# Exponential decay of two-dimensional rotating waves (Part 2)

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# Outline

# Problem

Consider the stationary problem

$$\alpha \triangle u + cD_\phi u + f(u) = 0,\ x \in \mathbb{R}^2$$

where $u : \mathbb{R}^2 \to \mathbb{R}^N$ is unknown, $\alpha \in \mathbb{R}$ with $\alpha > 0$, $c \in \mathbb{R}$ with $c \neq 0$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ are given and $D_\phi$ is defined as

$$D_\phi := -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$$

Under the assumptions

There exists a constant vector $u_\infty \in \mathbb{R}^N$ such that

(A1) $\lim_{R\to\infty} \sup_{|x|\geqslant R} |u(x) - u_\infty| = 0$,

(A2) $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $B_\infty := Df(u_\infty)$ is negative definite.

We want to show solvability and uniqueness of an exponential decaying solution, i.e.

$$|u(x) - u_\infty| \leqslant Ce^{-C|x|},$$
$$\left|D^\beta u(x)\right| \leqslant Ce^{-C|x|},\ 1 \leqslant |\beta| \leqslant 2.$$

# Motivation

Consider the stationary problem

$$\alpha \triangle u + cD_{\phi} u + f(u) = 0,\ x \in \mathbb{R}^2.$$

Let $u_\infty \in \mathbb{R}^N$ be a stationary point (satisfying (A1) and (A2))

$$\alpha \triangle u_\infty + cD_{\phi} u_\infty + f(u_\infty) = 0$$

i.e. $f(u_\infty) = 0$. Since $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ by Taylor's theorem we obtain for every $u = u(x) \in \mathbb{R}^N$

$$f(u) = \underbrace{f(u_\infty)}_{=0} + \underbrace{\int_0^1 Df(u_\infty + t(u - u_\infty))dt}_{=:a(x)} (u - u_\infty).$$

Using assumption (A1) we have

$$a(x) \to B_\infty,\ \text{as}\ |x| \to \infty$$

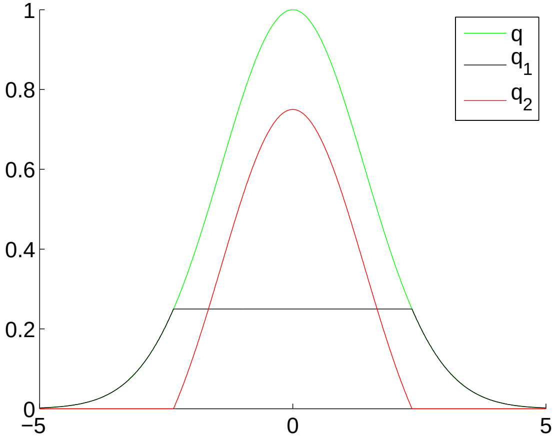
where $B_\infty := Df(u_\infty) \in \mathbb{R}^{N \times N}$. Define $q(x) := a(x) - B_\infty$, then

$$q(x) \to 0,\ \text{as}\ |x| \to \infty.$$

Now we decompose $q$ in the following way

$$q(x) = q_1(x) + q_2(x)$$

where $q_1$ is a small perturbation and $q_2$ is compactly supported.



From the preliminary idea we obtain (taking w.l.o.g. $u_\infty = 0$)

$$
\begin{aligned}
0 &= \alpha \triangle u + c D_\phi u + f(u) \\
&= \alpha \triangle u + c D_\phi u + a u \\
&= \alpha \triangle u + c D_\phi u + B_\infty u + q u \\
&= \alpha \triangle u + c D_\phi u + B_\infty u + q_1 u + q_2 u.
\end{aligned}
$$

Therefore, we must study the following operators

$$
\begin{aligned}
\mathcal{L}_\infty u &:= \alpha \triangle u + c D_\phi u + B_\infty u, && \text{(const. coeff. operator)} \\
\mathcal{L}_{q_1} u &:= \alpha \triangle u + c D_\phi u + B_\infty u + q_1 u, && \text{(small pert. of } \mathcal{L}_\infty\text{)} \\
\mathcal{L}_q u &:= \alpha \triangle u + c D_\phi u + B_\infty u + q u. && \text{(compact pert. of } \mathcal{L}_{q_1}\text{)}
\end{aligned}
$$

Today we will analyze the $\mathcal{L}_{q_1}$-operator.

# Assumptions

(A3) $\alpha \in \mathbb{R}$ with $\alpha > 0$ (diffusion coefficient)
(A4) $c \in \mathbb{R}$ with $c \neq 0$ (angular velocity)
(A5) $\delta \in \mathbb{R}$ with $\delta > 0$ (propagation constant)
(A6) $\eta \in \mathbb{R}$ with $\eta \geqslant 0$ (decay rate)
(A8) $g \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ with $p \in \mathbb{R}$ and $1 \leqslant p < \infty$ (inhomogenity)

# The operator $\mathcal{L}_\infty$

Consider the operator

$$\mathcal{L}_\infty u := \alpha \triangle u + c D_\phi u - \delta u.$$

**Theorem**

*Let the assumptions (A3)–(A8) be satisfied with* $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$. *Then* $\mathcal{L}_\infty : L^p_\eta(\mathbb{R}^2, \mathbb{R}) \supset \mathcal{D}(\mathcal{L}_\infty) \to L^p_\eta(\mathbb{R}^2, \mathbb{R})$ *is a linear, densely defined, closed operator and generates a* $C^0$*-semigroup. Moreover, let* $\bar{u}(x)$ *denote the solution of* $\mathcal{L}_\infty u = g$*, then we have* $\bar{u} \in W^{1,p}_\eta(\mathbb{R}^2, \mathbb{R})$ *with*

$$\|\bar{u}\|_{L^p_\eta} \leqslant C_5 \|g\|_{L^p_\eta},$$
$$\|D_i \bar{u}\|_{L^p_\eta} \leqslant C_6 \|g\|_{L^p_\eta}, \; i = 1, 2,$$

*where* $C_j = C_j(\alpha, \delta, \eta, p) > 0$, $j = 5, 6$.

# The operator $\mathcal{L}_{q_1}$

Consider the operator

$$\mathcal{L}_{q_1} u := \alpha \triangle u + c D_\phi u - \delta u + q_1 u$$

with a small perturbation $q_1 = q_1(x)$. To solve the stationary equation

$$\mathcal{L}_{q_1} u := \alpha \triangle u + c D_\phi u - \delta u + q_1 u = g$$

we make the following additional assumption:

(A9) $q_1 \in L^\infty(\mathbb{R}^2, \mathbb{R})$ (small perturbation, i.e. small w.r.t. $\|\cdot\|_{L^\infty}$)

Remark: $\mathcal{L}_{q_1}$ is a small perturbation of $\mathcal{L}_\infty$.

# Integral equation

Consider the stationary equation

$$\mathcal{L}_{q_1} u = \alpha \triangle u + c D_\phi u - \delta u + q_1 u = g$$

Putting the term $q_1 u$ on the r.h.s. we obtain

$$\mathcal{L}_\infty u = \alpha \triangle u + c D_\phi u - \delta u = g - q_1 u$$

Taking the solution representation for $\bar{u}$ (obtained by $\mathcal{L}_\infty$) we find the integral equation

$$\begin{aligned} u(x) &= - \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t} \left| e^{tQ} x - \xi \right|^2} \left( g(\xi) - q_1(\xi) u(\xi) \right) dt d\xi \\ &=: [Su](x) \end{aligned}$$

Motivation:

- self-mapping
- contraction mapping
- solvability (by contraction mapping principle)
- exponential decay (by roughness theorem)

# Self-mapping properties of $S$

**Lemma (Self-mapping on $L^p_\eta(\mathbb{R}^2,\mathbb{R})$)**

*Let the assumptions (A3)–(A9) be satisfied with $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$. If $u \in L^p_\eta(\mathbb{R}^2,\mathbb{R})$, then we have $Su \in L^p_\eta(\mathbb{R}^2,\mathbb{R})$ and it holds the estimate*

$$\|Su\|_{L^p_\eta} \leqslant C_5 \left( \|g\|_{L^p_\eta} + \|q_1\|_{L^\infty} \|u\|_{L^p_\eta} \right)$$

*where $C_5 = C_5(\alpha,\delta,\eta,p) > 0$ is from Theorem 1.*

**Proof**: Let $u \in L^p_\eta(\mathbb{R}^2,\mathbb{R})$. Using Hölder's inequality (with $\frac{1}{p} = \frac{1}{p} + \frac{1}{\infty}$) we obtain

$$\|Su\|_{L^p_\eta} \leqslant C_5 \|g - q_1 u\|_{L^p_\eta} \leqslant C_5 \left( \|g\|_{L^p_\eta} + \|q_1\|_{L^\infty} \|u\|_{L^p_\eta} \right)$$

i.e. $Su \in L^p_\eta(\mathbb{R}^2,\mathbb{R})$.

# Contraction properties of $S$

**Lemma (Contraction mapping on $L^p_\eta(\mathbb{R}^2, \mathbb{R})$)**

*Let the assumptions (A3)–(A9) be satisfied with* $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$ *and* $\|q_1\|_{L^\infty} < \frac{1}{C_5}$ *with* $C_5 > 0$ *from Theorem 1. Then we have*

$$\|Su - Sv\|_{L^p_\eta} \leqslant C_5 \|q_1\|_{L^\infty} \|u - v\|_{L^p_\eta} \;\forall\, u, v \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$$

**Proof**: Let $u, v \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$. Using Hölder's inequality (with $\frac{1}{p} = \frac{1}{p} + \frac{1}{\infty}$) we obtain

$$\|Su - Sv\|_{L^p_\eta} \leqslant C_5 \|q_1(u - v)\|_{L^p_\eta} \leqslant C_5 \|q_1\|_{L^\infty} \|u - v\|_{L^p_\eta}$$

Since $C_5 \|q_1\|_{L^\infty} < 1$ it follows that $S$ is a contraction mapping on $L^p_\eta(\mathbb{R}^2, \mathbb{R})$.

# Solvability by Contraction mapping principle

**Theorem (Solvability on $L^p_\eta(\mathbb{R}^2,\mathbb{R})$)**

*Let the assumptions (A3)–(A9) be satisfied with* $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$ *and* $\|q_1\|_{L^\infty} < \frac{1}{C_5}$ *with* $C_5 > 0$ *from Theorem 1. Then*

$$\exists_1 \bar{u} \in L^p_\eta(\mathbb{R}^2,\mathbb{R}) : \; S\bar{u} = \bar{u}$$

*Moreover,* $\bar{u}$ *solves* $\mathcal{L}_{q_1} u = g$ *and for every initial data* $u_0 \in L^p_\eta(\mathbb{R}^2,\mathbb{R})$ *the sequence* $u_{k+1} = Su_k$, $k \in \mathbb{N}_0$, *converges to* $\bar{u}$ *and it holds the apriori bound*

$$\|u_k - \bar{u}\|_{L^p_\eta} \leqslant \frac{C_5^k \|q_1\|^k_{L^\infty}}{1 - C_5 \|q_1\|_{L^\infty}} \|u_1 - u_0\|_{L^p_\eta} \; \forall\, k \in \mathbb{N}_0$$

**Proof**: Since $S$ is a self-mapping contraction, the aim follows by the contraction mapping principle.

# Roughness theorem

The Roughness theorem shows, that the solution for the (perturbed) variable coefficient operator $\mathcal{L}_{q_1}$ decays exponentially, if the solution of the constant coefficient operator $\mathcal{L}_\infty$ decays exponentially.

## Theorem (Roughness theorem)

*Let the assumptions (A3)–(A9) be satisfied with* $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$ *and* $\|q_1\|_{L^\infty} < \frac{1}{C_5}$. *Moreover, let* $\bar{u} \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ *denote the solution of* $\mathcal{L}_{q_1} u = g$, *then we have* $\bar{u} \in W^{1,p}_\eta(\mathbb{R}^2, \mathbb{R})$ *with*

$$\|\bar{u}\|_{L^p_\eta} \leqslant \frac{C_5}{1 - C_5 \|q_1\|_{L^\infty}} \|g\|_{L^p_\eta},$$

$$\|D_i \bar{u}\|_{L^p_\eta} \leqslant \frac{C_6}{1 - C_5 \|q_1\|_{L^\infty}} \|g\|_{L^p_\eta}, \ i = 1, 2,$$

*where* $C_5$ *and* $C_6$ *are from Theorem 1.*

# Proof

**Proof**: Consider $u_0(x) = 0$ for all $x \in \mathbb{R}^2$, i.e. $u_0 \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ and $\|u_0\|_{L^p_\eta} = 0$. Form the apriori bound follows that

$$
\begin{aligned}
\|\overline{u}\|_{L^p_\eta} &\leqslant \|\overline{u} - u_1\|_{L^p_\eta} + \|u_1\|_{L^p_\eta} \\
&\leqslant \frac{C_5 \|q_1\|_{L^\infty}}{1 - C_5 \|q_1\|_{L^\infty}} \|u_1 - u_0\|_{L^p_\eta} + \|u_1\|_{L^p_\eta} \\
&= \left( \frac{C_5 \|q_1\|_{L^\infty}}{1 - C_5 \|q_1\|_{L^\infty}} + 1 \right) \|u_1\|_{L^p_\eta} \\
&= \frac{1}{1 - C_5 \|q_1\|_{L^\infty}} \|Su_0\|_{L^p_\eta} \\
&\leqslant \frac{C_5}{1 - C_5 \|q_1\|_{L^\infty}} \left( \|g\|_{L^p_\eta} + \|q_1\|_{L^\infty} \|u_0\|_{L^p_\eta} \right) \\
&= \frac{C_5}{1 - C_5 \|q_1\|_{L^\infty}} \|g\|_{L^p_\eta} .
\end{aligned}
$$

# Consequence

Consider the operator

$$\mathcal{L}_{q_1} u := \alpha \triangle u + c D_\phi u - \delta u + q_1 u.$$

## Corollary

*Let the assumptions (A3)–(A9) be satisfied with* $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$ *and* $\|q_1\|_{L^\infty} < \frac{1}{C_5}$ *with* $C_5 > 0$ *from Theorem 1. Then*

$$\mathcal{L}_{q_1} : L^p_\eta(\mathbb{R}^2, \mathbb{R}) \supset \mathcal{D}(\mathcal{L}_{q_1}) \to L^p_\eta(\mathbb{R}^2, \mathbb{R})$$

*is a linear, densely defined, closed operator and generates a* $C^0$*-semigroup. Moreover, we have*

$$\mathcal{D}(\mathcal{L}_{q_1}) = \mathcal{D}(\mathcal{L}_\infty).$$

# The operator $\mathcal{L}_q$

Consider the operator

$$\mathcal{L}_q u := \alpha \triangle u + cD_\phi u - \delta u + \underbrace{(q_1 + q_2)}_{=q} u$$

with a compact perturbation $q_2 = q_2(x)$.
Remark: $\mathcal{L}_q$ is a compact perturbation of $\mathcal{L}_{q_1}$.
Motivation: Formally we decompose $\mathcal{L}_q - s$, $s \in \mathbb{C}$, as follows

$$\begin{aligned} & (\mathcal{L}_q - s)\, u = \alpha \triangle u + cD_\phi u - \delta u + qu - su = 0 \\ \Longleftrightarrow\ & \alpha \triangle u + cD_\phi u - \delta u + q_1 u - su = (q_1 - q)\, u \\ \Longleftrightarrow\ & (\mathcal{L}_{q_1} - s)\, u = (q_1 - q)\, u \\ \Longleftrightarrow\ & \left(I - (q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}\right)(\mathcal{L}_{q_1} - s)\, u = 0 \end{aligned}$$

i.e.

$$(\mathcal{L}_q - s) = \left(I - (q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}\right)(\mathcal{L}_{q_1} - s)$$

# Assumptions

Let $s \in \mathbb{C}$ and consider

$$(\mathcal{L}_q - s) = \left(I - (q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}\right)(\mathcal{L}_{q_1} - s)$$

We make the following additional assumptions

(A10) $q \in L^\infty(\mathbb{R}^2, \mathbb{R})$ (variable coefficient function)

(A11) $D_1 u$, $D_2 u$, $D_\phi u \in \mathcal{D}(\mathcal{L}_q)$ are nontrivial (where $\mathcal{L}_q u = 0$)

One can show that $\mathcal{L}_q$ possess the algebraic simple eigenvalues

$$s_1 = ic,\ s_2 = -ic,\ s_3 = 0$$

(i.e. $\dim \mathcal{N}(\mathcal{L}_q - s_j) = 1$, $j = 1, 2, 3$) with eigenfunctions

$$\varphi_1 = D_1 u + iD_2 u,\ \varphi_2 = D_1 u - iD_2 u,\ \varphi_3 = D_\phi u.$$

(A12) $\mathcal{L}_q$ has no eigenvalues $s \in \mathbb{C}$ with $\mathrm{Re}\ s > -\mathrm{Re}\ \delta$, except for $s_1, s_2, s_3$.

# Approach

Consider the decomposition

$$(\mathcal{L}_q - s) = \left(I - (q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}\right)(\mathcal{L}_{q_1} - s)$$

Fredholm theory:

- $\mathcal{L}_q - s$ is Fredholm of index 0 for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -\mathrm{Re}\, \delta$
  - $(\mathcal{L}_{q_1} - s)$ is a linear homeomorphism
  - $(q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}$ compact operator (Riesz-Frechet-Kolmogorov, compactness of multiplication operator)
  - $I - (q_1 - q)(\mathcal{L}_{q_1} - s)^{-1}$ Fredholm of index 0 (compact perturbation of the identity)
  - $(\mathcal{L}_{q_1} - s)$ is Fredholm of index 0
  - $(\mathcal{L}_q - s)$ is Fredholm of index 0
- Fredholm alternative
  - $\mathcal{L}_q'$ (formal/abstract) adjoint operator
  - solvability (by Fredholm alternative)
  - uniqueness of exponentially decaying function

# Main result

Consider the operator

$$\mathcal{L}_q u := \alpha \triangle u + cD_\phi u - \delta u + \underbrace{(q_1 + q_2)}_{=q} u$$

## Theorem

*Let the assumptions (A1)–(A12) be satisfied. Let $\bar{u} \in L^p(\mathbb{R}^2, \mathbb{R})$ denote the solution of $\mathcal{L}_q u = 0$, then we have $\bar{u} \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ for all $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$.*