Spatial decay of rotating waves in parabolic systems

Nonlinear Waves, CRC 701, Bielefeld, June 19, 2013

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CRC 701

Outline

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$, $A\in\mathbb{R}^{N,N}$, $f\in C^2(\mathbb{R}^N,\mathbb{R}^N)$.
Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^N$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.
Transformation (into a rotating frame): $v(x,t)=u(e^{tS}x,t)$ solves

$$(2)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx,\nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j D_i v(x)\ \textbf{(drift term)}.$$

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$$\langle Sx,\nabla v(x)\rangle := \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right)v(x)\ \textbf{(rotational term)}.$$

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$v_\star$ is a stationary solution of (2).

Question: How to show exponential decay of $v_\star$ at $|x|=\infty$?
Consequence: Exponentially small error by restriction to bounded domain.

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$d=2$: Spectral stability implies nonlinear stability.

[BL] W.-J. Beyn, J. Lorenz. 2008.

Example

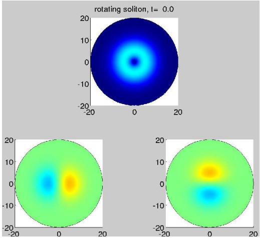
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

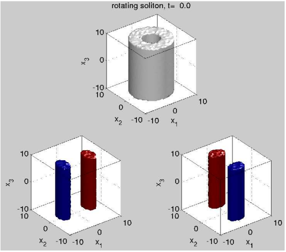
$$u_t = \alpha \triangle u + u \left(\mu + \beta |u|^2 + \gamma |u|^4 \right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{C}$, $d \in \{2, 3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2} i, \; \beta = \frac{5}{2} + i, \; \gamma = -1 - \frac{1}{10} i, \; \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





Applications: superconductivity, superfluidity, nonlinear optical systems.

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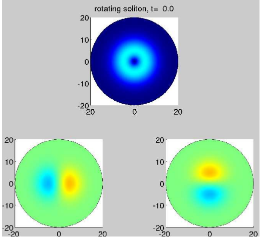

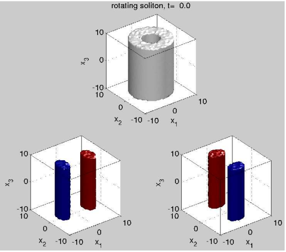

[CMM] L.-C. Crasovan, B.A. Malomed, D. Mihalache. 2001

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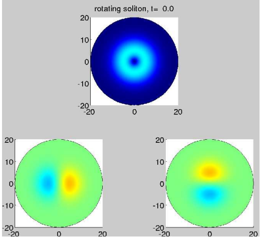

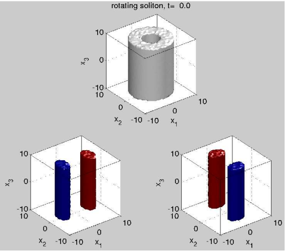

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Main result: Exponential decay of $v_\star$

Theorem: (Exponential Decay of $v_\star$)

Let $f(v_\infty)=0$ and $\mathrm{Re}\,\sigma(Df(v_\infty))<0$. Under further assumptions holds:
For every $1<p<\infty$, $0<\vartheta<1$ and for every radially nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with

$$0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$$

there exists $K_1=K_1(A,f,v_\infty,d,p,\theta,\vartheta)>0$ with the following property:
Every classical solution $v_\star$ of

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\; x\in\mathbb{R}^d,$$

such that $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and

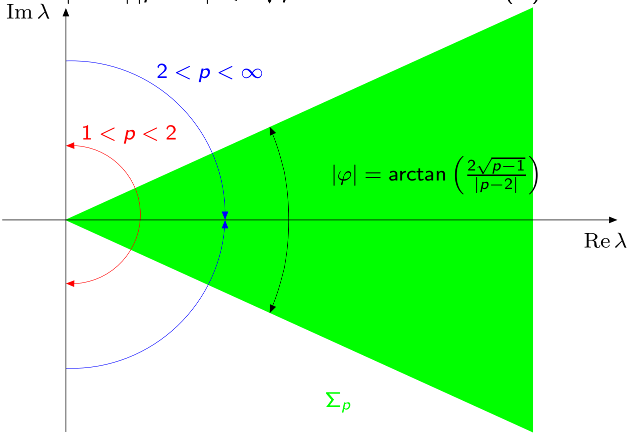
$$\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\leqslant K_1 \text{ for some } R_0>0$$

satisfies

$$v_\star-v_\infty\in W_\theta^{1,p}(\mathbb{R}^d,\mathbb{R}^N)\text{ (\textbf{weighted Sobolev space})}.$$

Main result: The assumptions

- $\mathrm{Re}\,\lambda > 0$ and $|\mathrm{Im}\,\lambda|\,|p-2| \leqslant 2\sqrt{p-1}\mathrm{Re}\,\lambda \quad \forall\,\lambda \in \sigma(A)$



  $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ simultaneously diagonalizable over $\mathbb{C}$
- $a_0 \leqslant \mathrm{Re}\,\lambda, \quad |\lambda| \leqslant a_{\max} \quad \forall\,\lambda \in \sigma(A)$
  $\mathrm{Re}\,\mu \leqslant -b_0 < 0 \quad \forall\,\mu \in \sigma(Df(v_\infty))$

Main result: The assumptions

- A positive function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a **weight function of exponential growth rate** $\eta \geqslant 0$ provided that

$$\exists\, C_\theta > 0 : \; \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \quad \forall\, x, y \in \mathbb{R}^d.$$

[ZM] S. Zelik, A. Mielke. 2009.

Examples: $\mu \in \mathbb{R}$, $x \in \mathbb{R}^d$

$$\theta_1(x) = \exp\left(-\mu|x|\right), \quad \theta_3(x) = \exp\left(-\mu\sqrt{|x|^2+1}\right),$$
$$\theta_2(x) = \cosh\left(\mu|x|\right), \quad \theta_4(x) = \cosh\left(\mu\sqrt{|x|^2+1}\right).$$

- **Exponentially weighted Sobolev spaces**: $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$

$$L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^N) \mid \|\theta v\|_{L^p} < \infty \right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \; \forall\, |\beta| \leqslant k \right\}.$$

Outline of proof: Exponential Decay of $v_\star$

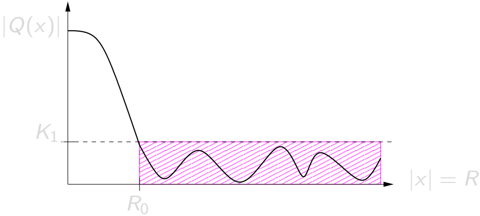
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

1. Far-Field Linearization: $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + a(x)w(x) = 0,\ x \in \mathbb{R}^d.$$



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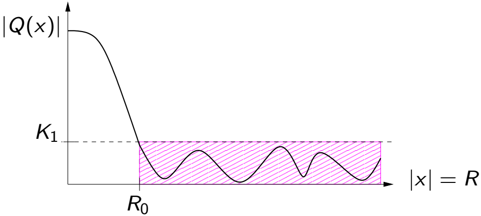
2. Decomposition of a:

$$Df(v_\infty) + Q(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q(x)\right) w(x) = 0,\ x \in \mathbb{R}^d.$$



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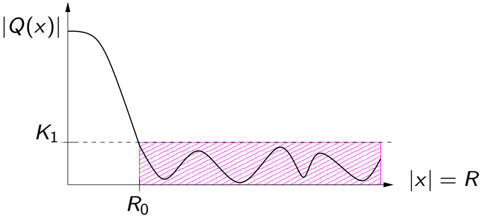

3. Decomposition of Q:

$Q(x) = Q_\varepsilon(x) + Q_c(x)$,

$Q, Q_\varepsilon, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

Q_ε small, i.e. $\|Q_\varepsilon\|_{L^\infty} < K_1$,

Q_c compactly supported.

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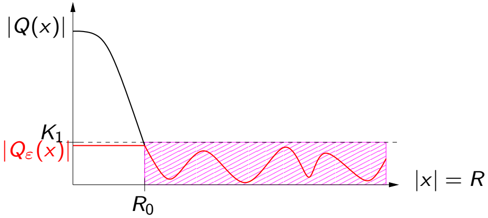

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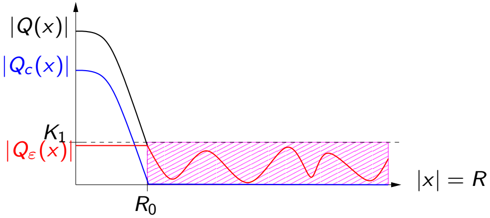

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$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Operators: Study the following operators

$$\begin{aligned}
\mathcal{L}_Q v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v + Q_c v,\\
\mathcal{L}_{Q_\varepsilon} v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v,\\
\mathcal{L}_\infty v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v,\\
\mathcal{L}_0 v &:= A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator).} \quad \text{(max. domain)}
\end{aligned}$$

Ornstein-Uhlenbeck Operator

Let $P, B \in \mathbb{R}^{d,d}$, $P = P^T$, $P > 0$ and $B \neq 0$.

$$\nabla^T P \nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij} D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x \in \mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\ x\in\mathbb{R}^d,\ d\geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty=0$)

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Let $P,B\in\mathbb{R}^{d,d}$, $P=P^T$, $P>0$ and $B\neq 0$.

$$\nabla^T P\nabla v(x)+\langle Bx,\nabla v(x)\rangle=\sum_{i=1}^{d}\sum_{j=1}^{d}D_i\left(P_{ij}D_j v(x)\right)+\sum_{i=1}^{d}\sum_{j=1}^{d}D_i v(x)B_{ij}x_j,\ x\in\mathbb{R}^d$$

Here: $P=I_d$ and $B=S$.

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Operators: Study the following operators

$$\begin{aligned}
\mathcal{L}_Q v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v + Q_c v, \\
\mathcal{L}_{Q_\varepsilon} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v, \\
\mathcal{L}_\infty v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, && \text{(exp. decay)} \\
\mathcal{L}_0 v :=& A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator).} && \text{(max. domain)}
\end{aligned}$$

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[MPV] G. Metafune, D. Pallara, V. Vespri. 2005.

[MPRS] G. Metafune. 2001.

The operator $\mathcal{L}_0$

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

$\downarrow$

Heat kernel

$$H_0(x,\xi,t) = (4\pi t A)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right), \; x,\xi \in \mathbb{R}^d, \; t > 0.$$

$\downarrow$

Semigroup in $L^p(\mathbb{R}^d, \mathbb{R}^N)$, $1 \leqslant p \leqslant \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t > 0.$$

strong $\downarrow$ continuity

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \; 1 \leqslant p < \infty.$$

semigroup theory $\swarrow$ | A-priori $\downarrow$ estimates | $\searrow$ $\mathcal{L}_0$: L^p-resolvent est.

unique solv. of resolvent equ., $1 \leqslant p < \infty$

$$(\lambda I - A_p)\, v_\star = g \in L^p.$$

exponential decay, $1 \leqslant p < \infty$

$$v_\star \in W^{1,p}_\theta.$$

max. domain and max. realization, $1 < p < \infty$

$$A_p = \mathcal{L}_0 \text{ on } \mathcal{D}(A_p) = \mathcal{D}^p(\mathcal{L}_0).$$

Spectrum of localized rotating waves

Linearization about the profile of the rotating wave

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Eigenvalue problem

$$[\mathcal{L}v](x) = \lambda v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2,\ \lambda \in \mathbb{C}.$$

A **rotating wave** solution $u_\star(x,t) = v_\star\left(e^{-tS}x\right)$ is **spectrally stable** if

$$\sigma(\mathcal{L}) \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \leqslant 0\}.$$

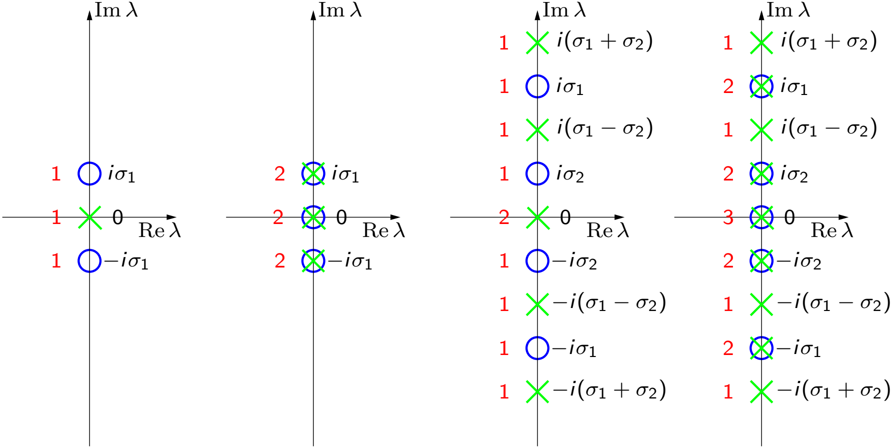
Decompose the **spectrum** $\sigma(\mathcal{L})$ into

$$\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}) \,\dot{\cup}\, \sigma_{\mathrm{pt}}(\mathcal{L}),$$

with essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ and point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$.

Illustration: Point spectrum of $\mathcal{L}$

$\lambda \in (\sigma(S) \cup \{\lambda + \mu \mid \lambda, \mu \in \sigma(S),\ \lambda \neq \mu\}) \subseteq \sigma_{\mathrm{pt}}(\mathcal{L})$ with algebraic multiplicity



$d = 2$	$d = 3$	$d = 4$	$d = 5$
$\dim \mathrm{SE}(2) = 3$	$\dim \mathrm{SE}(3) = 6$	$\dim \mathrm{SE}(4) = 10$	$\dim \mathrm{SE}(5) = 15$

Point spectrum of $\mathcal{L}$

Theorem: (Point spectrum of $\mathcal{L}$)

Let $A \in \mathbb{R}^{N,N}$ satisfy the main assumptions, $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, $1 < p < \infty$ and let $v_\star$ be a classical solution of $[\mathcal{L}_0 v](x) + f(v(x)) = 0$. Then

$$v(x) = \left\langle C^{rot} x + I_d C^{tra}, \nabla v_\star(x) \right\rangle, \; x \in \mathbb{R}^d, \; C^{rot} \in \mathrm{so}(d), \; C^{tra} \in \mathbb{R}^d$$

solves $\mathcal{L}v = \lambda v$, whenever $(\lambda, (C^{rot}, C^{tra}))$ solves

$$\begin{aligned} \lambda C^{rot} &= -SC^{rot} + \left(SC^{rot}\right)^T, \\ \lambda C^{tra} &= -SC^{tra}. \end{aligned}$$

In particular:

- $\sigma(S) \cup \{\lambda + \mu \mid \lambda, \mu \in \sigma(S), \lambda \neq \mu\} \subseteq \sigma_{\mathrm{pt}}(\mathcal{L})$,
- $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geqslant 2$,
- group action $\Rightarrow \sigma_{\mathrm{pt}}(\mathcal{L})$,
- Theorem also valid for spiral waves, scroll waves, scroll rings.

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Exponential decay of v

Theorem: (Exponential decay of v)

Let the assumptions of the main result be satisfied.
The every classical solution $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d,$$

with $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ satisfies

$$v \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$$

- $v_\star$ exp. localized $\Rightarrow$ v exp. localized (with same rate)

Outline of proof: Point spectrum of $\mathcal{L}$

(3) $$0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\; x \in \mathbb{R}^d,\; d \geqslant 2.$$

1. Group action: Apply $a(R,\tau)$ to (3)

$$0 = a(R,\tau)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$

2. Derivative $\frac{d}{d(R,\tau)}$ at $(R,\tau) = (I_d, 0)$ leads to $\frac{d(d+1)}{2}$ equations

$$0 = (x_j D_i - x_i D_j)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$
$$0 = D_l\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$

for $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$, $l = 1,\ldots,d$.

3. Commutator relations for differential operators yield, $D^{(ij)} := x_j D_i - x_i D_j$

$$0 = \mathcal{L}\left(D^{(ij)} v_\star(x)\right) + \sum_{\substack{n=1\\ n\neq j}}^{d} S_{in} D^{(jn)} v_\star(x) - \sum_{\substack{n=1\\ n\neq i}}^{d} S_{jn} D^{(in)} v_\star(x)$$

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4. Finite-dimensional eigenvalue problem: The ansatz

$$v(x) = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} C_{ij}^{rot}(x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} C_l^{tra} D_l v_\star(x),\ C_{ij}^{rot}, C_l^{tra} \in \mathbb{C}$$

reduces $\mathcal{L}v = \lambda v$ to a

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Note: S is unitary diagonalizable.

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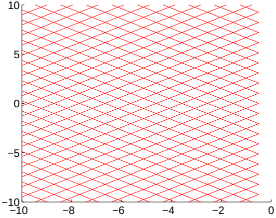
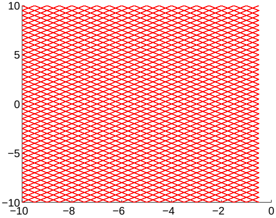
Illustration: Essential spectrum of $\mathcal{L}$

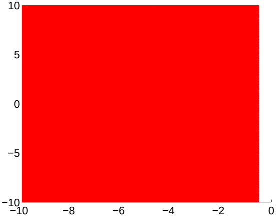
$$\left\{-\lambda(\omega)+i\sum_{l=1}^{k} n_l\sigma_l \;\middle|\; \lambda(\omega) \text{ eigenvalue of } \omega^2 A - Df(v_\infty)\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$



$d = 2$ or 3



$d = 4$ (not dense)



$d = 4$ (dense)

Essential spectrum of $\mathcal{L}$

Theorem: (Essential spectrum of v)

Let the assumptions of the main result be satisfied. Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S and let $\lambda(\omega)$ denote an eigenvalue of $\omega^2 A - Df(v_\infty)$ for some $\omega \in \mathbb{R}$. Then

$$\left\{ \lambda = -\lambda(\omega) - i \sum_{l=1}^{k} n_l \sigma_l \in \mathbb{C} \mid n_l \in \mathbb{Z}, \omega \in \mathbb{R} \right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.

- Far-field linearization $\Rightarrow \sigma_{\mathrm{ess}}(\mathcal{L})$

Dispersion relation: $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$ if

$$\det\left(\lambda I_N + \omega^2 A + i \sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty) \right) = 0 \text{ for some } \kappa \in \mathbb{R}.$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Linearization at the profile $v_\star$:

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\infty)v(x) + Q(x)v(x)$$

$$Q(x) := Df(v_\star(x)) - Df(v_\infty), \qquad \sup_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R \to \infty$$

1. Orthogonal transformation: $S \in \mathbb{R}^{d,d}$, $S^T = -S$, $S = P\Lambda^S_{\mathrm{block}}P^T$.
$T_1(x) = Px$ yields

$$[\mathcal{L}_1 v](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

with

$$\left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l}D_{2l-1} - x_{2l-1}D_{2l}\right) v(x).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Orthogonal transformation:

$$\left[\mathcal{L}_1 v\right](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x) \right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

$$\left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x) \right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l} D_{2l-1} - x_{2l-1} D_{2l} \right) v(x)$$

2. Several planar polar coordinates: Transformation

$$\begin{pmatrix} x_{2l-1} \\ x_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l \cos\phi_l \\ r_l \sin\phi_l \end{pmatrix},\ l = 1, \ldots, k,\ \phi_l \in]-\pi, \pi],\ r_l > 0.$$

yields for $\xi = (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d)$ with total transformation $T_2(\xi)$, $Q(\xi) := Q(T_1(T_2(\xi)))$

$$\left[\mathcal{L}_2 v\right](x) = A \left[\sum_{l=1}^{k} \left(\partial^2_{r_l} + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial^2_{\phi_l} \right) + \sum_{l=2k+1}^{d} \partial^2_{x_l} \right] v(\xi) - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Several planar polar coordinates:

$$[\mathcal{L}_2 v](\xi) = A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi)$$
$$-\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),$$

$$\xi = (r_1,\phi_1,\ldots,r_k,\phi_k,x_{2k+1},\ldots,x_d),\quad Q(\xi) := Q(T_1(T_2(\xi)))$$

3. Simplified operator (far-field linearization): Neglecting $\mathcal{O}\left(\frac{1}{r}\right)$–terms yields

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](x) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Simplified operator (far-field linearization):

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](\xi) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right] v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l} v(\xi) + Df(v_\infty)v(\xi)$$

4. Angular Fourier decomposition:

$$v(\xi) = \exp\left(i\omega\sum_{l=1}^{k} r_l\right)\exp\left(i\sum_{l=1}^{k} n_l\phi_l\right)\hat{v},\ n_l\in\mathbb{Z},\ \omega\in\mathbb{R},\ \hat{v}\in\mathbb{C}^N,\ |\hat{v}|=1$$

$$\phi_l \in]-\pi,\pi],\ r_l>0,\ l=1,\ldots,k,$$

yields

$$\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right) v\right](\xi) = \left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k} n_l\sigma_l I_N - Df(v_\infty)\right) v(\xi).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Angular Fourier decomposition:

$$\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right) v\right](\xi) = \left(\lambda I_N + \kappa^2 A + i \sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty)\right) v(\xi).$$

$$n_l \in \mathbb{Z}, \quad \kappa \in \mathbb{R}, \quad \pm i\sigma_l \text{ nonzero eigenvalues of } S \in \mathbb{R}^{d,d}$$

5. Finite-dimensional eigenvalue problem: $\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right) v\right](\xi) = 0$ for every ξ if $\lambda \in \mathbb{C}$ satisfies

$$\left(\omega^2 A - Df(v_\infty)\right) \hat{v} = -\left(\lambda + i \sum_{l=1}^{k} n_l \sigma_l\right) \hat{v}, \text{ for some } \omega \in \mathbb{R}.$$

Example

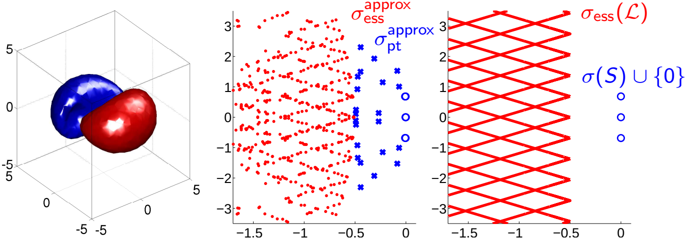
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.



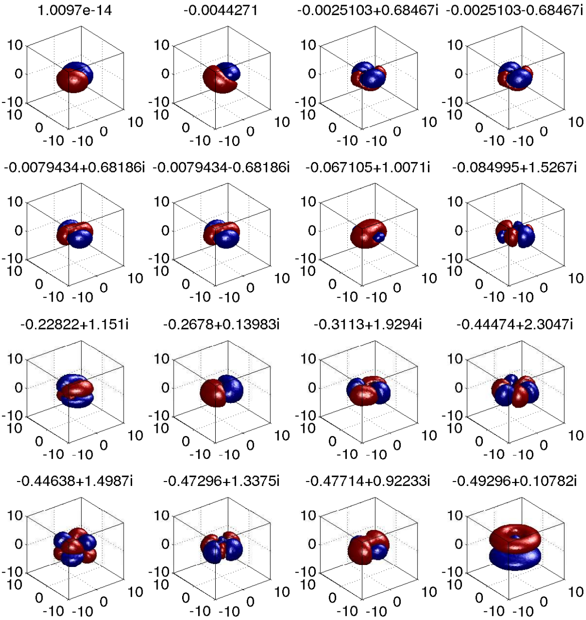


Figure: Eigenfunctions of QCGL for a spinning soliton with $d = 3$

Work in progress

- exponential decay in **space of continuous functions**
- rotating waves in **bounded domains**
- **approximation theorem** for rotating waves
- asymptotic boundary conditions
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