phi_n)_{n\in\mathbb{N}}$ is $\mathcal{P}$-compact. Thus there is a subsequence $(\phi_n)_{n\in\mathbb{N}'}$, $\mathbb{N}' \subset \mathbb{N}$, and $\phi \in E$ with $\|\phi\| = 1$ so that

$$\phi_n \xrightarrow{\mathcal{P}} \phi \ (n \in \mathbb{N}').$$

The definition of $\mathcal{PQ}$-convergence shows

$$\mathcal{A}_{J_n}(s_0)\phi_n \xrightarrow{\mathcal{Q}} \mathcal{A}(s_0)\phi \ (n \in \mathbb{N}') \text{ and also } \mathcal{A}_{J_n}(s_0)\phi_n \xrightarrow{\mathcal{Q}} 0 (n \in \mathbb{N}').$$

By Lemma 2.15 we have $\mathcal{A}(s_0)\phi = 0$ and the simplicity assumption of the eigenvalue s_0 implies $\phi = cv_0$ for some $0 \ne c \in \mathbb{C}$.
This form of ϕ now implies

$$\langle g_{J_n}, \phi_n\rangle \to \langle g_0, cv_0\rangle = c \ (n \in \mathbb{N}')$$

and therefore there is $n_0 \in \mathbb{N}$ with $|\langle g_{J_n}, \phi_n\rangle| \geq \frac{|c|}{2}$ for all $n \in \mathbb{N}'$ with $n \geq n_0$. Define

$$\tilde{\phi}_n := \begin{cases} \frac{\phi_n}{\langle g_{J_n},\phi_n\rangle}, & n \in \mathbb{N}',\ n \geq n_0, \\ \phi_n, & n \in \mathbb{N}',\ n < n_0. \end{cases}$$

For this sequence we obtain $\tilde{\phi}_n \xrightarrow{\mathcal{P}} v_0$ $(n \in \mathbb{N}')$ because of the inequality

$$\|\tilde{\phi}_n - p_{J_n} v_0\| \leq \left\| \left(\frac{1}{\langle g_{J_n}, \phi_n\rangle} - \frac{1}{c} \right) \phi_n \right\| + \left\| \frac{1}{c}(\phi_n - p_{J_n}\phi) \right\|$$

which holds for all $n \in \mathbb{N}'$ with $n \geq n_0$. Since the right hand side converges to zero and by the choice of the sequence λ_{J_n} follows that there is $N_0 \geq n_0$ such that

$$\|\tilde{\phi}_n - p_{J_n} v_0\| + |\lambda_n - s_0| < \delta \ \forall n \in \mathbb{N}' \text{ with } n \geq N_0. \tag{2.21}$$

Note that the choice of $(\lambda_n, \tilde{\phi}_n) \in \Sigma \times E_{J_n}$ for all $n \in \mathbb{N}'$ with $n \geq n_0$ leads to

$$B_{J_n}(\lambda_n, \tilde{\phi}_n) = \Big(\langle g_{J_n}, \tilde{\phi}_n\rangle - 1, \mathcal{A}_{J_n}(\lambda_n)\tilde{\phi}_n \Big) = 0. \tag{2.22}$$

In step 4 it is shown that a point $(\lambda_n, \tilde{\phi}_n)$ with (2.21) and (2.22) must equal (s_{J_n}, v_{J_n}) and therefore $\lambda_n = s_{J_n}$ and $\phi_n \in \operatorname{span}(v_{J_n})$ for all $n \in \mathbb{N}'$ with $n \geq N_0$ what contradicts the assumption. Step 5 is proven.

Step 6: There is $J_6 \in H$ with $J_6 \succ J_5$ so that for all $J \in H$ with $J \succ J_6$ the eigenvalue s_J is a simple eigenvalue.

Always assume $J \in H$ with $J \succ J_5$. First we show $DB_J(s_J, v_J) \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} DB^0$ stably. Note that for all $(\zeta, \phi) \in \mathbb{C} \times E_J$ holds

$$\begin{aligned}
&\big\|\big(DB_J^0 - DB_J(s_J, v_J)\big)(\zeta,\phi)\big\| \\
&\quad = \big\| \big(\langle g_J, \phi\rangle - \langle g_J, \phi\rangle, \zeta\big(\mathcal{A}_J'(s_0)p_J v_0 - \mathcal{A}_J'(s_J)v_J\big) + \big(\mathcal{A}_J(s_0) - \mathcal{A}_J(s_J)\big)\phi\big) \big\| \\
&\quad \leq \big\|\mathcal{A}_J'(s_0)p_J v_0 - \mathcal{A}_J'(s_J)v_J\big\||\zeta| + \big\|\mathcal{A}_J(s_0) - \mathcal{A}_J(s_J)\big\|\|\phi\| \\
&\quad \leq \big\{\|\mathcal{A}_J'(s_0)(p_J v_0 - v_J)\| + \|(\mathcal{A}_J'(s_0) - \mathcal{A}_J'(s_J))v_J\|\big\}\,|\zeta| \\
&\qquad + \Big\| \int_0^1 \mathcal{A}_J'(s_0 + t(s_J - s_0))(s_J - s_0)dt \Big\| \|\phi\| \\
&\quad \leq \big(\|\mathcal{A}_J'(s_0)\|\|p_J v_0 - v_J\| + \|\mathcal{A}_J'(s_0) - \mathcal{A}_J'(s_J)\|\|v_J\|\big)|\zeta| \\
&\qquad + \int_0^1 \|\mathcal{A}_J'(s_0 + t(s_J - s_0))\|\, dt\, |s_J - s_0|\|\phi\|.
\end{aligned} \tag{2.23}$$

Now the convergence result (2.13), proven in step 4, together with (2.23) imply

$$\|DB_J^0 - DB_J(s_J, v_J)\|_{L(\mathbb{C}\times E_J, \mathbb{C}\times F_J)} \to 0 \ (J \succ J_4), \tag{2.24}$$

where $(J \succ J_4)$ stands for $(J \in \{J \in H : J \succ J_4\})$.

The convergence $(DB_J(s_J,v_J)) \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} DB^0$ follows from

$$\begin{aligned}
&\|DB_J(s_J,v_J)\tilde{p}_J(\zeta,\phi) - \tilde{q}_J DB^0(\zeta,\phi)\| \\
&\qquad \le \|(DB_J(s_J,v_J) - DB^0_J)\tilde{p}_J(\zeta,\phi)\| + \|DB^0_J\tilde{p}_J(\zeta,\phi) - \tilde{q}_J DB^0(\zeta,\phi)\|
\end{aligned}$$

and (2.24) together with the boundedness of the nets $\tilde{p}_J(\zeta,\phi)$ and $(DB_J(s_J,v_J))$ (what is a result of (2.23)), and $DB^0_J \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} DB^0$.
Finally the stability of the convergence is obtained via Lemma A.1 from (2.17) and (2.24).

Therefore there is $J_6 \in H$ with $J_6 \succ J_5$ and a constant $C_0 > 0$ so that

$$\|(DB_J(s_J,v_J))^{-1}\|_{L(\mathbb{C}\times F_J,\mathbb{C}\times E_J)} \le C_0 \ \forall J \succ J_6.$$

This shows the stability inequality

$$\|(\zeta,\phi)\|_{\mathbb{C}\times E_J} \le C_0 \|DB_J(s_J,v_J)(\zeta,\phi)\|_{\mathbb{C}\times F_J} \ \forall J \succ J_6. \tag{2.25}$$

Now assume there is $J \in H$ with $J \succ J_6$ so that s_J is not a simple eigenvalue of $\mathcal{A}_J(\cdot)$. From step 5 we know that this implies $\mathcal{A}'_J(s_J)v_J \in \mathcal{R}(\mathcal{A}_J(s_J))$.
Let $\zeta = 1$ and choose $\phi_0 \in E_J$ with

$$\mathcal{A}'_J(s_J)v_J = -\mathcal{A}_J(s_J)\phi_0$$

and define $\phi := \phi_0 - \langle g_J, \phi_0\rangle v_J$. Then $0 \ne (\zeta,\phi) \in \mathbb{C}\times E_J$, but

$$\begin{aligned}
\|(\zeta,\phi)\|_{\mathbb{C}\times E_J} &\le C_0 \|DB_J(s_J,v_J)(\zeta,\phi)\|_{\mathbb{C}\times F_J} \\
&= C_0\big(|\langle g_J,\phi\rangle| + \|\zeta\mathcal{A}'_J(s_J)v_J + \mathcal{A}_J(s_J)\phi\|_{F_J}\big) \\
&= 0
\end{aligned}$$

since $J \succ J_1$. This is a contradiction and shows that s_J must be a simple eigenvalue which is the claim of step 6.

Taking s_J, v_J and $J_0 := J_6$ finishes the proof. □

2.5 Discrete approximations and exponential dichotomies

We consider the following setup of spaces and operators:
We take as index set $H := \{J = [x_-, x_+] : x_\pm \in \mathbb{R}, x_- \le 0 \le x_+, x_- < x_+\}$ with the direction $J_1 \succ J_2 :\Leftrightarrow J_1 \supset J_2$.

Consider the complex Banach-spaces

$$(E, \|\cdot\|_E) = (H^1(\mathbb{R},\mathbb{C}^l), \|\cdot\|_{H^1(\mathbb{R},\mathbb{C}^l)}),$$

$$(E_J, \|\cdot\|_{E_J}) = (H^1(J,\mathbb{C}^l), \|\cdot\|_{H^1(J,\mathbb{C}^l)}),$$

$$(F, \|\cdot\|_F) = (L_2(\mathbb{R},\mathbb{C}^l), \|\cdot\|_{L_2(\mathbb{R},\mathbb{C}^l)}),$$

$$(F_J, \|\cdot\|_{F_J}) = (L_2(J,\mathbb{C}^l)\times\mathbb{C}^l, \|\cdot\|_{L_2(J,\mathbb{C}^l)\times\mathbb{C}^l}),$$

where $\|\cdot\|_{L_2(J,\mathbb{C}^l)\times\mathbb{C}^l}$ is the usual product-norm given by

$$\|(h_J, s)\|_{L_2(J,\mathbb{C}^l)\times\mathbb{C}^l} := \|h_J\|_{L_2(J,\mathbb{C}^l)} + |s|.$$

Furthermore we define families of bounded linear operators $\mathcal{P} = (p_J)_{J\in H}$ and $\mathcal{Q} = (q_J)_{J\in H}$ by

$$p_J : \begin{array}{ccc} H^1(\mathbb{R},\mathbb{C}^l) & \to & H^1(J,\mathbb{C}^l), \\ z & \mapsto & z|_J \end{array}$$

and

$$q_J : \begin{array}{ccc} L_2(\mathbb{R},\mathbb{C}^l) & \to & L_2(J,\mathbb{C}^l)\times\mathbb{C}^l, \\ h & \mapsto & (h|_J, 0). \end{array}$$

Define the differential operator

$$L : \begin{array}{ccc} H^1(\mathbb{R},\mathbb{C}^l) & \to & L_2(\mathbb{R},\mathbb{C}^l), \\ z & \mapsto & Lz = z_x - M(\cdot)z, \end{array}$$

where $M \in \mathcal{C}(\mathbb{R},\mathbb{C}^{l,l})$ with hyperbolic limit matrices $M_\pm := \lim_{x\to\pm\infty} M(x)$. This operator has exponential dichotomies (ED)s* on $\mathbb{R}_+$ and $\mathbb{R}_-$ with data (K_+,β_+,π_+) and (K_-,β_-,π_-), respectively (see Theorem B.5).

Define the linear boundary operator

$$R : \begin{array}{ccc} H^1(J,\mathbb{C}^l) & \to & \mathbb{C}^l, \\ z & \mapsto & P_-z(x_-) + P_+z(x_+), \end{array}$$

where P_- and P_+ are fixed elements of $\mathbb{C}^{l,l}$. Finally denote by L_J, $J \in H'$, the differential operators

$$L_J : \begin{array}{ccc} H^1(J,\mathbb{C}^l) & \to & L_2(J,\mathbb{C}^l)\times\mathbb{C}^l, \\ z_J & \mapsto & (z_{J,x} - M(\cdot)z, Rz). \end{array}$$

The situation is summarized in the following diagram.

$$\begin{array}{ccc} H^1(\mathbb{R},\mathbb{C}^l) & \xrightarrow[z\mapsto z_x - M(\cdot)z]{L} & L_2(\mathbb{R},\mathbb{C}^l) \\ {\scriptstyle z\mapsto z|_J}\Big\downarrow{\scriptstyle p_J} & & {\scriptstyle q_J}\Big\downarrow{\scriptstyle r\mapsto(r|_J,0)} \\ H^1(J,\mathbb{C}^l) & \xrightarrow[z_J\mapsto(z_{J,x}-M(\cdot)z_J, Rz_J)]{L_J} & L_2(J,\mathbb{C}^l)\times\mathbb{C}^l \end{array}$$

Theorem 2.29. *Let M, L, L_J, and the data $K_\pm$, $\beta_\pm$, $\pi_\pm$ be given as above. Furthermore let $V_+^I \in \mathbb{C}^{l,r}$ be a basis of the unstable subspace of M_+ and let $V_-^{II} \in \mathbb{C}^{l,p}$ be a basis of the stable subspace of M_- and assume $p + r = l$. Finally assume that the boundary operator R satisfies*

$$\det\begin{pmatrix} P_-V_-^{II} & P_+V_+^I \end{pmatrix} \neq 0.$$

Then

$$L_J \xrightarrow{\mathcal{PQ}} L \text{ regularly } (J \in H').$$

*For a short review of the theory of exponential dichotomies see Appendix B.

Proof. First we show the $\mathcal{PQ}$-convergence of L_J to L.
For any $z \in E_J$ with $\|z_J\|_{E_J} \leq 1$ holds

$$\begin{aligned}\|L_J z_J\|_{F_J} &= \|z_{J,x} - M(\cdot)z_J\|_{L_2(J,\mathbb{C}^l)} + |R_J z_J| \\ &\leq \|z_{J,x}\|_{L_2(J,\mathbb{C}^l)} + \|M\|_\infty \|z_J\|_{L_2(J,\mathbb{C}^l)} + |P_- z_J(x_-)| + |P_+ z_J(x_+)|.\end{aligned}$$

By the Sobolev inequality (C.1) we have $\|z_J\|_\infty \leq \mathrm{const}\|z_J\|_{H^1}$ with a constant independent of J for all $J = [x_-, x_+]$ with $|x_+ - x_-| \geq 1$. Therefore it follows

$$\|L_J z_J\|_{F_J} \leq \mathrm{const}\|z_J\|_{H^1(J,\mathbb{C}^m)}$$

with a constant independent of J for all J with $|J| \geq 1$.

Hence by Lemma 2.18 it suffices to show for all $z \in E$

$$\|L_J p_J z - q_J L z\|_{F_J} \to 0 \ (J \in H). \tag{2.26}$$

But for $z \in E$ we have

$$\begin{aligned}\|L_J p_J z - q_J L z\|_{F_J} &= \|Lz|_J - (Lz)|_J\|_{L_2(J,\mathbb{C}^l)} + |P_- z(x_-) + P_+ z(x_+)| \\ &\leq |P_-||z(x_-)| + |P_+||z(x_+)| \to 0 \ (J \in H),\end{aligned}$$

by Lemma C.3.

Second we show the regularity of the convergence.
Let $(z_J)_{J\in H}$ be a bounded family, $z_J \in E_J$, $\|z_J\|_{E_J} \leq 1$, such that $(L_J z_J)_{J\in H}$ is $\mathcal{Q}$-compact.

Let $H' \subset H$ be any cofinal subset of H. Since H is an unbounded directed set there is a cofinal sequence $(J_n)_{n\in\mathbb{N}}$ in H'. We denote the endpoints of the interval J_n by x_-^n and x_+^n, i.e. $J_n = [x_-^n, x_+^n]$.

From the $\mathcal{Q}$-compactness of $(L_J z_J)_{J\in H}$ we obtain $\mathbb{N}' \subset \mathbb{N}$ and $h \in F$ with

$$L_{J_n} z_{J_n} = \begin{pmatrix} L z_{J_n} \\ R z_{J_n} \end{pmatrix} = \begin{pmatrix} L z_{J_n} \\ P_- z_{J_n}(x_-^n) + P_+ z_{J_n}(x_+^n) \end{pmatrix} =: \begin{pmatrix} h_n \\ s_n \end{pmatrix} \xrightarrow{\mathcal{Q}} h \ (n \in \mathbb{N}')$$

by definition this means

$$\|L_{J_n} z_{J_n} - q_{J_n} h\|_{F_{J_n}} = \|h_n - h|_{J_n}\|_{L_2(J_n,\mathbb{C}^l)} + |s_n| \to 0 \ (n \in \mathbb{N}'). \tag{2.27}$$

Because of the dichotomy property and the result of Theorem B.2 we can write $z_n := z_{J_n}$, using the notation from there, as

$$z_n(x) = \begin{cases} S(x,0)\pi_+(0)z_n(0) + S(x,x_+^n)(I - \pi_+(x_+^n))z_n(x_+^n) + \rho_+(x,x_+^n) & x \geq 0, \\ S(x,0)(I - \pi_-(0))z_n(0) + S(x,x_-^n)\pi_-(x_-^n)z_n(x_-^n) + \rho_-(x,x_-^n) & x \leq 0, \end{cases} \tag{2.28}$$

where

$$\rho_+(x,x_+^n) = \int_0^{x_+^n} G_+(x,y)h_n(y)dy, \ x \geq 0$$

and

$$\rho_-(x,x_-^n) = \int_{x_-^n}^0 G_-(x,y)h_n(y)dy, \ x \leq 0.$$

Here $G_\pm$ are the Green's functions from Theorem B.2. Notice that the right hand side of (2.28) is well-defined because of the uniqueness result in Theorem B.2.

By the Sobolev-inequality (C.1) and the boundedness of the sequence $\|z_n\|_{E(J_n)}$ there exist $\mathbb{N}'' \subset \mathbb{N}'$ and $\eta \in \mathbb{C}^l$ with

$$z_n(0) \to \eta \; (n \in \mathbb{N}'').$$

We define

$$z(x) := \begin{cases} S(x,0)\pi_+(0)\eta + \rho_+(x) & x > 0, \\ S(x,0)(I - \pi_-(0))\eta + \rho_-(x) & x < 0, \end{cases} \tag{2.29}$$

where

$$\rho_+(x) = \int_0^\infty G_+(x,y)r(y)dy, \; x \geq 0$$

and

$$\rho_-(x) = \int_{-\infty}^0 G_-(x,y)r(y)dy, \; x \leq 0.$$

By construction z is an element of $L_2(\mathbb{R}, \mathbb{C}^l)$.

Step 1: The subsequence $(z_n)_{n\in\mathbb{N}''}$ converges to z in the sense

$$\|z_n - z|_{J_n}\|_{L_2(J_n)} \to 0 \; (n \in \mathbb{N}''). \tag{2.30}$$

From the definition of η we find

$$\begin{aligned} &\int_{x_-^n}^0 |S(x,0)(I - \pi_-(0))(z_n(0) - \eta)|^2 dx \\ &\qquad \leq \int_{-\infty}^0 K_-^2 e^{-2\beta_-|x|}|z_n(0) - \eta|^2 dx = \frac{K_-^2}{2\beta_-}|z_n(0) - \eta|^2 \to 0 \; (n \in \mathbb{N}'') \end{aligned} \tag{2.31}$$

and similarly

$$\int_0^{x_+^n} |S(x,0)\pi_+(0)(z_n(0) - \eta)|^2 dx \to 0 \; (n \in \mathbb{N}''). \tag{2.32}$$

The estimates (2.31) and (2.32) show

$$\begin{aligned} &\|S(x,0)\pi_+(0)(z_n(0) - \eta)\|_{L_2([0,\infty))} + \|S(x,0)(I - \pi_-(0))(z_n(0) - \eta)\|_{L_2((-\infty,0])} \\ &\qquad \to 0 \; (n \in \mathbb{N}''). \end{aligned} \tag{2.33}$$

Furthermore, we find

$$\begin{aligned} &\int_{x_-^n}^0 \Big| \int_{x_-^n}^0 G_-(x,y)h_n(y)dy - \int_{-\infty}^0 G_-(x,y)h(y)dy \Big|^2 dx \\ &\leq 2\int_{x_-^n}^0 \left(\int_{x_-^n}^0 |G_-(x,y)||h_n(y) - h(y)|dy \right)^2 dx + 2\int_{x_-^n}^0 \left(\int_{-\infty}^{x_-^n} |G_-(x,y)||h(y)|dy \right)^2 dx, \end{aligned}$$

where we used $(a+b)^2 \le 2a^2 + 2b^2$. Using the Cauchy-Schwarz inequality, the Theorem of Fubini and the convergence (2.27) we obtain for the first summand

$$\begin{aligned}
&\int_{x_-^n}^0 \left(\int_{x_-^n}^0 |G_-(x,y)||h_n(y) - h(y)|dy \right)^2 dx \\
&\qquad \le \int_{x_-^n}^0 \left(\int_{x_-^n}^0 K_- e^{-\beta_-|x-y|}|h_n(y) - h(y)|dy \right)^2 dx \\
&\qquad \le \int_{x_-^n}^0 \int_{x_-^n}^0 K_-^2 e^{-\beta_-|x-y|}dy \int_{x_-^n}^0 e^{-\beta_-|x-y|}|h_n(y) - h(y)|^2 dydx \\
&\qquad \le \frac{2K_-^2}{\beta_-} \int_{x_-^n}^0 |h_n(y) - h(y)|^2 \int_{x_-^n}^0 e^{-\beta_-|x-y|}dx\, dy \\
&\qquad \le \frac{4K_-^2}{\beta_-^2} \|h_n - h|_{J_n}\|_{L_2(J_n)}^2 \to 0 \ (n \in \mathbb{N}'').
\end{aligned}$$

Similarly it holds

$$\int_{x_-^n}^0 \left(\int_{-\infty}^{x_-^n} |G_-(x,y)||h(y)|dy \right)^2 dx \le \frac{K_-^2}{\beta_-^2} \int_{-\infty}^{x_-^n} |h(y)|^2 dy \to 0 \ (n \in \mathbb{N}'')$$

since $h \in L_2(\mathbb{R}, \mathbb{C}^l)$. These computations show

$$\int_{x_-^n}^0 \Big| \int_{x_-^n}^0 G_-(x,y)h_n(y)dy - \int_{-\infty}^0 G_-(x,y)h(y)dy \Big|^2 dx \to 0 \ (n \in \mathbb{N}'') \qquad (2.34)$$

and in the same fashion

$$\int_0^{x_+^n} \Big| \int_0^{x_+^n} G_+(x,y)h_n(y)dy - \int_0^{+\infty} G_+(x,y)h(y)dy \Big|^2 dx \to 0 \ (n \in \mathbb{N}''). \quad (2.35)$$

The estimates (2.34) and (2.35) prove

$$\|\rho_-(x,x_-^n) - \rho_-(x)\|_{L_2([x_-^n,0])} + \|\rho_+(x,x_+^n) - \rho_+(x)\|_{L_2([0,x_+^n])} \to 0 \ (n \in \mathbb{N}''). \ (2.36)$$

Now we show

$$\int_{x_-^n}^0 |S(x,x_-^n)\pi_-(x_-^n)z_n(x_-^n)|^2 dx + \int_0^{x_+^n} \big|S(x,x_+^n)\{I - \pi_+(x_+^n)\}z_n(x_+^n)\big|^2 dx$$
$$\to 0 \ (n \in \mathbb{N}''). \quad (2.37)$$

Because of the estimates

$$\int_{x_-^n}^0 |S(x,x_-^n)\pi_-(x_-^n)z_n(x_-^n)|^2 dx \le \frac{K_-^2}{2\beta_-}|\pi_-(x_-^n)z_n(x_-^n)|^2$$

and

$$\int_0^{x_+^n} |S(x,x_+^n)(I - \pi_+(x_+^n))z_n(x_+^n)|^2 dx \le \frac{K_+^2}{2\beta_+}|(I - \pi_+(x_+^n))z_n(x_+^n)|^2$$

it suffices to prove

$$\pi_-(x_-^n)z_n(x_-^n) \to 0 \ (n \in \mathbb{N}''), \text{ and } (I - \pi_+(x_+^n))z_n(x_+^n) \to 0 \ (n \in \mathbb{N}''). \tag{2.38}$$

To get a hand on these terms we use $|s_n| = |P_- z_n(x_-) + P_+ z_n(x_+)| \to 0$ $(n \in \mathbb{N}')$ which follows from (2.27). We insert the representation (2.28) into the boundary operator and obtain

$$\begin{aligned} Rz_n &= P_- z_n(x_-^n) + P_+ z_n(x_+^n) \\ &= P_- \pi_-(x_-^n) z_n(x_-^n) + P_+ \big(I - \pi_+(x_+^n)\big) z_n(x_+^n) \\ &\quad + P_- S(x_-^n, 0)\big(I - \pi_-(0)\big) z_n(0) + P_+ S(x_+^n, 0)\pi_+(0) z_n(0) \\ &\quad + P_- \int_{x_-^n}^0 G_-(x_-^n, y) h_n(y) dy + P_+ \int_0^{x_+^n} G_+(x_+^n, y) h_n(y) dy. \end{aligned} \tag{2.39}$$

The uniform boundedness of $\|z_n\|_\infty$, $n \in \mathbb{N}''$, implies

$$\begin{aligned} &|P_- S(x_-^n, 0)\big(I - \pi_-(0)\big) z_n(0)| + \big|P_+ S(x_+^n, 0)\pi_+(0) z_n(0)\big| \\ &\quad \le \big(|P_-|\, K_- e^{-\beta_-|x_-^n|} + |P_+|\, K_+ e^{-\beta_+|x_+^n|}\big)\big)|z_n(0)| \to 0 \ (n \in \mathbb{N}''). \end{aligned} \tag{2.40}$$

Next we show

$$|P_- \int_{x_-^n}^0 G_-(x_-^n, y) h_n(y) dy| \to 0 \ (n \in \mathbb{N}''), \tag{2.41}$$

$$|P_+ \int_0^{x_+^n} G_+(x_+^n, y) h_n(y) dy| \to 0 \ (n \in \mathbb{N}''). \tag{2.42}$$

Before we prove these, note that because of (2.27) and the density of $\mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ in $L_2(\mathbb{R}, \mathbb{C}^l)$ (see [Alt99, Satz 2.14]), for every $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}''$ and a function $\tilde{h} \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ with

$$\|h_n - \tilde{h}|_{J_n}\|^2_{L_2(J_n, \mathbb{C}^l)} \le \varepsilon \ \forall n \ge n_0, \ n \in \mathbb{N}''.$$

Now use this to prove (2.41) and (2.42). The triangle inequality yields

$$\begin{aligned} &\left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n - y|} |h_n(y)| dy\right)^2 \\ &\quad \le 2\left\{\left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n - y|} |\tilde{h}(y) - h_n(y)| dy\right)^2 + \left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n - y|} |\tilde{h}(y)| dy\right)^2\right\}. \end{aligned}$$

With the Cauchy-Schwarz inequality one can estimate the first summand by

$$\left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n - y|} |\tilde{h}(y) - h_n(y)| dy\right)^2 \le \frac{1}{2\beta_-} \|\tilde{h}|_{J_n} - h_n\|^2_{L_2(J_n)} \le \frac{\varepsilon}{2\beta_-} \ \forall n \ge n_0, n \in \mathbb{N}''.$$

For the second summand we note that there is a constant $C_0 > 0$ with

$$|\tilde{h}(y)| \le C_0 e^{-\max(\beta_+, \beta_-)|y|},$$

thus

$$\left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n-y|}|\tilde{h}(y)|dy\right)^2 \le C_0^2\left(\int_{x_-^n}^0 e^{-\beta_-|x_-^n-y|}e^{-\beta_-|y|}dy\right)^2$$
$$\le C_0^2\left(\int_{x_-^n}^{\frac{x_-^n}{2}} e^{-\beta_-|x_-^n-y|}e^{-\beta_-\frac{|x_-^n|}{2}}dy + \int_{\frac{x_-^n}{2}}^0 e^{-\beta_-\frac{|x_-^n|}{2}}e^{-\beta_-|y|}dy\right)^2$$
$$\le \frac{4C_0^2}{\beta_-^2}e^{-\beta_-|x_-^n|} \to 0 \ (n \in \mathbb{N}'').$$

These estimates prove (2.41) and in a similar way one obtains (2.42).

Inserting the results of (2.27), (2.40), (2.41), and (2.42) into (2.39) shows

$$P_-\pi_-(x_-^n)z_n(x_-^n) + P_+\big(I - \pi_+(x_+^n)\big)z_n(x_+^n) \to 0 \ (n \in \mathbb{N}''). \tag{2.43}$$

Now we denote by $\bar{\pi}_\pm$ the projectors of the exponential dichotomies on $\mathbb{R}$ of the constant coefficient operators $L_\pm z = z_x - M_\pm z$. From Theorem B.5 we know

$$|\bar{\pi}_+ - \pi_+(x)| \xrightarrow[x\to\infty]{} 0$$

and

$$|\bar{\pi}_- - \pi_-(x)| \xrightarrow[x\to-\infty]{} 0.$$

This convergence of the projectors and the equalities $\mathcal{R}(V_-^{II}) = \mathcal{R}(\bar{\pi}_-)$ as well as $\mathcal{R}(V_+^I) = \mathcal{R}(I - \bar{\pi}_+)$ yield that there is $n_0 \in \mathbb{N}''$ so that for all $n \ge n_0$, $n \in \mathbb{N}''$, there are $\alpha_n \in \mathbb{C}^p$ and $\beta_n \in \mathbb{C}^r$ with

$$\pi_-(x_-^n)z_n(x_-^n) = \pi_-(x_-^n)V_-^{II}\alpha_n, \quad (I - \pi_+(x_+^n))z_n(x_+^n) = (I - \pi_+(x_+^n))V_+^I\beta_n.$$

Now we can write the left hand side of (2.43) in the form

$$\begin{aligned}
&P_-\pi_-(x_-^n)z_n(x_-^n) + P_+(I - \pi_+(x_+^n))z_n(x_+^n)\\
&= P_-\bar{\pi}_-V_-^{II}\alpha_n + P_+(I-\bar{\pi}_+)V_+^I\beta_n + P_-(\pi_-(x_-^n)-\bar{\pi}_-)V_-^{II}\alpha_n + P_+(\bar{\pi}_+-\pi_+(x_+^n))V_+^I\beta_n\\
&= \begin{pmatrix} P_-V_-^{II} & P_+V_+^I \end{pmatrix}\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} + \begin{pmatrix} P_-(\pi_-(x_-^n) - \bar{\pi}_-)V_-^{II} & P_+(\bar{\pi}_+ - \pi_+(x_+^n))V_+^I \end{pmatrix}\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}.
\end{aligned}$$

Because of the convergence

$$\Big(P_-(\pi_-(x_-^n) - \bar{\pi}_-)V_-^{II}\,,\, P_+(\bar{\pi}_+ - \pi_+(x_+^n))V_+^I\Big) \to 0 \ (J \in H)$$

the Banach-Lemma A.1 implies that there is $N_0 \ge n_0$ so that for all $n \in \mathbb{N}''$ with $n \ge N_0$ the matrix

$$A(x_-^n, x_+^n) := \begin{pmatrix} P_-V_-^{II} & P_+V_+^I \end{pmatrix} + \Big(P_-\big(\pi_-(x_-^n) - \bar{\pi}_-\big)V_-^{II}\,,\, P_+(\bar{\pi}_+ - \pi_+(x_+^n))V_+^I\Big)$$

is invertible and the inverse is uniformly bounded for all $n \ge N_0$, $n \in \mathbb{N}''$. Therefore α_n and β_n are uniquely determined for all $n \in \mathbb{N}''$ with $n \ge N_0$ and because of (2.43) we also obtain $(\alpha_n, \beta_n) \to 0$ $(n \in \mathbb{N}'')$.

Then the boundedness of $\pi_-(x)$ and $\pi_+(x)$ and the convergence of α_n, β_n imply

$$\pi_-(x_-^n)z_n(x_-^n) = \pi_-(x_-^n)V_-^{II}\alpha_n \xrightarrow[n\to\infty]{} 0,\ n \in \mathbb{N}'',$$
$$(I - \pi_+(x_+^n))z_n(x_+^n) = (I - \pi_+(x_+^n))V_+^I\beta_n \xrightarrow[n\to\infty]{} 0,\ n \in \mathbb{N}''.$$

Hence (2.38) follows. And this finishes the proof of (2.37) which states

$$\|S(x,x_-^n)\pi_-(x_-^n)z_n(x_-^n)\|_{L_2([x_-^n,0])} + \|S(x,x_+^n)(I-\pi_+(x_-^n))z_n(x_+^n)\|_{L_2([0,x_+^n])} \to 0\ (n \in \mathbb{N}''). \quad (2.44)$$

Now adding (2.33), (2.36), and (2.44) proves the assertion of step 1.

Step 2: The limit z from step 1 is an element of $H^1(\mathbb{R},\mathbb{C}^l)$ and satisfies the differential equation $Lz = h$ in $L_2(\mathbb{R},\mathbb{C}^l)$.

By construction z is an element of $L_2(\mathbb{R},\mathbb{C}^l)$ thus it remains to show that the distributional derivative can be represented by some element $w \in L_2(\mathbb{R},\mathbb{C}^l)$. Therefore define

$$w := M(\cdot)z + h. \quad (2.45)$$

From the boundedness of M and $z,h \in L_2(\mathbb{R},\mathbb{C}^l)$, w is an element of $L_2(\mathbb{R},\mathbb{C}^l)$. We show that w is the distributional derivative of z, i.e.

$$(w,\phi)_{L_2(\mathbb{R},\mathbb{C}^l)} = -(z,\phi')_{L_2(\mathbb{R},\mathbb{C}^l)}\ \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{C}^l).$$

Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{C}^l)$ be arbitrary. Because of the cofinality of the sequence $(J_n)_{n\in\mathbb{N}}$ there is $n_0 \in \mathbb{N}''$ with $J_n \supset \operatorname{supp}(\phi)\ \forall n \geq n_0, n \in \mathbb{N}''$. For all $n \geq n_0$, $n \in \mathbb{N}''$, holds

$$\begin{aligned}
|(w,\phi)_{L_2(\mathbb{R})} + (z,\phi')_{L_2(\mathbb{R})}| &= |(w|_{J_n},\phi)_{L_2(J_n)} + (z|_{J_n},\phi')_{L_2(J_n)}| \\
&\leq |(w|_{J_n},\phi)_{L_2(J_n)} - (z_{n,x},\phi)_{L_2(J_n)}| + |(z|_{J_n},\phi')_{L_2(J_n)} - (z_n,\phi')_{L_2(J_n)}| \\
&\leq \|w|_{J_n} - z_{n,x}\|_{L_2(J_n)}\|\phi\|_{L_2} + \|z|_{J_n} - z_n\|_{L_2(J_n)}\|\phi'\|_{L_2} \\
&\leq \|\phi\|_{H^1}\left\{\|w|_{J_n} - z_{n,x}\|_{L_2(J_n)} + \|z|_{J_n} - z_n\|_{L_2(J_n)}\right\} \\
&\leq \|\phi\|_{H^1}\left\{\|M\|_\infty\|z|_{J_n} - z_n\|_{L_2(J_n)} + \|h|_{J_n} - h_n\|_{L_2(J_n)} + \|z|_{J_n} - z_n\|_{L_2(J_n)}\right\}
\end{aligned}$$

where $(u,v)_{L_2}$ denotes the usual L_2-inner product. In the estimates we have used the definition of w and that $z_n \in H^1(J_n,\mathbb{C}^l)$ satisfies $z_{n,x} = M(\cdot)z_n + h_n$ in $L_2(J_n,\mathbb{C}^l)$. Now (2.30) and (2.27) imply

$$|(w,\phi)_{L_2} + (z,\phi')_{L_2}| \xrightarrow[n\to\infty]{} 0,\ n \in \mathbb{N}''.$$

Thus step 2 follows since the considerations from above show $z_x = w \in L_2(\mathbb{R},\mathbb{C}^l)$ and therefore

$$Lz = z_x - M(\cdot)z = w - M(\cdot)z = h \text{ in } L_2(\mathbb{R},\mathbb{C}^l).$$

Now the $\mathcal{P}$-convergence of $(z_n)_{n\in\mathbb{N}''}$ to z follows from

$$\begin{aligned}
\|z|_{J_n} - z_n\|^2_{H^1(J_n)} &= \|z|_{J_n} - z_n\|^2_{L_2(J_n)} + \|z_x|_{J_n} - z_{n,x}\|^2_{L_2(J_n)} \\
&\leq \|z|_{J_n} - z_n\|^2_{L_2(J_n)} + 2\left\{\|M\|^2_\infty\|z|_{J_n} - z_n\|^2_{L_2(J_n)} + \|h|_{J_n} - h_n\|^2_{L_2(J_n)}\right\} \\
&\xrightarrow[n\to\infty]{} 0,\ n \in \mathbb{N}''.
\end{aligned}$$

This finishes the proof of Theorem 2.29. □

3 The hyperbolic case

In this chapter we consider a linear strictly hyperbolic PDE of the form

$$v_t = Pv, \quad \text{in } [0,\infty) \times \mathbb{R}. \tag{3.1}$$

The operator P is given by

$$Pv = Bv_x + Cv. \tag{3.2}$$

For example (3.1) may be obtained as in the introduction by linearization at a travelling wave solution.

3.1 Assumptions

For the coefficients of P we make the following assumptions.

Assumption 1.

(H1) *The matrix-valued functions $B \in \mathcal{C}^2(\mathbb{R}, \mathbb{C}^{m,m})$ and $C \in \mathcal{C}^1(\mathbb{R}, \mathbb{C}^{m,m})$ satisfy*

$$\exists \lim_{x\to\pm\infty} B(x) =: B_\pm \ \text{ and } \exists \lim_{x\to\pm\infty} B_x(x) = 0,$$
$$\exists \lim_{x\to\pm\infty} C(x) =: C_\pm.$$

Furthermore we assume that their second respectively first derivative are uniformly bounded

$$\|B_{xx}\|_\infty < \infty, \|C_x\|_\infty < \infty.$$

(H2) *For every $x \in \mathbb{R}$ the matrix $B(x)$ is a real diagonal-matrix, where the first r entries are positive and the last $m-r$ entries are negative. The matrix satisfies a uniform invertibility condition, i.e. $\exists b_0 > 0$ with*

$$\begin{aligned} b_{ii}(x) &\geq b_0 \ \forall x \in \mathbb{R}, 1 \leq i \leq r, \\ -b_{ii}(x) &\geq b_0 \ \forall x \in \mathbb{R}, r+1 \leq i \leq m, \end{aligned} \tag{3.3}$$

as well as a uniform gap condition. There is $\gamma > 0$ with

$$|b_{ii}(x) - b_{jj}(x)| \geq \gamma \ \forall x \in \mathbb{R}, \ i \neq j. \tag{3.4}$$

(H3) *For the limit matrices $C_\pm$ the real parts of the diagonal elements are bounded from above by*

$$\operatorname{Re} C_{\pm jj} \leq -2\delta < 0$$

for some $\delta > 0$.

(H4) *For any $\omega \in \mathbb{R}$ it holds that*

$$s \in \sigma(i\omega B_+ + C_+)$$

or

$$s \in \sigma(i\omega B_- + C_-)$$

implies $\operatorname{Re} s \leq -\delta$. *Here* $\sigma(i\omega B_+ + C_+)$ *denotes the spectrum of the matrix* $i\omega B_+ + C_+$.

From assumption (H1) directly follows the boundedness of $\|B\|_\infty$, $\|B_x\|_\infty$ and $\|C\|_\infty$. We define

$$\|B\|_\infty =: B_0 < \infty. \tag{3.5}$$

Throughout the text we denote by $\mathcal{M}(\delta, c)$ the subset of the complex plane defined by

$$\mathcal{M}(\delta, c) := \{s \in \mathbb{C} : \operatorname{Re} s > -\delta, |s| > c\}. \tag{3.6}$$

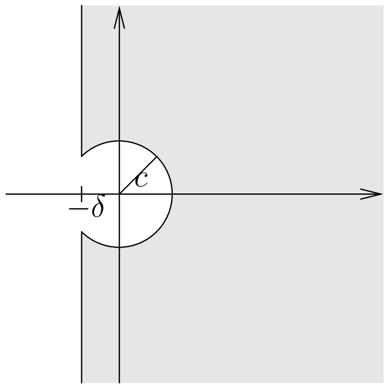


Figure 3.1: Illustration of the subset $\mathcal{M}(\delta, c)$ in the complex plane.

3.2 Resolvent estimates for large $|s|$

This section is in many parts a rigorous carry out of the proofs from the personal notes of J. Lorenz [Lor99]. At some places we also improved the estimates which was necessary for the application of the results in the mixed case in Chapter 4. A crucial part in this section is the application of Lemma A.7. We state and prove the lemma in Appendix A since the proof is quite long and would interrupt the line of argumentation.

The goal in this section is to show that the all line operator

$$Pv = Bv_x + Cv, \quad P : H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^m), \tag{3.7}$$

and its restriction to a finite interval $J = [x_-, x_+]$, J large enough, both have no spectrum in the region $\mathcal{M}(\delta, c)$ for large c. More specific we will show uniform resolvent estimates in this region. Analogous results for parabolic systems which we will review in Section 4.2 are given in [BL99, Chapter 2].

Denote by $P|_J$ the operator P on the finite interval $J = [x_-, x_+]$. For the boundary value problem on the infinite line the boundary conditions are given in the domain of the operator P, but for the finite interval boundary value problem one has to provide explicit boundary conditions. We will always assume linear two point boundary conditions. Therefore consider

$$\begin{aligned} P|_J v &= B v_x + C v, \ P|_J : H^1(J) \to L_2(J) \\ R v &= R_- v(x_-) + R_+ v(x_+), \ R : H^1(J) \to \mathbb{C}^m \end{aligned} \tag{3.8}$$

as an approximation of the all line operator on finite intervals. Here $R_\pm \in \mathbb{C}^{m,m}$ are constant matrices.

In the sequel we will often partition the boundary operator in the form $R_\pm = \left(R_\pm^I, R_\pm^{II}\right)$, where $R_\pm^I \in \mathbb{C}^{m,r}$ and $R_\pm^{II} \in \mathbb{C}^{m,m-r}$, corresponding to the splitting of B into positive and negative real parts.

Theorem 3.1. *Assume that (H1), (H2), and (H3) hold. Then there exist positive constants C_0, K such that for all $s \in \mathcal{M}(\delta, C_0)$ and every $F \in L_2(\mathbb{R}, \mathbb{C}^m)$ the all line problem*

$$(sI - P)v = F \text{ in } L_2(\mathbb{R}, \mathbb{C}^m) \tag{3.9}$$

has a unique solution $v \in H^1(\mathbb{R}, \mathbb{C}^m)$. Furthermore this solution can be estimated by

$$\|v\|^2 \le K \|F\|^2. \tag{3.10}$$

If in addition $F \in H^1(\mathbb{R}, \mathbb{C}^m)$, then the estimate can be improved to

$$\|v\|^2 + \|v_x\|^2 \le K(\|F\|^2 + \|F_x\|^2). \tag{3.11}$$

Moreover there exists $\delta' > 0$ such that for all $s \in \mathbb{C}$ with $|s| > C_0$ and $\mathrm{Re}(s) > \delta'$ the estimates (3.10) and (3.11) can be improved to

$$\mathrm{Re}(s)^2 \|v\|^2 \le K \|F\|^2 \tag{3.12}$$

and

$$\mathrm{Re}(s)^2 \left\{ \|v\|^2 + \|v_x\|^2 \right\} \le K \left\{ \|F\|^2 + \|F_x\|^2 \right\} \tag{3.13}$$

respectively.
In all inequalities the constant K is independent of s and F.

The analogous result for the restricted problem is given in the next theorem. In the theorem we consider functions at the endpoints of the intervals and use $|v|_\Gamma^2 := |v(x_-)|^2 + |v(x_+)|^2$ as a norm on the boundary. Note that the boundary norm also makes sense for functions $v \in H^1(J)$ because of Lemma C.2.

Theorem 3.2. *We assume (H1), (H2), (H3), as well as the determinant-condition*

$$D_\infty := \det(R_-^{II}, R_+^{I}) \neq 0. \tag{3.14}$$

Then there exist positive constants C_0, b, and K such that for all $s \in \mathcal{M}(\delta, C_0)$, all $J = [x_-, x_+] \supset [-b, b]$, all $F \in L_2(J, \mathbb{C}^m)$, and all $\eta \in \mathbb{C}^m$, there is a unique solution $v \in H^1(J, \mathbb{C}^m)$ of

$$(sI - P)v = F, \text{ in } L_2(J, \mathbb{C}^m), \quad Rv = \eta. \tag{3.15}$$

This solution can be estimated by

$$\|v\|^2 + |v|_\Gamma^2 \le K\{|\eta|^2 + \|F\|^2\}. \tag{3.16}$$

Furthermore there exists $\delta' > 0$ so that for all $s \in \mathbb{C}$ with $|s| > C_0$ and $\operatorname{Re} s > \delta'$, the estimate (3.16) can be improved to

$$\operatorname{Re}(s)^2\|v\|^2 + \operatorname{Re}(s)|v|_\Gamma^2 \le K\big\{\|F\|^2 + \operatorname{Re}(s)|\eta|^2\big\}. \tag{3.17}$$

In all inequalities the constant K does not depend on x_-, x_+, s, F, and η.

The inequalities (3.12), (3.13), and (3.17) are improvements of the resolvent estimates claimed in the notes [Lor99]. These estimates are also necessary for the operators P and $P|_J$ to generate C_0-semigroups on the respective spaces, but they are not sufficient since an estimate of this type is necessary for all powers of the resolvent and the constant K must be independent of the powers. For the theory of semigroups see for example [RR93, Chapter 11] or [Paz83].

Remark. *Note that the improved estimates only hold if one is far enough to the right of the imaginary axis, but the original estimates also hold in some parts to the left of the imaginary axis.*

In order to prove the Theorems we now follow the line of proof from [BL99, Chapter 2] and [Lor99]:
In a first step we transform the resolvent equation such that it is written as a perturbation of a diagonal system. For the diagonal system we show that it has an (ED) on $\mathbb{R}$ and with the Roughness Theorem for (ED)s B.4 we conclude the same for the perturbed system. This is done in 3.2.1.
In the second step we conclude in 3.2.2 from this result Theorem 3.1.
In 3.2.3 we follow the proof of [BL99][Theorem 2.1] to prove Theorem 3.2.

Until the end of this section we will always assume **(H1)**, **(H2)**, and **(H3)**.

3.2.1 Exponential dichotomies for large $|s|$

Rewrite the resolvent-equation

$$(sI - P)v = F \text{ in } L_2(\mathbb{R}, \mathbb{C}^m)$$

in the form

$$v_x = (sB^{-1} - B^{-1}C)v - B^{-1}F. \tag{3.18}$$

From the assumptions (H1) and (H2) one obtains the bounds

$$\|B^{-1}\|_\infty < \infty,\ \|B^{-1}C\|_\infty < \infty,\ \|(B^{-1})_x\|_\infty < \infty,\ \|(B^{-1}C)_x\|_\infty < \infty.$$

Furthermore by (H2) the matrix-valued function B^{-1} fulfills a uniform gap-condition of the form

$$\left|\frac{1}{b_i(x)} - \frac{1}{b_j(x)}\right| \geq \frac{\gamma}{\|B\|_\infty^2} > 0\ \forall x \in \mathbb{R}, i \neq j.$$

Therefore we can apply Lemma A.7 with $D = B^{-1}$ and $E = B^{-1}C$. This proves the existence of an $\varepsilon > 0$ such that there is a matrix-valued function

$$\tilde{T} : (x,r) \mapsto \tilde{T}(x,r) = I + r\tilde{T}_1(x,r) \in \mathcal{C}(\mathbb{R} \times \{|r| < \varepsilon\}, \mathrm{GL}_m(\mathbb{C}))$$

with for all $(x,r) \in \mathbb{R} \times \{|r| < \varepsilon\}$ is $\tilde{T}(x,r)^{-1}\big(B(x)^{-1} + rB(x)^{-1}C(x)\big)\tilde{T}(x,r)$ a diagonal-matrix and $\|\tilde{T}_1\|_\infty \leq C_{T,0} < \infty$. The Lemma also shows that $\tilde{T}_1$ is differentiable with respect to x, $\tilde{T}_{1,x}$ is continuous and also uniformly bounded $\|\tilde{T}_{1,x}\|_\infty =: C_{T,1} < \infty$.
Now choose

$$C_a \geq \max(\frac{1}{\varepsilon}, \frac{2}{C_{T,0}}) \tag{3.19}$$

and define as in (A.5)

$$T_1(x,s) := \tilde{T}_1(x, \frac{1}{s}), \quad \text{for } x \in \mathbb{R}, |s| > C_a,$$

and

$$T(x,s) := I + \frac{1}{s}T_1(x,s) \ \text{ for } x \in \mathbb{R}, |s| > C_a. \tag{3.20}$$

By the choice of C_a the Banach-Lemma (Lemma A.1) is applicable and yields (see (A.7))

$$\|T^{-1}\|_\infty \leq 2. \tag{3.21}$$

The construction of T shows that the matrix-valued function

$$\Lambda : (x,s) \mapsto \Lambda(x,s) := T(x,s)^{-1}\big(sB(x)^{-1} - B(x)^{-1}C(x)\big)T(x,s)$$

is diagonal-matrix-valued.
Moreover Lemma A.6 and Lemma A.7 imply the estimate

$$|\Lambda_{ii}(x,s) - (sb_{ii}(x)^{-1} - b_{ii}(x)^{-1}c_{ii}(x))| \leq \frac{C_\Lambda}{|s|} \tag{3.22}$$

for the eigenvalues. Here C_Λ is a positive constant independent of x and s.

Now we transform the system (3.18) by using the variables

$$Tz := v. \tag{3.23}$$

Then (3.18) becomes

$$z_x = (\Lambda - T^{-1}T_x)z + \bar{F}, \tag{3.24}$$

where $\bar{F}(x,s) = -T^{-1}(x,s)B^{-1}(x)F(x)$. We denote by $L(\cdot,s)$ the differential operator

$$L(\cdot,s)z = z_x - (\Lambda(\cdot,s) - T^{-1}(\cdot,s)T_x(\cdot,s))z \tag{3.25}$$

and write the transformed resolvent equation (3.24) in the form

$$L(x,s)z = \bar{F}. \tag{3.26}$$

We show the property of (ED)s for the diagonal part of (3.26).

Lemma 3.3. *There exists $C_b > C_a$ such that for all $s \in \mathcal{M}(\delta, C_b)$ the diagonal operators $\tilde{L}(x,s)$ given by*

$$\tilde{L}(x,s)z = z_x - \Lambda(x,s)z \tag{3.27}$$

have an (ED) on $\mathbb{R}$ where the data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ can be chosen independently of s. In particular on can choose

$$\tilde{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$$

as projector of the (ED) and

$$\tilde{\beta} = \frac{\delta}{4B_0}$$

as exponent of the (ED).

For the proof of Lemma 3.3 we need the following auxiliary result for scalar equations.

Lemma 3.4. *Let $\lambda \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ and let $\xi_- \leq \xi_+ \in \mathbb{R}$ with*

$$\operatorname{Re} \lambda(x) \leq -\alpha < 0, \ \forall x \in \mathbb{R} \setminus (\xi_-, \xi_+), \tag{3.28}$$

or

$$\operatorname{Re} \lambda(x) \geq \alpha > 0, \ \forall x \in \mathbb{R} \setminus (\xi_-, \xi_+). \tag{3.29}$$

then the scalar equation

$$Lu = u_x - \lambda u \tag{3.30}$$

has an (ED) on $\mathbb{R}$ with data (K, β, π), where we can choose

$$K = \max_{\xi_- \leq x \leq y \leq \xi_+} \exp\Big(\int_x^y \operatorname{Re} \lambda(\xi) d\xi\Big) \exp(\alpha(\xi_+ - \xi_-)), \tag{3.31}$$

$$\beta = \alpha, \tag{3.32}$$

$$\pi = 1, \tag{3.33}$$

in the case of (3.28) *and it is possible to choose*

$$K = \max_{\xi_- \leq x \leq y \leq \xi_+} \exp\Big(\int_x^y -\operatorname{Re} \lambda(\xi) d\xi\Big) \exp(\alpha(\xi_+ - \xi_-)), \tag{3.34}$$

$$\beta = \alpha, \tag{3.35}$$

$$\pi = 0 \tag{3.36}$$

in the case of (3.29).

Proof. The solution-operator of (3.30) is given by

$$S(x,y) = \exp(\int_y^x \lambda(\xi)d\xi).$$

First we assume the case of (3.28). Then for the data as in (3.31)–(3.33) we have

$$0 = |S(x,y)(1-\pi(y))| \leq Ke^{-\beta|x-y|} \; \forall x < y.$$

Therefore it remains to show that for all $y \leq x$ the inequality

$$|S(x,y)\pi(y)| \leq Ke^{-\beta|x-y|} \tag{3.37}$$

holds.

1. In the case $y \leq x < \xi_-$ we can estimate

$$\begin{aligned} |S(x,y)\pi(y)| &= |\exp(\int_y^x \lambda(\xi)d\xi)| = \exp(\int_y^x \operatorname{Re}\lambda(\xi)d\xi) \\ &\leq \exp(\int_y^x -\alpha d\xi) \leq K\exp(-\beta|x-y|) \end{aligned}$$

   and so inequality (3.37) holds.

2. In the case $y \leq \xi_- \leq x \leq \xi_+$ we split the integral and obtain the estimate

$$\begin{aligned} |S(x,y)\pi(y)| &= |\exp(\int_y^x \lambda(\xi)d\xi)| = \exp(\int_y^x \operatorname{Re}\lambda(\xi)d\xi) \\ &\leq \exp(\int_y^{\xi_-} -\alpha d\xi)\exp(\int_{\xi_-}^x \operatorname{Re}\lambda(\xi)d\xi) \leq K\exp(-\beta|x-y|). \end{aligned}$$

   This proves (3.37) for this case.

3. Similarly we split the integral in the case $y \leq \xi_- \leq \xi_+ < x$

$$\begin{aligned} |S(x,y)\pi(y)| &= |\exp(\int_y^x \lambda(\xi)d\xi)| = \exp(\int_y^x \operatorname{Re}\lambda(\xi)d\xi) \\ &\leq \exp(\int_y^{\xi_-} -\alpha d\xi)\exp(\int_{\xi_-}^{\xi_+} \operatorname{Re}\lambda(\xi)d\xi)\exp(\int_{\xi_+}^x -\alpha d\xi) \\ &\leq K\exp(-\beta|x-y|). \end{aligned}$$

4. The cases $\xi_- < y \leq x \leq \xi_+$ and $\xi_- < y \leq \xi_+ < x$ are shown in the same way.

The proof under the assumption (3.29) is similar. □

We could apply Lemma 3.4 directly to (3.27) and see that these operators have (ED)s on $\mathbb{R}$ for all $s \in \mathcal{M}(\delta, C_a)$, where C_a is the constant given in (3.19). Since we aim for estimates independent of s in Theorem 3.1 and Theorem 3.2 one has to be more accurate.

Proof of Lemma 3.3. By assumption (H3) for every $i = 1, \dots, m$, there exist $\xi_-(i) \le \xi_+(i) \in \mathbb{R}$ with

$$\operatorname{Re} c_{ii}(x) \le -\frac{3}{2}\delta \quad \forall x \in \mathbb{R} \setminus (\xi_-(i), \xi_+(i)).$$

Let $\xi_- := \min_{i=1,\dots,m} \xi_-(i)$ and $\xi_+ := \max_{i=1,\dots,m} \xi_+(i)$. By equation (3.22) one can choose $C_b > C_a$ such that for every $i = 1, \dots, m$ one has

$$|\Lambda_{ii}(x,s) - \frac{1}{b_{ii}(x)}(s - c_{ii}(x))| \le \frac{\delta}{4B_0}, \ \forall |s| \ge C_b, \ x \in \mathbb{R}.$$

Hence

$$|\operatorname{Re} \Lambda_{ii}(x,s) - \operatorname{Re} \frac{1}{b_{ii}(x)}(s - c_{ii}(x))| \le \frac{\delta}{4B_0}, \ \forall |s| \ge C_b, \ x \in \mathbb{R}. \tag{3.38}$$

This shows that for all $s \in \mathcal{M}(\delta, C_b)$ and all $x \in \mathbb{R} \setminus (\xi_-, \xi_+)$ one can estimate

$$\begin{aligned} \operatorname{Re} \Lambda_{ii}(x,s) &\ge \frac{1}{b_{ii}(x)} \operatorname{Re}(s - c_{ii}(x)) - \frac{\delta}{4B_0} \\ &\ge \frac{\delta}{2B_0} - \frac{\delta}{4B_0} = \frac{\delta}{4B_0}, \quad \text{for } 1 \le i \le r. \end{aligned} \tag{3.39}$$

In the case $r + 1 \le i \le m$ one obtains

$$\begin{aligned} \operatorname{Re} \Lambda_{ii}(x,s) &\le \operatorname{Re} \frac{1}{b_{ii}(x)}(s - c_{ii}(x)) + \frac{\delta}{4B_0} \\ &\le -\frac{1}{B_0}\frac{\delta}{2} + \frac{\delta}{4B_0} = -\frac{\delta}{4B_0}. \end{aligned} \tag{3.40}$$

With Lemma 3.4 follows that the diagonal operators $L(\cdot, s)$ have (ED)s on $\mathbb{R}$ with data $(\tilde{K}(s), \tilde{\beta}, \tilde{\pi})$, where

$$\begin{aligned} \tilde{K}(s) = \exp(\beta(\xi_+ - \xi_-)) \cdot \max \Bigg(&\max_{1 \le i \le r} \max_{\xi_- \le x \le y \le \xi_+} \exp(\int_x^y - \operatorname{Re} \Lambda_{ii}(\xi, s) d\xi), \\ &\max_{r+1 \le i \le m} \max_{\xi_- \le x \le y \le \xi_+} \exp(\int_x^y \operatorname{Re} \Lambda_{ii}(\xi, s) d\xi) \Bigg), \end{aligned}$$

$$\tilde{\beta} = \frac{\delta}{4B_0},$$

$$\tilde{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}.$$

We still have to show that $\tilde{K}(s)$ can be bounded uniformly in s. Therefore consider equation (3.38). By estimation in the other direction one obtains
for $1 \le i \le r$

$$-\operatorname{Re} \Lambda_{ii}(x,s) \le \frac{1}{b_0}\big(\delta + \max_{\xi_- \le x \le \xi_+} \operatorname{Re} c_{ii}(x)\big) + \frac{\delta}{4B_0} =: s_i, \tag{3.41}$$

and for $r+1 \leq i \leq m$

$$\operatorname{Re}\Lambda_{ii}(x,s) \leq \frac{1}{b_0}\big(\delta + \max_{\xi_- \leq x \leq \xi_+} \operatorname{Re} c_{ii}(x)\big) + \frac{\delta}{4B_0} =: s_i. \tag{3.42}$$

Hence

$$\tilde{K}(s) \leq \exp\left(\int_{\xi_-}^{\xi_+} \max(\max_{1\leq i\leq m} s_i, 0) d\xi\right) \exp(\beta|\xi_+ - \xi_-|) =: \tilde{K}.$$

□

For the proofs of the improved resolvent-estimates (3.12), (3.13), and (3.17) we need another version of Lemma 3.3. The resulting version will also be needed in the mixed parabolic-hyperbolic case for the proof of a Fredholm alternative.

Lemma 3.5. *There exists a $\tilde{\delta}' > 0$ such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \tilde{\delta}'$ the diagonal operators* (3.27) *have (ED)s on $\mathbb{R}$ with data $(\tilde{K}, \tilde{\beta}(s), \tilde{\pi})$, where*

$$\tilde{K} = 1,\ \tilde{\beta} = c\operatorname{Re}(s),\ \tilde{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}, \tag{3.43}$$

and $c > 0$ is some constant independent of s.

Proof. By assumption (H3) follows

$$\sup_{x\in\mathbb{R}} \max_{i=1,\dots,m} \operatorname{Re} c_{ii}(x) =: m_+ < \infty.$$

Choose $\tilde{\delta}' > C_a$, such that

$$\frac{1}{2B_0} > \frac{|m_+|}{b_0\tilde{\delta}'} + \frac{C_\Lambda}{\tilde{\delta}'^2},$$

where C_Λ is the constant from (3.22).
Then for $1 \leq i \leq r$ one can estimate by (3.22) for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq \tilde{\delta}'$

$$\begin{aligned}
\operatorname{Re}\Lambda_{ii}(x,s) &\geq \operatorname{Re}\left(\frac{1}{b_{ii}(x)}(s - c_{ii}(x))\right) - \frac{C_\Lambda}{\operatorname{Re} s} \\
&\geq \operatorname{Re}(s)\left(\frac{1}{B_0} - \frac{|m_+|}{b_0\operatorname{Re}(s)} - \frac{C_\Lambda}{\operatorname{Re}(s)^2}\right) \\
&\geq \operatorname{Re}(s)\left(\frac{1}{2B_0}\right).
\end{aligned}$$

In the same way one finds for $r+1 \leq i \leq m$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq \tilde{\delta}'$

$$\operatorname{Re}\Lambda_{ii}(x,s) \leq -\operatorname{Re}(s)\left(\frac{1}{2B_0}\right).$$

Now the Lemma follows from Lemma 3.4 with $\xi_- = \xi_+$. □

In the next step we conclude that the operators $L(\cdot, s)$ defined in (3.25) have exponential dichotomies on $\mathbb{R}$ by using the Roughness Theorem B.4. The aim is the following Lemma.

Lemma 3.6. *There exists $\tilde{C}_0 > 0$ such that for all $s \in \mathcal{M}(\delta, \tilde{C}_0)$ the operators $L(\cdot, s)$ have (ED)s on $\mathbb{R}$ with data $(K, \beta, \pi(x, s))$, where K and β do not depend on s. Moreover there is a constant independent of x and s so that*

$$|\pi(x,s) - \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}| \leq \frac{\text{const}}{|s|}. \tag{3.44}$$

Proof. Take $\tilde{C}_0 \geq C_b$ such that $3\tilde{K}\frac{2C_{T,1}}{\tilde{C}_0} < \tilde{\beta}$.
By construction of T and equations (3.21), (3.20), (3.19) follows with Lemma A.1 for all $|s| > \tilde{C}_0$ the estimate

$$\begin{aligned} |T^{-1}(x,s)T_x(x,s)| \leq 2|T_x(x,s)| &\leq \frac{2}{|s|}C_{T,1} \\ &\leq \frac{2C_{T,1}}{\tilde{C}_0} =: \nu. \end{aligned} \tag{3.45}$$

Then we can apply the Roughness Theorem B.4 and obtain that the operators $L(\cdot, s)$ also have (ED)s on $\mathbb{R}$ with data

$$\begin{aligned} K &= \tilde{K}\left(2 + \frac{4\nu\tilde{K}}{\beta - 3\nu\tilde{K}}\right), \\ \beta &= \tilde{\beta} - 2\nu\tilde{K}. \end{aligned}$$

And for the projectors holds the estimate

$$\begin{aligned} |\pi(x,s) - \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}| &\leq K\tilde{K}\int_{\mathbb{R}} e^{-(\beta+\tilde{\beta})|x-y|}|T^{-1}(y,s)T_x(y,s)|dy \\ &\leq \frac{K\tilde{K}2C_{T,1}}{|s|}\int_{\mathbb{R}} e^{-(\beta+\tilde{\beta})|x-y|}dy \\ &\leq \frac{4K\tilde{K}C_{T,1}}{(\beta+\tilde{\beta})|s|} = \frac{\text{const}}{|s|}. \end{aligned}$$

□

If we use Lemma 3.5 instead of Lemma 3.3 we find with much the same proof stronger estimates that will be important for the proof of an (ED) in the mixed hyperbolic-parabolic case in Chapter 4.

Lemma 3.7. *There is a positive constant δ', such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \delta'$ the operators $L(\cdot, s)$ have an (ED) on $\mathbb{R}$ with data $(K, \beta(s), \pi(\cdot, s))$, where the data can be chosen such that*

$$K = 3, \tag{3.46}$$

$$\beta(s) = \text{const}\operatorname{Re}(s), \tag{3.47}$$

and $\pi(x,s)$ can be estimated by

$$\left| \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} - \pi(x,s) \right| \leq \frac{\text{const}}{|s| \operatorname{Re}(s)}. \tag{3.48}$$

Proof. Take $\delta' \geq \tilde{\delta}'$ with $\tilde{\delta}'$ from Lemma 3.5 such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \delta'$ we have

$$3\nu(s) < \tilde{\beta}(s) = c \operatorname{Re}(s),$$

where $\nu(s)$ is given by

$$\nu(s) := \frac{2C_{T,1}}{\operatorname{Re} s} \geq |T^{-1}(x,s)T_x(x,s)|$$

(cf. (3.45)). Then we can apply Theorem B.4 because of the inequalities

$$\tilde{\beta}(s) > 3\nu(s) \geq 3 \cdot |T^{-1}(x,s)T_x(x,s)|$$

the required inequality (B.8) is satisfied. Hence Theorem B.4 yields an (ED) on $\mathbb{R}$ for the operators $L(\cdot,s)$ with data

$$K = 2 + \frac{8C_{T,1}}{\operatorname{Re}(s)(c \operatorname{Re}(s) - \frac{6C_{T,1}}{\operatorname{Re}(s)})}.$$

By enlarging δ' we can assume $K = 3$.
For the exponent of the dichotomy we obtain

$$\beta(s) = \tilde{\beta}(s) - 2\|T^{-1}T_x\|_\infty \tilde{K} > \frac{c}{3} \operatorname{Re}(s).$$

Finally the estimate (3.48) for the projectors are derived

$$\begin{aligned} |\pi(x,s) - \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}| &\leq K\tilde{K} \int_{\mathbb{R}} e^{-(\beta(s)+\tilde{\beta}(s))|x-y|} |T^{-1}(y,s)T_x(y,s)| dy \\ &\leq \frac{\text{const}}{|s|} \int_{\mathbb{R}} e^{-c \operatorname{Re}(s)|x-y|} dy \\ &\leq \frac{\text{const}}{|s| \operatorname{Re}(s)}. \end{aligned}$$

Here the constant const is generic, but does never depend on s. □

3.2.2 The all-line problem

Now we are ready to prove Theorem 3.1.

By Lemma 3.6 we know that for all $s \in \mathcal{M}(\delta, \tilde{C}_0)$ the operators $L(\cdot,s)$ from (3.25) have an (ED) on $\mathbb{R}$ with data (K,β,π), where K and β do not depend on s. Thus Theorem B.2 implies that for all $\bar{F} \in L_2(\mathbb{R})$ there is a unique solution $z \in H^1(\mathbb{R})$ of (3.26). This solution can be estimated by

$$\beta^2 \|z\|^2 \leq 5K^2 \|\bar{F}\|^2,$$

which leads to

$$\|z\|^2 \leq K_1 \|\bar{F}\|^2$$

with some constant $K_1 > 0$ which is independent of s and $\bar{F}$.

Transforming back to the original coordinates shows

$$\begin{aligned} \|v\|^2 &= \|Tz\|^2 \leq \|T\|_\infty^2 \|z\|^2 \leq \|T\|_\infty^2 K_1 \|\bar{F}\|^2 \\ &= K_2 \|T^{-1}(\cdot, s) B^{-1} F\|^2 \\ &\leq K_{alp} \|F\|^2. \end{aligned} \tag{3.49}$$

The constant K_{alp} thus depends on β, K, $\|T\|_\infty$, and $\|T^{-1}\|_\infty$, but these values do not depend on s and so (3.10) follows.

In order to prove equation (3.11), assume that F is an element of $H^1(\mathbb{R})$ and differentiate (3.9) with respect to x. This leads to the following equation for v_x

$$sv_x - Bv_{xx} - (C + B_x)v_x = C_x v + F_x \quad \text{in } L_2(\mathbb{R}, \mathbb{C}^m). \tag{3.50}$$

By assumption (H1) the operator P', given by

$$P'v = Bv_x + (C + B_x)v,$$

has the same properties as P. Thus we can find a new $\tilde{C}_0' \geq \tilde{C}_0$ such that for any s from $\mathcal{M}(\delta, \tilde{C}_0')$ we can estimate v_x in the same fashion as v in terms of the right hand side and find

$$\begin{aligned} \|v_x\|^2 &\leq K_{alp}' \|C_x v + F_x\|^2 \\ &\leq K_{alp}'' (\|v\|^2 + \|F_x\|^2), \end{aligned} \tag{3.51}$$

with a constant K_{alp}'' which does not depend on s.

Combination of the two equations (3.49) and (3.51) shows the claimed inequality (3.11).

For the proof of the improved estimates (3.12) and (3.13) we assume, that s is from $\mathcal{M}(-\delta', \tilde{C}_0')$. Then Lemma 3.7 implies that $K = 3$ and $\beta(s) = c\,\mathrm{Re}(s)$ can be chosen as dichotomy constant and dichotomy exponent. So we obtain by Theorem B.2

$$\mathrm{Re}(s)^2 \|z\|^2 \leq \mathrm{const} \|\bar{F}\|^2,$$

where const does not depend on s or $\bar{F}$.

Transforming back shows

$$\begin{aligned} \mathrm{Re}(s)^2 \|v\|^2 &\leq \mathrm{const}\, \mathrm{Re}(s)^2 \|z\|^2 \\ &\leq \mathrm{const} \|\bar{F}\|^2 \leq \mathrm{const} \|F\|^2, \end{aligned}$$

where const is a generic constant, but does not depend on s and F, again. This proves (3.12).

The H^1-estimate (3.13) is again obtained by differentiation of the original equation and applying the L_2-estimates. □

3.2.3 The finite interval problem

From now on we assume that J is a compact interval with endpoints x_- and x_+, i.e. $J = [x_-, x_+]$, and $s \in \mathcal{M}(\delta, C_a)$, such that the transformations (3.23)–(3.25) from 3.2.1 are justified. We rewrite (3.15) as before as

$$\begin{aligned} L(x,s) &= z_x - (\Lambda - T^{-1}T_x)z = \bar{F}, \text{ in } L_2(J, \mathbb{C}^m), \\ R_1 z &= \eta, \end{aligned} \tag{3.52}$$

where we have used the transformation $Tz := v$ and $\bar{F} := -T^{-1}B^{-1}F$ again. The boundary conditions in these new variables are given as

$$\begin{aligned} \eta &= R_- v(x_-) + R_+ v(x_+) \\ &= \bar{R}_-(x_-, s)z(x_-) + \bar{R}_+(x_+, s)z(x_+) =: R_1 z, \end{aligned} \tag{3.53}$$

where

$$\bar{R}_\pm(x_\pm, s) = R_\pm T(x_\pm, s). \tag{3.54}$$

Notice that although the original boundary operator does not depend on x_+, x_-, or s, the transformed operator R_1 does because of the x and s dependence of the transformation.

We show that the determinant-condition (3.14) for the boundary operator R implies a similar determinant-condition for the transformed operator R_1.

Lemma 3.8. *There is $C_c \geq C_a$ such that for all $s \in \mathcal{M}(\delta, C_c)$ and all $x_\pm \in \mathbb{R}$ holds*

$$\det \begin{pmatrix} \bar{R}_-^{II}(x_-, s) & \bar{R}_+^{I}(x_+, s) \end{pmatrix} \neq 0, \tag{3.55}$$

where $\bar{R}_\pm = \begin{pmatrix} \bar{R}_\pm^I & \bar{R}_\pm^{II} \end{pmatrix}$ is partitioned in the same way as $R_\pm$.

Proof. One obtains

$$\begin{aligned} \bar{R}_-^{II}(x_-, s) &= \bar{R}_-(x_-, s) \begin{pmatrix} 0 \\ I_{m-r} \end{pmatrix} = R_-(I + \frac{1}{s}T_1(x_-, s)) \begin{pmatrix} 0 \\ I_{m-r} \end{pmatrix} \\ &= R_-^{II} + \frac{1}{s}(R_- T_1(x_-, s)) \begin{pmatrix} 0 \\ I_{m-r} \end{pmatrix}, \end{aligned} \tag{3.56}$$

and similarly

$$\bar{R}_+^I(x_+, s) = R_+^I + \frac{1}{s}(R_+ T_1(x_+, s)) \begin{pmatrix} I_r \\ 0 \end{pmatrix}. \tag{3.57}$$

It follows

$$\begin{pmatrix} \bar{R}_-^{II}(x_-, s) & \bar{R}_+^{I}(x_+, s) \end{pmatrix} = \begin{pmatrix} R_-^{II} & R_+^{I} \end{pmatrix} + \frac{1}{s} \begin{pmatrix} (R_- T_1(x_-, s))^{II} & (R_+ T_1(x_+, s))^{I} \end{pmatrix}.$$

By choosing $C_c \geq 2C_{T,0}\big(|R_-|+|R_+|\big)\|(R_-^{II}, R_+^I)^{-1}\|_\infty$ the Banach-Lemma (Lemma A.1) implies (3.55). □

Remark 3.9. *Note that the definition of $\bar{R}_\pm$ in (3.54) directly implies that for all $J = [x_-, x_+]$ and all $s \in \mathcal{M}(\delta, C_c)$ holds the estimate*

$$|\bar{R}_\pm| = |R_\pm T(x_+, s)| \leq |R_\pm| \left(1 + \frac{C_{T,0}}{|s|}\right). \tag{3.58}$$

Furthermore from the proof of Lemma 3.8 follows

$$\left|\left(\bar{R}_-^{II}, \bar{R}_+^{I}\right)^{-1}\right| \leq 2 \left|\left(R_-^{II}, R_+^{I}\right)^{-1}\right|, \tag{3.59}$$

for all J and all $s \in M(\delta, C_c)$.

To simplify the argument, we assume that $\bar{R}_\pm \in \mathbb{C}^{m,m}$ are constant matrices. This is no restriction, since we will see that all estimates we derive only depend on the norms $|\bar{R}_\pm|$ and on $\left| \begin{pmatrix} \bar{R}_-^{II} & \bar{R}_+^{I} \end{pmatrix}^{-1} \right|$, but not on the exact entries. By Remark 3.9 we know that these are bounded independently of x and s if $|s|$ is large enough. So the same proofs also work for the case of non-constant matrices $\bar{R}_\pm$ defined in (3.54), as long as $|s|$ is sufficiently large.

By the Fredholm-alternative for boundary value problems, it suffices to show that there is a solution of (3.52) for arbitrary right hand sides. Then the unique solvability follows from the Fredholm property.

Now we proceed as usual. First we determine a particular solution $z_{sp} \in H^1(J, \mathbb{C}^m)$ of $L(x,s)z = \bar{F}$ in $L_2(J, \mathbb{C}^m)$ and then we solve the semi-homogeneous problem, i.e. for arbitrary $\zeta \in \mathbb{C}^m$ we look for a solution $z_{hom} \in H^1(J, \mathbb{C}^m)$ of

$$L(x,s)z = 0 \ \in L_2(J, \mathbb{C}^m), \quad R_1 z = \zeta.$$

In the following proofs we will denote the solution-operator for $L(\cdot, s)$ by $S(x,y)$, suppressing the s-dependence of the solution operator. As in Lemma 3.6 we denote the dichotomy data by $(K, \beta, \pi(\cdot, s))$. We will also write $\pi(\cdot)$ instead of $\pi(\cdot, s)$ in order to simplify notation.

Lemma 3.10. *For every $s \in \mathcal{M}(\delta, \tilde{C}_0)$, with $\tilde{C}_0$ from Lemma 3.6, and for every $\bar{F} \in L_2(J, \mathbb{C}^m)$ the differential equation*

$$L(\cdot, s)z = z_x - (\Lambda - T^{-1}T_x)z = \bar{F} \text{ in } L_2(J, \mathbb{C}^m) \tag{3.60}$$

has a solution $z_{sp} \in H^1(J, \mathbb{C}^m)$ that satisfies the estimates

$$\|z_{sp}\|^2 + |z_{sp}|_\Gamma^2 \leq \mathrm{const}_{K,\beta} \|\bar{F}\|^2 \tag{3.61}$$

and

$$|\eta_{sp}|^2 \leq \mathrm{const}_{R_1,K,\beta} \|\bar{F}\|^2, \tag{3.62}$$

where $\eta_{sp} := R_1 z_{sp}$.
The constant $\mathrm{const}_{K,\beta}$ only depends on the dichotomy-data K and β. Similarly the constant $\mathrm{const}_{R_1,K,\beta}$ only depends on the dichotomy-data K, β and on the norm of the matrices $\bar{R}_\pm$.
If in addition $\mathrm{Re}(s) \geq \delta'$, with δ' from Lemma 3.7, the estimate (3.61) can be improved to

$$\mathrm{Re}(s)^2 \|z_{sp}\|^2 + \mathrm{Re}(s)|z_{sp}|_\Gamma^2 \leq \mathrm{const}_K \|\bar{F}\|^2. \tag{3.63}$$

And the estimate (3.62) *becomes*

$$\operatorname{Re}(s)|\eta_{sp}|^2 \leq \text{const}_{R_1,K}\|\bar{F}\|^2. \tag{3.64}$$

In both equations the constants do not depend on s and J.

Proof. By Lemma 3.6 the operators $L(\cdot,s)$ have an (ED) on $\mathbb{R}$ with data (K,β,π) for all $s\in\mathcal{M}(\delta,\tilde{C}_0)$. The data K and β are independent of s. Therefore we can apply Theorem B.2. Note that $G(x,y)$, defined in (B.2), is the Green's function to $L(\cdot,s)u=0$ with boundary operator $u\mapsto \pi(x_-)u(x_-)+\pi(x_+)u(x_+)$.
Theorem B.2 shows that

$$z_{sp}(x) := \int_J G(x,y)\bar{F}(y)dy$$

is a solution of (3.60) and can be estimated by

$$\beta^2\|z_{sp}\|^2+\beta|z_{sp}|_\Gamma^2 \leq 5K^2\|\bar{F}\|^2 \tag{3.65}$$

which implies

$$\|z_{sp}\|^2+|z_{sp}|_\Gamma^2 \leq 5K^2\frac{\beta+1}{\beta^2}\|\bar{F}\|^2.$$

Thus (3.61) holds with a constant independent of s, J, $\bar{F}$.
If $s\in\mathbb{C}$ with $\operatorname{Re}s\geq\delta'$ we can apply Lemma 3.7 and from (3.65) follows with $\beta(s)=c\operatorname{Re}(s)$

$$c^2\operatorname{Re}(s)^2\|z_{sp}\|^2+c\operatorname{Re}(s)|z_{sp}|_\Gamma^2 \leq \text{const}\|\bar{F}\|^2.$$

This proves (3.63) and the constant is independent of s, J, and $\bar{F}$.
Inequality (3.62) is obtained from (3.61) via

$$\begin{aligned}|\eta_{sp}|^2 &= |\bar{R}_-z_{sp}(x_-)+\bar{R}_+z_{sp}(x_+)|^2\\ &\leq 2(|\bar{R}_-|^2+|\bar{R}_+|^2)(|z_{sp}(x_-)|^2+|z_{sp}(x_+)|^2)\\ &\leq \text{const}_{R_1,K,\beta}\|\bar{F}\|^2.\end{aligned}$$

For $\operatorname{Re}(s)\geq\delta'$ we can use (3.63) instead of (3.61) and find

$$\begin{aligned}\operatorname{Re}(s)|\eta_{sp}|^2 &\leq 2(|\bar{R}_-|^2+|\bar{R}_+|^2)\operatorname{Re}(s)(|z_{sp}(x_-)|^2+|z_{sp}(x_+)|^2)\\ &\leq \text{const}_{R_1,K}\|\bar{F}\|^2.\end{aligned}$$

So that in both estimates the constants are again independent of s, J, and $\bar{F}$. □

In the next step we show the existence of a solution for the homogeneous equation with inhomogeneous boundary conditions.

Lemma 3.11. *Assume that additional to (H1), (H2), and (H3) the determinant-condition*

$$\det \begin{pmatrix} \bar{R}_-^{II} & \bar{R}_+^{I} \end{pmatrix} \neq 0 \tag{3.66}$$

is satisfied. Then there are positive constants b and $C_0' \geq \tilde{C}_0$ such that for all $s \in \mathcal{M}(\delta, C_0')$ and all $J \supset [-b, b]$ the semi-homogeneous problem

$$\begin{aligned} L(\cdot, s)z &= 0, \ \text{in } L_2(J, \mathbb{C}^m), \\ R_1 z &= \zeta, \end{aligned} \tag{3.67}$$

has a unique solution $z_{hom} \in H^1(J, \mathbb{C}^m)$.
Moreover, the solution can be estimated by

$$\|z_{hom}\|^2 + |z_{hom}|_\Gamma^2 \leq \text{const}_{K,\beta,R_1} |\zeta|^2, \tag{3.68}$$

where the constant $\text{const}_{K,\beta,R_1}$ does not depend on s, J, or ζ.
If we in addition assume $\text{Re}(s) > \delta'$, with δ' as in Lemma 3.7, then equation (3.68) becomes

$$\text{Re}(s)\|z_{hom}\|^2 + |z_{hom}|_\Gamma^2 \leq \text{const}_{K,R_1} |\zeta|^2, \tag{3.69}$$

where the constant const_{K,R_1} is independent of s, J, and ζ.

The proof mainly follows the same lines as the proof of Lemma 2.5 in [BL99].

Proof. We make the ansatz

$$z_{hom}(x) = S(x, x_-)\alpha_- + S(x, x_+)\alpha_+, \tag{3.70}$$

where $\alpha_- \in \mathcal{R}(\pi(x_-))$ and $\alpha_+ \in \mathcal{R}(I - \pi(x_+))$. Then by construction z_{hom} is a solution of the homogeneous problem $L(\cdot)z = 0$ in J.

It remains to be shown that we can determine α_- and α_+ such that the boundary condition in (3.67) is satisfied, too. From Lemma 3.6 we know

$$\left| \pi(x) - \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} \right| \leq \frac{\text{const}}{|s|} \tag{3.71}$$

for all $x \in \mathbb{R}$ and all $s \in \mathcal{M}(\delta, \tilde{C}_0)$, where the constant const does not depend on s. Let $Q := \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$.

We choose $C' \geq \tilde{C}_0$ such that $|\pi(x) - Q| \leq \frac{1}{2}$ holds for all $x \in \mathbb{R}$ and all $s \in \mathcal{M}(\delta, C')$. Then Lemma A.2 implies the equalities

$$\mathcal{R}(\pi(x_-)) = \mathcal{R}\left(\pi(x_-)Q\right),$$

and

$$\mathcal{R}\left(I - \pi(x_+)\right) = \mathcal{R}\left(\left(I - \pi(x_+)\right)\left(I - Q\right)\right).$$

Because of these results we can write α_- and α_+ from (3.70) as

$$\begin{aligned}\alpha_- &= \pi(x_-)Q\begin{pmatrix}0\\ \beta_-\end{pmatrix} = Q\begin{pmatrix}0\\ \beta_-\end{pmatrix} + \big(\pi(x_-)-Q\big)Q\begin{pmatrix}0\\ \beta_-\end{pmatrix}\\ &= \begin{pmatrix}0\\ \beta_-\end{pmatrix} + \big(\pi(x_-)-Q\big)\begin{pmatrix}0\\ \beta_-\end{pmatrix},\end{aligned} \tag{3.72}$$

and

$$\begin{aligned}\alpha_+ &= \big(I-\pi(x_+)\big)(I-Q)\begin{pmatrix}\beta_+\\ 0\end{pmatrix}\\ &= \begin{pmatrix}\beta_+\\ 0\end{pmatrix} + \big(Q-\pi(x_+)\big)\begin{pmatrix}\beta_+\\ 0\end{pmatrix},\end{aligned} \tag{3.73}$$

where $\beta_- \in \mathbb{C}^{m-r}$ and $\beta_+ \in \mathbb{C}^r$. Insertion of the ansatz into the boundary term $R_1 z_{hom} = \zeta$ leads to

$$\begin{aligned}\zeta &= R_1 z_{hom} = \bar{R}_- z_{hom}(x_-) + \bar{R}_+ z_{hom}(x_+)\\ &= \bar{R}_-\alpha_- + \bar{R}_+\alpha_+ + \bar{R}_- S(x_-,x_+)\alpha_+ + \bar{R}_+ S(x_+,x_-)\alpha_-\\ &= \begin{pmatrix}\bar{R}_-^{II} & \bar{R}_+^{I}\end{pmatrix}\begin{pmatrix}\beta_-\\ \beta_+\end{pmatrix}\\ &\quad +\bar{R}_-(\pi(x_-)-Q)\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}\begin{pmatrix}\beta_-\\ \beta_+\end{pmatrix}\\ &\quad +\bar{R}_+(Q-\pi(x_+))\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\begin{pmatrix}\beta_-\\ \beta_+\end{pmatrix}\\ &\quad +\bar{R}_- S(x_-,x_+)(I-\pi(x_+))\big(I+(Q-\pi(x_+))\big)\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\begin{pmatrix}\beta_-\\ \beta_+\end{pmatrix}\\ &\quad +\bar{R}_+ S(x_+,x_-)\pi(x_-)\big(I+(\pi(x_-)-Q)\big)\begin{pmatrix}0 & 0\\ I & 0\end{pmatrix}\begin{pmatrix}\beta_-\\ \beta_+\end{pmatrix}.\end{aligned} \tag{3.74}$$

From Lemma 3.6 follow for all $x \in \mathbb{R}$ and all $s \in M(\delta, C')$ the estimates

$$|S(x,x_+)\big(I-\pi(x_+)\big)\big(I+(Q-\pi(x_+))\big)| \le \frac{3}{2}Ke^{-\beta|x_+-x|} \tag{3.75}$$

and

$$|S(x,x_-)\pi(x_-)\big(I+(\pi(x_-)-Q)\big)| \le \frac{3}{2}Ke^{-\beta|x-x_-|}, \tag{3.76}$$

where we used $\|(Q-\pi)\|_\infty \le \frac{1}{2}$. By the determinant-condition (3.66) the matrix $\begin{pmatrix}\bar{R}_-^{II} & \bar{R}_+^{I}\end{pmatrix}$ is nonsingular. Because of (3.71), (3.75), and (3.76) we can choose

$C_0 \geq C'$ and $b > 0$ so large that for all $s \in \mathcal{M}(\delta, C_0)$ and $x_- \leq -b$, $x_+ \geq b$ holds

$$
\begin{aligned}
|\bar{R}_-(\pi(x_-) - Q) \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} + \bar{R}_+(Q - \pi(x_+)) \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} + \\
\bar{R}_- S(x_-, x_+)(I - \pi(x_+))(I + (Q - \pi(x_+))) \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} + \\
\bar{R}_+ S(x_+, x_-)\pi(x_-)(I + (\pi(x_-) - Q)) \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} | \\
\leq \frac{1}{2} | \begin{pmatrix} \bar{R}_-^{II} & \bar{R}_+^{I} \end{pmatrix}^{-1} |. \quad (3.77)
\end{aligned}
$$

Then we can apply Lemma A.1 to (3.74) and obtain a unique solution $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} \in \mathbb{C}^m$. By equations (3.70), (3.72), (3.73), and the Fredholm alternative we thus have that z_{hom} from (3.70) is the unique solution of (3.67).

To show the estimate (3.68) we note that from the Banach-Lemma (Lemma A.1) together with inequality (3.77) we obtain the estimate

$$
\left| \begin{pmatrix} \beta_- \\ \beta_+ \end{pmatrix} \right| \leq 2 \left| \begin{pmatrix} \bar{R}_-^{II} & \bar{R}_+^{I} \end{pmatrix}^{-1} \right| |\zeta|.
$$

By the construction (3.70), (3.72), (3.73) we have

$$
\begin{aligned}
z_{hom}(x) = S(x, x_-)\pi(x_-)\Big(I + \big(\pi(x_-) - Q\big)\Big) \begin{pmatrix} 0 \\ \beta_- \end{pmatrix} \\
+ S(x, x_+)\big(I - \pi(x_+)\big)\Big(I + \big(Q - \pi(x_+)\big)\Big) \begin{pmatrix} \beta_+ \\ 0 \end{pmatrix}
\end{aligned}
$$

such that the estimates (3.71), (3.75), (3.76) imply the following pointwise bound for the solution

$$
|z_{hom}(x)| \leq \text{const}_{K, |[\bar{R}_-^{II}, \bar{R}_+^{I}]^{-1}|} \left(e^{-\beta|x - x_-|} + e^{-\beta|x - x_+|} \right) |\zeta|. \quad (3.78)
$$

From this we conclude

$$
|z_{hom}(x_\pm)|^2 \leq \text{const}_{K, R_1} |\zeta|^2,
$$

and by integration over J

$$
\|z_{hom}\|^2 \leq \text{const}_{K, \beta, R_1} |\zeta|^2.
$$

This shows the estimate (3.68).

If in addition s satisfies $\text{Re}(s) > \delta'$ with δ' from Lemma 3.7, then we can choose $K = 3$ and $\beta = \beta(s) = c\,\text{Re}(s)$. Using these constants in (3.78) we obtain by integration over J the improved L_2-estimate

$$
\text{Re}(s)\|z_{hom}\|^2 \leq \text{const}_{K, R_1} |\zeta|^2
$$

and as before the boundary-estimate

$$|z_{hom}(x_\pm)|^2 \leq \text{const}_{K,R_1}|\zeta|^2.$$

These two estimates imply inequality (3.69). □

Remark. *One sees from equations* (3.71), (3.75), *and* (3.76) *that by assuming* $\mathrm{Re}(s) > \delta'$, *it suffices to have* $b = 1$ *and to increase* δ'.

Now we can finish the proof of Theorem 3.2.

Proof of Theorem 3.2. Let b and C_0 be given as in Lemma 3.10 and in Lemma 3.11. Then for $s \in \mathcal{M}(\delta, C_0)$ and $J \supset [-b, b]$ both Lemmas apply.
Hence we find a particular solution $z_{sp} \in H^1(J, \mathbb{C}^m)$ of

$$L(\cdot, s)z = \bar{F} \text{ in } L_2(J, \mathbb{C}^m),$$

and a solution $z_{hom} \in H^1(J, \mathbb{C}^m)$ of

$$\begin{aligned} L(\cdot, s)z &= 0 \text{ in } L_2(J, \mathbb{C}^m), \\ R_1 z &= \eta - R_1 z_{sp}. \end{aligned}$$

By linearity of the boundary value problem we obtain by addition

$$z := z_{sp} + z_h \in H^1(J, \mathbb{C}^m)$$

the unique solution of (3.52).
To show the asserted inequalities we collect the estimates (3.61), (3.62), and (3.68) and find for $v = Tz \in H^1(J, \mathbb{C}^m)$, the unique solution of equation (3.15), the estimates

$$\begin{aligned} \|v\|^2 + |v|_\Gamma^2 &= \|Tz\|^2 + |Tz|_\Gamma^2 \leq \|T\|_\infty^2 \left\{\|z\|^2 + |z|_\Gamma^2\right\} \\ &\leq \text{const}\left\{\|z_{sp}\|^2 + |z_{sp}|_\Gamma^2 + \|z_{hom}\|^2 + |z_{hom}|_\Gamma^2\right\} \\ &\leq \text{const}_{K,\beta,R_1}\left\{\|\bar{F}\|^2 + |\eta|^2 + |R_1 z_{sp}|^2\right\} \\ &\leq \text{const}_{K,\beta,R_1}\left\{\|T^{-1}B^{-1}F\|^2 + |\eta|^2\right\} \\ &\leq \text{const}_{K,\beta,R}\left\{\|F\|^2 + |\eta|^2\right\}, \end{aligned}$$

where the constant has to be adapted during the estimates, but still does not depend on s, J, η, and F.
If in addition we assume $\mathrm{Re}(s) > \delta'$ and use the inequalities (3.63), (3.64), and (3.69) we find

$$\mathrm{Re}(s)^2\|z_{sp}\|^2 + \mathrm{Re}(s)|z_{sp}|_\Gamma^2 \leq \text{const}_K\|\bar{F}\|^2.$$

This implies $\mathrm{Re}(s)|R_1 z_{sp}|^2 \leq \text{const}_{R_1,K}\|\bar{F}\|^2$ which then leads to

$$\begin{aligned} \mathrm{Re}(s)^2\|z_{hom}\|^2 + \mathrm{Re}(s)|z_{hom}|_\Gamma^2 &\leq \mathrm{Re}(s)\text{const}_{K,R_1}\left(|R_1 z_{sp}|^2 + |\eta|^2\right) \\ &\leq \text{const}_{K,R_1}\left(\|\bar{F}\|^2 + \mathrm{Re}(s)|\eta|^2\right). \end{aligned}$$

Adding z_{hom} and z_{sp} and using the same calculations as before then shows inequality (3.17). □

3.3 Resolvent estimates in compact regions

In this section we consider the all line operator P defined in (3.7) and the operator defined in (3.8) which is obtained after truncation to a finite interval. We always assume that **Assumption 1** holds and our aim is to show that if the all line operator P has no spectrum in some compact set in $\{s \in \mathbb{C} : \operatorname{Re} s > -\delta\}$, then also the finite interval approximation $P|_J$ with suitable additional boundary conditions has this property. We will give a sufficient condition for the boundary operator under which we will show a uniform resolvent estimate in the compact set.

As in the previous section we rewrite the resolvent-equation

$$sv - Pv = F$$

in the form

$$L(x, s)v = v_x - M(x, s)v = -B^{-1}F \tag{3.79}$$

where $M(x, s) := B^{-1}(x)(sI - C(x))$.

For the formulation of the condition which we will require for the extra boundary operator, we prove hyperbolicity of the limit matrices.

Lemma 3.12. *For every $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ the limit matrices*

$$M_{\pm}(s) := \lim_{x \to \pm\infty} M(x, s) \tag{3.80}$$

exist and, counted with multiplicities, they have r eigenvalues with real parts larger than zero and $m - r$ eigenvalues with real parts less than zero.

Proof. The existence of the limit matrices immediately follows from assumptions (H1) and (H2). The mapping $s \mapsto M_{\pm}(s)$ is affine linear and so the eigenvalues of $M_{\pm}(s)$, which are the roots of $\kappa \mapsto \det(\kappa I - M_{\pm}(s))$, are algebraic functions of s and therefore depend continuously on s. Without loss of generality consider '+'.

For every $\kappa \in i\mathbb{R}$ with $\kappa \in \sigma(M_+(s))$ for some $s \in \mathbb{C}$ we have

$$\begin{aligned} &0 = \det(\kappa I - M_+(s)) = \det(\kappa I - B_+^{-1}(sI - C_+)) \\ \Leftrightarrow\ &0 = \det(\kappa B_+ + C_+ - sI), \end{aligned}$$

which implies $\operatorname{Re} s \leq -\delta$ by assumption (H4). Therefore for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ the dimensions of the stable and unstable subspaces of $M_+(s)$, i.e. the subspaces corresponding to the eigenvalues with negative real parts and with positive real parts, respectively, are constant, since the eigenvalues cannot cross the imaginary axis.

In Section 3.2 we saw that for large $|s|$ with $\operatorname{Re} s > -\delta$, the eigenspace to the eigenvalues with negative real parts has dimension $m - r$ and the eigenspace to the eigenvalues with positive real parts has dimension r. ☐

Justified by Lemma 3.12, we denote by $V_{\pm}^{I}(s)$ and $V_{\pm}^{II}(s)$ bases of the stable and unstable subspaces of $M_{\pm}(s)$, respectively. That means that for every $s \in \mathbb{C}$

with $\operatorname{Re} s > -\delta$ there are matrices $V_\pm^I(s) \in \mathbb{C}^{m,r}$ and $V_\pm^{II}(s) \in \mathbb{C}^{m,m-r}$ with the properties

$$M_\pm(s)V_\pm^I(s) = V_\pm^I(s)\Lambda_\pm^I(s) \tag{3.81}$$

for some $\Lambda_\pm^I(s) \in \mathbb{C}^{r,r}$ with $\operatorname{Re}\sigma(\Lambda_\pm^I(s)) > 0$ and

$$M_\pm(s)V_\pm^{II}(s) = V_\pm^{II}(s)\Lambda_\pm^{II}(s) \tag{3.82}$$

for some $\Lambda_\pm^{II}(s) \in \mathbb{C}^{m-r,m-r}$ with $\operatorname{Re}\sigma(\Lambda_\pm^{II}(s)) < 0$. Notice that we do not assume any smoothness of $V_\pm^{I,II}$ or $\Lambda_\pm^{I,II}$ although this is possible by Lemma A.4.

From (3.81) we obtain that for all $z_0 \in \mathbb{C}^r$ the function

$$z_+(x) = V_+(s)^I e^{\Lambda_+(s)x} z_0$$

is a backward decaying solution of the constant coefficient equation

$$L_+(s)z = z_x - M_+(s)z = 0.$$

Similar for all $z_1 \in \mathbb{C}^{m-r}$ the function

$$z_-(x) = V_-(s)^{II} e^{\Lambda_-(s)x} z_1$$

is a forward decaying solution of the constant coefficient equation

$$L_-(s)z = z_x - M_-(s)z = 0.$$

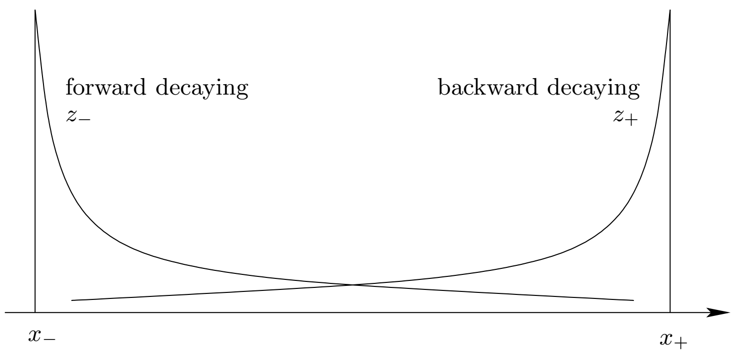


Figure 3.2: The growing and decaying modes of the differential operator. Heuristically these must be controlled by the boundary conditions if one wants to obtain uniform solution estimates of the boundary value problem on a finite line. This motivates the determinant-condition (3.86) in Theorem 3.14.

We assume linear boundary conditions of the form

$$R : \begin{array}{ccc} H^1(J,\mathbb{C}^m) & \to & \mathbb{C}^m \\ v & \mapsto & R_-v(x_-) + R_+v(x_+), \end{array} \tag{3.83}$$

with matrices $R_\pm \in \mathbb{C}^{m,m}$. Because of the Sobolev Embedding Theorem (Lemma C.3) the operator R is well defined.

Now we can state the main results of this section which correspond to the the main results of Section 3.2.

Theorem 3.13. *Let $\Omega \subset \{s \in \mathbb{C} : \operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set. Then for every $s \in \Omega$ and every $F \in L_2(\mathbb{R}, \mathbb{C}^m)$ there is a unique solution $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ of the resolvent equation*

$$(sI - P)v = F \text{ in } L_2(\mathbb{R}) \tag{3.84}$$

and the solution can be estimated in terms of the right hand side

$$\|v\|_{H^1} \leq \text{const}\|F\| \tag{3.85}$$

with a constant independent of F and s.

The analogous result for the finite interval problem is stated next.

Theorem 3.14. *Let $\Omega \subset \{s \in \mathbb{C} : \operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set. Let $V_\pm^{I,II}$ be given as in* (3.82) *and* (3.81) *and let R be of the form in* (3.83). *Assume the determinant-condition*

$$D(s) := \det \begin{pmatrix} R_- V_-^{II}(s) & R_+ V_+^{I}(s) \end{pmatrix} \neq 0 \ \forall s \in \Omega. \tag{3.86}$$

Then there is a compact interval J_0 such that for all compact intervals $J \supset J_0$, for every $s \in \Omega$, and for every right hand side, the finite interval boundary value problem

$$\begin{aligned} (sI - P)v &= F \text{ in } L_2(J), \\ Rv &= R_- v(x_-) + R_+ v(x_+) = \eta \end{aligned} \tag{3.87}$$

has a unique solution $v \in H^1(J)$. This solution can be estimated by

$$\|v\|_{H^1}^2 + |v|_\Gamma^2 \leq \text{const}\{\|F\|^2 + |\eta|^2\},$$

where const *does not depend on J, s, F, and η.*

Note, that the determinant-condition (3.86) does not depend on the choice of the actual bases $V_-^{II}(s)$, and $V_+^{II}(s)$.

Remark. *Recall that one only obtains L_2-estimates in Theorem 3.1 and Theorem 3.2 for $F \in L_2$, but in Theorems 3.13 and 3.14 one finds H^1-estimates. This was already observed in the paper [BL99].*

To prove the "all line" Theorem 3.13 we will mainly follow the same steps as in the case of large $|s|$ and for the "finite interval" Theorem 3.14 we use the theory of discrete approximations presented in Chapter 2.

3.3.1 Exponential dichotomies

First we prove (ED)s for the operators

$$L_{\pm}(s)v = v_x - M_{\pm}(s)v. \tag{3.88}$$

This follows directly from Lemma 3.12 and the result is stated in the following corollary.

Corollary 3.15. *For every $s \in \{\operatorname{Re} s > -\delta\}$ the constant coefficient operators $\tilde{L}_{\pm}(s)$ have exponential dichotomies on $\mathbb{R}$ with data $(\tilde{K}_{\pm}(s), \tilde{\beta}_{\pm}(s), \tilde{\pi}_{\pm}(s))$, where*

$$\mathcal{R}(\tilde{\pi}_{\pm}(s)) = \mathcal{R}(V_{\pm}^{II}(s)) \text{ and } \mathcal{R}(I - \tilde{\pi}_{\pm}(s)) = \mathcal{R}(V_{\pm}^{I}(s))$$

Proof. The corollary follows immediately from the hyperbolicity of the matrices $M_{\pm}(s)$ and the uniqueness of the range and kernel of the projectors for (ED)s on the whole real line. □

Lemma 3.16. *For every $s \in \{\operatorname{Re} s > -\delta\}$ the variable coefficient differential operators* (3.79) *have (ED)s on both half-lines $\mathbb{R}_-$ and on $\mathbb{R}_+$ with data $(K_-(s), \beta_-(s), \pi_-(x,s))$ and $(K_+(s), \beta_+(s), \pi_+(x,s))$, respectively. The projectors satisfy*

$$|\pi_-(x,s) - \tilde{\pi}_-(s)| \to 0, \text{ as } x \to -\infty,$$

and

$$|\pi_+(x,s) - \tilde{\pi}_+(s)| \to 0, \text{ as } x \to +\infty.$$

Proof. The Lemma follows from Corollary 3.15 and Lemma B.5. □

Next we show that if s does not only satisfy $\operatorname{Re} s > -\delta$, but also is an element of $\rho(P)$, the variable coefficient differential operator $L(\cdot, s)$ has an (ED) on the whole real line.

Lemma 3.17. *Let $s \in \{\operatorname{Re} s > -\delta\} \cap \rho(P)$. Then $L(\cdot, s)$ has an (ED) on $\mathbb{R}$ with data $(K(s), \beta(s), \pi(\cdot, s))$ where the projectors satisfy*

$$\lim_{x \to \pm\infty} \pi(x,s) = \tilde{\pi}_{\pm}(s). \tag{3.89}$$

Proof. Lemma 3.16 implies that $L(\cdot, s)$ has an (ED) on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with data $(K_+(s), \beta_+(s), \pi_+(\cdot, s))$ and $(K_-(s), \beta_-(s), \pi_-(\cdot, s))$, respectively. In the rest of the proof we suppress the s-dependence of the data and the solution-operator.

Since the mappings

$$\mathbb{R}_- \to \mathbb{C}^{m,m},\ x \mapsto \pi_-(x) \text{ and } \mathbb{R}_+ \to \mathbb{C}^{m,m},\ x \mapsto \pi_+(x)$$

are continuous in x, we obtain from Lemma A.3 and Lemma 3.16

$$\operatorname{rank}(I - \pi_-(0)) = \operatorname{rank}(I - \tilde{\pi}_-) = r \text{ and } \operatorname{rank} \pi_+(0) = \operatorname{rank} \tilde{\pi}_+ = m - r.$$

Hence by Theorem B.6 it suffices to show $\mathcal{R}(\pi_+(0)) \cap \mathcal{R}(I - \pi_-(0)) = \{0\}$.
Let $v_0 \in \mathcal{R}(\pi_+(0)) \cap \mathcal{R}(I - \pi_-(0))$. Then

$$v(x) := S(x,0)v_0$$

solves

$$v_x(x) - M(x,s)v(x) = 0 \;\forall x \in \mathbb{R}. \tag{3.90}$$

Furthermore v satisfies the estimates

$$\begin{aligned}|v(x)| &= |S(x,0)v_0| = |S(x,0)\pi_+(0)v_0| \\ &\leq K_+ e^{-\beta_+|x|}|v_0|, \text{ for all } x \geq 0,\end{aligned}$$

and

$$|v(x)| \leq K_- e^{-\beta_-|x|}|v_0|, \text{ for all } x \leq 0.$$

This shows that v is in fact an element of $L_2(\mathbb{R})$. The differential equation (3.90) then implies $v \in H^1(\mathbb{R})$ and v is a solution of $(sI - P)v = 0$ in $L_2(\mathbb{R}, \mathbb{C}^m)$. Since we assumed $s \in \rho(P)$ we obtain $v = 0$ in $L_2(\mathbb{R})$ and therefore $v_0 = 0$. Now Theorem B.6 is applicable and the Lemma follows. □

3.3.2 Proof of the all-line theorem

Let $s_0 \in \{\operatorname{Re} s > -\delta\} \cap \rho(P)$, let $F \in L_2(\mathbb{R})$. After multiplication with $-B^{-1}$, which is a homeomorphism of $L_2(\mathbb{R}, \mathbb{C}^m)$, the resolvent equation (3.84) reads

$$L(\cdot, s_0)v = -B^{-1}F =: \bar{F} \text{ in } L_2(\mathbb{R}, \mathbb{C}^m) \tag{3.91}$$

with $L(\cdot, s_0)$ from (3.79).

By Lemma 3.17 and Theorem B.2 there is a unique solution $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ of (3.91) and this satisfies the estimate

$$\beta(s_0)^2\|v\|^2 \leq 5K(s_0)^2\|\bar{F}\|^2.$$

Hence there exists a constant const with $\|v\| \leq \text{const}\|\bar{F}\|$. Then the differential equation (3.91)implies

$$\|v_x\| \leq \|M(\cdot, s_0)\|_\infty\|v\| + \|\bar{F}\| \leq \text{const}\|\bar{F}\|,$$

and therefore there is a constant $c_0(s_0) > 0$ with

$$\|v\|_{H^1} \leq c_0(s_0)\|\bar{F}\|. \tag{3.92}$$

To deduce the uniformity of the constant for compact sets Ω from the pointwise result (3.92), we recall that $M(x,s)$ is given by $M(x,s) = -B(x)^{-1}(sI - C(x))$. Assumption (H1) implies that we can find $\varepsilon_0 = \varepsilon_0(s_0) > 0$ with

$$\|M(\cdot, s) - M(\cdot, s_0)\|_\infty < \frac{1}{2c_0(s_0)} \text{ for all } s \in K_{\varepsilon_0}(s_0).$$

Hence we obtain

$$\begin{aligned} L(\cdot,s)v &= v_x - M(\cdot,s)v \\ &= \Big(I_{L_2(\mathbb{R})} + \big(M(\cdot,s_0) - M(\cdot,s)\big)L(\cdot,s_0)^{-1}\Big)L(\cdot,s_0)v, \end{aligned}$$

where the multiplication by $M(\cdot,s_0) - M(\cdot,s)$ is viewed as a mapping from $H^1(\mathbb{R})$ into $L_2(\mathbb{R})$.

Equation (3.92) and the choice of ε_0 imply for every $h \in L_2(\mathbb{R})$ the estimate

$$\|\big(M(\cdot,s_0) - M(\cdot,s)\big)L(\cdot,s_0)^{-1}h\|_{L_2} \le \frac{1}{2}\|h\|_{L_2}.$$

Now we can use Lemma A.1 to show the invertibility of $L(\cdot,s) : H^1(\mathbb{R}) \to L_2(\mathbb{R})$ for each $s \in K_{\varepsilon_0}(s_0)$ as well as the estimate

$$\begin{aligned} &\|L(\cdot,s)^{-1}\|_{L_2\to H^1} \\ &\quad\le \|L(\cdot,s_0)^{-1}\|_{L_2\to H^1}\Big\|\Big(I + \big(M(\cdot,s_0) - M(\cdot,s)\big)L(\cdot,s_0)^{-1}\Big)^{-1}\Big\|_{L_2\to L_2} \\ &\quad\le c_0(s_0)2 =: \tilde{c}(s_0). \end{aligned}$$

Now let Ω be given as in Theorem 3.13. Then the same considerations hold for every point $s_0 \in \Omega$ and the union of the resulting neighborhoods is an open covering of Ω. By compactness of Ω we can choose a finite sub-covering. Finally, taking c_1 as the maximum of the constants $\tilde{c}(s_0)$ from this finite sub-covering, we find a uniform resolvent estimate for all $s \in \Omega$

$$\|v\|_{H_1} \le c_1\|\bar{F}\| \le c_1\|B^{-1}\|_\infty\|F\|$$

and the assertion follows. □

3.3.3 The proof of the finite interval theorem

Before we can prove the Theorem we have to show some auxiliary results. Assume that the assumptions of Theorem 3.14 hold.

We analyze the finite interval problem (3.87) in the transformed form

$$\begin{aligned} L(\cdot,s)v &= \bar{F} \;\text{ in } L_2(J,\mathbb{C}^m), \\ Rv &= R_-v(x_-) + R_+v(x_+) = \eta. \end{aligned}$$

One can write this equation with the operator-matrix

$$L_J(s) := \begin{pmatrix} L(\cdot,s) \\ R \end{pmatrix} : H^1(J,\mathbb{C}^m) \to L_2(J,\mathbb{C}^m) \times \mathbb{C}^m$$

in the form

$$L_J(s)v = \begin{pmatrix} L(\cdot,s)v \\ Rv \end{pmatrix} = \begin{pmatrix} \bar{F} \\ \eta \end{pmatrix} \;\text{ in } L_2(J,\mathbb{C}^m) \times \mathbb{C}^m. \tag{3.93}$$

Consider the setting of spaces and operators as in Theorem 2.29. Let the index set be given by $H = \{J = [x_-, x_+] : 0 \in J, x_+ - x_- \geq 1\}$ with the direction $J_1 \succ J_2 :\Leftrightarrow J_1 \supset J_2$.

By the assumptions on the coefficients of P and on R the assumptions of Theorem 2.29 are satisfied and the Theorem implies that the finite interval operators $L_J(s)$ from (3.93) regularly $\mathcal{PQ}$-converge to the all line operator $L(\cdot, s)$ from (3.91).

By the Fredholm alternative for boundary value problems the operator $L_J(s)$ is Fredholm of index zero for all compact intervals $J \in H$ and all $s \in \mathbb{C}$. Finally in the proof of Theorem 3.13 it is shown that $L(\cdot, s) : H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^m)$ is a linear homeomorphism for all $s \in \rho(P)$.

Now Lemma 2.21 applies and shows for all $s_0 \in \rho(P) \cap \{\operatorname{Re} s > -\delta\}$

$$L_J(s_0) \xrightarrow{\mathcal{PQ}} L(s_0) \text{ stably.}$$

Hence there is a compact interval $J_0 = J_0(s_0) \in H$ and a positive constant $c_{s_0} > 0$, so that for all compact intervals $J = [x_-, x_+] \subset \mathbb{R}$ with $J_0 \subset [x_-, x_+]$ the operator $L_J(s_0) \in L\big(H^1(J, \mathbb{C}^m), L_2(J, \mathbb{C}^m) \times \mathbb{C}^m\big)$ is a linear homeomorphism and its inverse is bounded

$$\|L_J(s_0)^{-1}\|_{L_2 \times \mathbb{C}^m \to H^1} \leq c_{s_0} \; \forall J \subset J_0,$$

where we abbreviate $\|\cdot\|_{L_2(J,\mathbb{C}^m)\times\mathbb{C}^m}$ to $\|\cdot\|_{L_2\times\mathbb{C}^m}$. With application of the Sobolev inequality (C.1) this proves the following pointwise result.

Lemma 3.18. *For every $s_0 \in \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ there is a compact interval $J_0 \in H$ and a positive constant c_0 so that for all compact intervals $J = [x_-, x_+] \supset J_0$ there is for all $\bar{F} \in L_2(J, \mathbb{C}^m)$ and all $\eta \in \mathbb{C}^m$ a unique solution $v_J \in H^1(J, \mathbb{C}^m)$ of the transformed resolvent equation* (3.93) *and this satisfies the estimate*

$$\|v_J\|_{H^1} + |v_J|_\Gamma \leq c_0(\|\bar{F}\|_{L_2} + |\eta|). \tag{3.94}$$

Now it remains to prove that it is possible to choose a uniform minimal interval J_0 and a uniform constant c_0 in (3.94) for all s from the compact set Ω.

Let $s_0 \in \Omega$ be arbitrary and let J_0 and c_0 be the data obtained by Lemma 3.18. Then there is an $\varepsilon = \varepsilon(s_0) > 0$ with

$$\|M(\cdot, s_0) - M(\cdot, s)\|_\infty = \sup_{x \in \mathbb{R}} |M(x, s_0) - M(x, s)| \leq (2c_0)^{-1} \; \forall s \in K_\varepsilon(s_0).$$

This yields for every $v \in H^1(J, \mathbb{C}^m)$ the estimate

$$\begin{aligned}
\|(L_J(s) - L_J(s_0))v\|_{L_2(J)\times\mathbb{C}^m} &= \|(L(\cdot, s) - L(\cdot, s_0))v\|_{L_2(J)} + |Rv - Rv| \\
&= \|(M(\cdot, s) - M(\cdot, s_0))v\|_{L_2(J)} \\
&\leq \frac{1}{2c_0}\|v\|_{L_2(J)} \leq \frac{1}{2c_0}\|v\|_{H^1(J)}.
\end{aligned}$$

Thus Lemma A.1 implies that for all compact intervals J with $J \supset J_0$ and all $s \in K_\varepsilon(s_0)$ the operator

$$L_J(s) = \big(L_J(s_0) + (L_J(s) - L_J(s_0))\big) : H^1(J) \to L_2(J) \times \mathbb{C}^m$$

is a linear homeomorphism. Moreover, it also shows the estimate

$$\left\| L_J(s)^{-1} \right\|_{L_2(J)\times\mathbb{C}^m \to H^1(J)} \le 2c_0,$$

for all such s and J.

This construction therefore yields for every $s_0 \in \Omega$ an open neighborhood $K_{\varepsilon(s_0)}(s_0)$ so that for all s in this neighborhood the same minimal interval $J_0 = J_0(s_0)$ and the same constant $c(s_0) = 2c_0(s_0)$ can be chosen. The family of all these neighborhoods is an open covering of Ω and because of compactness one can choose a finite sub-covering. Let J' be the union of the finitely many compact intervals $J_0(s_0)$ corresponding to this sub-covering and let c' be the maximum of the finitely many $c(s_0)$ from this sub-covering. Note that J' is a compact interval since $0 \in J$ for all $J \in H$.

Then for every $s \in \Omega$ and every compact interval $J \supset J'$ there is a unique solution $v_J \in H^1(J, \mathbb{C}^m)$ of (3.93) for each choice of $\bar{F} \in L_2(J, \mathbb{C}^m)$ and $\eta \in \mathbb{C}^m$. This solution can be estimated by

$$\|v_J\|_{H^1(J)} + |v_J|_\Gamma \le c'\big(\|\bar{F}\| + |\eta|\big). \tag{3.95}$$

The assertions of Theorem 3.14 now follow by using $\bar{F} = -B^{-1}F$ since $\|B\|_\infty$ is bounded. □

Remark. *The uniformity of the minimal intervals and the independence of the resolvent constant from s could also be obtained by using the continuity of the dichotomy data in s and uniformity of the convergence of the projectors $\pi(x,s) \to \tilde{\pi}_\pm(s)$ as $x \to \pm\infty$. To show these one has to prove continuity of the data for the constant coefficient operators $\tilde{L}_\pm(s)$. The continuity of the data of the variable coefficient operators then follows from the Theorems B.4–B.6 since they carry over the continuity to the variable coefficient operators and also show uniformity of the convergence of the projectors in compact parameter sets. For results in this direction see [BL99] and [San93].*

3.3.4 Convergence of finite interval approximations

Here we briefly state a convergence result for the solutions of the finite interval problems to the solutions of the all line problem. It shows the consistency of the restricted problems in the sense that for $J \to \mathbb{R}$ the distances of the solution of the all line problem to the finite interval boundary value problems converges to zero.

Theorem 3.19. *Let the assumptions of Theorem 3.14 hold. Let $\Omega \subset \{\mathrm{Re}\, s > -\delta\} \cap \rho(P)$ be a compact subset of $\mathbb{C}$ and let $J \supset J_0$ be a compact interval, where J_0 is the interval obtained in Theorem 3.14. Let $v \in H^1(\mathbb{R})$ be the solution of the resolvent equation on the whole real line*

$$(sI - P)v = F \quad \text{in } L_2(\mathbb{R}) \tag{3.96}$$

and let $v_J \in H^1(J)$ denote the solution of the finite interval approximation

$$\begin{aligned} (sI - P)v_J &= F|_J \quad \text{in } L_2(J), \\ Rv_J &= 0. \end{aligned} \tag{3.97}$$

Then the approximation error $y_J := v|_J - v_J$ *satisfies*

$$\|y_J\|_{H^1(J)} + |y_J|_\Gamma \to 0 \quad \text{as } J \to \mathbb{R} \tag{3.98}$$

uniformly in $s \in \Omega$.

If there exists a constant $\kappa > 0$ *with* $\tilde{F} := e^{\kappa|\cdot|}F \in L_2(\mathbb{R})$ *the convergence is exponential, i.e. there exists a* $\beta > 0$ *so that for every* $0 < \alpha < \min(\beta, \kappa)$, *there exists a constant* $C > 0$, *independent of* $s \in \Omega$, *with*

$$\|y_J\|_{H_1(J)} + |y_J|_\Gamma \le C\, e^{-\alpha \min(x_+, -x_-)} \|\tilde{F}\|_{L_2(\mathbb{R})}. \tag{3.99}$$

Although the theorem is of some interest on its own we do not give a proof here because we do not need the result for the further analysis of the spectral properties. One can prove the theorem for example in the same way as Theorem 3.2 in [BL99].

3.4 Convergence of eigenvalues in the right half-plane

Throughout this section we always assume that **Assumption 1** holds. We will show that in the half-plane $\{\mathrm{Re}\, s > -\delta\}$ the eigenvalues and generalized eigenspaces of the finite interval approximation of the differential operator (3.2) with suitable boundary conditions converge to the eigenvalues and generalized eigenspaces of the all-line operator.

3.4.1 The general set-up of the eigenvalue-problem in the hyperbolic case

First of all we will explain that there is no essential spectrum* of the all line operator in this set and therefore it makes sense to consider isolated eigenvalues of finite algebraic multiplicity.

Lemma 3.20. *There is no essential spectrum of the all line operator* (3.2) *in* $\{\mathrm{Re}\, s > -\delta\}$.

We only sketch a proof of the Lemma which follows an idea from [San02, Remark 3.2].

Sketch of proof. The first step in the proof is to show that the operators $L(\cdot, s)$: $H^1(\mathbb{R}) \to L_2(\mathbb{R})$ from (3.79) are invertible for $s \in \{\mathrm{Re}\, s > -\delta\}$, except for isolated points. This can be done in the following steps:

- Show that the constant coefficient operators $L_\pm(s)$ from (3.88) have EDs on $\mathbb{R}_\pm$ with data analytical in s. This can be obtained similarly to the proof of Theorem A.7 with the Dunford-Taylor calculus from the analytic dependence of $M(\cdot, s)$ on s.

*See Definition C.1.

- Applying the Roughness Theorem B.4 on $\mathbb{R}_\pm$ shows that the variable coefficient operators also have EDs on $\mathbb{R}_\pm$ with data analytical in s. (A proof of a parameter dependent version of the Roughness Theorem can be found in [San93, Lemma 1.1]. The same proof also holds for analytic dependence on the parameter.)

- In Lemma 3.12 it is shown that $\dim \mathcal{R}(\pi_+(\cdot,s)) + \dim \mathcal{R}\big(I - \pi_-(\cdot,s)\big) = m$ and so by Lemma B.7 $L(\cdot,s)$ are Fredholm of index zero and therefore also $sI - P$ is Fredholm of index zero.

- In [Kat66, II §4.2] it is shown that one can construct bases

$$\operatorname{span}(\varphi_1(s),\dots,\varphi_{m-r}(s)) \ \text{ of } \ \mathcal{R}(\pi_+(0,s))$$

  and

$$\operatorname{span}(\varphi_{m-r+1}(s),\dots,\varphi_m(s)) \ \text{ of } \ \mathcal{R}\big(I - \pi_-(0,s)\big)$$

  which are analytical in s.

- Then one can define a holomorphic function (see also [Hen81, p. 139])

$$\triangle : \begin{array}{ccc} \{\operatorname{Re} s > -\delta\} & \to & \mathbb{C} \\ s & \mapsto & \det(\varphi_1(s),\dots,\varphi_m(s)). \end{array}$$

  The value $\triangle(s)$ is different from 0 if and only if $L(\cdot,s)$ has an (ED) on $\mathbb{R}$ (see Theorem B.6) and in this case $L(\cdot,s)$ is bijective (see Theorem B.2).

- The results of Section 3.2 imply $\triangle \not\equiv 0$ and the analyticity then shows that there are no limit points of $\{s \in \mathbb{C} : \triangle(s) = 0\}$ in $\{\operatorname{Re} s > -\delta\}$. Therefore $\sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ is discrete.

- From [Kat66, III §6.4, IV §5.4] then follows that any $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ is an eigenvalue of finite algebraic multiplicity since $s_0 I - P$ is Fredholm of index zero.

□

From now on we assume $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$. Note that Lemma 3.16 shows that the transformed operator $L(\cdot,s_0)$ has (ED)s on $\mathbb{R}_+$ and $\mathbb{R}_-$. Denote by β_+ and β_- the exponents of the (ED)s on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

By Lemma 3.20 s_0 is an isolated eigenvalue of finite algebraic multiplicity and so we can find $\varepsilon_0 > 0$ with $\overline{K_{\varepsilon_0}(s_0)} \subset \{\operatorname{Re} s > -\delta\}$ and $\sigma(P) \cap \overline{K_{\varepsilon_0}(s_0)} = \{s_0\}$.

Consider the directed set $H := \{J = [x_-, x_+] \subset \mathbb{R} : x_- \le 0 \le x_+, x_+ - x_- \ge 1\}$ with the direction $J_1 \succ J_2 :\Leftrightarrow J_1 \supset J_2$. In the sequel J will always stand for an element from H.

We analyze the spectrum of the all line operator P in a neighborhood of s_0 and write this problem as the eigenvalue problem (in the sense of Definition C.6) for the holomorphic operator-polynomial

$$\mathcal{A}(s) := sI - P \in L\big(H^1(\mathbb{R},\mathbb{C}^m), L_2(\mathbb{R},\mathbb{C}^m)\big).$$

We denote by W the root-subspace of $\mathcal{A}(\cdot)$ to the eigenvalue s_0. By Remark C.8 W is the set of all elements $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ so that there is a sequence of elements $v_0, \dots, v_k =: v$ in $H^1(\mathbb{R}, \mathbb{C}^m)$ with

$$(s_0 I - P)v_0 = 0 \tag{3.100}$$

$$(s_0 I - P)v_{i+1} = v_i \text{ in } L_2 \ \forall i = 0, \dots, k-1. \tag{3.101}$$

Since $v_i \in H^1(\mathbb{R}, \mathbb{C}^m)$ the equality (3.101) also holds in $H^1(\mathbb{R}, \mathbb{C}^m)$ and the mapping

$$(s_0 I - P)|_W : W \to W$$

is nilpotent. From Lemma C.7 one obtains that the length κ_J of the longest Jordan chain of $(s_0 I - P)|_W$, i.e. $\kappa_J = \min\{n \in \mathbb{N} : \mathcal{N}((s_0 I - P)|_W^n) = \mathcal{N}((s_0 I - P)|_W^{n+1})\}$, coincides with the highest order κ of all root-polynomials of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

Remark. *Another way of describing W is to define the space as the range $\mathcal{R}(\Pi_0)$, where Π_0 is the Riesz-Projector given by*

$$\Pi_0 = \frac{1}{2\pi i} \int_{|s-s_0|=\varepsilon_0} (sI - P)^{-1} ds \in L(L_2(\mathbb{R}, \mathbb{C}^m), L_2(\mathbb{R}, \mathbb{C}^m)).$$

Here P is viewed as a closed linear operator in $L_2(\mathbb{R}, \mathbb{C}^m)$ with domain $\mathcal{D}(P) = H^1(\mathbb{R}, \mathbb{C}^m)$. (See for example [Kat66, III §6.4 and III §6.5].)

The idea is to use Theorem 2.26 and Theorem 2.28 to obtain results about approximation of eigenvalues and eigenfunctions of the all line operator by finite interval approximations. First we describe the setting in which we will apply the theorems.

Consider the net of holomorphic operator valued functions

$$\mathcal{A}_J(s) := \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} : \begin{array}{ccc} H^1(J, \mathbb{C}^m) & \to & L_2(J, \mathbb{C}^m) \times \mathbb{C}^m \\ v & \mapsto & \big((sI - P|_J)v, Rv\big) \end{array}$$

as an approximation on the compact interval J of the all line operator on the compact. Here R is a boundary operator of the form (3.83) which satisfies the determinant-condition (3.86) for all s in some open neighborhood $\Sigma \subset \{\operatorname{Re} s > -\delta\}$ of $\overline{K_{\varepsilon_0}(s_0)}$. By Lemma A.4 this is always possible by choosing suitable R_- and R_+ for s_0, i.e. $\det\big(R_- V_-^{II}(s_0) \quad R_+ V_+^{I}(s_0)\big) \neq 0$ and then taking a sufficiently small ε_0.

Denote by σ_J the s_0–group of eigenvalues of $\mathcal{A}_J(\cdot)$ in $\overline{K_{\varepsilon_0}(s_0)}$, i.e. σ_J is the set of all $s \in \overline{K_{\varepsilon_0}(s_0)}$ for which there is $0 \neq v \in H^1(J, \mathbb{C}^m)$ with

$$(sI - P|_J)v = 0 \text{ and } Rv = 0.$$

Similar to the all line case we denote by W_J the closed linear hull of all root subspaces of $\mathcal{A}_J(\cdot)$ to eigenvalues $s_J \in \sigma_J$. Remark C.8 shows that v is an element of the root subspace $W_J(s_J)$ of $\mathcal{A}_J(\cdot)$ to an eigenvalue $s_J \in \sigma_J$ if and only if there is a sequence $v_0, \dots, v_k := v$ in $H^1(J, \mathbb{C}^m)$ with

$$(s_J I - P|_J)v_0 = 0 \text{ and } Rv_0 = 0, \tag{3.102}$$

$$(s_J I - P|_J)v_{i+1} = v_i \ \text{ in } L_2(J, \mathbb{C}^m) \text{ and } Rv_{i+1} = 0. \tag{3.103}$$

Since $v_i \in H^1(J,\mathbb{C}^m)$ the equality (3.103) holds in $H^1(J,\mathbb{C}^m)$.

REMARK. *If one considers the closed linear operator* $P_J : L_2(J,\mathbb{C}^m) \to L_2(J,\mathbb{C}^m)$, *where* $\mathcal{D}(P_J) = \{v \in H^1(J,\mathbb{C}^m) : Rv = 0\}$ *and* $P_J v = P|_J v$, *then one sees that* $W_J(s_J)$ *coincides with the generalized eigenspace of* P_J *to the eigenvalue* s_J.
By Theorem 3.14 there is $J_0 \in H$ *with* $\partial K_{\varepsilon_0}(s_0) \subset \rho(P_J)$ *for all compact intervals* J *with* $J \supset J_0$. *Then (see [Kat66, §6.4]) for all such* J *the space* W_J *can be written as* $W_J = \mathcal{R}(\Pi_J)$, *where* Π_J *is the Riesz-projector given by*

$$\Pi_J = \frac{1}{2\pi i}\int_{\partial K_{\varepsilon_0}(s_0)} (sI - P_J)^{-1} ds.$$

Define the families of operators (cf. Chapter 2.5)

$$\mathcal{P} := (p_J)_{J\in H}, \quad \text{where } \; p_J : \begin{array}{ccc} H^1(\mathbb{R},\mathbb{C}^m) & \to & H^1(J,\mathbb{C}^m), \\ v & \mapsto & v|_J, \end{array}$$

and

$$\mathcal{Q} := (q_J)_{J\in H}, \quad \text{where } \; q_J : \begin{array}{ccc} L_2(\mathbb{R},\mathbb{C}^m) & \to & L_2(J,\mathbb{C}^m)\times\mathbb{C}^m, \\ u & \mapsto & (u|_J, 0). \end{array}$$

These families of operators satisfy the properties (2.1) and (2.2). We use the same notations as in Section 2.5, namely

$$\begin{array}{rll} E := H^1(\mathbb{R},\mathbb{C}^m) & \text{with the norm} & \|v\|_E = \|v\|_{H^1}, \\ E_J := H^1(J,\mathbb{C}^m) & \text{with the norm} & \|v\|_{E_J} = \|v\|_{H^1(J)}, \\ F := L_2(\mathbb{R},\mathbb{C}^m) & \text{with the norm} & \|u\|_F = \|u\|_{L_2}, \\ F_J := L_2(J,\mathbb{C}^m)\times\mathbb{C}^m & \text{with the norm} & \|(u,\eta)\|_{F_J} = \|u\|_{L_2(J)} + |\eta|. \end{array}$$

Lemma 3.21. *For every* $0 < \beta' < \min(\beta_-, \beta_+)$ *there exists a constant* $C = C(\beta')$ *so that for all* $v \in W$ *with* $\|v\|_E = 1$ *holds*

$$|v(x)| \le Ce^{-\beta'|x|} \;\forall x \in \mathbb{R}.$$

Proof. For every $v \in W$ there is a (finite) sequence of elements $v_0, v_1, \dots, v_k = v$ in W so that

$$\begin{aligned} (s_0 I - P)v_0 = 0 \;\text{ in } L_2(J,\mathbb{C}^m) \\ \text{and } (s_0 I - P)v_{i+1} = v_i \;\text{ in } L_2(J,\mathbb{C}^m), \text{ for } i = 0,\dots,k-1. \end{aligned}$$

We prove by induction that for each $v \in W$ and $\beta' < \min(\beta_-, \beta_+)$ there is a constant $c = c(\beta', v)$ with $|v(x)| \le ce^{-\beta'|x|} \;\forall x \in \mathbb{R}$.

In the case $k = 0$ we have $(s_0 I - P)v = 0 \Leftrightarrow L(\cdot, s_0)v = 0$ and so Theorem B.3 implies that there is $c = c(v) > 0$ with

$$|v(x)| \le ce^{-\min(\beta_-,\beta_+)|x|} \;\forall x \in \mathbb{R}.$$

Now assume $k \ge 1$. Then $(s_0 I - P)v_k = v_{k-1}$ and by induction for each $0 < \tilde\beta < \min(\beta_-, \beta_+)$ there is $c = c(\tilde\beta, v_{k-1})$ with

$$|v_{k-1}(x)| \le ce^{-\tilde\beta|x|} \;\forall x \in \mathbb{R}.$$

Therefore $|B^{-1}(x)v_{k-1}(x)| \leq \|B^{-1}\|_\infty ce^{-\tilde{\beta}|x|}$ $\forall x \in \mathbb{R}$. It holds

$$L(\cdot, s_0)v_k = B^{-1}v_{k-1}$$

and so Theorem B.3 implies that for each $0 < \beta' < \tilde{\beta}$ there is $c_1 = c_1(\beta', c, v_k)$ with

$$|v_k(x)| \leq c_1 e^{-\beta'|x|} \;\forall x \in \mathbb{R}.$$

Since v_{k-1} is uniquely determined by v_k we can also write $c_1 = c_1(\beta', v_k)$.

Because of Lemma 3.20 it holds $\dim W < \infty$. Choose a basis $v_1, \dots, v_r$ of W, then every $v \in W$ can uniquely be written as $v = \sum_i \alpha_i(v)v_i$. Since the coefficients depend continuously on v one finds $\sum_i |\alpha_i(v)| \leq \text{const}$ for all $v \in W$ with $\|v\|_E = 1$. This leads to

$$\begin{aligned} |v(x)| &\leq |\sum_i \alpha_i(v)v_i(x)| \leq \sum_i |\alpha_i(v)||v_i(x)| \\ &\leq \left(\sum_i |\alpha_i(v)|c(v_i, \beta')\right) e^{-\beta'|x|} \leq \text{const}e^{-\beta'|x|} \;\forall x \in \mathbb{R}, \end{aligned}$$

where the constant depends on β', but does not depend on v. □

3.4.2 The convergence theorem in the hyperbolic case

Using the abstract theory of Chapter 2 we will now prove the following theorem about the approximation of the eigenvalues and eigenspaces in the right half-plane.

Theorem 3.22. *With the assumptions and notations from above, in particular Assumption 1 and the assumptions on Σ hold, there is a compact interval $J_0 \subset \mathbb{R}$ such that for all compact intervals $J = [x_-, x_+] \subset \mathbb{R}$ with $J \supset J_0$ the following properties hold.*

The s_0-group of eigenvalues σ_J converges to s_0 in the sense that for each $0 < \beta' < \min(\beta_-, \beta_+)$ there is a constant $\text{const} = \text{const}(\beta')$ *with*

$$\max_{s \in \sigma_J} |s - s_0| = \text{dist}(\sigma_J, s_0) \leq \text{const}e^{-\frac{\beta'}{\kappa}\min(x_+, x_-)}. \tag{3.104}$$

Each net $(v_J)_{J \succ J_0}$ of normalized eigenelements to eigenvalues $s_J \in \sigma_J$, i.e.

$$\mathcal{A}_J(s_J)v_J = 0, \; \|v_J\|_{H^1(J,\mathbb{C}^m)} = 1,$$

is $\mathcal{P}$-compact and it holds the estimate

$$\sup_{\substack{v_J \in E_J\colon \|v_J\|_{E_J}=1 \\ s_J \in \Sigma_J,\, Rv_J = 0 \\ (s_J I - P|_J)v_J = 0}} \; \inf_{\substack{v_0 \in E \\ (s_0 I - P)v_0 = 0}} \|v_J - v_0|_J\|_{H^1(J)} \leq \text{const}e^{-\frac{\beta'}{\kappa}\min(x_+, x_-)}. \tag{3.105}$$

Furthermore for the root-subspaces hold

$$\dim W_J = \dim W < \infty. \tag{3.106}$$

The root-subspace W_J approximates the root-subspace W in the sense

$$\vartheta(W_J, W) = \sup_{\substack{v_J \in W_J \\ \|v_J\|_{E_J}=1}} \operatorname{dist}(v_J, p_J W) \le \operatorname{const} e^{-\beta' \min(-x_-, x_+)} \tag{3.107}$$

and

$$\vartheta(W, W_J) = \sup_{\substack{v \in W \\ \|v\|_E=1}} \operatorname{dist}(p_J v, W_J) \le \operatorname{const} e^{-\beta' \min(-x_-, x_+)}. \tag{3.108}$$

The constants in (3.104)*,* (3.105)*,* (3.107)*, and* (3.108) *do not depend on J.*

Before we can prove the Theorem we show some properties of the operator functions $\mathcal{A}$ and $\mathcal{A}_J$ in the next two Lemmas. Recall that $\Sigma \subset \{\operatorname{Re} s > -\delta\}$ is an open neighborhood of $\overline{K_{\varepsilon_0}(s_)}$ such that

$$\det \begin{pmatrix} R_- V_-^{II}(s) & R_+ V_+^{I}(s) \end{pmatrix} \neq 0 \; \forall s \in \Sigma.$$

Lemma 3.23. *With the notations from above holds*

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \xrightarrow{\mathcal{PQ}} sI - P \text{ regularly for all } s \in \Sigma.$$

Proof. Let $s \in \Sigma$ be arbitrary and let $(v_J)_{J\in H}$ be a bounded net in (E_J) such that

$$\left(\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} v_J \right)_{J \in H} \text{ is } \mathcal{Q}\text{-compact.} \tag{3.109}$$

Then also the net

$$\left(\begin{pmatrix} B^{-1}(sI - P|_J) \\ R \end{pmatrix} v_J \right)_{J \in H} = \left(\begin{pmatrix} L(\cdot, s)|_J \\ R \end{pmatrix} v_J \right)_{J \in H} \tag{3.110}$$

is $\mathcal{Q}$-compact:
Let $H' \subset H$ be any cofinal subset of H. Then there is $h \in F$ and $H'' \subset H$ with

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} v_J \xrightarrow{\mathcal{Q}} h \; (J \in H''),$$

but then

$$\begin{aligned} &\left\| \begin{pmatrix} B^{-1}(sI - P|_J) v_J \\ R v_J \end{pmatrix} - \begin{pmatrix} (B^{-1} h)|_J \\ 0 \end{pmatrix} \right\|_{F_J} \\ &\qquad = \left\| B^{-1}\big((sI - P|_J) v_J - h|_J\big) \right\|_{L_2(J, \mathbb{C}^m)} + |R v_J| \\ &\qquad \le \operatorname{const} \Big(\left\| (sI - P|_J) v_J - h|_J \right\|_{L_2(J, \mathbb{C}^m)} + |R v_J| \Big) \to 0 \; (J \in H''). \end{aligned}$$

Since $B \in L(F, F)$ is a linear homeomorphism this proves (3.110). By the equality $L(\cdot, s) = -B^{-1}(sI - P)$, where $L(\cdot, s)$ is given in (3.79), Lemma 3.12, and Lemma 3.16 together with the determinant-condition for R show that the assumptions of Theorem 2.29 are fulfilled and hence $(v_J)_{J \in H}$ is $\mathcal{P}$-compact. ☐

Lemma 3.24. *For every $s \in \Sigma$ and all $J \in H$ the operators*

$$sI - P \in L(E,F) \quad \text{and} \quad \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \in L(E_J, F_J)$$

are Fredholm of index zero.

Proof. From assumption (H2) follows that $B^{-1} : F \to F$, $v \mapsto B^{-1}v$, is a linear homeomorphism. Therefore the operator $sI - P$ is Fredholm if and only if $-B^{-1}(sI - P) = L(\cdot, s) \in L(E,F)$ is Fredholm. In this case they have the same Fredholm indices. Lemma 3.12 and Lemma B.7 then yield the assertion.

For the analysis of the finite interval operator note that the bounded linear operator

$$\frac{d}{dx} : H^1(J, \mathbb{C}^m) \to L_2(J, \mathbb{C}^m)$$

is Fredholm of index m and therefore the assumption (H2) implies the same for

$$B\frac{d}{dx} : \begin{array}{ccc} H^1(J,\mathbb{C}^m) & \to & L_2(J,\mathbb{C}^m) \\ v & \mapsto & Bv_x. \end{array}$$

Since $sI|_J - P|_J \in L(H^1(J,\mathbb{C}^m), L_2(J,\mathbb{C}^m))$ can be viewed as a compact perturbation of $-B\frac{d}{dx}$ because of the Rellich embedding theorem [Rob01, Chapter 5], this operator is Fredholm of index m, too. Now by bordering with the boundary operator $R \in L(H^1(J,\mathbb{C}^m), \mathbb{C}^m)$ Lemma C.9 shows that

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \in L(E_J, F_J)$$

is a Fredholm operator of index zero. □

Proof of Theorem 3.22. By Lemma 3.24 the operator-valued functions $\mathcal{A}(\cdot)$ and $\mathcal{A}_J(\cdot)$ are Fredholm of index zero for all $s \in \Sigma$ and by Lemma 3.23 for every $s \in \Sigma$ the operators $\mathcal{A}_J(s)$ regularly $\mathcal{PQ}$-converge to $\mathcal{A}(s)$. Finally, because of Lemma 3.20 there is $\underline{s} \in \Sigma \backslash \overline{K_{\varepsilon_0}(s_0)}$ with $\underline{s} \in \rho(P)$. Therefore the assumptions of Theorem 2.26 are verified and the theorem implies that there is a compact interval $J_0 \in H$ such that (3.106) and the estimates

$$\max_{s_J \in \sigma_J} |s_J - s_0| \le C\epsilon_J^{\frac{1}{\kappa}}, \tag{3.104'}$$

$$\sup_{\substack{v_J \in \mathcal{N}(\mathcal{A}_J(s_J)) \\ s_J \in \sigma_J, \|v_J\|_{E_J}=1}} \inf_{v_0 \in \mathcal{N}(\mathcal{A}(s_0))} \|v_J - p_J v_0\|_{E_J} \le C\epsilon_J^{\frac{1}{\kappa}}, \tag{3.105'}$$

$$\vartheta(W_J, W) \le C\epsilon_J, \tag{3.107'}$$

$$\vartheta(W, W_J) \le C\epsilon_J, \tag{3.108'}$$

with

$$\epsilon_J = \max_{\substack{v,v' \in W \\ \|v\|_E = 1 \\ (s_0 I - P)v = v'}} \left\| \begin{pmatrix} s_0 I - P|_J \\ R \end{pmatrix} p_J v - \begin{pmatrix} I \\ 0 \end{pmatrix} p_J v' \right\|_{F_J},$$

hold. By application of Lemma 3.21 one can bound ϵ_J by

$$\begin{aligned}
\epsilon_J = &\max_{\substack{v,v'\in W\\ \|v\|_E=1\\ (s_0I-P)v=v'}} \left\| \begin{pmatrix} s_0 I - P|_J \\ R \end{pmatrix} p_J v - \begin{pmatrix} I \\ 0 \end{pmatrix} p_J v' \right\|_{F_J} \\
= &\max_{\substack{v,v'\in W\\ \|v\|_E=1\\ (s_0I-P)v=v'}} \left(\|(s_0 I - P|_J)v|_J - Iv'|_J\|_{H^1(J,\mathbb{C}^m)} + |Rv|_J| \right) \\
= &\max_{\substack{v,v'\in W\\ \|v\|_E=1\\ (s_0I-P)v=v'}} \left(\|0\|_{H^1(J,\mathbb{C}^m)} + |P_- v(x_-) + P_+ v(x_+)| \right) \\
\leq &\max_{\substack{v\in W\\ \|v\|_E=1}} \left(|P_-| + |P_+|\right)\left(|v(x_-)| + |v(x_+)|\right) \\
\leq &\operatorname{const} e^{-\beta' \min(|x_-|,|x_+|)}
\end{aligned}$$

with a constant which depends on $0 < \beta' < \min(\beta_-, \beta_+)$ only. □

3.4.3 Convergence in the case of simple eigenvalues

In the case of a simple eigenvalue $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ we prove a theorem similar to Theorem 3.22 that also allows for s-dependent boundary conditions. The aim is to have results similar to the ones from the previous theorem in the case of projection boundary conditions (see [Bey90]) which should allow for shorter intervals for the approximation of the eigenvalues.

We assume that the boundary operator R again is a linear two point boundary operator but depends holomorphically on s in an open neighborhood Σ of s_0. That means that the matrices $P_\pm$ depend holomorphic on s and

$$R : \Sigma \to L(H^1(J), \mathbb{C}^m),\ R(s)v = (P_-(s)v(x_-) + P_+(s)v(x_+)).$$

We consider the following determinant which is similar to the determinant from Theorem 3.22 (see (3.86)

$$D(s) := \begin{pmatrix} P_-(s)V_-(s)^{II} & P_+(s)V_+(s)^{I} \end{pmatrix}.$$

Here we assume that $V_\pm(s)^{I,II}$ are bases of the stable and unstable subspaces of $M_\pm(s)$ as above.

Theorem 3.25. *Let Assumption 1 hold and consider the same setting of spaces and operators as in 3.4.2. Furthermore assume*

$$D(s_0) \neq 0. \tag{3.111}$$

Let $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ be a simple eigenvalue, i.e. s_0 is a simple eigenvalue of the holomorphic operator-valued function $\mathcal{A}(s) = sI - P$, with eigenfunction $0 \neq v_0 \in \mathcal{N}(s_0 I - P)$.*

*See Definition C.6.

Then there is a compact interval $J_0 \in H$ and a positive constant δ_0 such that for all compact intervals $J \supset J_0$ there exists exactly one simple eigenvalue s_J with $|s_0 - s_J| \le \delta_0$ of the approximation of $\mathcal{A}$ on the finite interval J given by

$$\mathcal{A}_J(\cdot) : s \mapsto \begin{pmatrix} sI - P|_J \\ R(s) \end{pmatrix}.$$

Moreover, there is a corresponding eigenfunction $v_J \in E_J$ such that the estimate

$$|s_J - s_0| + \|v_J - p_J v_0\| \le \text{const}|R_-(s_0)v_0(x_-) + R_+(s_0)v_0(x_+)| \tag{3.112}$$

holds.

Proof. By the holomorphy assumption on R it is clear that $\mathcal{A}$ and $\mathcal{A}_J$ are holomorphic in s. Because of the determinant-condition (3.111) Lemma 3.23 shows

$$\begin{pmatrix} s_0 I - P|_J \\ R(s_0) \end{pmatrix} \xrightarrow{\mathcal{PQ}} (s_0 I - P) \text{ regularly}$$

and by Lemma 3.24 the operators $\mathcal{A}(s_0)$ and $\mathcal{A}_J(s_0)$ are Fredholm of index zero for all $J \in H$.

Furthermore, $\mathcal{A}'_J(s_0) = \begin{pmatrix} I \\ R'(s_0) \end{pmatrix} \xrightarrow{\mathcal{PQ}} I_{H^1 \to L_2}$. To see this, note that since the Sobolev-inequality (Lemma C.2) implies

$$\|\mathcal{A}'_J(s_0)\|_{L(H^1(J,\mathbb{C}^m), L_2(J,\mathbb{C}^m)\times\mathbb{C}^m)} \le \text{const } \forall J \in H, [-1,1] \subset J,$$

it suffices because of Lemma 2.18 to see

$$\|\mathcal{A}'_J(s_0)p_J v - q_J I v\|_{F_J} = \|v|_J - v|_J\|_{L_2(J)} + |R'(s_0)v|_J| \to 0.$$

This convergence in turn is a consequence of Lemma C.3.

Finally

$$\|\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0)\| = \left\| \begin{pmatrix} I \\ R'(s) \end{pmatrix} - \begin{pmatrix} I \\ R'(s_0) \end{pmatrix} \right\| = |R'(s) - R'(s_0)|_{H^1(J)\to\mathbb{C}^m}$$

implies because of the continuity of R' that for every $\varepsilon > 0$ there is $\delta > 0$ with

$$|R'_-(s) - R'_-(s_0)| + |R'_+(s) - R'_+(s_0)| < \varepsilon \text{ for all } |s - s_0| \le \delta.$$

Applying the Sobolev-inequality again shows that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|\mathcal{A}'_J(s) - \mathcal{A}'_J(s_0)\| \le \varepsilon \; \forall J \in H, \; J \supset [-1,1], \; |s - s_0| \le \delta.$$

Now the theorem follows directly by application of Lemma 2.28. □

Remark. *In [Bey90] it is shown that if one considers so called projection boundary conditions for the approximation of the zero eigenvalue the right hand side of (3.112) converges faster to zero with a factor two in the exponent compared to (3.104) and (3.105) from Theorem 3.22.*

Possibly the projection boundary conditions also lead to better estimates for the approximation of the other eigenvalues (cf. [HW80, Chapter 4]), but we do not know about a general Theorem which states this.

4 The mixed case

In this chapter we consider a linear mixed hyperbolic-parabolic PDE of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = P \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{in } [0,\infty) \times \mathbb{R}, \tag{4.1}$$

where the operator $P : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$ is of the form

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.2}$$

Such systems for example can arise by linearization of a nonlinear PDE (1.1) at a travelling wave solution as we will see in Chapter 5.3.

We analyze the system similarly to the analysis of Chapter 3.

We will see that the special structure of the operator P from (4.2) at least in the case of large $|s|$, allows to derive resolvent estimates by combining the results for the hyperbolic part presented in Section 3.2 and the results for the parabolic part shown in [BL99, Chapter 2]. Therefore we give a review of the results from [BL99] in Section 4.2 which we improve in one point. We also give an all line version of [BL99, Theorem 2.1] which is needed for the all line resolvent estimates of the mixed system. The coupling of the results for the hyperbolic and parabolic systems is done with a transformation argument. For the resolvent estimates in bounded regions of the complex plane we apply the abstract theory from Chapter 2 in the same fashion as in Chapter 3.

4.1 Assumptions

For the coefficients of the operator P from (4.2) we make the following assumptions.

Assumption 2. *The coefficients of the parabolic part, i.e. of the operator*

$$P_{par} : H^2(\mathbb{R}, \mathbb{C}^n) \to L_2(\mathbb{R}, \mathbb{C}^n), \; P_{par} u = A u_{xx} + B_{11} u_x + C_{11} u, \tag{4.3}$$

satisfy the following conditions.

(P1) *The coefficient matrices B_{11} and C_{11} belong to $\mathcal{C}(\mathbb{R}, \mathbb{C}^{n,n})$ and*

$$\exists \lim_{x\to\pm\infty} B_{11}(x) =: B_{11\pm},$$
$$\exists \lim_{x\to\pm\infty} C_{11}(x) =: C_{11\pm}.$$

(P2) *The matrix $A \in \mathbb{C}^{n,n}$ is constant and there is $\alpha > 0$ with $A + A^* \geq \alpha I$.*

The coefficients of the hyperbolic part, i.e. of the operator

$$P_{hyp} : H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^m),\ P_{hyp}u = B_{22}u_x + C_{22}u, \tag{4.4}$$

satisfy **(H1)**–**(H3)** *from Assumption 1.*
Finally,

(M1) *the coefficients B_{12}, C_{12}, and C_{21} are continuous matrix-valued functions and*

$$\exists \lim_{x\to\pm\infty} B_{12}(x) =: B_{12\pm},$$
$$\exists \lim_{x\to\pm\infty} C_{12}(x) =: C_{12\pm},$$
$$\exists \lim_{x\to\pm\infty} C_{21}(x) =: C_{21\pm},$$

and $B_{12} \in \mathcal{C}^1(\mathbb{R}, \mathbb{C}^{n,m})$ with $\|B_{12,x}\|_\infty < \infty$.

(M2) *There is $\delta > 0$ such that, for all $\omega \in \mathbb{R}$ and all $s \in \mathbb{C}$,*

$$\det\Big(-\omega^2 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + i\omega \begin{pmatrix} B_{11+} & B_{12+} \\ 0 & B_{22+} \end{pmatrix} + \begin{pmatrix} C_{11+} & C_{12+} \\ C_{21+} & C_{22+} \end{pmatrix} - sI_{n+m}\Big) = 0$$

or

$$\det\Big(-\omega^2 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + i\omega \begin{pmatrix} B_{11-} & B_{12-} \\ 0 & B_{22-} \end{pmatrix} + \begin{pmatrix} C_{11-} & C_{12-} \\ C_{21-} & C_{22-} \end{pmatrix} - sI_{n+m}\Big) = 0$$

imply

$$\operatorname{Re} s \leq -\delta < 0.$$

In the subsequent sections we will analyze the all line resolvent equation

$$(sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f + g_x \\ F \end{pmatrix} \quad \text{in } L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m) \tag{4.5}$$

and its restriction to a nonempty, compact interval $J = [x_-, x_+]$, $x_+ > x_-$,

$$(sI - P|_J)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f + g_x \\ F \end{pmatrix} \quad \text{in } L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m) \tag{4.6a}$$

As in the hyperbolic part one has to give additional boundary conditions for the restricted operator to obtain similar solvability properties. This is due to the fact that the boundary conditions of the all line operator P are hidden in its domain. Therefore we consider linear two point boundary conditions again.

$$R\begin{pmatrix} u \\ v \end{pmatrix} = \eta \in \mathbb{C}^{2n+m}. \tag{4.6b}$$

The exact assumptions for the boundary operator will be specified later.

As assumption (H4) the assumption (M2) is an assumption on the spectrum of P. We will see that it implies a Fredholm property for the operator $(sI - P)$ to the right of the algebraic curves defined by (M2).

4.2 Review of results from the parabolic case

We briefly review some results for the parabolic part of (4.2) for large $|s|$ which are presented in [BL99]. We always assume **(P1)** and **(P2)** in this section.

In [BL99, Chapter 2] the resolvent equation

$$(sI_n - P_{par})u = f + g_x \text{ in } L_2(\mathbb{R}, \mathbb{C}^n), \tag{4.7}$$

with $f \in L_2(\mathbb{R}, \mathbb{C}^n)$ and $g \in H^1(\mathbb{R}, \mathbb{C}^n)$ is transformed into a first order equation by using the variables

$$z = \begin{pmatrix} u \\ \frac{1}{\rho}(Au_x + g) \end{pmatrix},$$

with $\rho = |s|^{\frac{1}{2}}$. Then (4.7) can be rewritten as

$$L_{par}(s)z = z_x - M_{par}(\cdot, s)z = h \text{ in } L_2(\mathbb{R}, \mathbb{C}^{2n}) \tag{4.8}$$

with

$$M_{par}(x,s) = \rho \begin{pmatrix} 0 & A^{-1} \\ \frac{s-C}{\rho^2} & -\frac{1}{\rho}B_{11}A^{-1} \end{pmatrix} \text{ and } h(x,s) = \begin{pmatrix} -A^{-1}g \\ \frac{1}{\rho}B_{11}A^{-1}g - \frac{1}{\rho}f \end{pmatrix}. \tag{4.9}$$

Note that by the structure of h the equality (4.8) in fact is an equality in $H^1(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^n)$. The authors show some results about the asymptotic behavior of solutions of (4.8) which we briefly review and then apply in order to obtain resolvent estimates for the all line operator.

Lemma 4.1 ([BL99, Lemma 2.3])**.** *There are positive constants c_1, $\tilde{K}_1$, K_2, $\tilde{\beta}_1$, and ε so that the operators $L_{par}(s)$ have an (ED) on $\mathbb{R}$ if*

$$s = re^{2i\theta}, \; r \geq c_1, \; |\theta| \leq \frac{\pi}{4} + \varepsilon.$$

The dichotomy data are given as $(\tilde{K}_1, \rho\tilde{\beta}_1, \pi(x,s))$, where π is continuous in s.

Remark. *The proof presented in [BL99] only works for constant principal part A, since it uses a rescaling method which is not correct if A is not constant. Let $M_\theta(x) = \begin{pmatrix} 0 & A^{-1}(x) \\ e^{2i\theta}I & 0 \end{pmatrix}$ and let $z \in H^1(\mathbb{R}, \mathbb{C}^{2n})$ be a solution of $L_\theta z = z_x - M_\theta z = 0$. Then the function $u(x) := z(\rho x)$ which is considered in [BL99, p. 211] satisfies*

$$u_x(x) = \rho z_x(\rho x) = \rho M_\theta(\rho x) z(\rho x) = \rho M_\theta(\rho x) u(x),$$

which in general is not the same as $\rho M_\theta(x)u(x)$.

Since we will make use of the results from [BL99, Chapter 2], we assume that A is a constant matrix.

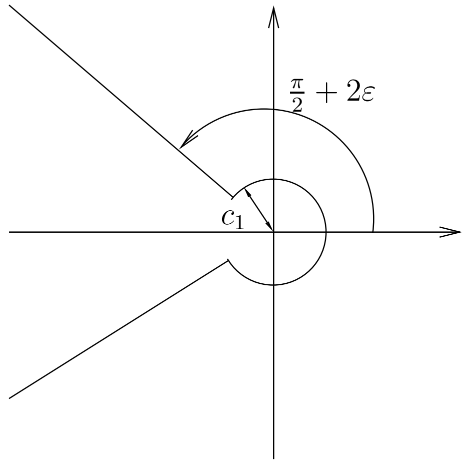


Figure 4.1: The sector of the complex plane defined in Lemma 4.1.

As in Section 3.2.2 we obtain from Lemma 4.1 that if s is restricted to the sector of the complex plane from Lemma 4.1, for each $h \in L_2(\mathbb{R}, \mathbb{C}^{2n})$ there is a unique solution $z \in H^1(\mathbb{R}, \mathbb{C}^{2n})$ of $L_{par}(s)z = h$. Moreover, the solution satisfies the estimate

$$\rho^2 \|z\|^2 \leq \text{const}\|h\|^2.$$

Applying the differential equation (4.8) one finds

$$\|z_x\|^2 \leq \text{const}\|h\|^2,$$

where the constant does not depend on h and s for all s from that region.

Therefore recalling the connection of (4.8) and (4.7) one obtains unique solvability of (4.7) for each $f \in L_2(\mathbb{R}, \mathbb{C}^n)$ and $g \in H^1(\mathbb{R}, \mathbb{C}^n)$. Furthermore, from the inequalities

$$\rho^2 \left\| \begin{pmatrix} u \\ \frac{1}{\rho}(Au_x + g) \end{pmatrix} \right\|^2 \leq \text{const} \left\| \begin{pmatrix} -A^{-1}g \\ \frac{1}{\rho}B_{11}A^{-1}g - \frac{1}{\rho}f \end{pmatrix} \right\|^2$$

and

$$\left\| \begin{pmatrix} u_x \\ \frac{1}{\rho}(Au_{xx} + g_x) \end{pmatrix} \right\|^2 \leq \text{const} \left\| \begin{pmatrix} -A^{-1}g \\ \frac{1}{\rho}B_{11}A^{-1}g - \frac{1}{\rho}f \end{pmatrix} \right\|^2$$

one derives the estimates

$$\begin{aligned} \rho^2\|u\|^2 &\leq \text{const}\big(\|g\|^2 + \frac{1}{\rho^2}\|f\|^2\big), \\ \|u_x\|^2 &\leq \text{const}\big(\|g\|^2 + \frac{1}{\rho^2}\|f\|^2\big), \\ \|u_{xx}\|^2 &\leq \text{const}\big(\rho^2\|g\|^2 + \|g_x\|^2 + \|f\|^2\big). \end{aligned}$$

This proves the following theorem (cf. [KKP94]).

Theorem 4.2. *Let $f \in L_2(\mathbb{R},\mathbb{C}^n)$ and $g \in H^1(\mathbb{R},\mathbb{C}^n)$. Then there are positive constants c_1, K, and ε such that for all $s \in \mathbb{C}$ with $s = re^{2i\theta}$, $r \geq c_1$, $|\theta| \leq \frac{\pi}{4} + \varepsilon$ the resolvent equation (4.7) has a unique solution $u \in H^2(\mathbb{R},\mathbb{C}^n)$. This solution satisfies estimates of the form*

$$|s|^2\|u\|^2 + |s|\|u_x\|^2 \leq K\big(\|f\|^2 + |s|\|g\|^2\big) \tag{4.10}$$

and

$$\|u_{xx}\|^2 \leq K\big(\|f\|^2 + |s|\|g\|^2 + \|g_x\|^2\big)$$

with a constant K independent of f, g, s.

In [BL99] the all line problem (4.7) is restricted to a finite interval with supplementary boundary conditions, i.e.

$$\begin{aligned} (sI_n - P_{par}|_J)u &= f + g_x \ \text{ in } L_2(J,\mathbb{C}^n), \\ R_{par}u &= \gamma. \end{aligned} \tag{4.11}$$

The boundary operator $R_{par} : H^2(J,\mathbb{C}^n) \to \mathbb{C}^{2n}$ is assumed to be of the form

$$R_{par}u = \begin{pmatrix} P_-^I & Q_-^I \\ P_-^{II} & 0 \end{pmatrix} \begin{pmatrix} u(x_-) \\ u_x(x_-) \end{pmatrix} + \begin{pmatrix} P_+^I & Q_+^I \\ P_+^{II} & 0 \end{pmatrix} \begin{pmatrix} u(x_+) \\ u_x(x_+) \end{pmatrix} = \begin{pmatrix} \gamma^I \\ \gamma^{II} \end{pmatrix}$$

with $\operatorname{rank}(Q_-^I\ Q_+^I) = r$, and matrices $P_-^I, P_+^I, Q_-^I, Q_+^I \in \mathbb{C}^{r,n}$, $P_-^{II}, P_+^{II} \in \mathbb{C}^{2n-r,n}$, $\gamma^I \in \mathbb{C}^r$, $\gamma^{II} \in \mathbb{C}^{2n-r}$. Note that this form can always be obtained by multiplication from the left with an invertible matrix.

The authors prove the following theorem.

Theorem 4.3 ([BL99, Theorem 2.1])**.** *Consider the BVP (4.11) for $s = re^{i\phi}$, $|\phi| \leq \frac{\pi}{2} + \varepsilon$, $r \geq c_1$, and assume*

$$\det \begin{pmatrix} Q_-^I & Q_+^I \\ -P_-^{II} A^{\frac{1}{2}} & P_+^{II} A^{\frac{1}{2}} \end{pmatrix} \neq 0.$$

Then there are positive constants c_1, K, ε so that for every $J = [x_-, x_+]$ with $x_+ - x_- \geq 1$ the BVP (4.11) has a unique solution $u \in H^2(J,\mathbb{C}^n)$. This solution can be estimated by

$$\rho^2\|u\|^2 + \|u_x\|^2 + \rho|u|_\Gamma^2 + \frac{1}{\rho}|u_x|_\Gamma^2 \leq K\Big(\frac{1}{\rho^2}\|f\|^2 + \|g\|^2 + \frac{1}{\rho}|\gamma^I|^2 + \rho|\gamma^{II}|^2 + \frac{1}{\rho}|g|_\Gamma^2\Big), \tag{4.12}$$

with $\rho = |s|^{\frac{1}{2}}$. The constants c_1, K, and ε are independent of the right hand side from (4.11), of s, and of J.

Remark. *In the estimate (4.12) we have the term $\frac{1}{\rho}|g|_\Gamma^2$ which is an improvement of the original term $\rho|g|_\Gamma^2$ derived in [BL99]. This is justified as follows. In*

[BL99, (2.12)] the values $g(x_-)$ and $g(x_+)$ enter the boundary value η only in the first r components since

$$\eta = \gamma + Q_- A^{-1} g(x_-) + Q_+ A^{-1} g(x_+) = \begin{pmatrix} \gamma^I + Q_-^I A^{-1} g(x_-) + Q_+^I A^{-1} g(x_+) \\ \gamma^{II} \end{pmatrix}.$$

Therefore in [BL99, (2.33)] one obtains

$$|\eta^I|^2 \le K_6 \big(|\gamma^I|^2 + |g|_\Gamma^2 \big) \text{ and } |\eta^{II}|^2 = |\gamma^{II}|^2.$$

Thus the estimate (4.12) *follows from*

$$\rho^2 \|z\|^2 + \rho |z|_\Gamma^2 \le K_3 \left(\|h\|^2 + \frac{1}{\rho} |\eta^I|^2 + \rho |\eta^{II}|^2 \right).$$

The improvement is essential for the analysis of coupled hyperbolic-parabolic systems (see (4.33) *and* (4.34)*).*

4.3 General properties of the mixed operator

In this section we show some general properties of the mixed operator

$$sI - P : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

from (4.5) and of its restriction to finite intervals. We always assume that **Assumption 2** holds.

The main results will be Fredholm properties of $sI - P$ and of the finite interval approximation of this operator. The idea is to transform the operator to a first order operator and then apply the result of Lemma B.7. This means that we transform the resolvent equation $(sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ F \end{pmatrix}$ into a first order equation. Note that this coincides with equation (4.5) if $g = 0$. By using the transformation $(u, v) \mapsto (u, Au_x, v)$ this equation can be rewritten as

$$L(s)z := z_x - M(x, s)z = \begin{pmatrix} 0 \\ -f + B_{12} B_{22}^{-1} F \\ -B_{22}^{-1} F \end{pmatrix}, \tag{4.13}$$

where

$$L(s) : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to H^1(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

and

$$M(\cdot, s) = \begin{pmatrix} 0 & A^{-1} & 0 \\ B_{12} B_{22}^{-1} C_{21} + (sI_n - C_{11}) & -B_{11} A^{-1} & -C_{12} - B_{12} B_{22}^{-1} (sI_m - C_{22}) \\ -B_{22}^{-1} C_{21} & 0 & B_{22}^{-1} (sI_m - C_{22}) \end{pmatrix}. \tag{4.14}$$

We make this unusual choice of domain of $L(s)$ since by this we can directly relate the Fredholm properties of $L(s)$ and of $sI - P$.

Before we analyze the properties of $L(s)$ and of $sI - P$ we show that assumption (M2) implies the hyperbolicity of the limit matrices $\lim_{x \to \pm\infty} M(x, s) =: M_\pm(s)$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$. This follows directly from the next lemma.

Lemma 4.4. *For s and κ in $\mathbb{C}$ the relation*

$$s \in \sigma\Big(\kappa^2 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \kappa \begin{pmatrix} B_{11\pm} & B_{12\pm} \\ 0 & B_{22\pm} \end{pmatrix} + \begin{pmatrix} C_{11\pm} & C_{12\pm} \\ C_{21\pm} & C_{22\pm} \end{pmatrix}\Big)$$

holds if and only if

$$\det(M_\pm(s) - \kappa I) = 0.$$

Proof. Consider the '+' case. The equation $\det(M_+(s) - \kappa I) = 0$ holds if and only if there are $u, w \in \mathbb{C}^n$, $v \in \mathbb{C}^m$ with $(u, w, v) \neq 0$ that satisfy

$$M_+(s) \begin{pmatrix} u \\ w \\ v \end{pmatrix} = \kappa \begin{pmatrix} u \\ w \\ v \end{pmatrix}.$$

This is equivalent to

$$\begin{aligned} w &= \kappa A u, \\ \kappa^2 A u + \kappa B_{12+} v + \kappa B_{11+} u + C_{11+} u + C_{12+} v &= s u, \\ \kappa B_{22+} v + C_{21+} u + C_{22+} v &= s v, \end{aligned}$$

which proves the Lemma. □

This result will now be used to show that $L(s)$ is Fredholm of index zero for all $\operatorname{Re} s > -\delta$, which in the end will be utilized to show that the operator $sI - P$ is Fredholm of index zero for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$.

In view of Lemma B.7 and its Corollary B.8 together with Lemma B.5 it suffices to show that the constant coefficient operators $L_\pm(s)z = z_x - M_\pm(s)z$ have an (ED) on $\mathbb{R}$ with $\dim \mathcal{R}(\pi_+(s)) + \dim \mathcal{R}(I - \pi_-(s)) = 2n + m$, where π_+ and π_- are the corresponding projectors of the constant coefficient operators $L_\pm(s)$.

From now on we assume $|s| \geq 1$. The main idea for the analysis of the constant coefficient operators $L_\pm(s)$ from (4.13) is to transform them such that the block-diagonal entries are of the forms which are already analyzed in the hyperbolic and in the parabolic case and simultaneously the outer block diagonal entries are small compared to the block diagonal entries.

We use the transformation of variables

$$\tilde{z} := S_{\frac{1}{\rho}} T_{B_{12\pm}} z,$$

where $\rho = |s|^{\frac{1}{2}}$ and the matrices are defined by

$$S_{\frac{1}{\rho}} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & \frac{1}{\rho} I_n & 0 \\ 0 & 0 & I_m \end{pmatrix} \quad \text{and} \quad T_{B_{12\pm}} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & B_{12\pm} \\ 0 & 0 & I_m \end{pmatrix}.$$

Note that $S_{\frac{1}{\rho}}^{-1} = S_\rho$ and $T_B^{-1} = T_{-B}$ holds. Thus we obtain the operators

$$\tilde{L}_\pm(s)z = z_x - \tilde{M}_\pm(s)z, \tag{4.15}$$

with

$$\tilde{M}_{\pm}(s) = \begin{pmatrix} 0 & \rho A^{-1} & -A^{-1}B_{12\pm} \\ \frac{1}{\rho}(sI - C_{11\pm}) & -B_{11\pm}A^{-1} & \frac{1}{\rho}B_{11\pm}A^{-1}B_{12\pm} - \frac{1}{\rho}C_{12\pm} \\ -B_{22\pm}^{-1}C_{21\pm} & 0 & B_{22\pm}^{-1}(sI - C_{22\pm}) \end{pmatrix}. \tag{4.16}$$

Until Lemma 4.6 we suppress the '±' in the notation.

First consider the block diagonal operators $\tilde{L}^d(s)$ given by

$$\tilde{L}^d(s)z = z_x - \begin{pmatrix} 0 & \rho A^{-1} & 0 \\ \frac{1}{\rho}(sI - C_{11}) & -B_{11}A^{-1} & 0 \\ 0 & 0 & B_{22}^{-1}(sI - C_{22}) \end{pmatrix} z.$$

In [BL99, Lemma 2.1–Lemma 2.3] it is shown that there are positive constants $\tilde{K}_1, \tilde{\beta}_1, c_1, \varepsilon$ such that for all $s \in \mathbb{C}$ with $s = re^{2i\theta}$, where $r \geq c_1$ and $|\theta| \leq \frac{\pi}{4} + \varepsilon$, the operators

$$\tilde{L}^{d1}(s) : z_1 \mapsto z_{1,x} - \begin{pmatrix} 0 & \rho A^{-1} \\ \frac{1}{\rho}(sI - C_{11}) & -B_{11}A^{-1} \end{pmatrix} z_1,$$

have an (ED) on $\mathbb{R}$. Moreover, the dichotomy data are given by $(\tilde{K}_1, \rho\tilde{\beta}_1, \tilde{\pi}_1(s))$ and satisfy $\dim \mathcal{R}(\tilde{\pi}_1(s)) = n$.

In Lemma 3.7 from Section 3.2 it is shown that there are positive constants $\tilde{K}_2, \tilde{\beta}_2$, and c_2 such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq c_2$ the operators

$$\tilde{L}^{d2}(s) : z_2 \mapsto z_{2,x} - B_{22}^{-1}(sI - C_{22})z_2,$$

have an (ED) on $\mathbb{R}$ with dichotomy data $(\tilde{K}_2, \operatorname{Re}(s)\tilde{\beta}_2, \tilde{\pi}_2(s))$ and the projectors satisfy $\dim \mathcal{R}(\tilde{\pi}_2(s)) = m - r$.

Since $\sqrt{\operatorname{Re} s} \leq |s|^{1/2}$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq \max(c_2, c_1, 1) =: \tilde{c}_0$, the block diagonal operators $\tilde{L}^d(s)$ have EDs on $\mathbb{R}$. The data can be chosen as

$$\big(\tilde{K}_0, \sqrt{\operatorname{Re}(s)}\tilde{\beta}_0, \begin{pmatrix} \tilde{\pi}_1 & 0 \\ 0 & \tilde{\pi}_2 \end{pmatrix}(s)\big),$$

where $\tilde{K}_0 = \max(\tilde{K}_1, \tilde{K}_2)$ and $\tilde{\beta}_0 = \min(\tilde{\beta}_1, \tilde{\beta}_2)$.

Let

$$\Delta(s) := \begin{pmatrix} 0 & 0 & -A^{-1}B_{12} \\ 0 & 0 & \frac{1}{\rho}B_{11}A^{-1}B_{12} - \frac{1}{\rho}C_{12} \\ -B_{22}^{-1}C_{21} & 0 & 0 \end{pmatrix}$$

If $\operatorname{Re} s \geq \tilde{c}_0$, then there is a constant C_Δ such that $|\Delta(s)| \leq C_\Delta$.

Therefore by choosing $c_0 =: \max\big(\tilde{c}_0, \big(\frac{7\tilde{K}_0 C_\Delta}{\tilde{\beta}_0}\big)^2, \big(\frac{2\tilde{K}_0^2 C_\Delta}{\tilde{\beta}_0}\big)^2\big)$ Theorem B.4 implies for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq c_0$ that the operators $\tilde{L}(s)$ have EDs on $\mathbb{R}$ with data $(\tilde{K}, \tilde{\beta}(s), \tilde{\pi})$ where one can choose

$$\tilde{K} = 3\tilde{K}_0,$$

$$\text{and } \tilde{\beta}(s) = \sqrt{\operatorname{Re} s}\frac{\tilde{\beta}_0}{2}.$$

Moreover, the projectors satisfy

$$\left|\tilde{\pi}(s) - \begin{pmatrix} \tilde{\pi}_1 & 0 \\ 0 & \tilde{\pi}_2 \end{pmatrix}(s)\right| \leq \frac{1}{\sqrt{\mathrm{Re}(s)}} \frac{\tilde{K}_0^2 C_\Delta}{\tilde{\beta}_0} \leq \frac{1}{2}. \tag{4.17}$$

Therefore it holds $\dim \mathcal{R}(\tilde{\pi}(s)) = \dim \mathcal{R}\left(\begin{smallmatrix} \tilde{\pi}_1 & 0 \\ 0 & \tilde{\pi}_2 \end{smallmatrix}\right) = n + m - r$. This yields that for all $\mathrm{Re}\, s \geq c_0$ the matrices $\tilde{M}_\pm(s)$ are hyperbolic and in particular also the matrices

$$M_\pm(s) = S_{\frac{1}{\rho}} T_{B_{12}} \tilde{M}(s) T_{-B_{12}} S_\rho$$

are hyperbolic for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > c_0$. Moreover the dimension of the stable subspaces of $M_\pm(s)$ is $n + m - r$ and the dimension of the unstable subspaces of $M_\pm(s)$ is $n + r$.

Recall that Lemma 4.4 shows that there is no $s \in \{\mathrm{Re}\, s \geq -\delta\}$ for which the matrix $M(s)$ has purely imaginary eigenvalues. Since the eigenvalues of a continuously parametrized matrix depend continuously on the parameter, it follows that the matrices $M(s)$ are hyperbolic for all $\mathrm{Re}\, s > -\delta$ and the dimension of the stable subspace is $n + m - r$ and the dimension of the unstable subspace is $n + r$.

This shows that the constant coefficient operators $L_\pm(s)$ have (EDs) on $\mathbb{R}$. Let

$$V_\pm^I(s) \in \mathbb{C}^{2n+m,n+r} \text{ be a basis of the unstable subspace of } M_\pm(s)$$

and let

$$V_\pm^{II}(s) \in \mathbb{C}^{2n+m,n+m-r} \text{ be a basis of the stable subspace of } M_\pm(s).$$

Finally let $\Lambda_\pm^I(s) \in \mathbb{C}^{n+r,n+r}$ and $\Lambda_\pm^{II}(s) \in \mathbb{C}^{n+m-r,n+m-r}$ with $\mathrm{Re}\, \sigma(\Lambda_\pm^I(s)) > 0$ and $\mathrm{Re}\, \sigma(\Lambda_\pm^{II}(s) < 0$ such that (3.81) and (3.82) hold. Note that we again do not assume any smoothness for $V_\pm$ or $\Lambda_\pm$ since we only aim for a point-wise result.

The considerations from above prove the following Lemma.

Lemma 4.5. *For every $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -\delta$ the constant coefficient operator*

$$L_\pm(s) : z \mapsto z_x - M_\pm(s) z$$

has an (ED) on $\mathbb{R}$. Moreover the corresponding projectors $\pi_\pm(s)$ are given by

$$\mathcal{R}(\pi_\pm(s)) = V_\pm^{II}(s) \text{ and } \mathcal{R}(I - \pi_\pm(s)) = V_\pm^I(s).$$

By application of Theorem B.5 and Lemma B.7 one obtains from Lemma 4.5 an (ED) for the variable coefficient operators.

Lemma 4.6. *For all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -\delta$ the variable coefficient operators $L(\cdot, s)$ from* (4.13) *have exponential dichotomies on $\mathbb{R}_+$ and on $\mathbb{R}_-$. The corresponding projectors $\pi_\pm(\cdot, s)$ satisfy*

$$\lim_{x \to +\infty} |\pi_+(x, s) - \pi_+(s)| = 0 \text{ and } \lim_{x \to -\infty} |\pi_-(x, s) - \pi_-(s)| = 0. \tag{4.18}$$

Moreover, for the ranges of the projectors one has

$$\dim \mathcal{R}(\pi_+(0,s)) = n + m - r$$
$$\dim \mathcal{R}(I - \pi_-(0,s)) = n + r.$$

Finally, the operators $L(\cdot, s) : H^2 \times H^1 \times H^1 \to H^1 \times L_2 \times L_2$ *from (4.13) are Fredholm of index* 0 *for all* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > -\delta$.

From this Lemma we now conclude that also the original operator $sI - P$ has a Fredholm property. This will be used in Section 4.4 to show unique solvability of the resolvent equation (4.5).

Lemma 4.7. *For all* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > -\delta$ *the operator* $sI - P$ *is Fredholm of index* 0.

Proof. We show that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ the operators $sI - P$ and $L(\cdot, s)$ are Fredholm of the same index. By Lemma 4.6 $L(\cdot, s)$ is Fredholm of index 0.

First we show $\dim \mathcal{N}(sI - P) = \dim \mathcal{N}(L(\cdot, s))$.
Let $(u, v)^T \in \mathcal{N}(sI - P)$, then $(u, Au_x, v)^T \in \mathcal{N}(L(\cdot, s))$ and it follows

$$\dim \mathcal{N}(sI - P) \leq \dim \mathcal{N}(L(\cdot, s)).$$

Now let $(z_1, z_2, z_3)^T \in \mathcal{N}(L(\cdot, s))$. By the definition of $L(\cdot, s)$ it holds $z_{1,x} = A^{-1} z_2$ and therefore $z_2 = A z_{1,x}$. One easily finds $(z_1, z_3)^T \in \mathcal{N}(sI - P)$. Let $(z_1^i, z_2^i, z_3^i)^T$, $i = 1, \dots, l$, be linearly independent elements in $\mathcal{N}(L(\cdot, s))$. Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{C}^l$ with

$$\sum_i \alpha_i \begin{pmatrix} z_1^i \\ z_3^i \end{pmatrix} = 0.$$

Then by linearity $\sum_i \alpha_i (z_1^i, A z_{1,x}^i, z_3^i)^T = 0$, but since the differential equation shows $A z_{1,x}^i = z_2^i$, it follows $\alpha = 0$ from the linear independency. Hence we find

$$\dim \mathcal{N}(sI - P) \geq \dim \mathcal{N}(L(\cdot, s)).$$

Second show $\operatorname{codim} \mathcal{R}(sI - P) = \operatorname{codim} \mathcal{R}(L(\cdot, s))$.
Since $(0, -f + B_{12} B_{22}^{-1} F, -B_{22}^{-1} F)^T \in \mathcal{R}(L(\cdot, s))$ implies $(f, F)^T \in \mathcal{R}(sI - P)$ we obtain

$$\operatorname{codim} \mathcal{R}(sI - P) \leq \operatorname{codim} \mathcal{R}(L(\cdot, s)).$$

Now let (f^i, g^i, h^i), $i = 1, \dots, l$, be a cobasis of $\mathcal{R}(L(\cdot, s))$. Then the elements

$$\begin{pmatrix} -A f_x^i - g^i - B_{11} f^i - B_{12} h^i \\ -B_{22} h^i \end{pmatrix} \in L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

are linearly independent elements of $[L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)] / \mathcal{R}(sI - P)$:
Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{C}^l$ and assume there is $\begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$ with

$$(sI - P) \begin{pmatrix} u \\ v \end{pmatrix} = \sum \alpha_i \begin{pmatrix} -A f_x^i - g^i - B_{11} f^i - B_{12} h^i \\ -B_{22} h^i \end{pmatrix}. \tag{4.19}$$

Consider $\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix}$ which is an element of $H^2(\mathbb{R},\mathbb{C}^n) \times H^1(\mathbb{R},\mathbb{C}^n) \times H^1(\mathbb{R},\mathbb{C}^m)$. Then

$$
\begin{aligned}
&L(s)\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix} \\
&= \begin{pmatrix} u_x \\ Au_{xx} - A\sum \alpha_i f^i_x \\ v_x \end{pmatrix} - M(\cdot,s)\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix} \\
&= \begin{pmatrix} \sum \alpha_i f^i \\ Au_{xx} - B_{12}B_{22}^{-1}C_{21}u + B_{12}B_{22}^{-1}(sI - C_{22})v - (sI - C_{11})u + B_{11}u_x + C_{12}v \\ v_x + B_{22}^{-1}C_{21}u - B_{22}^{-1}(sI - C_{22})v \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -B_{11}\sum \alpha_i f^i - A\sum \alpha_i f^i_x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \sum \alpha_i f^i \\ Au_{xx} + B_{11}u_x + B_{12}v_x + C_{11}u + C_{12}v - sIu \\ \sum \alpha_i h^i \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -B_{11}\sum \alpha_i f^i - A\sum \alpha_i f^i_x - B_{12}\sum \alpha_i h^i \\ 0 \end{pmatrix} \\
&= \sum \alpha_i \begin{pmatrix} f^i \\ g^i \\ h^i \end{pmatrix} \in \mathcal{R}(L(\cdot,s)),
\end{aligned}
$$

where we used the differential equation (4.19). Since the elements form a cobasis of $\mathcal{R}(L(\cdot,s))$ it follows $\alpha = 0$. This shows $\operatorname{codim}\mathcal{R}(sI - P) \geq \operatorname{codim}\mathcal{R}(L(\cdot,s))$. □

REMARK. *Note that the proof does not make use of the property* $\operatorname{Re}(s) > -\delta$, *but uses the Fredholm property of* $L(\cdot,s)$. *Hence the Lemma can be applied everywhere in the complex plane, where* $L(\cdot,s)$ *is Fredholm. This will be used in section 5.3 for the interpretation of the numerical results.*

For the approximation of the operator $sI - P$ on finite intervals $J = [x_-, x_+]$ with $x_+ > x_-$, as in (4.6) we consider the operators

$$
sI - P|_J : \begin{array}{ccc} H^2(J,\mathbb{C}^n) \times H^1(J,\mathbb{C}^m) & \to & L_2(J,\mathbb{C}^n) \times L_2(J,\mathbb{C}^m) \\ \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & (sI - P)\begin{pmatrix} u \\ v \end{pmatrix} \end{array} \tag{4.20}
$$

and assume that the supplementary linear boundary operator is of the form

$$R: \begin{array}{ccc} H^2(J,\mathbb{C}^n)\times H^1(J,\mathbb{C}^m) & \to & \mathbb{C}^{2n+m} \\ \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & R\begin{pmatrix} u \\ v \end{pmatrix} = R_- \begin{pmatrix} u(x_-) \\ u_x(x_-) \\ v(x_-) \end{pmatrix} + R_+ \begin{pmatrix} u(x_+) \\ u_x(x_+) \\ v(x_+) \end{pmatrix} \end{array} \tag{4.21}$$

with matrices $R_-, R_+ \in \mathbb{C}^{2n+m,2n+m}$. Then we obtain the Fredholm alternative for the mixed second order boundary value problem (4.6).

Lemma 4.8. *For every $s \in \mathbb{C}$ and every compact interval $J = [x_-, x_+]$ with $x_+ > x_-$ the operators*

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} : H^2(J,\mathbb{C}^n)\times H^1(J,\mathbb{C}^m) \to L_2(J,\mathbb{C}^n)\times L_2(J,\mathbb{C}^m)\times \mathbb{C}^{2n+m}$$

are Fredholm operators of index zero.

Proof. Since the embedding $H^1(J,\mathbb{C}^n) \hookrightarrow L_2(J,\mathbb{C}^n)$ is compact, the assertion follows by application of Lemma C.10 and using that the pure parabolic and pure hyperbolic part with their corresponding boundary operators are Fredholm operators of index 0. □

4.4 Resolvent estimates for large $|s|$

In this section we will combine the results from Chapter 3 and [BL99] as they are presented in Section 4.2 to obtain similar results for the hyperbolic-parabolic systems (4.5) and (4.6).

First we prove an all line result similar to the result from [KKP94, Section 4]. Note that in [KKP94] only resolvent estimates are shown. We improve the result from there in the sense that we give stronger estimates and also an existence and uniqueness result.

Theorem 4.9. *Under Assumption 2 there are positive constants K and C_0 such that for all $s \in M(\delta, C_0)$ the all line resolvent equation (4.5) has for all $f \in L_2(\mathbb{R},\mathbb{C}^n)$, $g \in H^1(\mathbb{R},\mathbb{C}^n)$, and $F \in L_2(\mathbb{R},\mathbb{C}^m)$ a unique solution $\begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R},\mathbb{C}^n)\times H^1(\mathbb{R},\mathbb{C}^m)$.*

Moreover, the solution can be estimated by

$$|s|\|u\|^2 + \|u_x\|^2 + \|v\|^2 \le K\big(\frac{1}{|s|}\|f\|^2 + \|F\|^2 + \|g\|^2\big). \tag{4.22}$$

If in addition $F \in H^1(\mathbb{R},\mathbb{C}^m)$ and $\|C_{21,x}\|_\infty < \infty$, there exists $K' > 0$ so that the solution satisfies the estimate

$$|s|\|u\|^2 + \|u_x\|^2 + \|v\|^2 + \|v_x\|^2 \le K'\big(\frac{1}{|s|}\|f\|^2 + \|g\|^2 + \|F\|^2 + \|F_x\|^2\big). \tag{4.23}$$

Proof. Assume that $\begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$ is a solution of (4.5). Let $C_1 > 0$ be so large that for all $s \in \mathcal{M}(\delta, C_1)$ the assumptions of Theorem 4.2 and of Theorem 3.1 are satisfied. Then by Theorem 4.2 there is a positive constant K_1 so that $u \in H^2(\mathbb{R}, \mathbb{C}^n)$ is the unique solution of

$$(sI_n - P_{par})u = (f + (C_{12} - B_{12,x})v) + (g + B_{12}v)_x \text{ in } L_2(\mathbb{R}, \mathbb{C}^n)$$

and satisfies the estimate

$$\begin{aligned} &|s|^2\|u\|^2 + |s|\|u_x\|^2 \\ &\quad \le K_1\Big\{\|f\|^2 + 2\big(\|C_{12}\|_\infty^2 + \|B_{12,x}\|_\infty^2\big)\|v\|^2 + 2|s|\big(\|g\|^2 + \|B_{12}\|_\infty^2\|v\|^2\big)\Big\} \\ &\quad \le K_1'\big(\|f\|^2 + |s|\|g\|^2 + |s|\|v\|^2\big). \end{aligned} \tag{4.24}$$

Similarly by Theorem 3.1 there is a positive constant K_2 so that $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ is the unique solution of

$$(sI_m - P_{hyp})v = F + C_{21}u \text{ in } L_2(\mathbb{R}, \mathbb{C}^m)$$

and satisfies

$$\|v\|^2 \le K_2\big(\|F\|^2 + \|u\|^2\big). \tag{4.25}$$

If $\|C_{21,x}\|_\infty < \infty$, the function v also satisfies the estimate

$$\|v\|^2 + \|v_x\|^2 \le K_2\big(\|F\|^2 + \|F_x\|^2 + \|u\|^2 + \|u_x\|^2\big). \tag{4.26}$$

Inserting (4.25) into (4.24) leads to

$$|s|^2\|u\|^2 + |s|\|u_x\|^2 \le K_1''\big(\|f\|^2 + |s|\|g\|^2 + |s|\|F\|^2 + |s|\|u\|^2\big).$$

By choosing $C_0 \ge \max(2K_1'', C_1, 1)$ one obtains for all $s \in \mathcal{M}(\delta, C_0)$

$$|s|^2\|u\|^2 + |s|\|u_x\|^2 \le 2K_1''\big(\|f\|^2 + |s|\|g\|^2 + |s|\|F\|^2\big) \tag{4.27}$$

which implies

$$\|u\|^2 \le 2K_1''\big(\frac{1}{|s|^2}\|f\|^2 + \frac{1}{|s|}\|g\|^2 + \frac{1}{|s|}\|F\|^2\big).$$

This inserted into (4.25) shows

$$\|v\|^2 \le K_2'\big(\frac{1}{|s|^2}\|f\|^2 + \frac{1}{|s|}\|g\|^2 + \|F\|^2\big). \tag{4.28}$$

Combination of (4.27) and (4.28) proves (4.22). Similarly combining the estimates (4.27) and (4.26) in the same fashion proves (4.23).

The solution estimates (4.22) and (4.23) imply that the operator $sI - P$ is one to one and the Fredholm alternative then shows that the operator is also onto. □

In a similar way we analyze the finite interval problem (4.6). Assume that the boundary operator R from (4.6b) is of the form

$$R\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}P_-^I & Q_-^I & R_-^a\\ P_-^{II} & 0 & R_-^b\\ R_-^c & R_-^d & R_-^{hyp}\end{pmatrix}\begin{pmatrix}u(x_-)\\u_x(x_-)\\v(x_-)\end{pmatrix}+\begin{pmatrix}P_+^I & Q_+^I & R_+^a\\ P_+^{II} & 0 & R_+^b\\ R_+^c & R_+^d & R_+^{hyp}\end{pmatrix}\begin{pmatrix}u(x_+)\\u_x(x_+)\\v(x_+)\end{pmatrix} \tag{4.29}$$

with $P_\pm^I, Q_\pm^I \in \mathbb{C}^{r,n}$, $P_\pm^{II} \in \mathbb{C}^{2n-r,n}$, $R_\pm^a \in \mathbb{C}^{r,m}$, $R_\pm^b \in \mathbb{C}^{2n-r,m}$, $R_\pm^c, R_\pm^d \in C^{m,n}$, $R_\pm^{hyp} \in \mathbb{C}^{m,m}$, and $\operatorname{rank}(Q_-\ Q_+) = r$. This form can always be achieved by multiplying (4.6b) with an invertible matrix from the left.

Theorem 4.10. *Assume the differential operator P from (4.2) satisfies (P1), (P2), (H1), (H2), (H3), (M1). Consider the boundary value problem* (4.6) *with $f \in L_2(J,\mathbb{C}^n)$, $g \in H^1(J,\mathbb{C}^n)$, $F \in L_2(J,\mathbb{C}^m)$, $\eta = \left(\eta^I\, \eta^{II}\, \eta^{III}\right)^T \in \mathbb{C}^{n+n+m}$, Assume the boundary operator R is of the form* (4.29) *with $(R_-^d, R_+^d) = 0$ and satisfies the determinant-condition Assume*

$$D_\infty := \det\begin{pmatrix}Q_-^I & Q_+^I & 0 & 0\\ -P_-^{II}A^{\frac12} & P_+^{II}A^{\frac12} & 0 & 0\\ 0 & 0 & R_-^{hyp,II} & R_+^{hyp,I}\end{pmatrix} \neq 0, \tag{4.30}$$

where $R_\pm^{hyp} = \left(R_\pm^{hyp,I}\ R_\pm^{hyp,II}\right)$ corresponds to the partition of B_{22} in (H2) of Assumption 1.

Then there are positive constants C_0, δ, b such that for all compact intervals $J \supset [-b,b]$, for all $s \in \mathcal{M}(\delta, C_0)$, and all choices of f, g, F, η the following properties hold.

(a) If $(R_-^c\ R_+^c) = 0$, then for every choice of $R_\pm^a$ and $R_\pm^b$ one obtains unique solvability of the resolvent equation (4.6)*. Moreover there is a positive constant K independent of J, s, η, f, g, and F so that for all $s \in \mathcal{M}(\delta, C_0)$ the solution (u,v) can be estimated by*

$$\begin{aligned}&\rho^2\|u\|^2 + \|u_x\|^2 + \rho|u|_\Gamma^2 + \frac{1}{\rho}|u_x|_\Gamma^2 + \|v\|^2 + |v|_\Gamma^2 \\ &\le K\left(\frac{1}{\rho^2}\|f\|^2 + \|g\|^2 + \rho\|F\|^2 + \frac{1}{\rho}|g|_\Gamma^2 + \frac{1}{\rho}|\eta^I|^2 + \rho|\eta^{II}|^2 + \rho|\eta^{III}|^2\right)\end{aligned} \tag{4.31}$$

with $\rho = \sqrt{|s|}$.

(b) If $(R_-^b\ R_+^b) = 0$ then for every choice of $R_\pm^a$ and $R_\pm^c$ one obtains unique solvability of the resolvent equation (4.6)*. Moreover, there is a positive constant K independent of J, s, η, f, g, and F such that the solution (u,v) can be estimated by*

$$\begin{aligned}&\rho^2\|u\|^2 + \|u_x\|^2 + \rho|u|_\Gamma^2 + \frac{1}{\rho}|u_x|_\Gamma^2 + \|v\|^2 + |v|_\Gamma^2 \\ &\le K\left(\frac{1}{\rho^2}\|f\|^2 + \|g\|^2 + \|F\|^2 + \frac{1}{\rho}|g|_\Gamma^2 + \frac{1}{\rho}|\eta^I|^2 + \rho|\eta^{II}|^2 + |\eta^{III}|^2\right)\end{aligned} \tag{4.32}$$

for all $s \in \mathcal{M}(\delta, C_0)$.

Proof. In Lemma 4.8 it is shown that for all $J = [x_-, x_+]$ with $x_+ > x_-$ the operator $\begin{pmatrix} sI - P|_J \\ R \end{pmatrix}$ from (4.6) is a Fredholm operator of index 0. Hence an estimate of the form (4.31) or (4.32) suffices to prove unique solvability.

Assume $(u, v) \in H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m)$ is a solution of (4.6). The determinant-condition (4.30) implies that there are positive constants C_0 and δ so that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ and $|s| > C_0$ the assertions of Theorem 3.2 and Theorem 4.3 hold for the hyperbolic and for the parabolic part, respectively.

By assumption (u, v) solve

$$(sI - P_{par}|_J)u = f + C_{12}v - B_{12,x}v + (g + B_{12}v)_x \tag{4.33a}$$

$$\begin{pmatrix} P_-^I & Q_-^I \\ P_-^{II} & 0 \end{pmatrix}\begin{pmatrix} u(x_-) \\ u_x(x_-) \end{pmatrix} + \begin{pmatrix} P_+^I & Q_+^I \\ P_+^{II} & 0 \end{pmatrix}\begin{pmatrix} u(x_+) \\ u_x(x_+) \end{pmatrix} = \begin{pmatrix} \eta^I - \left(R_-^a, R_+^a\right)\begin{pmatrix} v(x_-) \\ v(x_+) \end{pmatrix} \\ \eta^{II} - \left(R_-^b, R_+^b\right)\begin{pmatrix} v(x_-) \\ v(x_+) \end{pmatrix} \end{pmatrix}, \tag{4.33b}$$

and

$$(sI - P_{hyp}|_J)v = F + C_{21}u \tag{4.34a}$$

$$R_-^{hyp}v(x_-) + R_+^{hyp}v(x_+) = \eta^{III} - \begin{pmatrix} R_-^c & R_+^c \end{pmatrix}\begin{pmatrix} u(x_-) \\ u(x_+) \end{pmatrix} - \begin{pmatrix} R_-^d & R_+^d \end{pmatrix}\begin{pmatrix} u_x(x_-) \\ u_x(x_+) \end{pmatrix}. \tag{4.34b}$$

Thus by Theorem 4.3 there is a constant $K_p > 0$ independent of J, s, and the right hand side, so that the estimate

$$\begin{aligned} &\rho^2\|u\|^2 + \|u_x\|^2 + \rho|u|_\Gamma^2 + \frac{1}{\rho}|u_x|_\Gamma^2 \\ &\quad \le K_p\Big\{\frac{1}{\rho^2}\|f\|^2 + \|g\|^2 + \frac{1}{\rho}|g|_\Gamma^2 + \frac{1}{\rho}|\eta^I|^2 + \rho|\eta^{II}|^2 \\ &\qquad + \|v\|^2 + \frac{1}{\rho}|v|_\Gamma^2 + \frac{1}{\rho}|(R_-^a\ R_+^a)|^2|v|_\Gamma^2 + \rho|(R_-^b\ R_+^b)|^2|v|_\Gamma^2\Big\} \end{aligned} \tag{4.35}$$

holds.

Similarly one obtains from Theorem 3.2 a constant $K_h > 0$, independent of J, s, and the right hand side such that the estimate

$$\|v\|^2 + |v|_\Gamma^2 \le K_h\Big\{\|F\|^2 + |\eta^{III}|^2 + \|u\|^2 + |(R_-^c\ R_+^c)|^2|u|_\Gamma^2 + |(R_-^d\ R_+^d)|^2|u_x|_\Gamma^2\Big\} \tag{4.36}$$

is satisfied. Combining (4.35) and (4.36) similarly to the proof of Theorem 4.9 one obtains (a) and (b). □

Remark. *If the matrix-valued function B_{12} is equal to zero it is also possible to analyze the case with $R_\pm^a = 0$ and $R_\pm^b = 0$ and arbitrary choices of $R_\pm^c$ and $R_\pm^d$. Then one obtains for the unique solution of the resolvent equation (4.6) under the* *same assumptions as in Theorem 4.10 the estimate*

$$\begin{aligned}&\rho^2\|u\|^2+\|u_x\|^2+\rho|u|_\Gamma^2+\frac{1}{\rho}|u_x|_\Gamma^2+\frac{1}{\rho}(\|v\|^2+|v|_\Gamma^2)\\&\leq K\left(\frac{1}{\rho^2}\|f\|^2+\|g\|^2+\frac{1}{\rho}\|F\|^2+\frac{1}{\rho}|g|_\Gamma^2+\frac{1}{\rho}|\eta^I|^2+\rho|\eta^{II}|^2+\frac{1}{\rho}|\eta^{III}|^2\right)\end{aligned}\tag{4.37}$$

with a constant K independent of s, J, and the right hand side of (4.6).

For example in the FHN-System the function B_{12} is zero.*

4.5 Resolvent estimates in compact regions

In this section we always assume that Assumption 2 holds.

The main results will be uniform resolvent estimates for the all line problem (4.5) in compact subsets of the resolvent set $\rho(P)$ in the half-plane $\{\operatorname{Re} s > -\delta\}$. For the approximation (4.6) of the all line problem on finite intervals, we give sufficient conditions for the supplementary boundary operator (4.6b) such that the finite interval problem (4.6) satisfies resolvent estimates similar to the estimates for the all line problem. We will first state the main Theorems 4.11 and 4.12 and then provide the proofs.

Theorem 4.11. *Let $\Omega \subset \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set. Then for every $s \in \Omega$, $f \in L_2(\mathbb{R}, \mathbb{C}^n)$, $g \in H^1(\mathbb{R}, \mathbb{C}^n)$, and $F \in L_2(\mathbb{R}, \mathbb{C}^m)$ the resolvent equation* (4.5) *has a unique solution $(u, v) \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$.*

Moreover there is a positive constant K_0 independent of s, f, g, and F so that the solution can be estimated by

$$\|u\|^2+\|u_x\|^2+\|v\|^2+\|v_x\|^2 \leq K_0\big(\|f\|^2+\|g\|^2+\|F\|^2\big),\tag{4.38}$$

and if one includes the derivative of g one can also estimate u in the H^2-norm

$$\|u_{xx}\|^2 \leq K_0\big(\|f\|^2+\|g\|^2+\|g_x\|^2+\|F\|^2\big).\tag{4.39}$$

Before we formulate the analogous theorem for the restricted problem (4.6) we give a condition for the boundary operator R from (4.6b). We assume that the boundary term R in (4.6b) is linear and of the form

$$R\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}R_-^I & R_-^{II} & R_-^{III}\end{pmatrix}\begin{pmatrix}u(x_-)\\u_x(x_-)\\v(x_-)\end{pmatrix}+\begin{pmatrix}R_+^I & R_+^{II} & R_+^{III}\end{pmatrix}\begin{pmatrix}u(x_+)\\u_x(x_+)\\v(x_+)\end{pmatrix}.\tag{4.40}$$

By using the variable $z = (u, Au_x + g, v)$ the system (4.6a) transforms into the first order equation

$$L(\cdot, s)z = h \text{ in } H^1(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m),\tag{4.41a}$$

*The FitzHugh-Nagumo system will be analyzed as an example in Section 5.3.

where

$$L(\cdot,s)z = z_x - M(\cdot,s)z$$

with M from (4.14) and

$$h = \begin{pmatrix} -A^{-1}g \\ -B_{11}A^{-1}g - f + B_{12}B_{22}^{-1}F \\ -B_{22}^{-1}F \end{pmatrix} \in H^1(J,\mathbb{C}^n)\times L_2(J,\mathbb{C}^n)\times L_2(J,\mathbb{C}^m). \quad (4.41b)$$

The boundary condition (4.40) transforms into

$$R_1 z = \eta_1, \quad (4.41c)$$

where

$$R_1 z = \begin{pmatrix} R_-^I & R_-^{II}A^{-1} & R_-^{III} \end{pmatrix} z(x_-) + \begin{pmatrix} R_+^I & R_+^{II}A^{-1} & R_+^{III} \end{pmatrix} z(x_+)$$

and

$$\eta_1 = \eta + R_-^{II}A^{-1}g(x_-) + R_+^{II}A^{-1}g(x_+).$$

In the sequel we will simply write $H^1(J)$ or $L_2(J)$ the exact dimension of the image will be clear from the context.

As in (3.81) and (3.82) let

$$V_-^{II}(s) = \begin{pmatrix} X_-(s) \\ Y_-(s) \\ Z_-(s) \end{pmatrix} \quad \text{and} \quad V_+^{I}(s) = \begin{pmatrix} X_+(s) \\ Y_+(s) \\ Z_+(s) \end{pmatrix}$$

be bases of the stable subspace of $M_-(s)$ and of the unstable subspace of $M_+(s)$, respectively. The equations (3.81) and (3.82) show

$$A^{-1}Y_-(s) = X_-(s)\Lambda_-^{II}(s) \quad \text{and} \quad A^{-1}Y_+(s) = X_+(s)\Lambda_+^{I}(s).$$

Define the determinant

$$\begin{aligned} D(s) := \det &\left[\begin{pmatrix} R_-^I & R_-^{II}A^{-1} & R_-^{III} \end{pmatrix} \begin{pmatrix} X_-(s) \\ Y_-(s) \\ Z_-(s) \end{pmatrix}, \begin{pmatrix} R_+^I & R_+^{II}A^{-1} & R_+^{III} \end{pmatrix} \begin{pmatrix} X_+(s) \\ Y_+(s) \\ Z_+(s) \end{pmatrix} \right] \\ = \det &\left[\begin{pmatrix} R_-^I & R_-^{II} & R_-^{III} \end{pmatrix} \begin{pmatrix} X_-(s) \\ X_-(s)\Lambda_-^{II}(s) \\ Z_-(s) \end{pmatrix}, \begin{pmatrix} R_+^I & R_+^{II} & R_+^{III} \end{pmatrix} \begin{pmatrix} X_+(s) \\ X_+(s)\Lambda_+^{I}(s) \\ Z_+(s) \end{pmatrix} \right]. \end{aligned} \quad (4.42)$$

Now we can formulate the Theorem.

Theorem 4.12. *Let $\Omega \subset \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set and assume*

$$D(s) \neq 0 \; \forall s \in \Omega.$$

Then there is a compact interval J_0 and a constant $K_0 > 0$ so that for all $s \in \Omega$ and all compact intervals $J \supset J_0$ we have for every $f \in L_2(J)$, $g \in H^1(J)$, $F \in L_2(J)$,

$\eta \in \mathbb{C}^{2n+m}$ *a unique solution* $\begin{pmatrix} u \\ v \end{pmatrix} \in H^2(J,\mathbb{C}^n) \times H^1(J,\mathbb{C}^m)$ *of equation (4.6). Moreover the solution can be estimated by*

$$\|u\|_{H^2(J)} + \|v\|_{H^1(J)} + |u|_\Gamma + |u_x|_\Gamma + |v|_\Gamma \\ \le K_0\big(\|f\|_{L_2(J)} + \|g\|_{H^1(J)} + \|F\|_{L_2(J)} + |g|_\Gamma + |\eta|\big). \quad (4.43)$$

The proofs of the theorems are basically the same as in Section 3.3. We use the same transformation of variables for the all line problem as for the restricted problem, i.e. $z = (u, Au_x + g, v)$. Thus the all line problem (4.5) can be rewritten as the first order equation

$$L(\cdot,s)z = h \;\text{ in } H^1(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^m), \quad (4.44a)$$

where

$$L(\cdot,s)z = z_x - M(\cdot,s)z$$

with M from (4.14) and

$$h = \begin{pmatrix} -A^{-1}g \\ -B_{11}A^{-1}g - f + B_{12}B_{22}^{-1}F \\ -B_{22}^{-1}F \end{pmatrix} \in H^1(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^m). \quad (4.44b)$$

We show that for every $s \in \Omega$ the first order operator $L(\cdot,s)$ has an (ED) on $\mathbb{R}$.

Lemma 4.13. *For all* $s_0 \in \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ *the operator* $L(\cdot,s_0)$ *has an (ED) on* $\mathbb{R}$ *and the projectors* π *from the dichotomy data satisfy*

$$\lim_{x\to+\infty} \pi(x,s_0) = \pi_+(s_0) \quad \textit{and} \quad \lim_{x\to-\infty} \pi(x,s_0) = \pi_-(s_0),$$

where $\pi_\pm(s_0)$ *are the projectors of the constant coefficient operators in Lemma 4.5.*

Proof. By Lemma 4.6 we know that for all $s \in \{\operatorname{Re} s > -\delta\}$ the variable coefficient operator $L(\cdot,s)$ from (4.44a) has an (ED) on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with projectors π_+ and π_- respectively. Similar to the proof of Lemma 3.17 it suffices to show that

$$z_0 \in \mathcal{R}(\pi_+(0,s_0)) \cap \mathcal{R}(I - \pi_-(0,s_0)) \text{ implies } z_0 = 0.$$

Let $S(x,y)$ denote the solution operator of $L(\cdot,s_0)$ and define

$$z(x) := S(x,0)z_0,\ \forall x \in \mathbb{R}.$$

It follows that $z \in L_2(\mathbb{R},\mathbb{C}^{2n+m})$ and $z_x = M(\cdot,s_0)z \in L_2(\mathbb{R},\mathbb{C}^{2n+m})$. So by the boundedness of M one obtains $z \in H^1(\mathbb{R},\mathbb{C}^{2n+m})$. The structure of M implies

$$z = \begin{pmatrix} z_1 \\ Az_{1,x} \\ z_3 \end{pmatrix} \in H^1(\mathbb{R},\mathbb{C}^{n+n+m})$$

and therefore $\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$ and furthermore

$$(sI - P) \begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = 0 \text{ in } L_2(\mathbb{R}, \mathbb{C}^{n+m}).$$

From the assumption $s_0 \in \rho(P)$ then follows $\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = 0$ and so $z = 0$. Thus $z_0 = z(0) = 0$ and by application of Theorem B.6 the assertion follows. □

We prove Theorem 4.11 in a similar way as Theorem 3.13.

Proof of Theorem 4.11. Let $s_0 \in \Omega$ and write the resolvent equation (4.5) in the form (4.44). By Lemma 4.13 and Theorem B.2 there is a constant c_{s_0} so that for all f, g, and F there is a unique solution z of (4.44) and this satisfies the estimate

$$\|z\|^2 \le c_{s_0} \|h\|^2. \tag{4.45}$$

By application of the differential equation

$$z_x = M(\cdot, s_0) z + h \text{ in } L_2$$

we obtain that its derivative satisfies

$$\|z_x\|^2 \le c'_{s_0} \|h\|^2 \tag{4.46}$$

where c'_{s_0} only depends on c_{s_0} and $\|M(\cdot, s_0)\|_\infty$.

With the same argumentation as in the proof of Theorem 3.13 one uses the compactness of Ω to derive from the pointwise estimates (4.45) and (4.46) an estimate independent of $s \in \Omega$. Thus there is a positive constant K so that for all $s \in \Omega$ and all $h \in L_2 \times L_2 \times L_2$ there is a unique solution $z \in H^1 \times H^1 \times H^1$ of

$$L(\cdot, s) z = h \text{ in } L_2 \times L_2 \times L_2$$

and this satisfies

$$\|z\|^2 + \|z_x\|^2 \le K \|h\|^2. \tag{4.47}$$

If in addition $h \in H^1 \times L_2 \times L_2$ one obtains from the structure of M and the differential equation, that z in fact is an element of $H^2 \times H^1 \times H^1$ and the equality

$$L(\cdot, s) z = h \text{ holds in } H^1 \times L_2 \times L_2.$$

Now recall the structure of h (4.44b) and M. These imply that z is of the form $z = \begin{pmatrix} z_1 \\ Az_{1,x} + g \\ z_3 \end{pmatrix}$. We set $u := z_1 \in H^2(\mathbb{R}, \mathbb{C}^n)$ and $v := z_3 \in H^1(\mathbb{R}, \mathbb{C}^m)$.

Then (4.47) implies the inequality

$$\left\| \begin{pmatrix} u \\ Au_x + g \\ v \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} u_x \\ Au_{xx} + g_x \\ v_x \end{pmatrix} \right\|^2 \le K \left\| \begin{pmatrix} -A^{-1} g \\ -B_{11} A^{-1} g - f + B_{12} B_{22}^{-1} F \\ -B_{22}^{-1} F \end{pmatrix} \right\|^2$$

which leads to

$$\|u\|^2 + \|u_x\|^2 + \|v\|^2 + \|v_x\|^2 \le K_0\big(\|f\|^2 + \|g\|^2 + \|F\|^2\big).$$

In order to estimate the second derivative of u one has to move the g_x-term from the left to the right hand side. Then one obtains

$$\|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|v\|^2 + \|v_x\|^2 \le K_0\big(\|f\|^2 + \|g\|^2 + \|g_x\|^2 + \|F\|^2\big).$$

Furthermore by the form of M and h we obtain that (u, v) is a solution of (4.5). This finishes the proof. □

As in the hyperbolic case we use the general convergence result Theorem 2.29 to prove Theorem 4.12.

Proof of Theorem 4.12. As above we use the variables $z = (u, Au_x + g, v)$ and consider the transformed system (4.41) as one equation

$$L_J(s)z = \begin{pmatrix} L(\cdot, s)z \\ R_1 z \end{pmatrix} = \begin{pmatrix} h \\ \eta_1 \end{pmatrix} \quad \text{in } L_2(J, \mathbb{C}^{n+n+m}) \times \mathbb{C}^{2n+m}. \tag{4.48}$$

Here we view $L(\cdot, s)$ as an operator defined on

$$L(\cdot, s) : H^1(J, \mathbb{C}^{2n+m}) \to L_2(J, \mathbb{C}^{2n+m})$$

or

$$L(\cdot, s) : H^1(\mathbb{R}, \mathbb{C}^{2n+m}) \to L_2(\mathbb{R}, \mathbb{C}^{2n+m}).$$

In notation we do not distinguish between the operator on the whole real line and the finite interval operator, but it will always be clear which definition is considered.

From the assumption $D(s) \neq 0$ and the assumptions on the coefficients of P, we see that for every $s \in \Omega$ we are in the setting of Theorem 2.29 with $l = 2n+m$. With the notation from there we therefore obtain

$$L_J(s) \xrightarrow{\mathcal{PQ}} L(\cdot, s) \text{ regularly } (J \in H)\ \forall s \in \Omega.$$

As in the proof of Theorem 3.14 we know $L_J(s)$ is a Fredholm operator of index zero for all $s \in \Omega$ and all $J \in H$ by the Fredholm alternative for boundary value problems. Furthermore we know from the proof of Theorem 4.11 that for all $s \in \Omega$ the operator $L(\cdot, s)$ is a linear homeomorphism.

Now we can use the same arguments as in the hyperbolic case* and obtain that there is a positive constant K and a compact interval J_0 such that for all compact intervals $J \supset J_0$ we obtain that for every $s \in \Omega$ one has for every choice of $f \in L_2(J, \mathbb{C}^n)$, $g \in H^1(J, \mathbb{C}^n)$, $F \in L_2(J, \mathbb{C}^m)$, and $\eta \in \mathbb{C}^{2n+m}$ a unique solution $z \in H^1(J, \mathbb{C}^{2n+m})$ of (4.41). This solution can be estimated by

$$\|z\|^2_{H^1(J,\mathbb{C}^{2n+m})} + |z|^2_\Gamma \le K\big(\|h\|^2_{L_2(J,\mathbb{C}^{2n+m})} + |\eta_1|^2\big). \tag{4.49}$$

*See Section 3.3.3.

We split

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in H^1(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m).$$

By using the structure of M we obtain from

$$L(\cdot, s)z = h \text{ in } L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m)$$

and (4.41b) the equality

$$z_{1,x} = A^{-1}z_2 - A^{-1}g \text{ in } L_2(J, \mathbb{C}^n).$$

This shows that $z_{1,x}$ is in fact an element of $H^1(J, \mathbb{C}^n)$ and so $z_1 \in H^2(J, \mathbb{C}^n)$ and $z_2 = Az_{1,x} + g$.

With the definition $u := z_1$ and $v := z_3$ we obtain that $(u, v) \in H^2(J) \times H^1(J)$ is a solution of the finite interval problem (4.6).

Finally from (4.49) we obtain by rewriting z in terms of u and v the estimate

$$\begin{aligned} &\left\| \begin{pmatrix} u \\ Au_x + g \\ v \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} u_x \\ Au_{xx} + g_x \\ v_x \end{pmatrix} \right\|^2 + \left| \begin{pmatrix} u \\ Au_x + g \\ v \end{pmatrix} \right|_\Gamma^2 \\ &\leq K \left(\left\| \begin{pmatrix} -A^{-1}g \\ -B_{11}A^{-1}g - f + B_{12}B_{22}^{-1}F \\ -B_{22}^{-1}F \end{pmatrix} \right\|^2 + \left| R_-^{II} A^{-1} g(x_-) + R_+^{II} A^{-1} g(x_+) \right|^2 \right). \end{aligned}$$

This shows

$$\begin{aligned} &\|u\|_{H^1(J)}^2 + \|v\|_{H^1(J)}^2 + |u|_\Gamma^2 + |u_x|_\Gamma^2 + |v|_\Gamma^2 \\ &\qquad \leq K_0 \left(\|f\|_{L_2(J)}^2 + \|g\|_{L_2(J)}^2 + \|F\|_{L_2(J)}^2 + |g|_\Gamma^2 + |\eta|^2 \right) \end{aligned}$$

and

$$\begin{aligned} &\|u\|_{H^2(J)}^2 + \|v\|_{H^1(J)}^2 + |u|_\Gamma^2 + |u_x|_\Gamma^2 + |v|_\Gamma^2 \\ &\qquad \leq K_0' \left(\|f\|_{L_2(J)}^2 + \|g\|_{H^1(J)}^2 + \|F\|_{L_2(J)}^2 + |g|_\Gamma^2 + |\eta|^2 \right). \end{aligned}$$

□

4.5.1 Convergence of the finite interval approximations

As in the hyperbolic part we present a 'consistency result' about the approximation of the all line problem by the finite interval problems. Consistency in our setting means that the error of the all line solution inserted into the truncated problem converges to zero as the interval converges to $\mathbb{R}$.

Theorem 4.14. *Let $\Omega \subset \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set and let J_0 be the compact interval from Theorem 4.12. Let $f \in L_2(\mathbb{R}, \mathbb{C}^n)$, $g \in H^1(\mathbb{R}, \mathbb{C}^n)$, and $F \in L_2(\mathbb{R}, \mathbb{C}^m)$ be arbitrary. Let $(u, v) \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$ be the unique solution of*

$$(sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f + g_x \\ F \end{pmatrix} \quad \text{in } L_2(\mathbb{R})$$

which one obtains from Theorem 4.11. Finally let $(u_J, v_J) \in H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m)$ be the unique solution of

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix}\begin{pmatrix} u_J \\ v_J \end{pmatrix} = \begin{pmatrix} f|_J + g_x|_J \\ F|_J \\ 0 \end{pmatrix} \quad \text{in } L_2(J) \times L_2(J) \times \mathbb{C}^{2n+m}$$

one obtains from Theorem 4.12. Then

$$\begin{aligned} \|u|_J - u_J\|_{H^2(J)} + \|v|_J - v_J\|_{H^1(J)} + |u|_J - u_J|_\Gamma + |u_x|_J - u_{J,x}|_\Gamma + \big|v|_J - v_J\big|_\Gamma \\ \to 0 \quad \text{if } J \to \mathbb{R} \end{aligned} \tag{4.50}$$

and the convergence is uniform in $s \in \Omega$.

Assume there is some $\kappa > 0$ such that $\tilde{f} = e^{\kappa|x|} f \in L_2(\mathbb{R}, \mathbb{C}^n)$, $\tilde{g} = e^{\kappa|x|} g \in L_2(\mathbb{R}, \mathbb{C}^n)$, $\tilde{F} = e^{\kappa|x|} F \in L_2(\mathbb{R}, \mathbb{C}^m)$. Then for every $\alpha < \min(\kappa, \beta)$, where β is a uniform dichotomy exponent for $L(\cdot, s)$ in Ω, there is a constant const > 0 *independent of s, f, g, F, J such that for all $J \supset J_0$ holds the quantitative version of* (4.50) *given by*

$$\begin{aligned} \|u|_J - u_J\|_{H^2(J)} + \|v|_J - v_J\|_{H^1(J)} + \big|u|_J - u_J\big|_\Gamma + \big|u_x|_J - u_{J,x}\big|_\Gamma + \big|v|_J - v_J\big|_\Gamma \\ \le \operatorname{const}\Big\{\|\tilde{f}\| + \|\tilde{g}\| + \|\tilde{F}\| + |\tilde{g}|_\Gamma\Big\}\, e^{-\alpha \min(x_+, -x_-)}. \end{aligned}$$

For the proof one can use the same methods as mentioned in the hyperbolic case (see 3.3.4).

4.6 Convergence of eigenvalues in the right half-plane

As in the hyperbolic case we now show that the eigenvalues and eigenspaces of the finite interval operators approximate the eigenvalues and eigenspaces of the operator on the whole real line. We will mainly follow the proof of the hyperbolic case. Throughout the whole section we assume that **Assumption 2** holds.

4.6.1 The general setup of the eigenvalue problem in the mixed case

First of all we show that in the half-plane $\{\operatorname{Re} s > -\delta\}$ the operator P has isolated eigenvalues of finite algebraic multiplicity only.

Lemma 4.15. *There is no essential spectrum of the all line operator P in the half-plane $\{\operatorname{Re} s > -\delta\}$.*

Indication of a proof. As in the proof of Lemma 3.20 one shows that all eigenvalues of the transformed operator $L(\cdot, s)$ in $\{\operatorname{Re} s > -\delta\}$ are isolated points. Then one concludes by using the Fredholm property from Lemma 4.7 that all eigenvalues of P in the right half-plane are eigenvalues of finite algebraic multiplicity (cf. [Kat66, III §6.4 and IV §5.4]). □

We use the same notations as in the hyperbolic case.
Let $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ and let β_+ and β_- denote the exponents of the (ED) of $L(\cdot, s_0)$ on $\mathbb{R}_+$ and on $\mathbb{R}_-$, respectively.
Choose $\varepsilon_0 > 0$ so that $\overline{K_{\varepsilon_0}(s_0)} \subset \{\operatorname{Re} s > -\delta\}$ and $\overline{K_{\varepsilon_0}(s_0)} \cap \sigma(P) = \{s_0\}$.

Let $\mathcal{A}(s)$ be the operator-polynomial defined by

$$\mathcal{A}(s) := sI - P \in L(H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m), L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)).$$

As in the hyperbolic case let W denote the root-subspace* of $\mathcal{A}(\cdot)$ to the eigenvalue s_0. Let κ be the highest order of all root-polynomials of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

Furthermore consider the same directed set $(H, \succ)$ as in Section 3.4, i.e. $H = \{J = [x_-, x_+] \subset \mathbb{R} : 0 \in J, |J| \geq 1\}$ with $|[x_-, x_+]| := x_+ - x_-$.

Finally, the finite interval approximation $\mathcal{A}_J(\cdot)$ of $\mathcal{A}(\cdot)$ is given by

$$\mathcal{A}_J(s) := \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} : \begin{array}{ccc} H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m) & \to & L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m) \times \mathbb{C}^{2n+m} \\ \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{array}. \tag{4.51}$$

Denote by σ_J the s_0-group of eigenvalues of $\mathcal{A}_J(\cdot)$ in $K_{\varepsilon_0}(s_0)$ and by W_J denote the closed linear hull of all root-subspaces of $\mathcal{A}_J(\cdot)$ to the eigenvalues $s_J \in \sigma_J$.

The following lemma is a quantitative result about the decaying of the elements from W. It is the analogon to Lemma 3.21, used in the hyperbolic case.

Lemma 4.16. *For every $0 < \beta' < \min(\beta_-, \beta_+)$ there is a constant $c = c(\beta)$ so that for all $\begin{pmatrix} u \\ v \end{pmatrix} \in W$ with $\| \begin{pmatrix} u \\ v \end{pmatrix} \|_{H^2 \times H^1} = 1$ holds*

$$|u(x)| + |u_x(x)| + |v(x)| \leq c e^{-\beta' |x|} \; \forall x \in \mathbb{R}. \tag{4.52}$$

Proof. Let $\begin{pmatrix} u \\ v \end{pmatrix} \in W$ then there are $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \ldots, \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ in W with

$$\begin{aligned} (s_0 I - P) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} &= 0 \\ \text{and} \;\; (s_0 I - P) \begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix} &= \begin{pmatrix} u_i \\ v_i \end{pmatrix} \quad \text{for } i = 0, \ldots, k-1. \end{aligned} \tag{4.53}$$

*See Definition C.6.

With the transformation $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ Au_x \\ v \end{pmatrix}$ one can rewrite (4.53) as

$$L(\cdot, s_0) \begin{pmatrix} u_0 \\ Au_{0,x} \\ v_0 \end{pmatrix} = 0$$

$$\text{and } \; L(\cdot, s_0) \begin{pmatrix} u_{i+1} \\ Au_{i+1,x} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ u_i \\ v_i \end{pmatrix} \quad \text{for } i = 0, \dots, k-1.$$

The rest of the proof is the same induction argument as in the hyperbolic case (see Lemma 3.21). ☐

4.6.2 The convergence theorem in the mixed case

For the original problem (4.5) and its approximation on finite intervals (4.6) we use the spaces

$$\begin{aligned} \tilde{E} &:= H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m), \\ \tilde{F} &:= L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m), \\ \tilde{E}_J &:= H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m), \\ \tilde{F}_J &:= L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m) \times C^{2n+m} \end{aligned}$$

with the families of restrictions

$$\tilde{\mathcal{P}} := \{\tilde{p}_J : J \in H\} \qquad \tilde{p}_J : \tilde{E} \to \tilde{E}_J, \qquad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u|_J \\ v|_J \end{pmatrix},$$

$$\tilde{\mathcal{Q}} := \{\tilde{q}_J : J \in H\} \qquad \tilde{q}_J : \tilde{F} \to \tilde{F}_J, \qquad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} f|_J \\ g|_J \\ 0 \end{pmatrix}.$$

They satisfy the properties (2.1). The next lemma is the main ingredient for the proofs of the convergence Theorems 4.18 and 4.19.

Lemma 4.17. *Let $s \in \{\operatorname{Re} s > -\delta\}$ and assume that the boundary-operator R satisfies $D(s) \neq 0$ with $D(s)$ defined in (4.42).*
Then the finite interval approximation $\mathcal{A}_J(s)$ regularly $\tilde{\mathcal{P}}\tilde{\mathcal{Q}}$ converges to the all line operator $\mathcal{A}$.

Proof. First we show the convergence

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} sI - P.$$

Let $J \in H$ and let $(u_J, v_J) \in \tilde{E}_J$ be arbitrary. Then Lemma C.2 implies

$$\begin{aligned}
&\left\| \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix} \right\| \\
&\leq \left\| (sI - P|_J) \begin{pmatrix} u_J \\ v_J \end{pmatrix} \right\|_{L_2(J)} + \left| R_- \begin{pmatrix} u_J(x_-) \\ u_{J,x}(x_-) \\ v_J(x_-) \end{pmatrix} + R_+ \begin{pmatrix} u_J(x_+) \\ u_{J,x}(x_+) \\ v_J(x_+) \end{pmatrix} \right| \\
&\leq \left\| \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix}_{xx} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix}_x + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix} \right\|_{L_2(J)} \\
&\quad + \left| R_- \begin{pmatrix} u_J(x_-) \\ u_{J,x}(x_-) \\ v_J(x_-) \end{pmatrix} + R_+ \begin{pmatrix} u_J(x_+) \\ u_{J,x}(x_+) \\ v_J(x_+) \end{pmatrix} \right| \\
&\leq c_0 \left\| \begin{pmatrix} u_J \\ v_J \end{pmatrix} \right\|_{H^2(J) \times H^1(J)},
\end{aligned}$$

with a constant c_0 independent of J and (u_J, v_J). Because of Lemma 2.18 it hence suffices to show the convergence

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u|_J \\ v|_J \end{pmatrix} \xrightarrow{\tilde{\mathcal{Q}}} (sI - P) \begin{pmatrix} u \\ v \end{pmatrix} \quad (J \in H)$$

for every $\begin{pmatrix} u \\ v \end{pmatrix} \in \tilde{E}$. By definition of $\mathcal{A}$, $\mathcal{A}_J$, $\tilde{\mathcal{P}}$, and $\tilde{\mathcal{Q}}$ it holds

$$\left\| \begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u|_J \\ v|_J \end{pmatrix} - \begin{pmatrix} (sI - P) \begin{pmatrix} u \\ v \end{pmatrix} \Big|_J \\ 0 \end{pmatrix} \right\|_{\tilde{F}_J} = \left| R \begin{pmatrix} u|_J \\ v|_J \end{pmatrix} \right|, \ \forall J \in H,$$

and Lemma C.2 shows

$$\left| R \begin{pmatrix} u|_J \\ v|_J \end{pmatrix} \right| \to 0 \ (J \in H).$$

It remains to show the regularity of the convergence. Consider the auxiliary spaces

$$\begin{aligned}
E &:= H^1(\mathbb{R}, \mathbb{C}^{n+n+m}) = H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m), \\
F &:= L_2(\mathbb{R}, \mathbb{C}^{n+n+m}) = L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m), \\
E_J &:= H^1(J, \mathbb{C}^{n+n+m}) = H^1(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m), \\
F_J &:= L_2(J, \mathbb{C}^{n+n+m}) \times \mathbb{C}^{2n+m} = L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m) \times \mathbb{C}^{2n+m},
\end{aligned}$$

and families of bounded linear operators given by

$$\begin{aligned}
\mathcal{P} &:= \{p_J : J \in H\} \qquad p_J : E \to E_J, \qquad \begin{pmatrix} u \\ w \\ v \end{pmatrix} \mapsto \begin{pmatrix} u|_J \\ w|_J \\ v|_J \end{pmatrix}, \\
\mathcal{Q} &:= \{q_J : J \in H\} \qquad q_J : F \to F_J, \qquad \begin{pmatrix} h \\ f \\ g \end{pmatrix} \mapsto \begin{pmatrix} h|_J \\ f|_J \\ g|_J \\ 0 \end{pmatrix}.
\end{aligned}$$

Note that these spaces and operators are the same as in Theorem 2.29.

Using the usual transformation, we rewrite the second order equation

$$(sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{in } \tilde{F}$$

as the first order equation

$$L(\cdot, s)\begin{pmatrix} u \\ Au_x \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix} \quad \text{in } F.$$

Similarly the finite interval problem

$$\begin{pmatrix} (sI - P|_J) \\ R \end{pmatrix}\begin{pmatrix} u_J \\ v_J \end{pmatrix} = \begin{pmatrix} f_J \\ g_J \\ \eta_J \end{pmatrix} \quad \text{in } \tilde{F}_J$$

becomes

$$L_J(s)\begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} = \begin{pmatrix} L(\cdot, s) \\ R_1 \end{pmatrix}\begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} = \begin{pmatrix} 0 \\ -f_J + B_{12}B_{22}^{-1}g_J \\ -B_{22}^{-1}g_J \\ \eta_J \end{pmatrix} \quad \text{in } F_J.$$

In both cases $L(\cdot, s)$ is the operator defined in (4.13) with the appropriate domains, i.e. $L(\cdot, s) : E \to F$ and $L_J(s) : E_J \to F_J$, which differs from the definition in (4.13). The boundary operator R is of the form (4.21) and R_1 is given by $R_1 z = \begin{pmatrix} R_-^I & R_-^{II}A^{-1} & R_-^{III} \end{pmatrix} z(x_-) + \begin{pmatrix} R_+^I & R_+^{II}A^{-1} & R_+^{III} \end{pmatrix} z(x_+)$. Finally we have the inclusions

$$\iota_E : \tilde{E} \to E, \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ Au_x \\ v \end{pmatrix}, \qquad \iota_F : \tilde{F} \to F, \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix},$$

$$\iota_{E_J} : \tilde{E}_J \to E, \begin{pmatrix} u_J \\ v_J \end{pmatrix} \mapsto \begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix}, \quad \iota_{F_J} : \tilde{F}_J \to F_J, \begin{pmatrix} f_J \\ g_J \\ \eta_J \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -f_J + B_{12}B_{22}^{-1}g_J \\ -B_{22}^{-1}g_J \\ \eta_J \end{pmatrix}.$$

The whole situation is presented in Figure 4.2. By the determinant-condition $D(s) \neq 0$ and the assumptions on the coefficients of P, Theorem 2.29 implies

$$L_J(s) \xrightarrow{\mathcal{PQ}} L(\cdot, s) \text{ regularly.}$$

Let $\begin{pmatrix} u_J \\ v_J \end{pmatrix}_{J \in H}$ be a bounded net in $\tilde{E}_J$ such that the net $\left\{\begin{pmatrix} sI - P|_J \\ R \end{pmatrix}\begin{pmatrix} u_J \\ v_J \end{pmatrix}\right\}_{J \in H}$ is $\tilde{\mathcal{Q}}$-compact. Then by construction of $L_J(s)$ the net $\left\{L_J(s)\big(u_J, Au_{J,x}, v_J\big)\right\}_{J \in H}$ in F_J is $\mathcal{Q}$-compact.

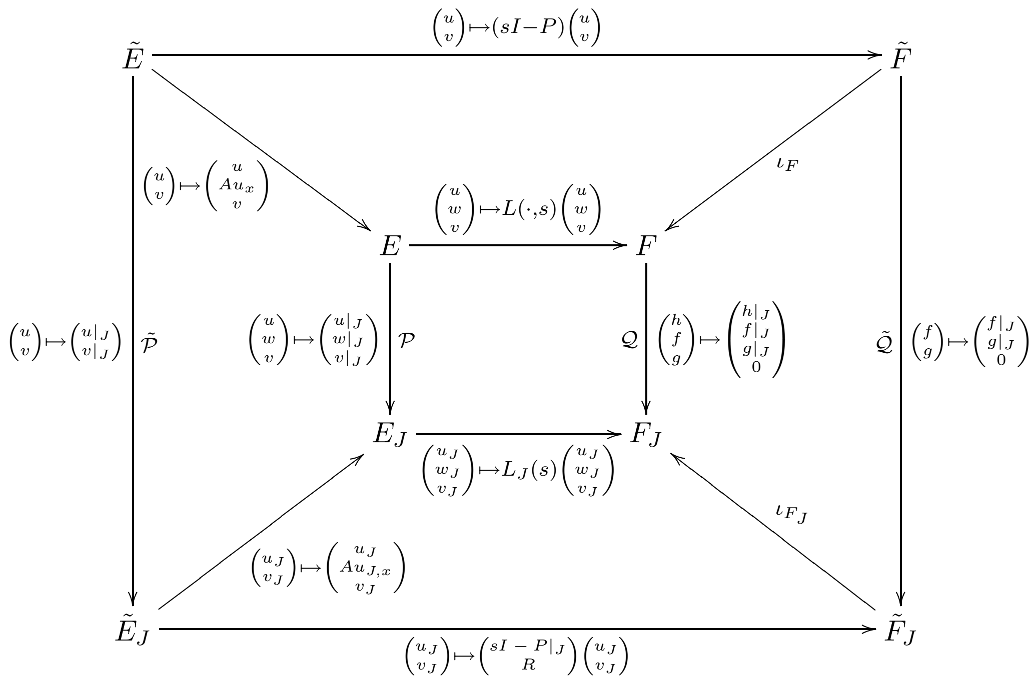


Figure 4.2: The setting of spaces and mappings in Lemma 4.17.

This is proven as follows. Let $H' \subset H$ be any cofinal subset. Then there are $H'' \subset H'$ and $(f,g) \in \tilde{F}$ so that

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix} := \begin{pmatrix} f_J \\ g_J \\ \eta_J \end{pmatrix} \xrightarrow{\tilde{\mathcal{Q}}} \begin{pmatrix} f \\ g \end{pmatrix} \quad (J \in H'').$$

By the definitions of $L(\cdot,s)$ and R_1 one obtains

$$L_J(s) \begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} = \begin{pmatrix} 0 \\ -f_J + B_{12}B_{22}^{-1}g_J \\ -g_J \\ \eta_J \end{pmatrix} \xrightarrow{\mathcal{Q}} \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix} \quad (J \in H'')$$

and hence the $\mathcal{Q}$-compactness.

Let $H' \subset H$ be any cofinal subset. By the regular convergence of $L_J(s)$ to $L(\cdot,s)$ there is a cofinal subset $H'' \subset H'$ and an element $(u,w,v) \in E$ such that

$$\begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} \xrightarrow{\mathcal{P}} \begin{pmatrix} u \\ w \\ v \end{pmatrix} \quad (J \in H''). \tag{4.54}$$

In addition, by the $\tilde{\mathcal{Q}}$-compactness of $\left(\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix}\right)_{J \in H}$ there also are a cofinal subset $H''' \subset H''$ and $(f,g) \in \tilde{F}$ so that

$$\left(\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u_J \\ v_J \end{pmatrix}\right) \xrightarrow{\tilde{\mathcal{Q}}} \begin{pmatrix} f \\ g \end{pmatrix} \quad (J \in H''').$$

The construction of $L_J(s)$ yields

$$L_J(s)\begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} \xrightarrow{\mathcal{Q}} \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix} \ (J \in H'''). \tag{4.55}$$

With (4.54) and (4.55) the $\mathcal{PQ}$-convergence $L_J(s) \to L(\cdot, s)$ implies the equality

$$L(\cdot, s)\begin{pmatrix} u \\ w \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix} \ \text{ in } L_2(\mathbb{R}, \mathbb{C}^{n+n+m}). \tag{4.56}$$

Application of the differential equation (4.56) implies

$$w = Au_x \in H^1(\mathbb{R}, \mathbb{C}^n),\ u \in H^2(\mathbb{R}, \mathbb{C}^n),\ \text{ and } (sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \ \text{ in } L_2(\mathbb{R}, \mathbb{C}^{n+m}).$$

Finally, the convergence (4.54) shows

$$\begin{aligned} &\|u_J - u|_J\|_{H^1(J,\mathbb{C}^n)} + \|u_{J,x} - u_x|_J\|_{H^1(J,\mathbb{C}^n)} + \|v_J - v|_J\|_{H^1(J,\mathbb{C}^m)} \\ &\quad \le \text{const} \left\| \begin{pmatrix} u_J \\ Au_{J,x} \\ v_J \end{pmatrix} - \begin{pmatrix} u|_J \\ Au_x|_J \\ v|_J \end{pmatrix} \right\|_{E_J} \\ &\quad \to 0 \ (J \in H'''), \end{aligned}$$

which by the definition of $\tilde{\mathcal{P}}$ means

$$\begin{pmatrix} u_J \\ v_J \end{pmatrix} \xrightarrow{\tilde{\mathcal{P}}} \begin{pmatrix} u \\ v \end{pmatrix} \ (J \in H''').$$

Since H' was arbitrary the $\tilde{\mathcal{P}}$-compactness of the net $\begin{pmatrix} u_J \\ v_J \end{pmatrix}_{J \in H}$ follows. □

As in Chapter 3 we can now prove quantitative results about the convergence of eigenvalues and eigenfunctions of the finite interval approximations by using the abstract theory from Chapter 2. With the notations and assumptions from above we have the following Theorem.

Theorem 4.18. *With the assumptions and notations from above, in particular Assumption 2 hold. Let Σ be an open neighborhood of the isolated eigenvalue s_0 with $D(s) \neq 0$ for all $s \in \Sigma$ and assume that ε_0 is so small that $\overline{K_{\varepsilon_0}(s_0)} \subset \Sigma$.*

Then there is a compact interval $J_0 \subset \mathbb{R}$ such that for all compact intervals $J = [x_-, x_+] \subset \mathbb{R}$ with $J \supset J_0$ the following properties hold.

The s_0-group of eigenvalues σ_J converges to the eigenvalue s_0 in the sense that for every $0 < \beta' < \min(\beta_-, \beta_+)$ there is a constant $\text{const} = \text{const}(\beta') > 0$ *with*

$$\max_{s \in \sigma_J} |s - s_0| = \operatorname{dist}(\sigma_J, s_0) \le \text{const} e^{-\frac{\beta'}{\kappa}\min(x_+, -x_-)}, \tag{4.57}$$

*where κ is the maximal order of the eigenelements of $\mathcal{A}(\cdot)$ to the eigenvalue s_0**. *Each net* $\begin{pmatrix} u_J \\ v_J \end{pmatrix}_{J \succ J_0}$ *of normalized eigenelements to eigenvalues $s_J \in \sigma_J$, i.e.* $\mathcal{A}_J(s_J)\begin{pmatrix} v_J \\ u_J \end{pmatrix} = 0$, $\| \begin{pmatrix} u_J \\ v_J \end{pmatrix} \|_{\tilde{E}_J} = 1$, *is $\tilde{\mathcal{P}}$-compact and the estimate*

$$\sup_{\substack{\| \binom{u_J}{v_J} \|_{E_J} = 1 \\ s_J \in \Sigma_J, \mathcal{A}_J(s_J)\binom{u_J}{v_J} = 0}} \inf_{\binom{u_0}{v_0} \in \mathcal{N}(\mathcal{A}(s_0))} \| \begin{pmatrix} u_J \\ v_J \end{pmatrix} - \begin{pmatrix} u_0|_J \\ v_0|_J \end{pmatrix} \|_{\tilde{E}_J} \leq \mathrm{const} e^{-\frac{\beta'}{\kappa} \min(x_+, x_-)} \tag{4.58}$$

holds.

Furthermore, for the root-subspaces we have

$$\dim W_J = \dim W < \infty, \tag{4.59}$$

and the family of root-subspaces W_J approximates the root-subspace W in the following sense.

$$\vartheta(W_J, W) = \sup_{\substack{\binom{u_J}{v_J} \in W_J \\ \| \binom{u_J}{v_J} \|_{\tilde{E}_J} = 1}} \mathrm{dist}(\begin{pmatrix} u_J \\ v_J \end{pmatrix}, \tilde{p}_J W) \leq \mathrm{const} e^{-\beta' \min(-x_-, x_+)}, \tag{4.60}$$

and

$$\vartheta(W, W_J) = \sup_{\substack{\binom{u}{v} \in W \\ \| \binom{u}{v} \|_{\tilde{E}} = 1}} \mathrm{dist}(\begin{pmatrix} u|_J \\ v|_J \end{pmatrix}, W_J) \leq \mathrm{const} e^{-\beta' \min(-x_-, x_+)}. \tag{4.61}$$

The constants in (4.57), (4.58), (4.60), *and* (4.61) *do not depend on J.*

All the necessary properties of the operators $\mathcal{A}(\cdot)$ and $\mathcal{A}_J(\cdot)$ one needs for the application of Theorem 2.26 are already shown in the Lemmas 4.7, 4.8, 4.17 so that Theorem 4.18 can be deduced from Theorem 2.26. The speed of the convergence is obtained by using the exponential decay of the functions from the root-subspace W proven in Lemma 4.16. Since the details of the proof are already given in the proof of Theorem 3.22, we do not carry out the proof.

4.6.3 Convergence in the case of simple eigenvalues

As in the hyperbolic case we finish this section with a theorem for the case of simple eigenvalues where we also allow for s-dependent boundary conditions.

Assume that the boundary operator R is of the usual form (4.40), but depends holomorphic on s in an open neighborhood Σ of the simple eigenvalue s_0. That means that the matrices $R_\pm$ depend holomorphic on s and so the mapping

$$R : \Sigma \to L(\tilde{E}_J, \mathbb{C}^{n+n+m}),$$

*See Definition C.6.

given by

$$R(s)\begin{pmatrix}u\\v\end{pmatrix} = (R_-(s)\begin{pmatrix}u(x_-)\\u_x(x_-)\\v(x_-)\end{pmatrix} + R_+(s)\begin{pmatrix}u(x_+)\\u_x(x_+)\\v(x_+)\end{pmatrix})$$

is holomorphic. Recall the determinant $D(s)$ from (4.42) which has the form

$$D(s) = \det\left[R_-(s)\begin{pmatrix}X_-(s)\\X_-(s)\Lambda_-^{II}(s)\\Z_-(s)\end{pmatrix} \quad R_+(s)\begin{pmatrix}X_+(s)\\X_+(s)\Lambda_+^{I}(s)\\Z_+(s)\end{pmatrix}\right]$$

where $X_\pm(s)$, $\Lambda_\pm(s)$, and $Z_\pm(s)$ are as in Section 4.5.

Theorem 4.19. *Let the assumptions from above hold.*
Let $s_0 \in \sigma(P) \cap \{\operatorname{Re} s > -\delta\}$ be a simple eigenvalue of the holomorphic operator-valued function† $\mathcal{A}(s) = sI - P$. Let (u_0, v_0) be a nontrivial eigenelement of $\mathcal{A}(\cdot)$ for the eigenvalue s_0. Furthermore assume $D(s_0) \neq 0$.*

Then there is a compact interval $J_0 \in H$ and $\delta_0 > 0$ so that for all compact intervals $J \supset J_0$ there exists exactly one simple eigenvalue s_J with $|s_0 - s_J| \leq \delta_0$ of the finite interval approximation $\mathcal{A}_J(\cdot) : s \mapsto \begin{pmatrix}sI - P|_J\\R(s)\end{pmatrix}$. Moreover there is a corresponding eigenfunction $\begin{pmatrix}u_J\\v_J\end{pmatrix} \in \tilde{E}_J$ so that we have the estimate

$$|s_J - s_0| + \left\|\begin{pmatrix}u_J\\v_J\end{pmatrix} - \begin{pmatrix}u_0|_J\\v_0|_J\end{pmatrix}\right\|_{\tilde{E}_J} \leq C\left|R_-(s_0)\begin{pmatrix}u_0(x_-)\\u_{0,x}(x_-)\\v_0(x_-)\end{pmatrix} + R_+(s_0)\begin{pmatrix}u_0(x_+)\\u_{0,x}(x_+)\\v_0(x_+)\end{pmatrix}\right| \tag{4.62}$$

with a constant C independent of J.

One proves Theorem 4.19 in the same way as Theorem 3.25 in the hyperbolic case. Therefore we do not give the proof here.

*See Definition C.6.
†See Definition C.5.

5 Analysis of the boundary conditions and an application

In this Chapter we will briefly look at the Assumptions 1 and 2. We give conditions that imply the validity of (H4) of Assumption 1 and (M2) of Assumption 2. Furthermore, we show that the determinant-conditions $D_\infty \neq 0$ and $D(s) \neq 0$ are satisfied for some natural choices of boundary conditions. We finish the chapter with an application of our theory to the FitzHugh-Nagumo equation.

5.1 Boundary conditions in the hyperbolic case

Consider the operator P from (3.2). We analyze the validity of the spectral assumption (H4) of Assumption 1 in this section and show that the characteristic and in some important cases also the periodic boundary conditions satisfy the determinant-conditions (3.14) and (3.86).

The characteristic boundary conditions prescribe the values of the ingoing variables at the endpoints of the interval.

EXAMPLE 3. *Consider the equation $u_t = \lambda u_x$ on $\mathbb{R} \times [0, \infty)$ subject to boundary conditions $u(x, 0) = f(x)$. From the differential equation follows $u(x, t) = f(\lambda t + x)$, thus the solution "propagates" along the characteristics given by $x(t) = -\lambda t + x_0$.*

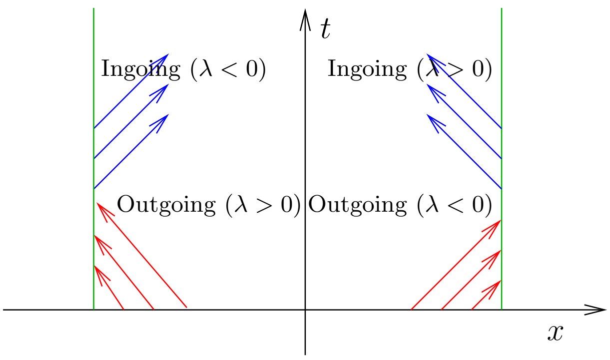


Figure 5.1: In and outgoing characteristics for different values of λ in Example 3.

REMARK. *In [KL89, Chapter 7.6] it is shown that linear hyperbolic initial boundary value problems subject to characteristic boundary conditions are well-posed if the boundary is not characteristic.*

Because of assumption (H2) the equation (3.1) is already in characteristic variables and the boundary operator R has the simple form

$$R_{char}v = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} v(x_-) + \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} v(x_+). \tag{5.1}$$

If one considers periodic boundary conditions, the boundary operator is of the form

$$R_{per}v = Iv(x_-) + (-I)v(x_+). \tag{5.2}$$

Next we give a further assumption which implies the spectral assumption (H4). We will also show that under this assumption the characteristic boundary conditions satisfy the determinant-conditions from Chapter 3.

Assumption 3. *There is a positive definite diagonal matrix $H = H^*$ such that*

$$HC_\pm + C_\pm^* H < 0.$$

Let Assumption 3 hold. By $\hat{P}$ we denote the symbol of P and $\hat{P}_H$ denotes the symbol of $HBv_x + HCv$. Then there is a constant $\delta > 0$ such that for every $u, v \in \mathbb{C}^m$ holds the estimate

$$-u^*(HB - B^*H^*)v + v^*(HC + C^*H)v = v^*(HC + C^*H)v < -\delta v^* v. \tag{5.3}$$

With the choice $u = i\omega v$ this implies

$$\begin{aligned} & v^* i\omega HBv + v^*HCv + v^* - i\omega B^*Hv + v^*C^*Hv \\ =& v^*\hat{P}_H(i\omega)v + v^*\hat{P}_H(i\omega)^*v \\ =& v^*H\hat{P}(i\omega)v + v^*\hat{P}(i\omega)^*Hv. \end{aligned} \tag{5.4}$$

Now assume $\hat{P}(i\omega)v = sv$ then it follows from (5.4) and (5.3) the equality

$$v^*H\hat{P}(i\omega)v + v^*\hat{P}(i\omega)^*Hv = 2\operatorname{Re}(s)v^*Hv < -\delta v^*v$$

what implies (H4).

Before we analyze the boundary conditions we need another auxiliary result which shows that the characteristic parts of the bases of the stable and unstable subspaces, which appear in the determinant-condition (3.86) are nonsingular. The Lemma follows the ideas of [BL99, Lemma 5.1], but we directly included a rescaling since it will be necessary for the application to the FHN-system.

Lemma 5.1. *Let the assumptions 1 and 3 hold and assume s is an element of $\mathbb{C}$ such that*

$$HC + C^*H - 2\operatorname{Re}(s)H < 0.$$

Let $V_-^{II}(s) = \begin{pmatrix} V_{-,1}^{II}(s) \\ V_{-,2}^{II}(s) \end{pmatrix} \in \mathbb{C}^{(m-r)+r,r}$ be a basis of the stable subspace of $M_-(s)$ as in (3.82).

Let $V_+^{I}(s) = \begin{pmatrix} V_{+,1}^{I}(s) \\ V_{+,2}^{I}(s) \end{pmatrix} \in \mathbb{C}^{(m-r)+r,m-r}$ be a basis of the unstable subspace of $M_+(s)$ as in (3.81).

Then the matrices $V_{-,2}^{II}(s)$ and $V_{+,1}^{I}(s)$ are nonsingular.

Proof. Without loss of generality consider $V_{-,2}^{II}(s)$.
Let $\phi \in \mathbb{C}^r$ with $V_{-,2}^{II}(s)\phi = 0$. The function

$$z(x) := V_-^{II}(s)e^{\lambda_-(s)x}\phi$$

is an exponentially decaying solution of

$$Bz_x + (C - sI)z = 0 \text{ in } [0, \infty).$$

Therefore by multiplication from the left with z^*H, integration by parts, and taking real parts one obtains

$$\begin{aligned} 0 &= \int_0^\infty z^*HBz_x + z^*(HC - sH)z + z_x^*B^*Hz + z^*(C^*H - \bar{s}H)zdx \\ &= \int_0^\infty z^*(HB - B^*H)z_x + z^*(HC + C^*H - 2\operatorname{Re}(s)H)zdx + [z^*B^*Hz]_0^\infty , \end{aligned}$$

where $\bar{s}$ stands for the complex conjugate of s.

From the assumptions on B, H, and ϕ it follows $z^*(0)B^*Hz(0) \geq c_0z^*(0)z(0)$ with a positive constant c_0 which implies

$$\int_0^\infty z^*(HC + C^*H - 2\operatorname{Re}(s)H)zdx = z^*(0)B^*Hz(0) \geq c_0z^*(0)z(0) \geq 0.$$

Since $HC + C^*H - 2\operatorname{Re}(s)H < 0$ we obtain $z(0) = V_-^{II}(s)\phi = 0$ and therefore $\phi = 0$.

For the proof of the non-singularity of $V_{+,1}^{I}(s)$ one has to consider the left half-line. □

Now we analyze the determinant-conditions from 3.2 and 3.3 for the case of periodic and characteristic boundary conditions.

Characteristic BC. In the case of characteristic boundary conditions the boundary operator R from (3.8) is of the form (5.1) and the determinant-conditions read

$$D_\infty = \det \begin{pmatrix} 0 & I_r \\ I_{m-r} & 0 \end{pmatrix}$$

and

$$D(s) = \left(\begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} V_-^{II}(s) \quad \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V_+^{I}(s) \right) = \det \begin{pmatrix} 0 & V_{+,1}^{I}(s) \\ V_{-,2}^{II}(s) & 0 \end{pmatrix},$$

where $V_\pm^{I,II}$ are as in Lemma 5.1. Obviously $D_\infty \neq 0$ is satisfied and by Lemma 5.1 also $D(s) \neq 0$ for all $\operatorname{Re} s > -\delta$ for some positive δ.

Periodic BC. Periodic boundary conditions in (3.8) which are of the form (5.2) are a natural choice if the coefficients of the operator P from (3.2) satisfy $B_- = B_+$ and $C_- = C_+$. For example this is the case if P is obtained by linearization of (1.3) at a pulse solution. The determinant-conditions then take the form

$$D_\infty = \det \begin{pmatrix} 0 & -I_r \\ I_{m-r} & 0 \end{pmatrix}$$

and

$$D(s) = \det \begin{pmatrix} V_-^{II}(s) & -V_+^{I}(s) \end{pmatrix},$$

where $V_\pm^{I,II}$ are as in Lemma 5.1. As in the case of characteristic boundary conditions one directly obtains $D_\infty \neq 0$. If $B_- = B_+$ and $C_- = C_+$ one finds $\mathcal{R}(V_-^{II}(s)) = \mathcal{R}(V_+^{II}(s))$ as well as $\mathcal{R}(V_-^{I}(s)) = \mathcal{R}(V_+^{I}(s))$ and since $V_-(s)$ and $V_+(s)$ are bases of $\mathbb{C}^m$ one also obtains $D(s) \neq 0$.

5.2 Boundary conditions in the mixed case

For the mixed case we proceed as in the hyperbolic case. We consider the operator P from (4.1) which is of the form (4.2). Similar to the analysis of the boundary conditions for the hyperbolic case in Section 5.1 we give a sufficient condition for assumption (M2) that will also imply that some typical choices of boundary conditions satisfy the determinant-conditions (4.30) and (4.42). As in the hyperbolic case the assumption is easier to check than the original condition (H4).

Assumption 4. *Assume there is a matrix* $H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \in \mathbb{C}^{n+m,n+m}$ *with* $H = H^* > 0$ *and*

$$H_1 A + A^* H_1 > 0, \ H_2 \text{ is a diagonal matrix}$$

such that

$$HB = B^* H \ \text{ and } HC + C^* H < -2\delta H \text{ for some } \delta > 0.$$

From Assumption 4 follows (M2) of Assumption 2.
Let $u, v \in \mathbb{C}^{n+m}$, Assumption 4 yields

$$-u^* \begin{pmatrix} H_1 A + A^* H_1 & 0 \\ 0 & 0 \end{pmatrix} u - u^*(HB - B^*H)v + v^*(HC + C^*H)v \\ \leq -2\delta v^* H v. \quad (5.5)$$

Therefore assume $\hat{P}(i\omega)v = sv$ and let $u = i\omega v$. Now (5.5) implies

$$\begin{aligned} -2\delta v^* H v \geq & -\omega^2 v^* \begin{pmatrix} H_1 A & 0 \\ 0 & 0 \end{pmatrix} v + i\omega v^* H B v + v^* H C v \\ & + \left(-\omega^2 v^* \begin{pmatrix} A^* H_1 & 0 \\ 0 & 0 \end{pmatrix} v - i\omega v^* B^* H v + v^* C^* H v \right) \\ = & v^* H \hat{P}(i\omega) v + v^* \hat{P}(i\omega)^* H v = 2\,\mathrm{Re}(s) v^* H v. \end{aligned}$$

Therefore (M2) follows.

Next we prove a result similar to Lemma 5.1 in this case. It will be used to show that the theory is applicable to the FHN-System (see Section 5.3).

Lemma 5.2. *Let the Assumptions 2 and 4 hold. Assume that s is an element of $\mathbb{C}$ such that $HC + C^*H - 2\,\mathrm{Re}(s)H < 0$ holds.*

Let $\begin{pmatrix} X_-(s) \\ X_-(s)\Lambda_-^{II}(s) \\ Z_-(s) \end{pmatrix} \in \mathbb{C}^{n+n+m,n+(m-r)}$ *be a basis of the stable subspace of $M_-(s)$ and let* $\begin{pmatrix} X_+(s) \\ X_+(s)\Lambda_+^{II}(s) \\ Z_+(s) \end{pmatrix} \in \mathbb{C}^{n+n+m,n+r}$ *be a basis of the unstable subspace of $M_+(s)$. Partition the last m rows of these matrices in the form $Z_\pm(s) = \begin{pmatrix} Z_\pm^I(s) \\ Z_\pm^{II}(s) \end{pmatrix}$ with the partitioning corresponding to the partitioning of B_{22} in (H4) of Assumption 2. Then the matrices*

$$\begin{pmatrix} X_-(s) \\ 0 \\ Z_-^{II}(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_+(s) \\ Z_+^I(s) \\ 0 \end{pmatrix}$$

have maximum rank.

Proof. Without loss of generality consider $\begin{pmatrix} X_-(s) \\ 0 \\ Z_-^{II}(s) \end{pmatrix}$. Let $\phi \in \mathbb{C}^{n+m-r}$ with

$$\begin{pmatrix} X_-(s) \\ Z_-^I(s) \\ Z_-^{II}(s) \end{pmatrix} \phi = \begin{pmatrix} 0 \\ Z_-^I(s)\phi \\ 0 \end{pmatrix}.$$

Then the function $z(x) := \begin{pmatrix} X_-(s) \\ Z_-(s) \end{pmatrix} e^{-\Lambda_-^{II}(s)}\phi$ is an exponentially decaying solution of

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} z_{xx} + Bz_x + Cz - sIz = 0 \ \text{ in } [0,\infty).$$

Multiplication from the left with z^*H, integration, and taking real parts yields

$$\begin{aligned} 0 = \int_0^\infty & z^*H \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} z_{xx} + z_{xx}^* \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} Hz \\ &+ z^*HBz_x + z_x^*B^*Hz + z^*\big(HC + C^*H - 2\,\mathrm{Re}(s)H\big)zdx. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} \int_0^\infty & -z_x^* \begin{pmatrix} H^1A + A^*H^1 & 0 \\ 0 & 0 \end{pmatrix} z_x \\ &\quad + z^*\big(HB - B^*H\big)z_x + z^*\big(HC + C^*H - 2\,\mathrm{Re}(s)H\big)zdx \\ &= -\left[z^*H \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} z_x + z_x^* \begin{pmatrix} A^*H_1 & 0 \\ 0 & 0 \end{pmatrix} z\right]_0^\infty - \big[z^*B^*Hz\big]_0^\infty \\ &\qquad\qquad = z^*(0)B^*Hz(0) \geq b_0 z^*(0)Hz(0), \end{aligned}$$

where b_0 is given (H2). Therefore the left hand side must be zero and this is only possible if $z \equiv 0$. This implies $\begin{pmatrix} X_-(s) \\ X_-(s)\Lambda_-^{II}(s) \\ Z_-(s) \end{pmatrix} \phi = 0$ and therefore $\phi = 0$. □

Now we can analyze some natural choices for the boundary operator R from (4.6b). The determinants from Chapter 4 are

$$D_\infty = \det \begin{pmatrix} Q_-^I & Q_+^I & 0 & 0 \\ -P_-^{II} A^{\frac{1}{2}} & P_+^{II} A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & R_-^{hyp,II} & R_+^{hyp,I} \end{pmatrix}$$

and

$$D(s) = \det \left((R_-^I, R_-^{II}, R_-^{III}) \begin{pmatrix} X_-(s) \\ X_-(s)\Lambda_-^{II}(s) \\ Z_-(s) \end{pmatrix}, (R_+^I, R_+^{II}, R_+^{III}) \begin{pmatrix} X_+(s) \\ X_+(s)\Lambda_+^{I}(s) \\ Z_+(s) \end{pmatrix} \right).$$

The first given name stands for the boundary conditions for the parabolic part and the second for the boundary conditions for the hyperbolic part of the equation.

Dirichlet-characteristic boundary conditions. In this case the matrices read

$$R_- = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} \end{pmatrix} \quad \text{and} \quad R_+ = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

We directly obtain

$$D_\infty = \det \begin{pmatrix} -A^{\frac{1}{2}} & 0 & 0 \\ 0 & A^{\frac{1}{2}} & 0 \\ 0 & 0 & \begin{pmatrix} 0 & I_r \\ I_{m-r} & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

And for bounded $|s|$ with $\operatorname{Re} s > -\delta$ one obtains

$$D(s) = \det \begin{pmatrix} X_-(s) & 0 \\ 0 & X_+(s) \\ 0 & Z_+^I(s) \\ Z_-^{II}(s) & 0 \end{pmatrix}$$

which also satisfies $D(s) \neq 0$ by Lemma 5.2.

Periodic boundary conditions. Again this is the natural choice if $B_- = B_+$ and $C_- = C_+$. The boundary operator is of the form

$$R_- = I \quad \text{and} \quad R_+ = -I$$

and so the determinants are

$$D_\infty = \det \begin{pmatrix} -A^{\frac{1}{2}} & -A^{\frac{1}{2}} & 0 \\ I & -I & 0 \\ 0 & 0 & \begin{pmatrix} 0 & -I_r \\ I_{m-r} & 0 \end{pmatrix} \end{pmatrix} \neq 0$$

and

$$D(s) = \det \begin{pmatrix} X_-(s) & -X_+(s) \\ X_-(s)\Lambda_-^{II}(s) & -X_+(s)\Lambda_+^{I}(s) \\ Z_-(s) & -Z_+(s) \end{pmatrix} \neq 0.$$

5.3 The FitzHugh-Nagumo System

In this section we will show that our theoretical results apply to the FitzHugh-Nagumo System (FHN) (see [Fit61] and [Mur93]). Furthermore we also present some numerical results which in some sense seem to validate the theoretically predicted behavior in practice.

5.3.1 Theoretical embedding of the FitzHugh-Nagumo System

The FitzHugh-Nagumo system (FHN) is a simplification of the Hodgkin-Huxley system [HH52] which models the electrical signalling in nerve cells. The intention of the (FHN) system is not to approximate the Hodgkin-Huxley system itself, but to model its behavior [Fit61]. The (FHN) system reads

$$u_t = u_{xx} + f_1(u,v), \tag{5.6a}$$

$$v_t = f_2(u,v), \tag{5.6b}$$

where $f_1(u,v) = u - \frac{1}{3}u^3 - v$ and $f_2(u,v) = \Phi(u + a - bv)$, and a, b, Φ are positive constants.

We consider the parameter values $a = 0.7$, $b = 0.8$, $\Phi = 0.08$ which were already chosen in [Miu82] for the computation of a stable travelling wave solution.

For these values the System (5.6) has a stable and an unstable pulse. A proof of the stability of the fast travelling pulse using centre manifold theory is presented in [BJ89].

Let $\begin{pmatrix} \bar{u}_s \\ \bar{v}_s \end{pmatrix}$ be a stable pulse with speed c_s and with profile W_s this means

$$\begin{pmatrix} \bar{u}_s \\ \bar{v}_s \end{pmatrix}(t,x) = W_s(x - c_s t)$$

and let $\begin{pmatrix} \bar{u}_u \\ \bar{v}_u \end{pmatrix}$ be an unstable pulse with speed c_u and profile W_u.

As in the introduction the pulses lead to steady states of the equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + cI \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}, \tag{5.7}$$

where $c = c_s$ for the stable pulse and $c = c_u$ for the unstable pulse.

Now let $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$ be a pulse solution of (5.6) with speed c. Linearization of (5.7) at the pulse leads to the linear PDE

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = P \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} B \begin{pmatrix} u \\ v \end{pmatrix}_x + C \begin{pmatrix} u \\ v \end{pmatrix}, \tag{5.8}$$

where $A = 1$, $B = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, $C = \begin{pmatrix} 1 - \bar{u}^2 & -1 \\ \Phi & -\Phi b \end{pmatrix}$.

With an abuse of notation we will also write A for the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Here $\bar{u}$ stands for some shifted version of the first component of the profile W, i.e. $\bar{u}(x) = W_1(x + x_0)$.

Furthermore, the pulses are homoclinic connecting orbit of the stationary point $\begin{pmatrix} \bar{u}_\infty \\ \bar{v}_\infty \end{pmatrix} = \begin{pmatrix} -1.1994 \\ -0.6243 \end{pmatrix}$. Therefore one easily sees that the assumptions (P1), (P2), (H1), (H2), (H3), and (M1) are satisfied.

For the validation of assumption (M2) we consider the positive definite hermitian matrix $H = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\Phi} \end{pmatrix} \in \mathbb{C}^{2,2}$. It holds

$$1A + A^*1 > 0,\ HB_\infty = \begin{pmatrix} c & 0 \\ 0 & \frac{c}{\Phi} \end{pmatrix} = B_\infty^* H,$$

and

$$HC_\infty + C_\infty^* H = \begin{pmatrix} 1 - \bar{u}_\infty^2 & -1 \\ 1 & -b \end{pmatrix} + \begin{pmatrix} 1 - \bar{u}_\infty^2 & 1 \\ -1 & -b \end{pmatrix} = \begin{pmatrix} 2 - 2\bar{u}_\infty^2 & 0 \\ 0 & -2b \end{pmatrix} < -2\delta H$$

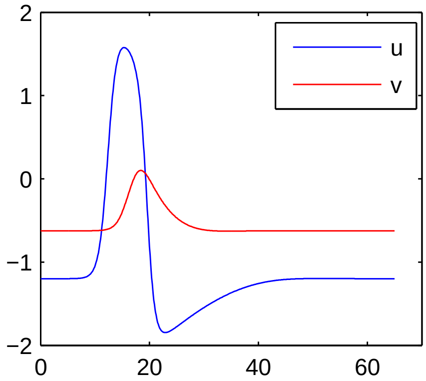
for some $\delta > 0$. Thus Assumption 4 is satisfied and implies (M2) of Assumption 2 and so our results from Chapter 4 are applicable. Furthermore the analysis of the boundary conditions from Section 5.2 show that the Dirichlet-characteristic and the periodic boundary conditions are suitable for analyzing the spectrum at least in the right half-plane $\{\mathrm{Re}(s) > -\varepsilon\}$, where ε is some positive constant.

5.3.2 Numerical experiments

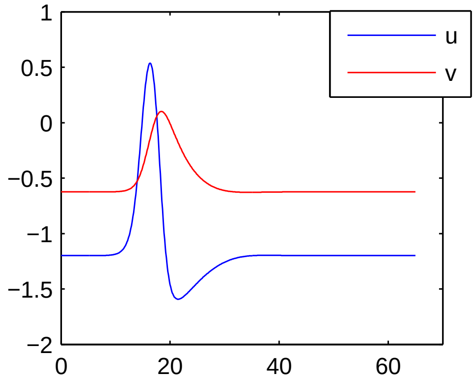
For the numerical experiments we used an approximation of the profile of the stable pulse provided by V. Thümmler who used the interval $[-80, 80]$ with stepsize $h = 0.2$ for the numerical computation of the stable travelling wave and its speed. We also used an approximation obtained by Claudia Nölker who chose the interval $[0, 65]$ with stepsize $h = 0.1$. The latter person also computed the unstable pulse's profile for the given parameter values on the same lattice by a continuation method. The shapes of the profiles of the stable and unstable pulses computed by C. Nölker are shown in Figure 5.2.

In the theory of Chapters 3 and 4 we did not analyze the spectral behavior in the left half-plane, where the essential spectra* of the operators lie. In the

*See Definition C.1.



(a) The stable pulse, $c \approx -0.8117$.



(b) The unstable pulse, $c \approx -0.5414$.

Figure 5.2: Numerical approximations of the stable and the unstable pulse of the FHN system, on the interval $[0, 65]$, stepsize $h = 0.1$.

explanations of the following figures we do not intend to give rigorous proves, we only give an idea of why the spectra obtained in the numerical experiments look as they do.

Dispersion relations

By Lemma 4.7 and its remark we know that the mixed operator $sI - P$ has a Fredholm property if and only if the operator $L(\cdot, s)$ obtained by rewriting the resolvent equation as a first order system, has one. Moreover the Fredholm indices coincide.

By the results of K. Palmer (Lemma B.7 and [Pal88]) about the relation of the Fredholm property of $L(\cdot, s)$ and the presence of exponential dichotomies, one sees with the help of Lemma B.5 that the operator $L(\cdot, s)$ is Fredholm if and only if the limit matrices $M_\pm(s)$ are hyperbolic. In the special case of pulses one obtains $M_+(s) = M_-(s)$ and so directly has that $L(\cdot, s)$ is Fredholm index 0 if and only if the limit matrix is hyperbolic. This shows that σ_Δ, the part of the spectrum where the operator is not Fredholm of index 0^*, exactly coincides with the set of all $s \in \mathbb{C}$ for which there is a real solution ω of the characteristic equation

$$\det(M_\pm(s) - i\omega I) = 0.$$

Thus we define the set $S \subset \mathbb{C}$ as the set of all $s \in \mathbb{C}$ for which the quadratic eigenvalue problem

$$\det\left(\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \omega \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 1 - \bar{u}_\infty^2 - s & -1 \\ \Phi & -\Phi b - s \end{pmatrix}\right) = 0 \qquad (5.9)$$

has a purely imaginary solution ω and this set is the same as the spectral set σ_Δ.

Note that in [Hen81, 5.4 Theorem A.2] a very similar result is shown for elliptic second order operators with constant principle part.

*See the remark after Definition C.1.

The relation (5.9) is sometimes known as **dispersion relation** (for example see [Kev00, p. 216] or [Zau89, Chapter 3.5]). The solutions of the dispersion relation correspond to solutions of the constant coefficient differential equation

$$\begin{aligned} u_t &= u_{xx} + cu_x + (1 - \bar{u}_\infty^2)u - v \\ v_t &= cv_x + \Phi u - \Phi b v \end{aligned}$$

of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} (t, x) = e^{st - i\omega x} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

which are spatially not elements of $L_2(\mathbb{R})$.

In Chapters 3 and 4 we only analyzed the influence of the restriction to finite intervals on the spectra, but for the numerical computations it is also necessary to consider the effect the discretization has on the spectra. We do not analyze this here, but give a discrete analogon to the dispersion relation (see also [Thü98, Chapter 4.3]).

After discretization of the operator P one obtains an operator $\tilde{P}$ on the grid functions $\Omega_h = \{U : J_h = h \cdot \mathbb{Z} \to \mathbb{R}^2\}$. We write the grid functions as $U = (U_j)_{j \in \mathbb{Z}}$ with $U_j \in \mathbb{R}^2$.

Consider the constant coefficient limit operator $P_\pm$ which is given by

$$P_\pm \begin{pmatrix} u \\ v \end{pmatrix} = AU_{xx} + BU_x + CU$$

with A and B from (5.8) and C is of the form

$$C = \begin{pmatrix} 1 - \bar{u}_\infty & -1 \\ \Phi & -\Phi b \end{pmatrix}.$$

We next describe the discretization $\tilde{P}_\pm$ of $P_\pm$ with finite differences to the grid $J_h = h\mathbb{Z}$ for several different approximations of the first order derivatives. Then we insert the discrete analogon of the continuous functions from the dispersion relation. This means we make the **ansatz** of grid functions of the form

$$U_j = \begin{pmatrix} u \\ v \end{pmatrix}_j = e^{i\omega jh} \begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix} = e^{i\omega jh} U_\omega, \; j \in \mathbb{Z} \tag{5.10}$$

and insert these into the resulting formulas. As in the continuous case these functions are bounded elements in Ω_h, but they are not in the discrete analogons of the spaces H^1 or L_2. They are not even decaying. (One usually uses the square summable sequences as a discrete analogon of the space L_2.)

- If one uses central differences for the approximation of u_x and v_x one obtains

$$(\tilde{P}_\pm U)_j = A \frac{U_{j+1} - 2U_j + U_{J-1}}{h^2} + B \frac{U_{J+1} - U_{J-1}}{2h} + CU_J.$$

Inserting the functions from the ansatz into this formula leads to

$$\begin{aligned}(\tilde{P}_{\pm}U)_j &= \frac{e^{i\omega h}+e^{-i\omega h}-2}{h^2}AU_J + B\frac{e^{i\omega h}-e^{-i\omega h}}{2h}U_J + CU_J \\ &= \left\{\frac{2}{h^2}(\cos(\omega h)-1)A + \frac{i}{h}B\sin(\omega h)+C\right\}U_J.\end{aligned}$$

Hence there is an eigenfunction $(U_j)_{j\in\mathbb{Z}}$ of $\tilde{P}_{\pm}$ to the eigenvalue s of the form (5.10) if and only if

$$\det\left\{A\frac{2}{h^2}(\cos(\omega h)-1)+\frac{i}{h}B\sin(\omega h)+C-sI\right\}=0. \tag{5.11}$$

- Similar, if one uses forward differences for the approximation of u_x and v_x one obtains

$$(\tilde{P}_{\pm}U)_j = A\frac{U_{j+1}-2U_j+U_{J-1}}{h^2}+B\frac{U_{J+1}-U_J}{h}+CU_J.$$

Now inserting the functions from the ansatz as before implies

$$\begin{aligned}(\tilde{P}_{\pm}U)_j &= \frac{e^{i\omega h}+e^{-i\omega h}-2}{h^2}AU_J + B\frac{e^{i\omega h}-1}{h}U_J + CU_J \\ &= \left\{\frac{2}{h^2}(\cos(\omega h)-1)A + \frac{e^{i\omega h}-1}{h}B+C\right\}U_J\end{aligned}$$

and so in this cases there is an eigenfunction $(U_j)_{j\in\mathbb{Z}}$ of $\tilde{P}_{\pm}$ to the eigenvalue s of the form (5.11) if and only if

$$\det\left\{\frac{2}{h^2}(\cos(\omega h)-1)A+\frac{e^{i\omega h}-1}{h}B+C-sI\right\}=0. \tag{5.12}$$

- Finally, using central differences for the approximation of u_x and forward differences for the hyperbolic part leads to

$$\left(\tilde{P}_{\pm}\begin{pmatrix}u\\v\end{pmatrix}\right)_j = \begin{pmatrix}\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}+c\frac{u_{j+1}-u_{j-1}}{2h}+(1-\bar{u}_{\infty}^2)u_j-v_j, \\ c\frac{v_{j+1}-v_j}{h}+\Phi u_j-\Phi b v_j\end{pmatrix}.$$

Inserting as before shows the relation

$$\left(\tilde{P}_{\pm}\begin{pmatrix}u\\v\end{pmatrix}\right)_j = \begin{pmatrix}\frac{2}{h^2}(\cos(\omega h)-1)u_j+\frac{ci}{h}\sin(\omega h)u_j+(1-\bar{u}_{\infty}^2)u_j-v_j \\ \frac{c}{h}(e^{i\omega h}-1)v_j+\Phi u_j-\Phi b v_j\end{pmatrix}$$

and so there is an eigenfunction of $\tilde{P}_{\pm}$ to the eigenvalue s of the assumed type if and only if

$$\begin{aligned}\det\Bigg[\Big\{\frac{2}{h^2}(\cos(\omega h)-1)+\frac{ci}{h}\sin(\omega h)\Big\}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{c}{h}(e^{i\omega h}-1)\begin{pmatrix}0&0\\0&1\end{pmatrix} \\ +\begin{pmatrix}1-u_{\infty}^2&-1\\ \Phi&-\Phi b\end{pmatrix}-sI\Bigg]=0.\end{aligned} \tag{5.13}$$

We call the equations (5.11)–(5.13) **dispersion relations for the discrete operator**.

Numerical results

We computed the spectra of the spatially discretized operators $\tilde{P}$ as follows. First we discretized the operator P on a finite interval $[x_-, x_+]$ using either periodic or Dirichlet-characteristic boundary conditions, which are a reasonable choice in view of the results from Section 5.2. Then we computed the spectrum of the resulting matrix using the matlab-*eig*- function. This function uses the *lapack DGEEV-routine* see the matlab-help and [ABB⁺99]*.

As an example we describe the structure of the resulting matrix in the case of Dirichlet-characteristic boundary conditions with downwind discretization.

We choose the interval $[x_-, x_+]$ with an equidistant grid of stepsize $h = \frac{x_+ - x_-}{N}$ and $N+1$ grid points. Under the boundary conditions we have $u_0 = v_0 = u_N = 0$. We will write $\bar{u}_j$ for the value $\bar{u}(x_- + jh)$ of the u-component of the approximation of the wave profile.

Semi-discretization with downwind ($c < 0$) of the linear PDE (5.8) on the grid leads to

$$\begin{aligned} u_j' &= \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + c\frac{u_j - u_{j-1}}{h} + (1 - \bar{u}_j^2)u_j - v_j, \; j = 1, \ldots, N-1 \\ v_j' &= c\frac{v_j - v_{j-1}}{h} + \Phi u_j - \Phi b v_j, \; j = 1, \ldots, N. \end{aligned}$$

By using the Dirichlet-characteristic boundary conditions we obtain

$$\frac{d}{dt}\begin{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\ \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} u_{N-1} \\ v_{N-1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} Y_1 & Z_1 & 0 & \cdots & & & \\ X_2 & Y_2 & Z_2 & \cdots & & & \\ & & & & X_{N-2} & Y_{N-2} & Z_{N-2} \\ & & & & & X_{N-1} & Y_{N-1} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\ \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} u_{N-1} \\ v_{N-1} \end{pmatrix} \end{pmatrix} \tag{5.14}$$

where we already eliminated v_N which one directly obtains from v_{N-1}. If we do not eliminate this, it leads to an additional eigenvalue of the operator at $\frac{c}{h} - \Phi b < 0$.

The X, Y, and Z-parts of the big matrix from (5.14) are defined as

$$\begin{aligned} X_j &= \begin{pmatrix} \frac{1}{h^2} - \frac{c}{h} & 0 \\ 0 & -\frac{c}{h} \end{pmatrix}, \; j = 2, \ldots, N-1, \\ Y_j &= \begin{pmatrix} -\frac{2}{h^2} + \frac{c}{h} + 1 - \bar{u}_j^2 & -1 \\ \Phi & -\Phi b + \frac{c}{h} \end{pmatrix}, \; j = 1, \ldots, N, \\ Z_j &= \begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & 0 \end{pmatrix}, \; j = 1, \ldots, N-2. \end{aligned}$$

*The function therefore first reduces the matrix to upper Hessenberg form and then reduces to Schur form. The eigenvalues then are obtained as the diagonal elements or as conjugate pairs of eigenvalues of 2×2 submatrices.

In Figures 5.3 and 5.4 we compare the spectrum of the discrete operator (downwind and periodic boundary conditions) with the dispersion relation and the dispersion relation for the discretized operator. Here we used the approximation of the wave profile of the stable wave on the interval $[0, 65]$ which was obtained by C. Nölker (speed $c \approx -0.8117$).

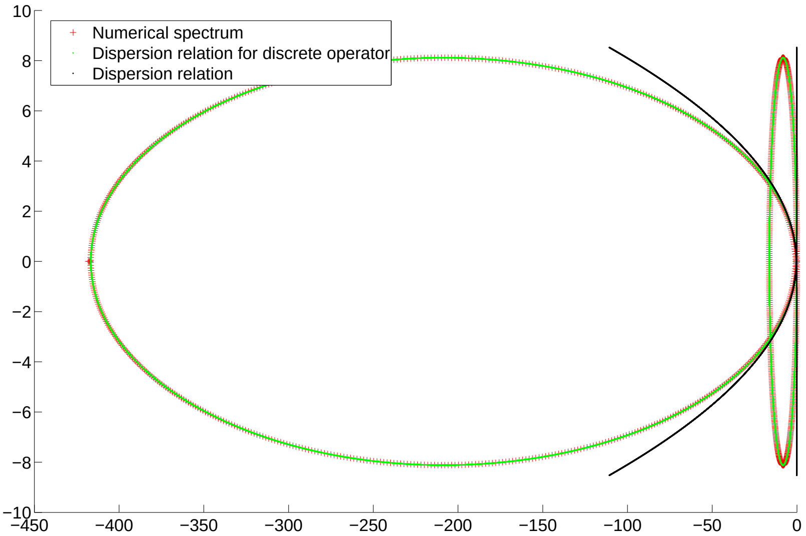


Figure 5.3: The spectrum of the numerical operator for backward differences, compared with the dispersion relation.

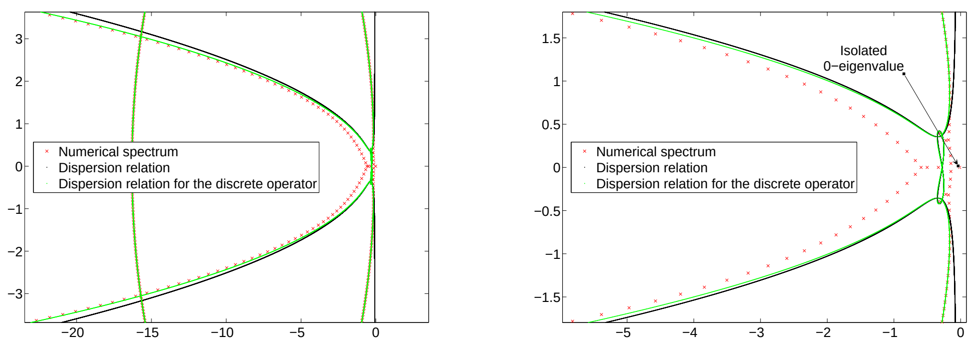


Figure 5.4: Zoom into the origin for the spectra from Figure 5.3.

One obtains quite different pictures if one uses Dirichlet-characteristic boundary conditions for the numerical spectrum as one sees in Figure 5.5. This phenomenon is analyzed in [SS00]. There the authors show that under so called separated boundary conditions the spectrum of the finite interval operator approaches the

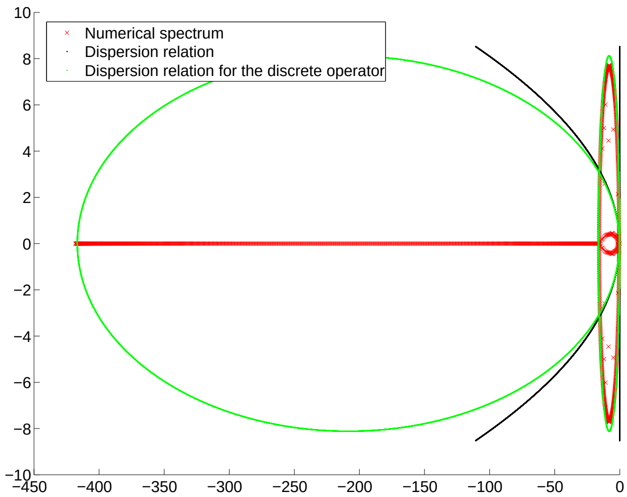


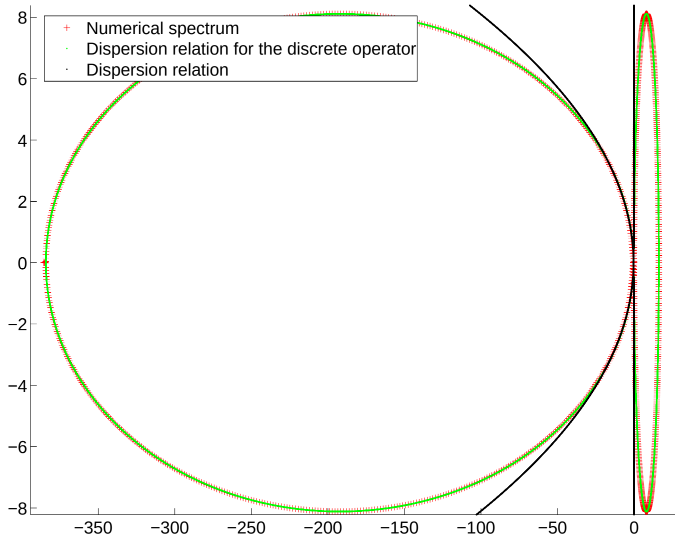
Figure 5.5: Dispersion relations and the numerical spectrum, downwind, Dirichlet-characteristic bc.

absolute spectrum*. They also analyzed the behavior of the spectrum under periodic boundary conditions and show that under certain conditions the spectrum of the finite interval operator approaches the spectral set σ_Δ.

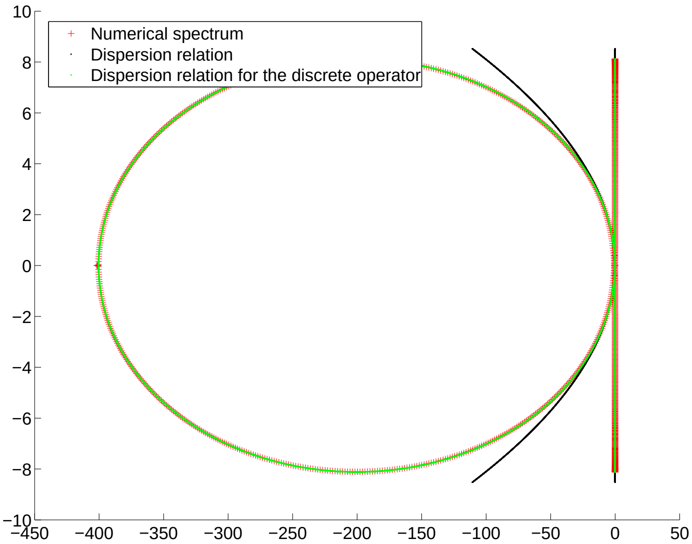
Not only the boundary conditions, but also the choice of discretization have a big influence on the spectrum. This can be seen if one compares Figures 5.3 and 5.6, where we chose the constant interval $[0, 65]$, constant stepsize 0.1, and always periodic boundary conditions, but varied the discretization of the derivative. Note that it is nicely seen that in the case of the "wrong" choice of discretization, i.e. upwind in the case $c < 0$, one obtains an operator with spectrum to the right of the imaginary axis although the original operator does not have spectrum in this region. This leads to an unstable steady state of the semi-discretization.

Finally, for fixed step-sizes we computed the convergence of the eigenvalue with the largest real part. We have used up- respectively downwind for the discretization of the first derivative and Dirichlet-characteristic boundary conditions. In the case of the stable pulse this eigenvalue corresponds to the zero eigenvalue and in the case of the unstable pulse it corresponds to the unstable eigenvalue. The results can be seen in Figure 5.7. One observes that the eigenvalues seem to converge exponentially, but the (numerical) limit is different from the limit of the continuous problem. This effect is due to discretization errors. For the stable pulses we have chosen the intervals J symmetric around the grid-point where $\bar{u}$ is maximal, so that we can compare the results for the two approximations for different step-sizes. For the unstable pulse we are gone three times as fast to the right than to the left, what was intended to reflect the profile of the pulse, see Figure 5.2.]

*See definition 3.5 in [SS00].

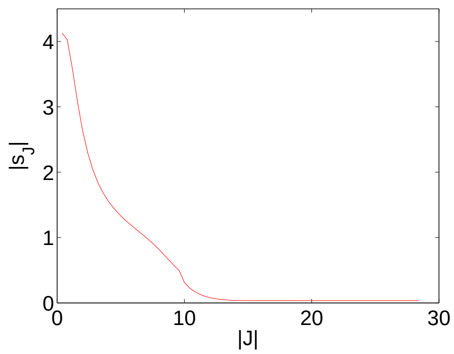


(a) Upwind although $c < 0$.

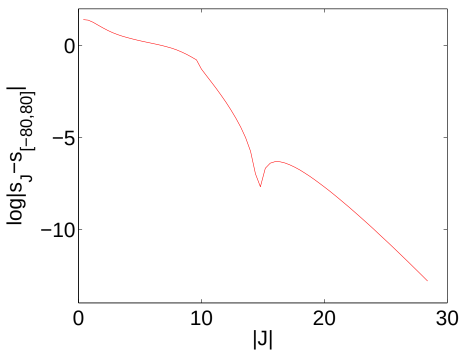


(b) Central differences.

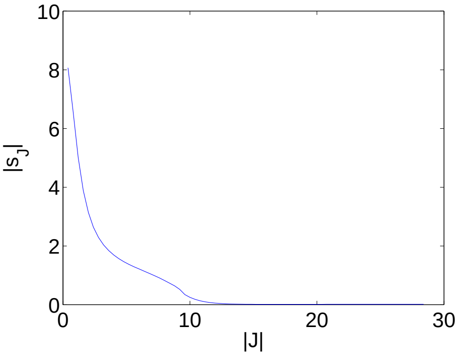
Figure 5.6: The spectrum of the numerical operator for different possibilities of discretizing the first derivative.



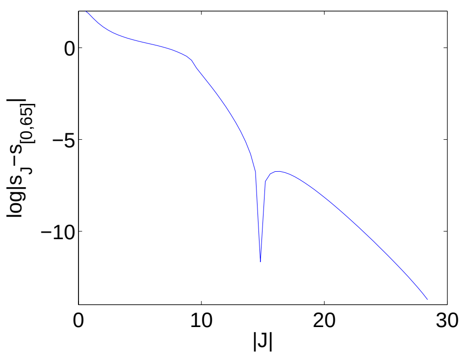
(a) Convergence of the zero eigenvalue for the approximation of the wave profile obtained by V. Thümmler ($c \approx 0.8126$, $h = 0.2$). Here we plot the absolute value of the eigenvalue closest to zero.



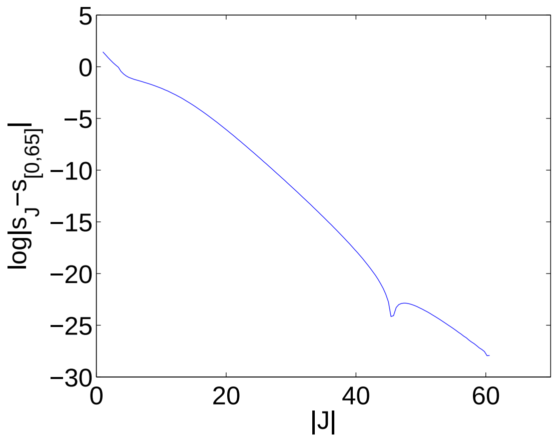
(b) One sees exponential convergence if one compares the eigenvalues closest to zero with the eigenvalue closest to zero of the numerical operator on the interval $[-80, 80]$.



(c) Convergence of the zero eigenvalue for the approximation of the wave profile obtained by C. Nölker ($c = -0.8117$, $h = 0.1$).



(d) Convergence of the 0 eigenvalue. Here we plot the logarithm of the distance of the (numerical) eigenvalue on the interval $[0, 65]$ with an interval J of the length $|J|$.



(e) Convergence of the unstable eigenvalue for the approximation of the profile of the unstable pulse computed by C. Nölker ($c = -0.5414$, $h = 0.1$)) to the unstable eigenvalue for the interval $[0, 65]$.

Figure 5.7: Convergence of the isolated eigenvalues.

A Perturbation theory

An easy, but very useful result which one obtains with the help of a Neumann-series argument is Lemma A.1. We do not give the proof.

Lemma A.1 (Banach-Lemma)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Assume that $A : X \to Y$ is a linear homeomorphism.*

Then for every bounded linear operator $B : X \to Y$ with

$$\|B\|_{X \to Y} < \frac{1}{\|A^{-1}\|_{Y \to X}}$$

the operator $A + B : X \to Y$ is a linear homeomorphism and

$$\left\|(A+B)^{-1}\right\|_{Y \to X} \leq \left\|A^{-1}\right\|_{Y \to X} \frac{1}{1 - \|A^{-1}\|_{Y \to X} \|B\|_{X \to Y}}.$$

We apply Lemma A.1 for the proof of two easy results about perturbations of continuous projectors.

Lemma A.2. *Let $(X, \|\cdot\|)$ be a Banach space, let P and Q be two continuous projectors on X. If $\|P - Q\| < 1$ then $\mathcal{R}(P) = \mathcal{R}(PQ)$.*

Proof. The Banach-Lemma A.1 implies that $I - (P - Q)$ is a linear homeomorphism so that $\mathcal{R}(I - (P - Q)) = X$. This shows $\mathcal{R}(P(I - (P - Q))) = \mathcal{R}(P)$, but $P(I - (P - Q)) = PQ$ which proves the Lemma. □

Corollary A.3. *Let $P : \mathbb{R} \to \mathbb{C}^{l,l}$ be continuously parametrized projectors. Then we have*

$$\dim \mathcal{R}(P(x)) = \dim \mathcal{R}(P(y)) \ \forall x, y \in \mathbb{R}.$$

Proof. This follows from Lemma A.2, since

$$\mathcal{R}\,P = \mathcal{R}\,PQ \ \Rightarrow \dim \mathcal{R}\,P \leq \dim \mathcal{R}\,Q.$$

Symmetry implies $\dim \mathcal{R}\,Q \leq \dim \mathcal{R}\,P$ and this shows that $x \mapsto \dim \mathcal{R}(P(x))$ is locally constant. Finally, since $\mathbb{R}$ is connected, the map must be constant. □

The next Lemma is mainly taken from [Bey90]. It states that for smoothly parametrized hyperbolic matrices the bases of the stable and unstable subspaces can be chosen as smooth in the parameter as the matrices are. In the paper [Bey90] it is only presented for matrix-valued functions from the category $\mathcal{C}^k$ ($k < \infty$), but the proof also holds in the analytic case since the implicit function Theorem conserves analytic dependence. For an analytic version of the implicit function Theorem see [Hen81, p. 15].

Lemma A.4 ([Bey90, Appendix C]). *Let $\Sigma \subset \mathbb{C}$ and assume $M : \Sigma \to \mathbb{C}^{l,l}$ is an analytic matrix-valued function and there is some $s_0 \in \Sigma$ such that $M(s_0)$ is a hyperbolic matrix. Then there is an open neighborhood $U \subset \Sigma$ of s_0 and there are analytic matrix functions $B_u : U \to \mathbb{C}^{l,l_u}$, $B_s : U \to \mathbb{C}^{l,l_s}$, $\Lambda_u : U \to \mathbb{C}^{l_u,l_u}$, and $\Lambda_s : U \to \mathbb{C}^{l_s,l_s}$, where $B_s(s)$ and $B_u(s)$ are bases of the stable and unstable subspaces of $M(s)$ for $s \in \Sigma$, respectively. The matrices satisfy*

$$M(s)B_s(s) = B_s(s)\Lambda_s(s)$$

and

$$\Lambda_u(s)$$

where $\Lambda_s(s)$ is a matrix whose spectrum coincides with the stable spectrum of $M(s)$ and $\Lambda_u(s)$ is a matrix whose spectrum coincides with the unstable spectrum of $M(s)$.

Next we show two results about matrices for which the diagonal elements are large compared to the outer diagonal elements. Both lemmas heavily rely on a gap-condition of the diagonal entries.

In the proofs of the lemmas we will use the following theorem. It is a combination of Theorem 3 and Theorem 4 from [Wil65, p.71].

Until the end of this section we will use the maximum norm $|v|_\infty = \max_j |v_j|$ for vectors in $\mathbb{C}^l$. The matrix norm $|M|_\infty$ is the corresponding operator norm and we will use $\|M\|_\infty = \sup_x |M(x)|_\infty$ for matrix-valued functions M. Note that we usually use the Euclidean norm in $\mathbb{C}^l$, but since all norms are equivalent in a finite dimensional vector space this only introduces constant factors.

Theorem A.5 (Gershgorin). *Let $A \in \mathbb{C}^{l,l}$. Every eigenvalue of A lies in at least one of the disks*

$$\mathcal{D}_i := \{\lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{i \neq j} |a_{ij}|\},\ i = 1 \ldots, l.$$

If k of these disks form a connected domain in $\mathbb{C}$ which is isolated from all other disks, then there are exactly k eigenvalues of A within this domain.

The disks $\mathcal{D}_i$ defined in Theorem A.5 are sometimes called **Gershgorin disks**.

Lemma A.6. *Let $D, E \in \mathbb{C}^{l,l}$ with $D = \operatorname{diag}(d_1, \ldots, d_l)$, where $|d_i - d_j| \geq \delta_0 > 0$, $\forall i, j = 1, \ldots, l$, $i \neq j$. Then there is a positive constant C_0 such that for all $s \in \mathbb{C}$ with $|s| > C_0$ the eigenvalues λ_k, $k = 1, \ldots, l$ of $sD + E$ can be sorted so that they can be estimated in the form*

$$|\lambda_k - (sd_k + e_{kk})| \leq c\frac{1}{|s|},\ k = 1, \ldots, l, \tag{A.1}$$

where c does not depend on s.

The idea of the proof is a scaling trick that shrinks some Gershgorin disks and increases others so that they still do not overlap.

Proof. Take $C_0 := \frac{5|E|_\infty}{\delta_0}$ and let $0 < \varepsilon = \varepsilon(s) < \frac{1}{2}$ which will be specified later.

Define $R_k := \mathrm{diag}(r_1, \dots, r_l)$ where $r_k = \varepsilon$ and $r_j = 1$ for $j \neq k$. The matrix $M := sD + E$ has the same eigenvalues as the matrix

$$(m_{ij}^k)_{ij} := M^k := R_k M R_k^{-1}.$$

The elements of the matrices M and M^k are related in the form

$$m_{ij}^k = \begin{cases} \varepsilon m_{kj} & , \text{ if } i = k, j \neq k, \\ \varepsilon^{-1} m_{ik} & , \text{ if } j = k, i \neq k, \\ m_{ij} & , \text{ otherwise.} \end{cases}$$

This implies for the Gershgorin disks $\mathcal{D}_i^k$ of M^k

$$\begin{aligned} \mathcal{D}_i^k &= \{\lambda \in \mathbb{C} : |\lambda - m_{ii}^k| \leq \sum_{j \neq i} |m_{ij}^k|\} \\ &\subset \{\lambda \in \mathbb{C} : |\lambda - (sd_i + e_{ii})| \leq \varepsilon^{-1} |E|_\infty\}. \end{aligned}$$

if $i \neq k$. In the case $i = k$ one obtains

$$\begin{aligned} \mathcal{D}_k^k &= \{\lambda \in \mathbb{C} : |\lambda - m_{kk}^k| \leq \sum_{j \neq k} |m_{kj}^k|\} \\ &\subset \{\lambda \in \mathbb{C} : |\lambda - (sd_k + e_{kk})| \leq \varepsilon |E|_\infty\}. \end{aligned}$$

Let $\lambda \in \mathcal{D}_i^k$ with $i \neq k$. We can estimate the distance of λ to the center of $\mathcal{D}_k^k$ by

$$\begin{aligned} |\lambda - m_{kk}^k| &\geq |m_{kk}^k - m_{ii}^k| - |\lambda - m_{ii}^k| \\ &\geq |sd_k + e_{kk} - sd_i - e_{ii}| - \varepsilon^{-1} |E|_\infty \\ &\geq |s|\delta_0 - 2|E|_\infty - \varepsilon^{-1} |E|_\infty. \end{aligned}$$

Now let $|s| > C_0$ and choose $\varepsilon = \frac{|E|_\infty}{|s|\delta_0 - 3|E|_\infty}$. Then it follows

$$\begin{aligned} |\lambda - m_{kk}^k| &\geq |s|\delta_0 - 2|E|_\infty - |s|\delta_0 + 3|E|_\infty \\ &= |E|_\infty > \frac{1}{2}|E|_\infty \geq \varepsilon |E|_\infty \end{aligned}$$

which proves $\mathcal{D}_i^k \cap \mathcal{D}_k^k = \emptyset$, $\forall i \neq k$. Therefore Theorem A.5 shows that there is exactly one eigenvalue λ_k of M in the disk $\mathcal{D}_k^k$ and the choice of $\varepsilon(s)$ implies the estimate

$$|\lambda_k - (sd_k + e_{kk})| \leq \frac{|E|_\infty^2}{|s|\delta_0 - 3|E|_\infty} \leq \frac{5}{2\delta_0} |E|_\infty^2 \frac{1}{|s|}$$

for this eigenvalue.

Since the choice of ε and C_0 was independent of k the Lemma follows. □

Next we state a result that is essential for the proof of the Theorems 3.1 and 3.2. It is used to show that one can smoothly (in the parameter s) diagonalize the matrix $sB^{-1} - B^{-1}C$ from (3.18).

Lemma A.7. *Let $D, E \in \mathcal{C}^1(\mathbb{R}, \mathbb{C}^{l,l})$ where D is a diagonal matrix with a uniform gap condition, i.e. there is a $\delta_0 > 0$ such that*

$$|d_i(x) - d_j(x)| > \delta_0 \ \forall x \in \mathbb{R}. \tag{A.2}$$

Furthermore, we assume

$$\|D\|_\infty =: C_{d,0} < \infty, \ \|D_x\|_\infty =: C_{d,1} < \infty, \tag{A.3}$$

and

$$\|E\|_\infty =: C_{e,0} < \infty, \ \|E_x\|_\infty =: C_{e,1} < \infty. \tag{A.4}$$

Then there exist $\varepsilon > 0$ and $T \in \mathcal{C}(\mathbb{R} \times \{|s| < \varepsilon\}, \mathbb{C}^{l,l})$ of the form

$$T(x,s) = I + sT_1(x,s) \tag{A.5}$$

with the property that for all $(x,s) \in \mathbb{R} \times \{|s| < \varepsilon\} =: G_\varepsilon$ the inverse $T(x,s)^{-1}$ exists and satisfies for all $(x,s) \in G_\varepsilon$

$$T(x,s)^{-1}(D(x) + sE(x))T(x,s) = \Lambda(x,s) = \operatorname{diag}(\lambda_{11}(x,s), \ldots, \lambda_{ll}(x,s)). \tag{A.6}$$

Moreover, the matrix-valued function T_1 is differentiable with respect to x for all $(x,s) \in \mathbb{R} \times \{|s| < \varepsilon\}$ and can be estimated in the form

$$\|T_1\|_\infty =: C_{T,0} < \infty, \ \|T_{1,x}\|_\infty =: C_{T,1} < \infty. \tag{A.7}$$

Proof. Choose $\varepsilon_0 := \frac{\delta_0}{4}$ and $\varepsilon := \frac{\varepsilon_0}{3C_{e,0}}$ and define

$$G := G_\varepsilon, \qquad M(x,s) := D(x) + sE(x).$$

From Theorem A.5 follows that for every $(x,s) \in G$ there is exactly one eigenvalue inside each of the disks

$$\begin{aligned}
\mathcal{D}_j(x,s) &= \{\lambda \in \mathbb{C} : |\lambda - d_j(x) - se_{jj}(x)| \le |s| \sum_{i \neq j} |e_{ji}(x)|\} \\
&\subset \{\lambda \in \mathbb{C} : |\lambda - d_j(x)| \le |s| \sum_{i} |e_{ji}(x)|\} \\
&\subset \{\lambda \in \mathbb{C} : |\lambda - d_j(x)| \le \frac{\varepsilon_0}{3}\} \\
&\subset \{\lambda \in \mathbb{C} : |\lambda - d_j(x)| \le \frac{\delta_0}{3} = \frac{4}{3}\varepsilon_0\} =: G_j(x),
\end{aligned} \tag{A.8}$$

since for $i \neq j$ it holds $\mathcal{D}_i(x,s) \cap \mathcal{D}_j(x,s) \subset G_i(x) \cap G_j(x) = \emptyset$.

Denote by $\Gamma_j(x)$ the positively oriented contour with winding number 1 on the boundary of $G_j(x)$. Since $\mathcal{D}_j(x,s)$ lies in the interior of $\Gamma_j(x)$ and there is no spectrum of $M(x,s)$ on $\Gamma_j(x)$ one can define the Riesz-projectors (cf. [Kat66, I §5.3], [BSU96, Ch. 10 4.2], and for the general case of closed operators [Kat66, III §6.4])

$$\Pi_j(x,s) := \frac{1}{2\pi i} \int_{\Gamma_j(x)} (zI - D(x) - sE(x))^{-1} dz, \ (x,s) \in G, \ j = 1, \ldots, l.$$

The matrix $\Pi_j(x,s)$ projects $\mathbb{C}^l$ onto the eigenspace of $M(x,s)$ corresponding to the eigenvalue $\lambda_j(x,s)$ of $M(x,s)$ that lies inside $G_j(x)$.

To see the continuity of the projector-valued functions in (x,s) note that we have

$$\begin{aligned}
&|\big(zI - D(x_0) - s_0E(x_0)\big)^{-1} - \big(zI - D(x) - sE(x)\big)^{-1}| \\
&\le |\big(zI - D(x_0) - s_0E(x_0)\big)^{-1}| \cdot |D(x) - D(x_0) + sE(x) - s_0E(x_0)| \\
&\quad \cdot |\big(zI - D(x) - sE(x)\big)^{-1}| \\
&\xrightarrow[(x,s)\to(x_0,s_0)]{} 0,
\end{aligned} \tag{A.9}$$

where we used the equality

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1} \tag{A.10}$$

for invertible operators A and B on a Banach space, the Banach-Lemma A.1, and the continuity of the matrix-valued functions E, D, and M.

Let $(x_0, s_0) \in G$ be given. The continuity of D implies that there is an $\eta > 0$ such that for all $(x,s) \in G$ with $|x - x_0| < \eta$ it holds

$$\mathcal{D}_j(x,s) \subset \{\lambda \in \mathbb{C} : |\lambda - d_j(x_0)| \le \frac{\varepsilon_0}{2}\} \subset \{\lambda \in \mathbb{C} : |\lambda - d_j(x_0)| \le \varepsilon_0\} \subset G_j(x_0).$$

Thus the Cauchy-Integral-Formula implies that one can take the same contour for $\Pi_j(x,s)$ and $\Pi_j(x_0,s_0)$. Together with (A.9) this implies

$$\begin{aligned}
&|\Pi_j(x,s) - \Pi_j(x_0,s_0)| \\
&\quad = |\frac{1}{2\pi i}\int_{\Gamma_j(x_0)} \big(zI_D(x) - sE(x)\big)^{-1} - \big(zI - D(x_0) - s_0E(x_0)\big)^{-1} dz| \\
&\qquad\qquad \to 0 \text{ as } (x,s) \to (x_0,s_0)
\end{aligned}$$

which proves the continuity.

Show that $\Pi_j(x,s)e_j \neq 0$, where e_j denotes the j-th unit vector.
From $\Pi_j(x,0)e_j = e_j$ we obtain

$$\begin{aligned}
|\Pi_j(x,s)e_j - e_j|_\infty &= |\frac{1}{2\pi i}\int_{\Gamma_j(x)} (zI - D(x) - sE(x))^{-1} - (zI - D(x))^{-1} dz e_j|_\infty \\
&\le \frac{4}{3}\varepsilon_0 |(I - (zI - D(x))^{-1}sE(x))^{-1} - I|_\infty |(zI - D(x))^{-1}|_\infty.
\end{aligned}$$

In order to estimate the integral we analyze the integrand and find

$$\begin{aligned}
|(zI - D(x))^{-1}sE(x)|_\infty &\le |s||(zI - D(x))^{-1}|_\infty |E(x)|_\infty \\
&\le \frac{\varepsilon_0}{3C_{e,0}} \max_{j=1,\dots,l} |\frac{1}{z - d_j(x)}| C_{e,0} \\
&\le \frac{\varepsilon_0}{3C_{e,0}} \frac{1}{\varepsilon_0} C_{e,0} = \frac{1}{3}.
\end{aligned}$$

Then Lemma A.1 implies that $(I-(zI-D(x))^{-1}sE(x))^{-1}$ exists and the integrand can be estimated by

$$\left|\big(I-(zI-D(x))^{-1}sE(x)\big)^{-1}-I\right|_\infty \le \sum_{j=1}^{\infty}\left|(zI-D(x))^{-1}sE(x)\right|^j \le \sum_{j=1}^{\infty}\big(\frac{1}{3}\big)^j = \frac{1}{2}.$$

Therefore

$$\begin{aligned}\left|\Pi_j(x,s)e_j - e_j\right|_\infty &\le \frac{1}{2\pi}2\pi\frac{\delta_0}{3}\frac{1}{2}\left|(zI-D(x))^{-1}\right|_\infty \\ &\le \frac{1}{2\pi}2\pi\frac{\delta_0}{3}\cdot\frac{1}{2}\cdot\frac{4}{\delta_0} = \frac{2}{3} < 1\end{aligned} \tag{A.11}$$

and so $\Pi_j(x,s)e_j \neq 0$.

Define

$$T(x,s) := \big[\Pi_1(x,s)e_1,\dots,\Pi_l(x,s)e_l\big] \in \mathbb{C}^{l,l}. \tag{A.12}$$

We see $T(x,s)\in \mathrm{GL}_l(\mathbb{C})$ since the columns $\Pi_i(x,s)e_i$ are nonzero eigenvectors to pairwise different eigenvalues and hence they form a basis of $\mathbb{C}^l$. The continuity of the columns implies $T\in\mathcal{C}(G,\mathrm{GL}_l(\mathbb{C}))$.

Now set $T_1(x,s) = \big[T_{1,1}(x,s),\dots,T_{1,l}(x,s)\big]$, where the columns are defined by

$$T_{1,j}(x,s) := \frac{1}{2\pi i}\int_{\Gamma_j(x)}\big(zI-D(x)-sE(x)\big)^{-1}E(x)\big(zI-D(x)\big)^{-1}dz\,e_j,\ j=1,\dots,l. \tag{A.13}$$

Then the equality

$$\begin{aligned}T(x,s)e_j - e_j &= \Pi_j(x,s)e_j - e_j \\ &= \frac{1}{2\pi i}\int_{\Gamma_j(x)}\big(zI-D(x)-sE(x)\big)^{-1} - \big(zI-D(x)\big)^{-1}dze_j \\ &= \frac{s}{2\pi i}\int_{\Gamma_j(x)}\big(zI-D(x)-sE(x)\big)^{-1}E(x)\big(zI-D(x)\big)^{-1}dze_j\end{aligned}$$

implies $T(x,s) = I + sT_1(x,s)$. Here we used (A.10) again.

It remains to show the differentiability of T_1 with respect to x and the bounds asserted in (A.7).

For $j=1,\dots,l$, it holds

$$|T_{1,j}(x,s)|_\infty \le \frac{1}{2\pi}\frac{\delta_0}{3}2\pi\left|\big(zI-D(x)-sE(x)\big)^{-1}\right|_\infty\left|E(x)\right|_\infty\left|\big(zI-D(x)\big)^{-1}\right|_\infty.$$

One can estimate the factors uniformly for $(x,s)\in G$

$$\begin{aligned}\left|\big(zI-D(x)-sE(x)\big)^{-1}\right|_\infty &\le \left|\big(I-\big(zI-D(x)\big)^{-1}sE(x)\big)^{-1}\right|_\infty\left|\big(zI-D(x)\big)^{-1}\right|_\infty \\ &\le \frac{3}{2}\max_{l=1,\dots,l}\left|\frac{1}{z-d_l(x)}\right| \\ &\le \frac{6}{\delta_0}, \\ \left|E(x)\right|_\infty &\le C_{e,0}, \\ \left|\big(zI-D(x)\big)^{-1}\right|_\infty &\le \frac{4}{\delta_0}\end{aligned}$$

and this shows the estimate uniform in $(x,s)\in G$

$$|T_{1,j}(x,s)|_\infty \le \frac{1}{2\pi}2\pi\frac{\delta_0}{3}\frac{6}{\delta_0}C_{e,0}\frac{4}{\delta_0} = \frac{8C_{e,0}}{\delta_0} < \infty.$$

Because this inequality holds for every $(x,s)\in G$ and every $j=1,\dots,l$, the first bound in (A.7) follows.

In the last step we show the differentiability of T_1 with respect to x and the uniform boundedness of the derivative. Let $x_0\in\mathbb{R}$ be arbitrary. With the same $\eta>0$ as above we can choose the same contour for all $(x,s)\in G$ with $|x-x_0|\le\eta$.

To justify differentiation under the integral sign with respect to x we must estimate the derivative of the integrand uniformly in $z\in\Gamma_j(x_0)$ for all $(x,s)\in G$ with $|x-x_0|<\eta$.

For every $z\in\Gamma_j(x_0)$ and $(x,s)\in G$ with $|x-x_0|<\eta$ hold

$$\frac{d}{dx}E(x) = E_x(x) \tag{A.14}$$

$$\frac{d}{dx}\big(zI-D(x)-sE(x)\big)^{-1} = \big(zI-D(x)-sE(x)\big)^{-1} \cdot \big(D_x(x)+sE_x(x)\big)\big(zI-D(x)-sE(x)\big)^{-1} \tag{A.15}$$

$$\frac{d}{dx}\big(zI-D(x)\big)^{-1} = \big(zI-D(x)\big)^{-1}D_x(x)\big(zI-D(x)\big)^{-1}. \tag{A.16}$$

The bounds of the matrices and their derivatives assumed in (A.3) and (A.4) lead to the following bounds, independent of $z\in\Gamma_j(x_0)$ and $(x,s)\in G$ with $|x-x_0|<\eta$,

$$\big|D_x(x)+sE_x(x)\big|_\infty \le C_{d,1}+\frac{\varepsilon_0}{3C_{e,0}}C_{e,1} < \infty, \tag{A.17}$$

$$\Big|\big(zI-D(x)\big)^{-1}\Big| \le \frac{2}{\varepsilon_0} = \frac{8}{\delta_0} < \infty, \tag{A.18}$$

$$\Big|\big(zI-D(x)-sE(x)\big)^{-1}\Big|_\infty \le \frac{6}{\varepsilon_0} = \frac{24}{\delta_0} < \infty, \tag{A.19}$$

where (A.18) and the Banach-Lemma A.1 is used for (A.19).

The equations (A.3), (A.4), and (A.17)–(A.19) imply the estimate

$$\Big|\frac{d}{dx}\Big(\big(zI-D(x)-sE(x)\big)^{-1}E(x)\big(zI-D(x)\big)^{-1}\Big)\Big|_\infty \le \text{const} < \infty \tag{A.20}$$

with a constant const independent of $z\in\Gamma_j(x_0)$ and $(x,s)\in G$ with $|x-x_0|<\eta$. Thus by [Bau92, §16 Lemma 2] we can differentiate under the integral sign and obtain

$$\frac{d}{dx}T_{1,j}(x_0,s) = \frac{1}{2\pi i}\int_{\Gamma_j(x_0)}\frac{d}{dx}\Big\{\big(zI-D(x)sE(x)\big)^{-1}E(x)\big(zI-D(x)\big)^{-1}\Big\}\Big|_{x=x_0} dze_j. \tag{A.21}$$

The estimate (A.20) also implies $|\frac{d}{dx}T_{1,j}(x_0,s_0)| \le \text{const} < \infty$ with a constant independent of $(x_0,s_0)\in G$. From this follows the existence of a $C_{T,1}<\infty$ with

$$\|T_{1,x}\|_\infty \le C_{T,1} \qquad \forall (x,s)\in G.$$

□

Remark. *From the construction of* $T(x,s)$ *we see that the entries* $\lambda_{ii}(x,s)$, $i = 1, \ldots, l$, *of* $\Lambda(x,s)$ *from* (A.6) *satisfy*

$$|\lambda_{ii}(x,s) - d_i(x)| = \min_j |\lambda_{ii}(x,s) - d_j(x)|, \ i = 1, \ldots, l.$$

B Exponential dichotomies

In this section we give the definition of an exponential dichotomy and some results which are essentially used in the proofs of Chapters 3 and 4. The results are mainly presented in [BL99, Appendix] other references are [Cop78] and the papers by Palmer [Pal84], [Pal88].

We will use in this section the notation $\langle x_-, x_+ \rangle$ for intervals in $\mathbb{R}$. Where $\langle x_-, x_+ \rangle$ with $x_- \in \mathbb{R} \cup \{-\infty\}$ and $x_+ \in \mathbb{R} \cup \{\infty\}$ is defined by

$$\langle x_-, x_+ \rangle = \begin{cases} \emptyset, & x_- > x_+, \\ [x_-, x_+], & x_- \in \mathbb{R},\, x_+ \in \mathbb{R},\, x_- \leq x_+, \\ (-\infty, x_+], & x_- = -\infty,\, x_+ \in \mathbb{R}, \\ [x_-, \infty), & x_- \in \mathbb{R},\, x_+ = \infty, \\ \mathbb{R}, & x_- = -\infty,\, x_+ = \infty. \end{cases}$$

Let $M \in \mathcal{C}(J, \mathbb{C}^{l,l})$. We denote by $S(\cdot,\cdot)$ the solution-operator of

$$Lz = z_x - M(x)z, \qquad x \in J. \tag{B.1}$$

First we give the definition of an exponential dichotomy.

Definition B.1. We say, that the operator L has an exponential dichotomy (ED) on J if there are positive constants K, β and for every $x \in J$ there is a projection $\pi(x) : \mathbb{C}^l \to \mathbb{C}^l$ such that

$$\begin{aligned} S(x,y)\pi(y) &= \pi(x)S(x,y) && \forall x, y \in J, \\ |S(x,y)\pi(y)| &\leq Ke^{-\beta(x-y)} && \forall x \geq y \in J, \\ |S(x,y)(I - \pi(y))| &\leq Ke^{-\beta(y-x)} && \forall x < y \in J. \end{aligned}$$

We call $(K, \beta, \pi(\cdot))$ the data of the dichotomy and refer to K as dichotomy constant, β as dichotomy exponent, and π as projectors of the dichotomy.

The data of the dichotomy are in general not unique, as one easily sees if J is a finite interval. The benefit of (ED)s lie in semi-infinite or infinite interval problems. If $J = \langle x_-, x_+ \rangle$ contains an interval of the form $[x_0, \infty)$, the range of the projectors are unique and if it contains an interval of the form $(-\infty, x_0]$, the kernel of the projectors is unique. In particular, if the operator L has an (ED) on the whole real line, the projectors are uniquely determined. For results in this direction see [Cop78, Chapter 2].

Example 4. *The equation* $z_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z$ *on* $\mathbb{R}$ *has an (ED) on* $\mathbb{R}$ *and the data can be chosen as* $K = 1$, $\beta = 1$, $\pi(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Next we state a result about the solvability solution estimates for boundary value problems in the presence of an (ED).

Theorem B.2 ([BL99, Theorem A.1]). *Assume the operator L has an (ED) on $J = \langle x_-, x_+ \rangle$, $x_- < x_+$, with data (K, β, π).*

Define the Green's function G with respect to π for all $x, y \in J$ by

$$G(x,y) = \begin{cases} S(x,y)\pi(y), & y \le x \\ S(x,y)(\pi(y) - I), & x < y. \end{cases} \tag{B.2}$$

Then for every $h \in L_2(J)$, $\gamma_- \in \mathcal{R}(\pi(x_-))$, $\gamma_+ \in \mathcal{R}(I - \pi(x_+))$ there is a unique solution $z \in H^1(J, \mathbb{C}^l)$ of the boundary value problem

$$\begin{aligned} Lz &= h, \quad \textit{in } L_2(J), \\ (I - \pi(x_+))z(x_+) &= \gamma_+, \\ \pi(x_-)z(x_-) &= \gamma_-. \end{aligned}$$

In the case $x_- = -\infty$ the boundary condition for $z(x_-)$ is hidden in the space and there is no explicit boundary condition. The same is true for the case $x_+ = +\infty$. The solution can be written in the form $z = z_{sp} + z_h$, where z_{sp} and z_h are given by

$$z_{sp}(x) = \int_J G(x,y)h(y)dy \tag{B.3}$$

and

$$z_h(x) = S(x, x_-)\gamma_- + S(x, x_+)\gamma_+. \tag{B.4}$$

For the particular solution of the inhomogeneous equation given by (B.3) *the following estimate holds*

$$\beta^2 \|z_{sp}\|^2 + \beta |z_{sp}|_\Gamma^2 \le 5K^2 \|h\|^2. \tag{B.5}$$

The solution of the homogeneous equation with inhomogeneous boundary conditions z_h from (B.4) *satisfies the estimate*

$$\beta \|z_h\|^2 + |z_h|_\Gamma^2 \le (2 + 3K^2)(|\gamma_-|^2 + |\gamma_+|^2). \tag{B.6}$$

Recall that $G(x,y)$ is the Green's function for $Lz = h$ in $\mathbb{R}$ with $z(-\infty) = 0$, $z(+\infty) = 0$.

The next theorem, presented in [BL99, Theorem A.2], shows that the solution of an initial value problem is exponentially decaying in forward time if the initial data comes from the right subspace and the inhomogeneity is also exponentially decaying.

Theorem B.3. *Let L have an (ED) on $J = [0, \infty)$ with data (K, β, π). Then every solution $z \in H^1(J)$ of the homogeneous initial value problem*

$$Lz = 0 \textit{ in } L_2(J),$$

with initial data $z(0) \in \mathcal{R}(\pi(0))$ *satisfies*

$$|z(x)| \leq Ke^{-\beta x}|z(0)|. \tag{B.7}$$

Let $z \in H^1([0,\infty), \mathbb{C}^l)$ *be the solution of the inhomogeneous initial value problem*

$$Lz = h \text{ in } L_2(J),$$

with initial data $z(0) \in \mathcal{R}\pi(0)$ *and inhomogeneity* $h \in L_2([0,\infty))$ *which is exponentially bounded, i.e.*

$$|h(x)| \leq C_1 e^{-\beta_1 x}$$

with some constants $C_1, \beta_1 > 0$. *Then for every* $0 < \beta' < \min(\beta, \beta_1)$ *there is a constant* $C > 0$, *with* C *depending on* $z(0)$ *and* β' *only such that*

$$|z(x)| \leq C(|\pi(0)z(0)|e^{-\beta x} + e^{-\beta' x})), \ \forall x \geq 0.$$

The same is true for $J = (-\infty, 0]$.

An important property of an (ED) is its roughness under perturbations, which is stated in the next Theorem [BL99, Theorem A.3]. It is an improved version of [Cop78, p. 34].

Theorem B.4. *Let* $\tilde{L}z = z_x - M(\cdot)z$ *have an (ED) on* $J = \langle x_-, x_+ \rangle$, $x_- < x_+$, *with data* $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$. *Assume* $\Delta \in \mathcal{C}(J, \mathbb{C}^{m,m})$ *and this fulfills*

$$3\tilde{K}\|\Delta\|_\infty < \tilde{\beta}. \tag{B.8}$$

Then the operator

$$Lz = z_x - (M + \Delta)z$$

has an (ED) on J, *too. The data* (K, β, π) *of this dichotomy can be chosen so that*

$$\begin{aligned} K &= \tilde{K}(2 + \frac{4\|\Delta\|_\infty \tilde{K}}{\tilde{\beta} - 3\|\Delta\|_\infty \tilde{K}}), \\ \beta &= \tilde{\beta} - 2\|\Delta\|_\infty \tilde{K}, \\ |\tilde{\pi}(x) - \pi(x)| &\leq \tilde{K}K \int_J e^{-(\tilde{\beta}+\beta)|x-y|}|\Delta(y)|dy \end{aligned}$$

are fulfilled.

If the matrices $M(x)$ *and* $\Delta(x)$ *as well as the dichotomy data* $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ *depend continuously on some additional parameter and the inequality* (B.8) *holds for all parameter values, then the dichotomy data of the perturbed operator can be chosen continuously in the parameter.*

Note that the constant K for the perturbed equation differs from the constant given in [BL99]. This reflects that by carrying out the proof indicated in [BL99] we only obtained the weaker estimate.

The strength of this theorem lies in the estimate of the projectors, which shows, that for small $\|\Delta\|_\infty$ the projectors of the unperturbed and of the perturbed operator are close to each other.

Using this Theorem one can proof the following Theorem (cf. [Pal84, Lemma 3.4] and [BL99, Theorem A.4]).

Theorem B.5. *Let $\tilde{L}z = z_x - M(\cdot)z$ have an exponential dichotomy on a semi-infinite interval $J = [x_0, \infty)$ with data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ and let $\Delta \in \mathcal{C}(J, \mathbb{C}^{m,m})$ with*

$$|\Delta(x)| \to 0, \text{ as } x \to \infty.$$

Then for every fixed $0 < \beta < \tilde{\beta}$ the operator $Lz = z_x - (M+\Delta)z$ has an exponential dichotomy on J with data (K, β, π). Moreover for every allowable data (K, β, π) the projector π satisfies

$$|\pi(x) - \tilde{\pi}(x)| \to 0, \text{ as } x \to \infty. \tag{B.9}$$

If the data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$, the matrices $M(x)$, and $\Delta(x)$ depend continuously on some parameter from a compact set, then the perturbed data (K, β, π) can be chosen continuously in this parameter so that the convergence (B.9) *for these projectors is uniform in the parameter.*

The next theorem we extensively use for the proofs of exponential dichotomies of the first order operators, obtained in Chapters 3 and 4, in compact intervals.

Theorem B.6 ([BL99, Theorem A.5])**.** *Let $Lz = z_x - M(x)z$ have exponential dichotomies on $\mathbb{R}_-$ with data (K_-, β_-, π_-) and on $\mathbb{R}_+$ with data (K_+, β_+, π_+). Then L has an (ED) on the whole real line $\mathbb{R}$ if and only if*

$$\mathcal{R}(\pi_+(0)) \oplus \mathcal{R}(I - \pi_-(0)) = \mathbb{C}^l.$$

In this case the data (K, β, π) for the ED on $\mathbb{R}$ can be chosen such that $\beta = \min(\beta_-, \beta_+)$ and

$$|\pi(x) - \pi_\pm(x)| \le Ce^{-2\beta|x|} \text{ for } x \in \mathbb{R}_\pm.$$

We also need a result about the Fredholm properties of ordinary differential operators on the whole real line. On bounded intervals a similar property is easy to verify by integration, but on unbounded domains it is not so easy. A general result is given by K. J. Palmer in [Pal84, Lemma 4.2]. It is presented for bounded and continuously differentiable functions, but the proof directly applies in the case of the Sobolev space H^1.

Lemma B.7. *Let $M \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{l,l})$ be a bounded matrix-valued function so that the differential operator*

$$L(\cdot) : \begin{array}{ccc} H^1(\mathbb{R}, \mathbb{C}^l) & \to & L_2(\mathbb{R}, \mathbb{C}^l) \\ z & \mapsto & z_x - M(\cdot)z \end{array}$$

has an (ED) on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with projectors $\pi_\pm(\cdot)$. Then L is Fredholm and $f \in \mathcal{R}(L)$ if and only if

$$\int_{-\infty}^{\infty} u^*(t) f(t) dt = 0$$

for all solutions $u \in H^1(\mathbb{R}, \mathbb{C}^l)$ of the adjoint equation

$$L^{ad}(\cdot)u = u_x + M(\cdot)^* u = 0.$$

Furthermore the index of L is $\dim \mathcal{R}(\pi_+(0)) + \dim \mathcal{R}(I - \pi_-(0)) - n$.

In the Lemma '*' stands for transposed conjugated.

Remark. *It is shown in [Pal88] that also the opposite direction holds, i.e. a differential operator of the form considered in Lemma B.7 that is semi-Fredholm has (ED)s on $\mathbb{R}_+$ and on $\mathbb{R}_-$.*

When we show the Fredholm property for certain linear first order differential operators obtained after transformation in Chapter 4 we need a slight modification of the domain and image spaces in Lemma B.7. Therefore we will use the characterization of the co-range to show a corollary of this Lemma.

Corollary B.8. *Let $M = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$ be an $l \times l$ matrix-valued function with the same properties as in Lemma B.7 and assume that $A \in \mathbb{C}^{r,l-r}$ is constant.*
Then the ordinary differential operator

$$L_a : H^2(\mathbb{R}, \mathbb{C}^r) \times H^1(\mathbb{R}, \mathbb{C}^{l-r}) \to H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r}),\ z \mapsto z_x - M(\cdot)z,$$

is a Fredholm operator of the same index as

$$L_b : H^1(\mathbb{R}, \mathbb{C}^l) \to L_2(\mathbb{R}, \mathbb{C}^l),\ z \mapsto z_x - M(\cdot)z.$$

Proof. Let $(u, v) \in \mathcal{N}(L_b) \subset H^1(\mathbb{R}, \mathbb{C}^l)$. Then $u_x = Av$, where the equality holds in $L_2(\mathbb{R}, \mathbb{C}^r)$. Since the right hand side is an element of $H^1(\mathbb{R}, \mathbb{C}^r)$ the equality also holds in $H^1(\mathbb{R}, \mathbb{C}^r)$. Hence $(u, v) \in H^2(\mathbb{R}, \mathbb{C}^r) \times H^1(\mathbb{R}, \mathbb{C}^{l-r})$ and $(u, v) \in \mathcal{N}(L_a)$. This shows $\mathcal{N}(L_a) = \mathcal{N}(L_b)$ and especially

$$\dim \mathcal{N}(L_a) = \dim \mathcal{N}(L_b).$$

By Lemma B.7 $(g, f) \in \mathcal{R}(L_b)$ if and only if

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} g \\ f \end{pmatrix} \rangle dx = 0 \ \forall \psi \in \mathcal{N}(L^{ad}),$$

where $L^{ad} : H^1(\mathbb{R}, \mathbb{C}^l) \to L_2(\mathbb{R}, \mathbb{C}^l)$ is the same operator as in Lemma B.7.
Let $(g, f) \in \mathcal{R}(L_a) \subset H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r})$. Then

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} g \\ f \end{pmatrix} \rangle dx = 0 \ \forall \psi \in \mathcal{N}(L^{ad}),$$

since $\mathcal{R}(L_a) \subset \mathcal{R}(L_b)$.
Let $(g, f) \in H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r})$ and assume

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} g \\ f \end{pmatrix} \rangle dx = 0 \ \forall \psi \in \mathcal{N}(L^{ad}).$$

By Lemma B.7 there is $(z_1, z_2) \in H^1(\mathbb{R}, \mathbb{C}^r) \times H^1(\mathbb{R}, \mathbb{C}^{l-r})$ with

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x - M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_{1,x} - Az_2 \\ z_{2,x} - Bz_1 - Cz_2 \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix}.$$

This implies that z_1 is an element of $H^2(\mathbb{R}, \mathbb{C}^r)$.

Therefore for $(g, f) \in H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r})$ we have

$$(g, f) \in \mathcal{R}(L_a) \Leftrightarrow \int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} g \\ f \end{pmatrix} \rangle dx = 0 \; \forall \psi \in \mathcal{N}(L^{ad}).$$

Thus $\mathcal{R}(L_a)$ is closed and $\operatorname{codim}(\mathcal{R}(L_a)) = \dim \mathcal{N}(L^{ad}) = \operatorname{codim}(\mathcal{R}(L_b))$, as in the end of the proof of [Pal84, Lemma 4.2]. □

Following an idea from [Thü98], one can use the roughness Theorem B.4 to show uniformity of the data of an (ED) in compact parameter sets for parameter dependent problems.

Lemma B.9. *Let $\Omega \subset \mathbb{C}$ be a compact set. Let $M \in \mathcal{C}(J \times \Omega, \mathbb{C}^{l,l})$ and assume for every $s_0 \in \Omega$ there exist* $\text{const}_{s_0} > 0$ *and $\delta_{s_0} > 0$ such that*

$$\|M(\cdot, s) - M(\cdot, s_0)\|_\infty \le \text{const}_{s_0} |s - s_0| \; \forall s, s_0 \in \Omega \text{ with } |s - s_0| < \delta_{s_0}.$$

Furthermore we assume that the operators $L(\cdot, s)$, as above, have an (ED) on J with data $(K(s), \beta(s), \pi(\cdot, s))$. Then the dichotomy constant $K(s)$ and the dichotomy exponent $\beta(s)$ can be chosen independently of s for all $s \in \Omega$.

Proof. Let $s_0 \in \Omega$ be arbitrary and let $\varepsilon = \varepsilon(s_0) > 0$ so that

$$\|M(\cdot, s_0) - M(\cdot, s)\|_\infty \le \frac{1}{2} \frac{\beta(s_0)}{3K(s_0)} \; \forall s \in K_\varepsilon(s_0) \cap \Omega.$$

Theorem B.4 then implies that for every $s \in \Omega \cap K_\varepsilon(s_0)$ also the operator

$$L(\cdot, s)v = L(\cdot, s_0)v + (M(\cdot, s_0) - M(\cdot, s))v$$

has an (ED) on J with data (K, β, π), which can be estimated by

$$K \le K(s_0) \left(2 + \frac{4\beta(s_0)}{3(\beta(s_0) - \frac{1}{2}\beta(s_0))} \right) = \frac{8}{3} K(s_0) =: K_{s_0}$$

and

$$\beta \ge \beta(s_0) - \frac{\beta(s_0)}{3} = \frac{2}{3}\beta(s_0) =: \beta_{s_0}.$$

Since $s_0 \in \Omega$ was arbitrary the same argumentation is possible for every $s \in \Omega$ and so a compactness argument shows the assertion. □

C Results from functional analysis

Throughout the text we use the following definitions of spectra.

Definition C.1. Let X and Y be complex Banach spaces. Let P be a bounded linear operator from X to Y.

The **resolvent set** $\rho(P)$ of the operator P is defined as the set of all $s \in \mathbb{C}$ for which $sI - P$ is a linear homeomorphism.

The **spectrum** $\sigma(P)$ is defined as $\mathbb{C} \setminus \rho(P)$. We split the spectrum into two parts. The **eigen spectrum** $\sigma_{eig}(P)$ is the set of all $s \in \sigma(P)$ which are isolated points in $\sigma(P)$ and eigenvalues of finite algebraic multiplicity of P. The **essential spectrum** is then defined as $\sigma_{ess} := \sigma(P) \setminus \sigma_{eig}(P)$.

Remark. *In the literature there are quite a few definitions of the essential spectrum. Our definition is taken from [Hen81, p. 136]. Another widely used definition is by defining the* **point spectrum** σ_p *as the set of all* $s \in \sigma$ *for which the operator* $sI - P$ *is Fredholm of index* 0*. The essential spectrum is then defined as the set* $\sigma_\Delta := \sigma \setminus \sigma_p$*. For example this definition is used in [SS00].*

Lemma C.2. *For every compact interval* $J = [x_-, x_+]$, $|x_+ - x_-| \geq 1$, *every function* $f \in H^1(J, \mathbb{C}^l)$ *is an element of* $\mathcal{C}^0(J, \mathbb{C}^l)$ *and satisfies the Sobolev inequality*

$$\|f\|_\infty \leq \text{const}\|f\|_{H^1} \tag{C.1}$$

with const *independent of* J *and* f.

Proof. Let $f \in \mathcal{C}^1(J, \mathbb{C}^l) \cap H^1(J, \mathbb{C}^l)$ and choose $x_m \in J$ with

$$|f(x_m)| = \min_{x \in J} |f(x)|.$$

The Fundamental Theorem of Calculus implies

$$|f(x)|^2 \leq |f(x_m)|^2 + 2\int_{x_m}^{x} |\langle f(\xi), f'(\xi)\rangle| d\xi \leq \frac{1}{|x_+ - x_-|}\|f\|_{L_2}^2 + \|f\|_{L_2}^2 + \|f'\|_{L_2}^2,$$

where $\langle u, v\rangle := u^* v$ is the Euclidean inner product in $\mathbb{C}^l$. Therefore

$$|f(x)| \leq \text{const}\|f\|_{H^1}$$

with const independent of the length of the interval as long as $|x_+ - x_-| \geq 1$ holds. This shows that there is a constant independent of the length of the interval J for $|J| \geq 1$ such that

$$\|f\|_\infty \leq \text{const}\|f\|_{H^1(J,\mathbb{C}^l)}\ \forall f \in \mathcal{C}^1(J, \mathbb{C}^l) \cap H^1(J, \mathbb{C}^l).$$

The density of $\mathcal{C}^1(J, \mathbb{C}^l) \cap H^1(J, \mathbb{C}^l)$ in $H^1(J, \mathbb{C}^l)$ (cf. [Rob01, Theorem 5.21]) is then used to finish the proof. ☐

For H^1-functions on the whole real axis one has a similar result (cf. [Rau91, §2.6 Theorem 7] and [RR93, Theorem 6.91]):

Lemma C.3. *Every $f \in H^1(\mathbb{R}, \mathbb{C}^l)$ is an element of $\mathcal{C}^0(\mathbb{R}, \mathbb{C}^l)$ and satisfies the Sobolev inequality*

$$\|f\|_\infty \leq \text{const}\|f\|_{H^1(\mathbb{R},\mathbb{C}^l)}. \tag{C.2}$$

Furthermore for each element $f \in H^1(\mathbb{R}, \mathbb{C}^l)$ holds $\lim_{x\to\pm\infty} |f(x)| = 0$.

Proof. We only prove the second part. Assume there is $f \in H^1(\mathbb{R}, \mathbb{C}^l)$ such that there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = \infty$, but $|f(x_n)| \geq c > 0$ for all $n \in \mathbb{N}$ and some $c \in \mathbb{R}$. Since $\mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ is dense in $H^1(\mathbb{R}, \mathbb{C}^l)$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ with $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Choose N_0 so large that $\|f_n - f\|_\infty < \frac{c}{2}\ \forall n \geq N_0$. Then for every $n \geq N_0$ there is $n' \in \mathbb{N}$ with $x_{n'} \notin \operatorname{supp} f_n$. And so $\|f_n - f\|_\infty \geq |f_n(x_{n'}) - f(x_{n'})| = |f(x_{n'})| > c$ which contradicts the choice of N_0. Therefore the second assertion follows. ☐

From now on E, F always denote separable Banach spaces and Σ is an open subset of the complex plane.

We next state a quantative version of the contraction mapping Theorem presented in [Vai76, §3 Hilfssatz 18].

Lemma C.4. *Let $\mathcal{A} : \Omega \to F$ with Ω an open subset of E be (Fréchet-)differentiable in $\overline{K_{\delta_0}(x_0)} \subset \Omega$ and assume that $(\mathcal{A}'(x_0))^{-1} \in L(F, E)$ exists and satisfies*

1. $\|\mathcal{A}'(x_0)\| < \tau$ *and* $\|(\mathcal{A}'(x_0))^{-1}\| \leq \kappa$,
2. $\sup_{x\in\overline{K_{\delta_0}(x_0)}} \|\mathcal{A}'(x) - \mathcal{A}'(x_0)\| \leq \frac{q}{\kappa}$,
3. $\|\mathcal{A}(x_0) - y\| \leq \delta_0 \frac{1-q}{\kappa}$,

for some constants $\tau > 0$, $\kappa > 0$, and $0 \leq q < 1$.

Then $\mathcal{A}(x) = y$ has a unique solution $\bar{x}$ in $\overline{K_{\delta_0}(x_0)}$ and the following estimates (i) and (ii) hold.

$$\frac{1}{1+q}\|(\mathcal{A}'(x_0))^{-1}(\mathcal{A}(x_0) - y)\| \leq \|\bar{x} - x_0\| \leq \frac{1}{1-q}\|(\mathcal{A}'(x_0))^{-1}(\mathcal{A}(x_0) - y)\|. \tag{i}$$

$$\frac{1}{\tau}\|\mathcal{A}(x_0) - y\| \leq \|(\mathcal{A}'(x_0))^{-1}(\mathcal{A}(x_0) - y)\| \leq \kappa\|\mathcal{A}(x_0) - y\|. \tag{ii}$$

Definition C.5. [DS58] A function $\mathcal{A} : \Sigma \to E$ is called **holomorphic** on Σ iff $\mathcal{A}$ is continuous and the first partial derivative exists at each point in Σ.

We need some definitions about root-spaces and root-elements. The following definition is taken from [Vai76].

Definition C.6. Let $\mathcal{A} : \Sigma \to L(E, F)$ be a holomorphic operator-valued function. The **resolvent set** $\rho(\mathcal{A})$ is the set of all $s \in \Sigma$ so that $\mathcal{A}(s)$ is a linear homeomorphism. A number $s_0 \in \Sigma$ is called an **eigenvalue** of $\mathcal{A}(\cdot)$ if and only if

$\mathcal{N}(\mathcal{A}(s_0)) \neq \{0\}$. Every nonzero element of $\mathcal{N}(\mathcal{A}(s_0))$ is called an **eigenelement** of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

Let v_0 be an eigenelement of $\mathcal{A}(\cdot)$ to the eigenvalue s_0. Then we call a polynomial $v(s) = \sum_{i=0}^{k}(s-s_0)^i v_i$ with coefficients $v_1, \ldots, v_k$ in E a **root-polynomial** iff

$$\frac{d^j}{ds^j}(\mathcal{A}(s)v(s))_{s=s_0} = 0 \qquad j = 0, \ldots, k.$$

The coefficients are called **root-elements** and the closed linear hull of all possible root-elements of $\mathcal{A}(\cdot)$ to the eigenvalue s_0 we denote by $\mathcal{W}(\mathcal{A}, s_0)$ and this space is called the **root-subspace** of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

For a root-polynomial v we denote by $\nu(v) \in \mathbb{N}$ the **order of the polynomial**, where $\nu(v)$ is defined as the unique integer that satisfies

$$\begin{aligned} \frac{d^j}{ds^j}(\mathcal{A}(s)v(s))_{s=s_0} &= 0, \; j = 0, \ldots, \nu(v)-1, \\ \frac{d^j}{ds^j}(\mathcal{A}(s)v(s))_{s=s_0} &\neq 0, \qquad j = \nu(v). \end{aligned}$$

For an eigenelement v_0 of $\mathcal{A}$ to the eigenvalue s_0 we call

$$\nu(v_0) := \sup\{\nu(v) : v \text{ is a rootpolynomial of } \mathcal{A}(\cdot) \text{ to } s_0 \text{ with } v(s_0) = v_0\}$$

the **order of the eigenelement** v_0.

An eigenvalue s_0 of $\mathcal{A}(\cdot)$ is called **simple eigenvalue**, iff $\dim \mathcal{N}(\mathcal{A}(s_0)) = 1$ and $\nu(v_0) = 1$ for all eigenelements v_0 of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

REMARK. *Note that in the definition $\nu(v_0) = \infty$ is allowed. A trivial example with this property is given by the constant matrix-polynomial $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenelement for every eigenvalue $s_0 \in \mathbb{C}$.*

The above concept of eigenvalues and eigenfunctions generalizes the usual concept in the following way: Instead of the linear operator $A \in L(E, E)$ one considers the operator polynomial $\mathcal{A}(s) = sI - A$. Then the eigenvalues of $\mathcal{A}(\cdot)$ coincide with those of A and the root-subspaces coincide with the generalized eigenspaces (cf. Lemma 2.27). There is a whole theory about such polynomials in the finite dimensional case, for example see [GLR82] and [WRL95, Part I].

The next characterization of root-vectors and root-polynomials directly follows from the definition and we omit a proof.

Lemma C.7. *Let $v_0 \neq 0, v_1, \ldots, v_k \in E$. Then the polynomial*

$$v(s) = \sum_{i=0}^{k}(s-s_0)^i v_i$$

is a root-polynomial of $\mathcal{A}(\cdot)$ for v_0 if and only if

$$\sum_{i=0}^{j} \frac{1}{i!}\mathcal{A}^{(i)}(s_0)v_{j-i} = 0, \; j = 0, \ldots, k.$$

Remark C.8. *In the simple case that $\mathcal{A}(s) = sB - A$ with $A, B \in L(E,F)$ the Lemma implies that $v \neq 0$ is a root vector of $\mathcal{A}(\cdot)$ to the eigenvalue s_0 if and only if there are elements $v_0, \dots, v_{k-1}, v_k$ of E with $v_0 \neq 0$ and $v_k = v$ so that*

$$\begin{aligned} \mathcal{A}(s_0)v_0 &= 0 \text{ and} \\ \mathcal{A}(s_0)v_{i+1} &= Bv_i, \ i = 0, \dots, k-1. \end{aligned}$$

In the special case that $\mathcal{A}(s) = sI - B$ with $B \in L(E,E)$ this implies that the root-subspace of $\mathcal{A}(\cdot)$ to the eigenvalue s_0 simply is the generalized eigenspace of B to the eigenvalue s_0.

We finish the review of results from functional analysis with the presentation of some results about operator matrices with Fredholm operators. For the definitions and general theory see the textbook by Kato [Kat66, Chapter IV].

One result we use is a bordering lemma for Fredholm operators see [Bey90, Lemma 2.3].

Lemma C.9. *Let X and Y be complex Banach spaces and consider the operator*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(X \times \mathbb{C}^r, Y \times \mathbb{C}^s),$$

with bounded linear operators $A \in L(X,Y)$, $B \in L(\mathbb{C}^r, Y)$, $C \in L(X, \mathbb{C}^s)$, $D \in L(\mathbb{C}^r, \mathbb{C}^s)$. If A is Fredholm of index p then S is Fredholm of index $p + r - s$.

The next Lemma is in some sense a simple generalization of the previous lemma. It might by of some interest by itself.

Lemma C.10. *Let X_1, X_2, Y_1, Y_2 be complex Banach spaces. Consider bounded linear operators $A \in L(X_1,Y_1)$, $B \in L(X_2,Y_1)$, $C \in L(X_1,Y_2)$, $D \in L(X_2,Y_2)$, and assume that A is a Fredholm operator of index $r \geq 0$, D is a Fredholmoperator of index $s \in \mathbb{Z}$ and C is a compact operator.*

Then the operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X_1 \times X_2 \to Y_1 \times Y_2 \tag{C.3}$$

is Fredholm of index $r + s$.

Proof. Since A is Fredholm of index r there is a basis $x_1^a, \dots, x_m^a$ of $\mathcal{N}(A)$ and a cobasis $y_1^a, \dots, y_{m-r}^a$ of $\mathcal{R}(A)$ in Y_1, i.e. $y_1^a, \dots, y_{m-r}^a$ are linearly independent in $Y_1/\mathcal{R}(A)$ and $\mathcal{R}(A) \oplus \operatorname{span}(y_1^1, \dots, y_{m-r}^a) = Y_1$. Let $x_1^*, \dots, x_m^* \in X_1^*$, be a biorthogonal basis for $x_1^a, \dots, x_m^a$, where X_i^* denotes the dual space of X_i. Define the compact operator

$$K_A := \sum_{i=1}^{m-r} \langle x_i^*, \cdot \rangle y_i^a.$$

Hence the operator $A + K_A$ is also Fredholm of index r and by construction it holds

$$\mathcal{R}(A + K_A) = Y_1 \text{ and } \dim \mathcal{N}(A + K_A) = r.$$

From the stability of the Fredholm property under compact perturbations (cf. [Kat66, IV Theorem 5.26]) follows that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is Fredholm of index $r+s$ if and only if $M := \begin{pmatrix} A + K_A & B \\ 0 & D \end{pmatrix}$ is Fredholm of index $r+s$. Since $A + K_A$ is onto we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \notin \mathcal{R}(M) \Leftrightarrow \begin{pmatrix} 0 \\ v \end{pmatrix} \notin \mathcal{R}(M) \Leftrightarrow v \notin \mathcal{R}(D).$$

This shows the equality $\operatorname{codim} \mathcal{R}(M) = \operatorname{codim} \mathcal{R}(D)$.
Let $x_1^1, \ldots, x_r^1$ be a basis of $\mathcal{N}(A + K_A)$ and let $x_1^2, \ldots, x_p^2$ be a basis of $\mathcal{N}(D)$. By choosing $\tilde{x}_i \in X_1$ so that

$$(A + K_A)\tilde{x}_i = -Bx_i^2, \ i = 1, \ldots, p$$

we find that $\begin{pmatrix} x_1^1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_r^1 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{x}_1 \\ x_1^2 \end{pmatrix}, \ldots, \begin{pmatrix} \tilde{x}_p \\ x_p^2 \end{pmatrix}$ are linearly independent elements of $\mathcal{N}(M)$. A simple computation shows that they are also a basis of $\mathcal{N}(M)$. Therefore

$$\operatorname{ind}(M) = \dim \mathcal{N}(M) - \operatorname{codim} \mathcal{R}(M) = r + \dim \mathcal{N}(D) - \operatorname{codim} \mathcal{R}(D) = r + s.$$

□

Bibliography

[ABB$^+$99] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen. *LAPACK Users' Guide, Third Edition.* SIAM, Philadelphia, August 1999. http://www.netlib.org/lapack/lug/lapack_lug.html.

[Alt99] H. W. Alt. *Lineare Funktionalanalysis.* Springer, Berlin-Heidelberg, 3 edition, 1999.

[Bau92] H. Bauer. *Maß- und Integrationstheorie.* Walter de Gruyter & Co., Berlin, 2. edition, 1992.

[Bey80] W.-J. Beyn. On discretizations of bifurcation problems. In *Bifurcation problems and their numerical solution (Proc. Workshop, Univ. Dortmund, Dortmund, 1980)*, pages 46–73, Basel, 1980. Birkhäuser.

[Bey90] W.-J. Beyn. The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Anal.*, 10:379–405, 1990.

[BJ89] P.W. Bates and C.K.R.T. Jones. Invariant manifolds for semilinear partial differential equations. In *Dynamics reported*, volume 2 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 1–38. Wiley, Chichester, 1989.

[BL99] W.-J. Beyn and J. Lorenz. Stability of traveling waves: Dichotomies and eigenvalue conditions on finite intervals. *Numer. Funct. Anal. and Optimiz*, 20(3&4):201–244, 1999.

[BSU96] J. M. Berezanski, Z. G. Sheftel, and G. F. Us. *Functional Analysis*, volume I. Birkhäuser, Basel, 1996.

[Cop78] W. A. Coppel. *Dichotomies in Stability Theory.* Number 629 in Lecture Notes in Mathematics. Springer, Berlin Heidelberg, 1978.

[DS58] N. Dunford and J. T. Schwartz. *Linear Operators, Part I: General Theory.* Interscience Publishers, Inc., New York, 1958.

[Fit61] R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1:445–466, 1961.

[GLR82] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix polynomials.* Academic Press Inc., New York, 1982.

[Gri75a] R. D. Grigorieff. Diskrete Approximation von Eigenwertproblemen. I. Qualitative Konvergenz. *Numer. Math.*, 24(4):355–374, 1975.

[Gri75b] R. D. Grigorieff. Diskrete Approximation von Eigenwertproblemen. II. Konvergenzordnung. *Numer. Math.*, 24(5):415–433, 1975.

[Gri76] R. D. Grigorieff. Diskrete Approximation von Eigenwertproblemen. III. Asymptotische Entwicklungen. *Numer. Math.*, 25(1):79–97, 1975/76.

[Hen81] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[HH52] A.L. Hodgkin and A.F. Huxley. A quantitative description of ion currents and its applications to conduction and excitation in nerve membranes. *Journal of Physiology*, 117:500–544, 1952.

[HW80] F.R. de Hoog and R. Weiss. An approximation theory for boundary value problems on infinite intervals. *Computing*, 24(2-3):227–239, 1980.

[Kat66] T. Kato. *Perturbation Theory for Linear Operators*. Springer, New York, 1966.

[Kel75] J. L. Kelley. *General Topology*. Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.

[Kev00] J. Kevorkian. *Partial differential equations*, volume 35 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2000.

[KKP94] G. Kreiss, H.-O. Kreiss, and A. Petersson. On the convergence to steady state of solutions of nonlinear hyperbolic-parabolic systems. *SIAM J. Numer. Anal.*, 31:1577–1604, 1994.

[KL89] H.-O. Kreiss and J. Lorenz. *Initial-boundary value problems and the Navier-Stokes equations*, volume 136 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1989.

[KS94] R. Kapral and K. Showalter, editors. *Chemical Waves and Patterns*, volume 10 of *Understanding Chemical Reactivity*, Dordrecht, 1994. Kluwer Academic Publishers.

[Lor99] J. Lorenz. Notes on resolvent estimates for hyperbolic systems. Personal Notes, 1999.

[Miu82] R.M. Miura. Accurate computation of the stable solitary wave for the FitzHugh-Nagumo equations. *J. Math. Biol.*, 13(3):247–269, 1982.

[Mur93] J.D. Murray. *Mathematical Biology*, volume 19. Springer-Verlag, Berlin, second edition, 1993.

[Pal84] K.J. Palmer. Exponential dichotomies and transversal homoclinic points. *J. Differential Equations*, 55, 1984.

[Pal88] K.J. Palmer. Exponential dichotomies and Fredholm operators. *Proc. Amer. Math. Soc.*, 104(1):149–156, 1988.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[Rau91] J. Rauch. *Partial Differential Equations*. Springer, New York, 1991.

[Rob01] J. C. Robinson. *Infinite Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge University Press, Cambridge, 2001.

[RR93] M. Renardy and R. C. Roberts. *An Introduction to Partial Differential Equations*. Springer, New York, 1993.

[San93] B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen*. Dissertation, Stuttgart, 1993.

[San02] B. Sandstede. Stability of travelling waves. In B. Fiedler, editor, *Handbook of Dynamical Systems*, volume 2, chapter 18, pages 983–1055. Elsevier, 2002.

[SS00] B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Physica D*, 145(3-4):233–277, 2000.

[Stu71a] F. Stummel. Diskrete Konvergenz linearer Operatoren. I. *Math. Ann.*, 190:45–92, 1971.

[Stu71b] F. Stummel. Diskrete Konvergenz linearer Operatoren. II. *Math. Z.*, 120:231–264, 1971.

[Stu72] F. Stummel. Diskrete Konvergenz linearer Operatoren. III. In *Linear operators and approximation (Proc. Conf., Oberwolfach, 1971)*, volume 20 of *Internat. Ser. Numer. Math.*, pages 196–216. Birkhäuser, Basel, 1972.

[Thü98] V. Thümmler. Numerische Stabilitätsindikatoren für wandernde Wellen. Diplomarbeit, Universität Bielefeld, 1998.

[Vai76] G. Vainikko. *Funktionalanalysis der Diskretisierungsmethoden*. Teubner, Leipzig, 1976.

[Vai77a] G. Vainikko. Über die Konvergenz und Divergenz von Näherungsmethoden bei Eigenwertproblemen. *Math. Nachr.*, 78:145–164, 1977.

[Vai77b] G. Vainikko. Über Konvergenzbegriffe für lineare Operatoren in der numerischen Mathematik. *Math. Nachr.*, 78:145–164, 1977.

[VVV94] A. I. Volpert, Vi. A. Volpert, and Vl. A. Volpert. *Traveling wave solutions of parabolic systems*, volume 140 of *Translations of Mathematical Monographs*. AMS, Providence, RI, 1994.

[Wil65] J. H. Wilkinson. *The Algebraic Eigenvalue Problem.* Clarendon Press, Oxford, 1965.

[Wil70] S. Willard. *General Topology.* Addison-Wesley Publishing Co., Massachusetts-London-Don Mills, 1970.

[WRL95] J. Wloka, B. Rowley, and B. Lawruk. *Boundary value problems for elliptic systems.* Cambridge University Press, Cambridge, 1995.

[Yos78] K. Yosida. *Functional Analysis.* Springer, Berlin-Heidelberg-New York, fifth edition, 1978.

[Zau89] E. Zauderer. *Partial differential equations of applied mathematics.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1989.