Energy Estimates for Ornstein-Uhlenbeck Operators in Exponentially Weighted L^p-Spaces

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joint work with: **Wolf-Jürgen Beyn** (Bielefeld University)

- W.-J. Beyn, D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. *Dyn. Partial Differ. Equ.*, 13(3):191-240, 2016.
- D. Otten. The identification problem for complex Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. *Semigroup Forum*, DOI: `http://dx.doi.org/10.1007/s00233-016-9804-y`, 2016.
- D. Otten. Spatial decay and spectral properties of rotating waves in parabolic systems. PhD thesis, Bielefeld University, *Shaker Verlag*, 2014.

Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. Energy estimates in exponentially weighted L^p-spaces

4. L^p-dissipativity condition vs. L^p-antieigenvalue bound

5. Explicit representations of the first antieigenvalue

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5. Explicit representations of the first antieigenvalue

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1) \qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$, $A\in\mathbb{R}^{m,m}$, $f:\mathbb{R}^m\to\mathbb{R}^m$, $u_0:\mathbb{R}^d\to\mathbb{R}^m$.

Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^m$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x,t)=u(e^{tS}x,t)$ solves

$$(2) \qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx,\nabla v(x)\rangle = Dv(x)Sx = \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_jD_iv(x) \overset{-S=S^\top}{=} \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_jD_i - x_iD_j\right)v(x)$$

(drift term) **(rotational term)**

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Note: $v_\star$ is a stationary solution of (2), i.e. $v_\star$ solves the **rotating wave equation**

$$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle$: **Ornstein-Uhlenbeck operator**.

Rotating Patterns in $\mathbb{R}^d$

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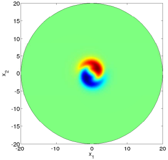
Questions and Ingredients: I1: exp. decay of $v_\star$, I2: spectral properties
Q1: Nonlinear stability of rotating waves on $\mathbb{R}^d$? (**Tools**: I1+I2)
Q2: Truncations of rotating waves to bounded domains? (**Tools**: I1+...)
Q3: Spatial approximation (e.g. with finite element method)? (**open problem**)
Q4: Temporal approximation (e.g. with Euler or BDF)? (**open problem**)

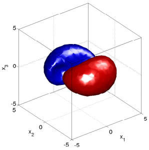
Examples for rotating waves

Cubic-quintic complex Ginzburg-Landau equation: (spinning solitons)

$$u_t = \alpha \triangle u + u\left(\delta + \beta |u|^2 + \gamma |u|^4\right)$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\alpha > 0$, $\delta \in \mathbb{R}$, $d \in \{2,3\}$.

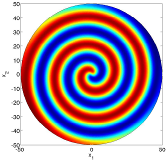


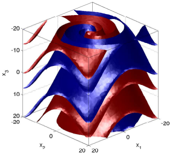


λ-ω system: (spiral waves, scroll waves)

$$u_t = \alpha \triangle u + \left(\lambda(|u|^2) + i\omega(|u|^2)\right) u$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\lambda, \omega : [0,\infty[\to \mathbb{R}$, $\alpha \in \mathbb{C}$, $\mathrm{Re}\alpha > 0$, $d \in \{2,3\}$.

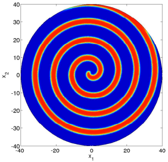




Barkley model: (spiral waves, also scroll waves)

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle u + \begin{pmatrix} \frac{1}{\varepsilon} u_1 (1 - u_1)(u_1 - \frac{u_2 + b}{a}) \\ u_1 - u_2 \end{pmatrix}$$

with $u(x,t) \in \mathbb{R}^2$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $0 \leqslant D \ll 1$, $\varepsilon, a, b > 0$.



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Outline

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$)

Let $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^m)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

Spatial decay of rotating waves

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Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

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$$0 \leqslant \mu \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$: higher regularity)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^m,\mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p\geqslant\frac{d}{2}$ (if $k\geqslant 3$).

Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ with the following property:

Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))=0,\; x\in\mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\leqslant K_1 \text{ for some } R_0>0$$

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$$v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m)$$

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$$0\leqslant\mu\leqslant\varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max}p}. \qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$: pointwise estimates)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^m,\mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p\geqslant\frac{d}{2}$ (if $k\geqslant 3$).

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satisfies

$$v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m),\ |D^\alpha(v_\star(x)-v_\infty)| \leqslant C\exp\left(-\mu\sqrt{|x|^2+1}\right)\ \forall x\in\mathbb{R}^d$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p} \qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

and for every multiindex $\alpha\in\mathbb{N}_0^d$ satisfying $d<(k-|\alpha|)p$.

Spatial decay of eigenfunctions

Theorem 2: (Exponential decay of eigenfunctions v)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2+1}\right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j = 1, 2$, $k \in \mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k \geqslant 2$).
Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following property holds: Every classical solution $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$ of

$$\text{(EVP)} \qquad A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d,$$

with $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty$, such that

$$v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m) \quad \text{for } \textbf{some} \text{ exp. growth rate} \quad -\sqrt{\varepsilon \frac{\gamma_A \beta_\infty}{2d|A|^2}} \leqslant \mu_1 < 0$$

satisfies

$$v \in W^{k,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m) \quad \text{for } \textbf{every} \text{ exp. decay rate} \quad 0 \leqslant \mu_2 \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$$

and

$$|D^\alpha v(x)| \leqslant C \exp\left(-\mu_2 \sqrt{|x|^2+1}\right) \ \forall x \in \mathbb{R}^d$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$.

Exponentially weighted Sobolev spaces and assumptions

Exponentially weighted Sobolev spaces: For $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d,\mathbb{R}^m) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^m) \mid \|\theta v\|_{L^p} < \infty \right\},$$

$$W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m) := \left\{ v \in L^p_\theta(\mathbb{R}^d,\mathbb{R}^m) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d,\mathbb{R}^m)\ \forall\, |\beta| \leqslant k \right\}.$$

Assumptions:

(A1) (L^p-dissipativity condition): For $A \in \mathbb{R}^{m,m}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2\ \forall\, z, w \in \mathbb{R}^m$$

(A2) (System condition): $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) (Rotational condition): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

Note: Assumption (A1) is equivalent with

(A1') (L^p-antieigenvalue condition): $A \in \mathbb{R}^{m,m}$ is invertible and

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{R}^m\\ w\neq 0\\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p} \text{ for some } 1 < p < \infty$$

($\mu_1(A)$: first antieigenvalue of A)

(to be read as $A > 0$ in case $m = 1$).

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + (Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x))\, w_\star(x) = 0,\ x \in \mathbb{R}^d.$$

Operators: Study the following operators

$$\mathcal{L}_{\mathrm{c}} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v + Q_{\mathrm{c}} v, \qquad \text{(exp. decay)}$$

$$\mathcal{L}_{\mathrm{s}} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v, \qquad \text{(exp. decay)}$$

$$\mathcal{L}_{\infty} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, \quad \text{(far-field operator)} \qquad \text{(exp. decay)}$$

$$\mathcal{L}_{0} v := A\triangle v + \langle S\cdot, \nabla v\rangle. \quad \text{(Ornstein-Uhlenbeck operator)} \qquad \text{(max. domain)}$$

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Spatial Decay of Rotating Waves in Reaction Diffusion Systems, 2016.

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\; x\in\mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\; x\in\mathbb{R}^d.$$

Operators: Study the following operators

$$\mathcal{L}_{\mathrm{c}} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}}v + Q_{\mathrm{c}}v, \qquad \text{(exp. decay)}$$

$$\mathcal{L}_{\mathrm{s}} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}}v, \qquad \text{(exp. decay)}$$

$$\mathcal{L}_{\infty} v := A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, \quad \text{(far-field operator)} \qquad \text{(exp. decay)}$$

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Maximal domain of $\mathcal{L}_0$ given by

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \left\{v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m) \cap L^p(\mathbb{R}^d,\mathbb{C}^m) :\ \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^m)\right\},\ 1 < p < \infty$$

satisfies $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$.

The operator $\mathcal{L}_0$

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

$$\downarrow$$

Heat kernel

$$H_0(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right), \; x,\xi \in \mathbb{R}^d, \; t > 0.$$

$$\downarrow$$

Semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 \leqslant p \leqslant \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t > 0.$$

strong $\downarrow$ continuity

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \; 1 \leqslant p < \infty.$$

semigroup theory $\swarrow$ | $\searrow$ identification problem

unique solv. of resolvent equ. for A_p, $1 \leqslant p < \infty$, $\mathrm{Re}\lambda > 0$

$(\lambda I - A_p)\, v_\star = g \in L^p$.

A-priori $\rightarrow$ estimates

exponential decay, $1 \leqslant p < \infty$

$v_\star \in W^{1,p}_\theta$.

max. domain and max. realization, $1 < p < \infty$

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.

Identification problem of $\mathcal{L}_0$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m) \cap L^p(\mathbb{R}^d,\mathbb{C}^m) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^m) \right\},\ 1<p<\infty.$$

Infinitesimal generator

$(A_p, \mathcal{D}(A_p))$, $1 \leqslant p < \infty$.

$\downarrow$

$\mathcal{S}$ is a core for $(A_p, \mathcal{D}(A_p))$

$\downarrow$

Identification of $\mathcal{L}_0$

maximal domain and maximal realization for $1<p<\infty$:

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$

$\leftarrow$

Ornstein-Uhlenbeck operator

$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle$, $x \in \mathbb{R}^d$, $d \geqslant 2$.

$\downarrow$

$\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \to L^p(\mathbb{R}^d,\mathbb{C}^m)$ is a closed operator, $1<p<\infty$

$\downarrow$

L^p-resolvent estimates and **unique solv. of resolvent equ.** for $\mathcal{L}_0$ in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$, $1<p<\infty$

$\uparrow$

L^p-dissipativity condition: $\exists\, \gamma_A > 0$

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w,z\rangle\, \mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \ \forall\, z,w \in \mathbb{K}^m$$

$\updownarrow$

L^p-first antieigenvalue condition

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{K}^m\\ w\neq 0\\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}, \quad 1<p<\infty$$

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Outline

Energy estimates in exponentially weighted L^p-spaces

Theorem 1: (Resolvent estimates in weighted L^p-spaces)

Let $A \in \mathbb{C}^{m,m}$ satisfy (A1) for some $1 < p < \infty$, let $S \in \mathbb{R}^{d,d}$ satisfy (A3), and let $B \in L^\infty(\mathbb{R}^d, \mathbb{C}^{m,m})$ satisfy the strict accretivity condition

(3) $\quad \mathrm{Re}\,\langle w, B(x)w\rangle \geqslant c_B|w|^2 \; \forall\, x \in \mathbb{R}^d \; \forall\, w \in \mathbb{C}^m$, for some $c_B \in \mathbb{R}$.

Moreover, let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda + c_B > 0$ and let $\theta_1, \theta_2 \in C(\mathbb{R}^d, \mathbb{R})$ be positive with

(4) $\quad \theta_1(x) = \exp\left(-\mu_1\sqrt{|x|^2+1}\right) \quad \text{for} \quad 0 \leqslant |\mu_1| \leqslant \sqrt{\frac{(\mathrm{Re}\lambda + c_B)\gamma_A}{d|A|^2}},$

(5) $\quad \theta_1(x) \leqslant C\theta_2(x) \; \forall\, x \in \mathbb{R}^d$ for some $C > 0$,

Finally, let $g \in L^p_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m)$ and $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m)$ be a solution of

(RE) $\quad (\lambda I - \mathcal{L}_B)\, v = g \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m).$

Then, v is the unique solution of (RE) in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m)$. It holds:

1. $\|v\|_{L^p_{\theta_1}} \leqslant \frac{2C^{\frac{1}{p}}}{\mathrm{Re}\lambda + c_B}\|g\|_{L^p_{\theta_2}}$,
2. $\|D_i v\|_{L^p_{\theta_1}} \leqslant \frac{2C^{\frac{1}{p}}\gamma_A^{-\frac{1}{2}}}{(\mathrm{Re}\lambda + c_B)^{\frac{1}{2}}}\|g\|_{L^p_{\theta_2}}$, if $1 < p \leqslant 2$,

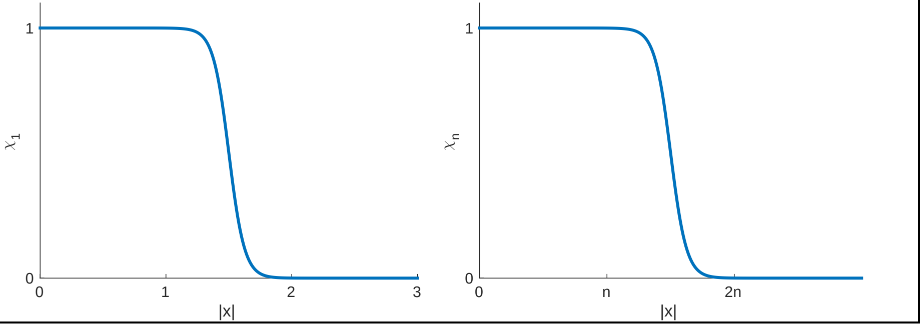
with C from (5), γ_A from (A1) and c_B from (3).

Proof of Theorem 1

Cut-off functions: Let $v \in W^{2,p}_{\mathrm{loc}} \cap L^p_{\theta_1}$ satisfy (RE) for some $g \in L^p_{\theta_2}$.
Introduce cut-off functions: $n \in \mathbb{N}$, $n > 0$

$$\chi_n(x) = \chi_1\left(\frac{x}{n}\right), \quad \chi_1 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}), \quad \chi_1(x) = \begin{cases} 1 & , |x| \leqslant 1, \\ \in [0,1], \text{ smooth} & , 1 < |x| < 2, \\ 0 & , |x| \geqslant 2. \end{cases}$$

(RE) $$g = (\lambda I - \mathcal{L}_B)v = \lambda v - A\triangle v - \langle Sx, \nabla v\rangle + B(x)v$$



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(RE) $$g = (\lambda I - \mathcal{L}_B)v = \lambda v - A\triangle v - \langle Sx, \nabla v\rangle + B(x)v$$

Step 1: Multiply (RE) by $\chi_n^2\theta_1\overline{v}^\top |v|^{p-2}$, integrate over $\mathbb{R}^d$, and take real parts

$$\begin{aligned} \chi_n^2\theta_1 |v|^{p-2}\overline{v}^\top g = \quad & \lambda \quad \chi_n^2\theta_1 |v|^p - \quad \chi_n^2\theta_1\overline{v}^\top |v|^{p-2} A\triangle v \\ & - \quad \chi_n^2\theta_1\overline{v}^\top |v|^{p-2}\sum_{j=1}^d (Sx)_j D_j v \\ & + \quad \chi_n^2\theta_1\overline{v}^\top |v|^{p-2} Bv. \end{aligned}$$

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$$\begin{aligned}\int_{\mathbb{R}^d} \chi_n^2\theta_1 |v|^{p-2}\,\overline{v}^\top g = & \quad \lambda \int_{\mathbb{R}^d} \chi_n^2\theta_1 |v|^p - \int_{\mathbb{R}^d} \chi_n^2\theta_1\overline{v}^\top |v|^{p-2} A\triangle v \\ & - \int_{\mathbb{R}^d} \chi_n^2\theta_1\overline{v}^\top |v|^{p-2} \sum_{j=1}^d (Sx)_j D_j v \\ & + \int_{\mathbb{R}^d} \chi_n^2\theta_1\overline{v}^\top |v|^{p-2} Bv.\end{aligned}$$

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(RE) $$g = (\lambda I - \mathcal{L}_B)v = \lambda v - A\triangle v - \langle Sx, \nabla v\rangle + B(x)v$$

Step 1: Multiply (RE) by $\chi_n^2 \theta_1 \overline{v}^\top |v|^{p-2}$, integrate over $\mathbb{R}^d$, and take real parts

$$\begin{aligned} \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} \overline{v}^\top g =& (\mathrm{Re}\lambda) \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p - \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^\top |v|^{p-2} A\triangle v \\ &- \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^\top |v|^{p-2} \sum_{j=1}^d (Sx)_j D_j v \\ &+ \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \overline{v}^\top |v|^{p-2} Bv. \end{aligned}$$

Proof of Theorem 1

Step 1:

$$\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\overline{v}^{\top}g=(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p}-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^{\top}\left|v\right|^{p-2}A\triangle v$$

$$-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^{\top}\left|v\right|^{p-2}\sum_{j=1}^{d}(Sx)_jD_jv+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^{\top}\left|v\right|^{p-2}Bv.$$

Step 2: Rewrite the 3rd term on the RHS. (A3) and integration by parts imply

$$0=\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\theta_1\Big(\sum_{j=1}^{d}S_{jj}\Big)\left|v\right|^{p}=\frac{1}{p}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2D_j\left((Sx)_j\right)\theta_1\left|v\right|^{p}$$

$$=-\frac{2}{p}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n(D_j\chi_n)(Sx)_j\theta_1\left|v\right|^{p}-\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1(Sx)_j\mathrm{Re}\left(\overline{D_jv}^{\top}v\right)\left|v\right|^{p-2}$$

$$-\frac{1}{p}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2(Sx)_j(D_j\theta_1)\left|v\right|^{p}\quad(\text{use: } D_j\left(|v|^p\right)=p|v|^{p-2}\mathrm{Re}(\overline{D_jv}^{\top}v))$$

$$=-\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\left|v\right|^{p}\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^{\top}\left|v\right|^{p-2}\sum_{j=1}^{d}(Sx)_jD_jv$$

$$-\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\left|v\right|^{p}\sum_{j=1}^{d}(Sx)_j(D_j\theta_1).$$

Proof of Theorem 1

Step 2:

$$\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\overline{v}^\top g=(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p}-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\left|v\right|^{p-2}A\triangle v$$

$$+\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\left|v\right|^{p}\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j+\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\left|v\right|^{p}\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)$$

$$+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\left|v\right|^{p-2}Bv.$$

Proof of Theorem 1

Step 2:

$$\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\overline{v}^\top g=(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\left|v\right|^{p-2}A\triangle v$$

$$+\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\left|v\right|^p\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j+\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\left|v\right|^p\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)$$

$$+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\left|v\right|^{p-2}Bv.$$

Step 3: To the 2nd term apply the following formula with $\Omega=B_{2n}(0)$, $\eta=\chi_n^2\theta_1$

$$-\mathrm{Re}\int_\Omega\eta\overline{v}^\top\left|v\right|^{p-2}A\triangle v$$

$$\geqslant\mathrm{Re}\int_\Omega\eta|v|^{p-2}\sum_{j=1}^{d}\overline{D_jv}^\top AD_jv\mathbb{1}_{v\neq0}+\mathrm{Re}\int_\Omega\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}D_j\eta AD_jv$$

$$+(p-2)\mathrm{Re}\int_\Omega\eta|v|^{p-4}\sum_{j=1}^{d}\mathrm{Re}\left(\overline{D_jv}^\top v\right)\overline{v}^\top AD_jv\mathbb{1}_{v\neq0}.$$

Note: $\chi_n(x)=0$, if $\left|\frac{x}{n}\right|\geqslant2$. $[|v|^q\mathbb{1}_{\{v\neq0\}}](x)=\begin{cases}|v(x)|^q, & |v(x)|>0,\\ 0, & v(x)=0,\end{cases}$ if $q<0$.

Proof of Theorem 1

Step 3:

$$\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\overline{v}^\top g \geqslant (\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p+\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\left|v\right|^p\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j$$

$$+\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\left|v\right|^p\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)+\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}D_j\chi_n AD_jv$$

$$+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}(D_j\theta_1)AD_jv+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\sum_{j=1}^{d}\overline{D_jv}^\top AD_jv\mathbb{1}_{v\neq 0}$$

$$+(p-2)\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\sum_{j=1}^{d}\mathrm{Re}\left(\overline{D_jv}^\top v\right)\overline{v}^\top AD_jv\mathbb{1}_{v\neq 0}+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\left|v\right|^{p-2}Bv.$$

Proof of Theorem 1

Step 3:

$$\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,\overline{v}^\top g \geqslant (\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p+\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\,|v|^p\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j$$

$$+\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\,|v|^p\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)+\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}D_j\chi_n AD_jv$$

$$+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}(D_j\theta_1)AD_jv+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\sum_{j=1}^{d}\overline{D_jv}^\top AD_jv\mathbb{1}_{v\neq 0}$$

$$+(p-2)\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\sum_{j=1}^{d}\mathrm{Re}\left(\overline{D_jv}^\top v\right)\overline{v}^\top AD_jv\mathbb{1}_{v\neq 0}+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\,|v|^{p-2}\,Bv.$$

Step 4: Substract the 2nd, 3rd, 4th and 5th term of the RHS.

Proof of Theorem 1

Step 4:

$$(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\sum_{j=1}^d\overline{D_jv}^\top AD_jv\mathbb{1}_{v\neq 0}$$

$$+(p-2)\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\sum_{j=1}^d\mathrm{Re}\left(\overline{D_jv}^\top v\right)\overline{v}^\top AD_jv\mathbb{1}_{v\neq 0}+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\,|v|^{p-2}\,Bv$$

$$\leqslant\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,\overline{v}^\top g-\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^d D_j\chi_nAD_jv$$

$$-\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\,|v|^p\sum_{j=1}^d(D_j\chi_n)(Sx)_j-\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\,|v|^p\sum_{j=1}^d(Sx)_j(D_j\theta_1)$$

$$-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^d(D_j\theta_1)AD_jv.$$

Proof of Theorem 1

Step 4:

$$(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\sum_{j=1}^d\overline{D_jv}^\top AD_jv\mathbb{1}_{v\neq 0}$$

$$+(p-2)\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\sum_{j=1}^d\mathrm{Re}\left(\overline{D_jv}^\top v\right)\overline{v}^\top AD_jv\mathbb{1}_{v\neq 0}+\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\overline{v}^\top\,|v|^{p-2}\,Bv$$

$$\leqslant\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,\overline{v}^\top g-\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^d D_j\chi_nAD_jv$$

$$-\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1\,|v|^p\sum_{j=1}^d(D_j\chi_n)(Sx)_j-\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2\,|v|^p\sum_{j=1}^d(Sx)_j(D_j\theta_1)$$

$$-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^d(D_j\theta_1)AD_jv.$$

Step 4: Write the LHS in terms of inner products.

Proof of Theorem 1

Step 4:

$$(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\,\mathrm{Re}\,\langle v,Bv\rangle$$

$$+\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\,\mathbb{1}_{\{v\neq 0\}}\sum_{j=1}^{d}\left[|v|^2\,\mathrm{Re}\,\langle D_jv,AD_jv\rangle+(p-2)\mathrm{Re}\,\langle D_jv,v\rangle\,\mathrm{Re}\,\langle v,AD_jv\rangle\right]$$

$$\leqslant \mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\,\overline{v}^\top g-\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}D_j\chi_nAD_jv$$

$$-\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1|v|^p\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j-\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2|v|^p\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)$$

$$-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}(D_j\theta_1)AD_jv.$$

Proof of Theorem 1

Step 4:

$$(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\,\mathrm{Re}\,\langle v,Bv\rangle$$

$$+\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\,\mathbb{1}_{\{v\neq 0\}}\sum_{j=1}^{d}\left[|v|^2\,\mathrm{Re}\,\langle D_jv,AD_jv\rangle+(p-2)\mathrm{Re}\,\langle D_jv,v\rangle\,\mathrm{Re}\,\langle v,AD_jv\rangle\right]$$

$$\leqslant\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\,\overline{v}^\top g-\mathrm{Re}\int_{\mathbb{R}^d}2\chi_n\theta_1\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}D_j\chi_nAD_jv$$

$$-\frac{2}{p}\int_{\mathbb{R}^d}\chi_n\theta_1|v|^p\sum_{j=1}^{d}(D_j\chi_n)(Sx)_j-\frac{1}{p}\int_{\mathbb{R}^d}\chi_n^2|v|^p\sum_{j=1}^{d}(Sx)_j(D_j\theta_1)$$

$$-\mathrm{Re}\int_{\mathbb{R}^d}\chi_n^2\overline{v}^\top|v|^{p-2}\sum_{j=1}^{d}(D_j\theta_1)AD_jv=:\sum_{j=1}^{5}T_j.$$

Step 5: Next estimate the terms $T_1,\ldots,T_5$ successively.

Proof of Theorem 1

Estimate on T_1:

$$T_1 = \mathrm{Re} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \left|v\right|^{p-2} \overline{v}^\top g$$

Apply $\mathrm{Re} z \leqslant |z|$, (5) (i.e. $\theta_1(x) \leqslant C\theta_2(x)\ \forall\, x \in \mathbb{R}^d$), and Hölder's inequality

$$\begin{aligned} T_1 =& \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \left|v\right|^{p-2} \mathrm{Re}\left(\overline{v}^T g\right) \leqslant \int_{\mathbb{R}^d} \chi_n^2 \theta_1 \left|v\right|^{p-1} |g| \\ \leqslant& \left(\int_{\mathbb{R}^d} \left(\chi_n^{\frac{2(p-1)}{p}} \theta_1^{\frac{p-1}{p}} \left|v\right|^{p-1} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \left(\chi_n^{\frac{2}{p}} \theta_1^{\frac{1}{p}} |g| \right)^p \right)^{\frac{1}{p}} \\ \leqslant& C^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 \left|v\right|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_2 \left|g\right|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hölder's inequality: If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $1 = \frac{1}{p} + \frac{1}{q}$, $p, q \in [1, \infty]$, then $fg \in L^1(\mathbb{R}^d)$ and

$$\left\|fg\right\|_{L^1} = \int_{\mathbb{R}^d} |fg| \leqslant \left(\int_{\mathbb{R}^d} |f|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |g|^q \right)^{\frac{1}{q}} = \left\|f\right\|_{L^p} \left\|g\right\|_{L^q}.$$

Proof of Theorem 1

Estimate on T_2:

$$T_2 = -\mathrm{Re}\int_{\mathbb{R}^d} 2\chi_n\theta_1\overline{v}^\top |v|^{p-2}\sum_{j=1}^{d} D_j\chi_n A D_j v$$

Apply Hölder's inequality with $p=q=2$ and Young's inequality with $\delta>0$

$$\begin{aligned}
T_2 \leqslant & 2|A|\int_{\mathbb{R}^d}\chi_n\theta_1 |v|^{p-1}\sum_{j=1}^{d}|D_j\chi_n|\,|D_j v| \leqslant \frac{2|A|\|\chi_1\|_{1,\infty}}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n\theta_1|D_j v|\,|v|^{p-1}\\
\leqslant & \frac{2|A|\|\chi_1\|_{1,\infty}}{n}\sum_{j=1}^{d}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1|D_j v|^2|v|^{p-2}\mathbb{1}_{\{v\neq 0\}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\theta_1|v|^p\right)^{\frac{1}{2}}\\
\leqslant & \frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1|D_j v|^2|v|^{p-2}\mathbb{1}_{\{v\neq 0\}} + \frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1|v|^p.
\end{aligned}$$

Here we used that for every $x\in\mathbb{R}^d$ and $j=1,\ldots,d$

$$|D_j\chi_n(x)| = \left|D_j\left(\chi_1\left(\tfrac{x}{n}\right)\right)\right| \leqslant \frac{1}{n}\max_{j=1,\ldots,d}\max_{y\in\mathbb{R}^d}|D_j\chi_1(y)| = \frac{\|\chi_1\|_{1,\infty}}{n}.$$

Proof of Theorem 1

Estimate on T_3:

$$T_3 = -\frac{2}{p}\int_{\mathbb{R}^d} \chi_n \theta_1 |v|^p \sum_{j=1}^{d} (D_j\chi_n)(Sx)_j$$

Use $\chi_n(x) = 0$ for $|x| \geqslant 2n$ and $D_j\chi_n(x) = 0$ for $|x| \leqslant n$

$$T_3 \leqslant \frac{2}{p}\sum_{j=1}^{d}\int_{\mathbb{R}^d} \chi_n\theta_1 |v|^p |(Sx)_j| |D_j\chi_n|$$

$$= \frac{2}{p}\sum_{j=1}^{d}\int_{n\leqslant|x|\leqslant 2n} \chi_n\theta_1 |v|^p |(Sx)_j| |D_j\chi_n| \leqslant \frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n} \theta_1 |v|^p .$$

For the last estimate note that $\chi_n(x) \leqslant 1$ and

$$|(Sx)_j| |D_j\chi_n(x)| = \frac{1}{n}|(Sx)_j| \left|(D_j\chi_1)\left(\frac{x}{n}\right)\right| \leqslant \frac{1}{n}|S||x| \left|(D_j\chi_1)\left(\frac{x}{n}\right)\right|$$

$$\leqslant \frac{|S|}{n}\Big(\sup_{n\leqslant|\xi|\leqslant 2n}|\xi|\Big)\max_{j=1,\ldots,d}\max_{y\in\mathbb{R}^d}|D_j\chi_1(y)| = 2\,|S|\,\|\chi_1\|_{1,\infty} .$$

Proof of Theorem 1

Estimate on T_4:

$$T_4 = -\frac{1}{p}\int_{\mathbb{R}^d} \chi_n^2 \,|v|^p \sum_{j=1}^{d} (Sx)_j (D_j\theta_1)$$

The 4th term vanishes due to (4) and (A3)

$$T_4 = -\frac{1}{p}\int_{\mathbb{R}^d} \chi_n^2 \frac{-\mu_1}{\sqrt{|x|^2+1}}\theta_1 \,|v|^p \underbrace{\sum_{j=1}^{d} x_j (Sx)_j}_{=x^\top Sx=0} = 0.$$

Note that skew-symmetry of $S \in \mathbb{R}^{d,d}$ from (A3) implies

$$x^\top Sx = \tfrac{1}{2}x^\top Sx + \tfrac{1}{2}(x^\top Sx)^\top = \tfrac{1}{2}x^\top (S+S^\top)x = 0,\ x \in \mathbb{R}^d.$$

Proof of Theorem 1

Estimate on T_5:

$$T_5 = -\mathrm{Re}\int_{\mathbb{R}^d} \chi_n^2 \overline{v}^\top |v|^{p-2} \sum_{j=1}^{d} (D_j\theta_1) A D_j v$$

Apply $\mathrm{Re}z \leqslant |z|$, Hölder's inequality with $p = q = 2$ and Young's inequality with some $\rho > 0$, (4) and $|\mu_1| \leqslant \mu_0$ for some $\mu_0 \geqslant 0$ that will be specified below

$$T_5 \leqslant \int_{\mathbb{R}^d} \chi_n^2 |v|^{p-1} \sum_{j=1}^{d} \left| \frac{-\mu_1 x_j}{\sqrt{|x|^2+1}} \right| \theta_1 |A| \, |D_j v| \leqslant |\mu_1||A| \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-1} |D_j v|$$

$$\leqslant |\mu_1||A| \sum_{j=1}^{d} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} |D_j v|^2 \mathbb{1}_{\{v \neq 0\}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p \right)^{\frac{1}{2}}$$

$$\leqslant \frac{\mu_0 |A|}{4\rho} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} |D_j v|^2 \mathbb{1}_{\{v \neq 0\}} + \mu_0 |A| \rho d \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p .$$

Proof of Theorem 1

Step 5: Summarizing, we arrive at the following estimate

$$
\begin{aligned}
&(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}\,\mathrm{Re}\,\langle v,Bv\rangle\\
&+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-4}\,\mathbb{1}_{\{v\neq 0\}}\Big[|v|^2\,\mathrm{Re}\,\langle D_jv,AD_jv\rangle+(p-2)\mathrm{Re}\,\langle D_jv,v\rangle\,\mathrm{Re}\,\langle v,AD_jv\rangle\Big]\\
\leqslant{}& C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2|g|^p\right)^{\frac{1}{p}}+\frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1|v|^p\\
&+\frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1|v|^p+\frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\,\mathbb{1}_{\{v\neq 0\}}\\
&+\frac{\mu_0|A|}{4\rho}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\,\mathbb{1}_{\{v\neq 0\}}+\mu_0|A|\rho d\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p.
\end{aligned}
$$

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&(\mathrm{Re}\lambda)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p+\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,\mathrm{Re}\,\langle v,Bv\rangle\\
&+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-4}\,\mathbb{1}_{\{v\neq 0\}}\Big[|v|^2\,\mathrm{Re}\,\langle D_jv,AD_jv\rangle+(p-2)\mathrm{Re}\,\langle D_jv,v\rangle\,\mathrm{Re}\,\langle v,AD_jv\rangle\Big]\\
\leqslant\, & C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\,|g|^p\right)^{\frac{1}{p}}+\frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\,|v|^p\\
&+\frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\,|v|^p+\frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,|D_jv|^2\,\mathbb{1}_{\{v\neq 0\}}\\
&+\frac{\mu_0|A|}{4\rho}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,|D_jv|^2\,\mathbb{1}_{\{v\neq 0\}}+\mu_0|A|\rho d\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\,.
\end{aligned}
$$

- L^p**-dissipativity** for $A\in\mathbb{C}^{m,m}$: There is $\gamma_A>0$ such that

(A1) $\quad |z|^2\,\mathrm{Re}\,\langle w,Aw\rangle+(p-2)\mathrm{Re}\,\langle w,z\rangle\,\mathrm{Re}\,\langle z,Aw\rangle\geqslant\gamma_A|z|^2|w|^2\;\forall\,z,w\in\mathbb{C}^m$

- **strict accretivity** for $B\in L^\infty(\mathbb{R}^d,\mathbb{C}^{m,m})$: There is $c_B\in\mathbb{R}$ such that

(3) $\quad \mathrm{Re}\langle v,B(x)v\rangle\geqslant c_B|v|^2\;\forall\,x\in\mathbb{R}^d\;\forall\,v\in\mathbb{C}^m$

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&(\mathrm{Re}\lambda + c_B)\int_{\mathbb{R}^d} \chi_n^2\theta_1 \left|v\right|^p \\
&+ \gamma_A \sum_{j=1}^{d}\int_{\mathbb{R}^d} \chi_n^2\theta_1 \left|v\right|^{p-2}\left|D_j v\right|^2 \mathbb{1}_{\{v\neq 0\}} \\
\leqslant\, & C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\left|g\right|^p\right)^{\frac{1}{p}} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p \\
&+ \frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\left|v\right|^p + \frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_j v\right|^2\mathbb{1}_{\{v\neq 0\}} \\
&+ \frac{\mu_0|A|}{4\rho}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_j v\right|^2\mathbb{1}_{\{v\neq 0\}} + \mu_0|A|\rho d\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p.
\end{aligned}
$$

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&(\mathrm{Re}\lambda + c_B)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p \\
&+\gamma_A\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}} \\
\leqslant\, & C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\left\|\chi_1\right\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p \\
&+\frac{4d\left|S\right|\left\|\chi_1\right\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\left|v\right|^p+\frac{2|A|\left\|\chi_1\right\|_{1,\infty}\delta}{n}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}} \\
&+\frac{\mu_0|A|}{4\rho}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}}+\mu_0|A|\rho d\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p.
\end{aligned}
$$

Subtracting the 4th, 5th and 6th term of the RHS.

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&\left(\mathrm{Re}\lambda + c_B - \mu_0|A|\rho d\right)\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p \\
&+\left(\gamma_A - \frac{\mu_0|A|}{4\rho} - \frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\right)\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}} \\
\leqslant{}& C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\left|g\right|^p\right)^{\frac{1}{p}} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p \\
&+\frac{4d\left|S\right|\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\left|v\right|^p
\end{aligned}
$$

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&(\mathrm{Re}\lambda + c_B - \mu_0|A|\rho d)\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p \\
&+\left(\gamma_A - \frac{\mu_0|A|}{4\rho} - \frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\right)\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^{p-2}\,|D_j v|^2\,\mathbb{1}_{\{v\neq 0\}} \\
\leqslant{}& C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\,|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\,|g|^p\right)^{\frac{1}{p}} + \frac{2d|A|\,\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\,|v|^p \\
&+\frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\,|v|^p
\end{aligned}
$$

Choose $\rho = \sqrt{\frac{\mathrm{Re}\lambda + c_B}{4d\gamma_A}}$, $\mu_0 = \sqrt{\frac{(\mathrm{Re}\lambda + c_B)\gamma_A}{d|A|^2}}$ so that

$$
\mathrm{Re}\lambda + c_B - \mu_0|A|\rho d = \frac{\mathrm{Re}\lambda + c_B}{2} \quad \text{and} \quad \gamma_A - \frac{\mu_0|A|}{4\rho} = \frac{\gamma_A}{2}.
$$

Proof of Theorem 1

Step 5:

$$
\begin{aligned}
&\frac{\mathrm{Re}\lambda + c_B}{2} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p \\
&+ \left(\frac{\gamma_A}{2} - \frac{2|A| \, \|\chi_1\|_{1,\infty} \, \delta}{n} \right) \sum_{j=1}^{d} \int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^{p-2} |D_j v|^2 \, \mathbb{1}_{\{v \neq 0\}} \\
\leqslant\, & C^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_1 |v|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \chi_n^2 \theta_2 |g|^p \right)^{\frac{1}{p}} + \frac{2d|A| \, \|\chi_1\|_{1,\infty}}{4n\delta} \int_{\mathbb{R}^d} \theta_1 |v|^p \\
&+ \frac{4d\,|S|\,\|\chi_1\|_{1,\infty}}{p} \int_{n \leqslant |x| \leqslant 2n} \theta_1 |v|^p
\end{aligned}
$$

Proof of Theorem 1

Step 5:

$$\frac{\mathrm{Re}\lambda + c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p$$
$$+\left(\frac{\gamma_A}{2}-\frac{2|A|\left\|\chi_1\right\|_{1,\infty}\delta}{n}\right)\sum_{j=1}^{d}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_j v\right|^2\mathbb{1}_{\{v\neq 0\}}$$
$$\leqslant C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\left\|\chi_1\right\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p$$
$$+\frac{4d\left|S\right|\left\|\chi_1\right\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\left|v\right|^p$$

Step 6: Apply **Fatou's lemma** & **Lebesgue's dominated convergence theorem**.

6.a. Apply limit inferior as $n\to\infty$ on both sides

6.b. Apply Lebesgue's dominated convergence to the integrals on the RHS.

6.c. Apply Fatou to the integrals on the LHS.

Note: Assumptions of **Fatou** are satisfied thanks to **Lebesgue**!!!

Proof of Theorem 1

Step 6.a: Apply limit inferior as $n \to \infty$

$$\liminf_{n\to\infty}\left[\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\,\mathbb{1}_{\{v\neq 0\}}\right]$$

$$\overset{5.}{\leqslant}\liminf_{n\to\infty}\left[C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2|g|^p\right)^{\frac{1}{p}}+\frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1|v|^p\right.$$

$$\left.+\frac{4d|S|\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1|v|^p\right]$$

Proof of Theorem 1

Step 6.b: Apply Lebesgue's dominated convergence (L)

$$\liminf_{n\to\infty}\left[\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\mathbb{1}_{\{v\neq 0\}}\right]$$

$$\overset{5.}{\leqslant}\liminf_{n\to\infty}\left[C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2|g|^p\right)^{\frac{1}{p}}+\frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1|v|^p\right.$$
$$\left.+\frac{4d|S|\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1|v|^p\right]$$

$$=C^{\frac{1}{p}}\left(\lim_{n\to\infty}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\lim_{n\to\infty}\int_{\mathbb{R}^d}\chi_n^2\theta_2|g|^p\right)^{\frac{1}{p}}+\lim_{n\to\infty}\int_{\mathbb{R}^d}\frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\theta_1|v|^p$$
$$+\frac{4d|S|\|\chi_1\|_{1,\infty}}{p}\lim_{n\to\infty}\int_{n\leqslant|x|\leqslant 2n}\theta_1|v|^p$$

$$\overset{L}{=}C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\theta_2|g|^p\right)^{\frac{1}{p}}=C^{\frac{1}{p}}\|v\|_{L^p_{\theta_1}}^{p-1}\|g\|_{L^p_{\theta_2}}$$

Lebesgue's dominated convergence: $f_n, f: S\to Y$ measurable, $g\in L^1(S,Y)$, $|f_n|\leqslant g$ a.e. $\forall\, n\in\mathbb{N}$, $f_n\to f$ a.e. as $n\to\infty$. Then

$$f_n, f\in L^1(S,Y)\quad\text{and}\quad f_n\to f \text{ in } L^1 \text{ as } n\to\infty.$$

Proof of Theorem 1

Step 6.c: Apply Fatou's lemma (F)

$$\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p+\frac{\gamma_A}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}}$$

$$=\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\liminf_{n\to\infty}\chi_n^2\theta_1\left|v\right|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\liminf_{n\to\infty}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}}$$

$$\overset{F}{\leqslant}\liminf_{n\to\infty}\left[\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1\left|v\right|^{p-2}\left|D_jv\right|^2\mathbb{1}_{\{v\neq 0\}}\right]$$

$$\overset{5.}{\leqslant}\liminf_{n\to\infty}\left[C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1\left|v\right|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2\left|g\right|^p\right)^{\frac{1}{p}}+\frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1\left|v\right|^p\right.$$

$$\left.+\frac{4d|S|\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1\left|v\right|^p\right]$$

$$\overset{L}{=}C^{\frac{1}{p}}\left\|v\right\|_{L^p_{\theta_1}}^{p-1}\left\|g\right\|_{L^p_{\theta_2}}$$

Fatou's lemma: $f_n\in L^1(S,Y)$, $f_n\geqslant 0$, $\liminf_{n\to\infty}\int_S f_n dx<\infty$. Then

$$\liminf_{n\to\infty}f_n\in L^1(S,Y)\quad\text{and}\quad\int_S\liminf_{n\to\infty}f_n dx\leqslant\liminf_{n\to\infty}\int_S f_n dx$$

Proof of Theorem 1

Step 6.c: Apply Fatou's lemma (F)

$$\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\theta_1|v|^p+\frac{\gamma_A}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\theta_1|v|^{p-2}|D_jv|^2\mathbb{1}_{\{v\neq 0\}}$$

$$=\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\liminf_{n\to\infty}\chi_n^2\theta_1|v|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\liminf_{n\to\infty}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\mathbb{1}_{\{v\neq 0\}}$$

$$\overset{F}{\leqslant}\liminf_{n\to\infty}\left[\frac{\mathrm{Re}\lambda+c_B}{2}\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p+\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(\frac{\gamma_A}{2}-\frac{2|A|\|\chi_1\|_{1,\infty}\delta}{n}\right)\chi_n^2\theta_1|v|^{p-2}|D_jv|^2\mathbb{1}_{\{v\neq 0\}}\right]$$

$$\overset{5.}{\leqslant}\liminf_{n\to\infty}\Bigg[C^{\frac{1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_1|v|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\chi_n^2\theta_2|g|^p\right)^{\frac{1}{p}}+\frac{2d|A|\|\chi_1\|_{1,\infty}}{4n\delta}\int_{\mathbb{R}^d}\theta_1|v|^p$$
$$+\frac{4d|S|\|\chi_1\|_{1,\infty}}{p}\int_{n\leqslant|x|\leqslant 2n}\theta_1|v|^p\Bigg]$$

$$\overset{L}{=}C^{\frac{1}{p}}\|v\|_{L^p_{\theta_1}}^{p-1}\|g\|_{L^p_{\theta_2}}$$

Choose $\delta>0$ such that $\frac{\gamma_A}{2}-2|A|\,\|\chi_1\|_{1,\infty}\,\delta>0$, then

$$\frac{\gamma_A}{2}\geqslant 2|A|\,\|\chi_1\|_{1,\infty}\,\delta\geqslant\frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\;\forall\,n\in\mathbb{N}\quad\Rightarrow\quad\frac{\gamma_A}{2}-\frac{2|A|\,\|\chi_1\|_{1,\infty}\,\delta}{n}\geqslant 0$$

Proof of Theorem 1

Step 6:

$$\frac{\mathrm{Re}\lambda + c_B}{2}\int_{\mathbb{R}^d}\theta_1 |v|^p + \frac{\gamma_A}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\theta_1 |v|^{p-2}|D_j v|^2 \mathbb{1}_{\{v\neq 0\}} \leqslant C^{\frac{1}{p}} \|v\|_{L^p_{\theta_1}}^{p-1}\|g\|_{L^p_{\theta_2}}$$

Step 7: From Step 6 we obtain

$$\frac{\mathrm{Re}\lambda + c_B}{2}\|v\|_{L^p_{\theta_1}}^p \leqslant \frac{\mathrm{Re}\lambda + c_B}{2}\int_{\mathbb{R}^d}\theta_1 |v|^p + \frac{\gamma_A}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\theta_1 |v|^{p-2}|D_j v|^2 \mathbb{1}_{\{v\neq 0\}}$$
$$\leqslant C^{\frac{1}{p}} \|v\|_{L^p_{\theta_1}}^{p-1}\|g\|_{L^p_{\theta_2}}$$

Dividing both sides by $\frac{\mathrm{Re}\lambda+c_B}{2}$ and $\|v\|_{L^p_{\theta_1}}^{p-1}$ yields the $L^p_{\theta_1}$-resolvent estimate

$$\|v\|_{L^p_{\theta_1}} \leqslant \frac{2C^{\frac{1}{p}}}{\mathrm{Re}\lambda + c_B}\|g\|_{L^p_{\theta_2}}$$

Proof of Theorem 1

Step 7: $L^p_{\theta_1}$-resolvent estimate

$$\|v\|_{L^p_{\theta_1}} \leqslant \frac{2C^{\frac{1}{p}}}{\mathrm{Re}\lambda + c_B} \|g\|_{L^p_{\theta_2}} .$$

Unique solvability of $(\lambda I - \mathcal{L}_B)v = g$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m)$:

Let $g \in L^p_{\theta_2}$ and let $v_1, v_2 \in W^{2,p}_{\mathrm{loc}} \cap L^p_{\theta_1}$ satisfy

$$(\lambda I - \mathcal{L}_B)v_1 = g, \quad (\lambda I - \mathcal{L}_B)v_2 = g, \quad \text{in } L^p_{\mathrm{loc}}.$$

Then $w = v_1 - v_2 \in W^{2,p}_{\mathrm{loc}} \cap L^p_{\theta_1}$ satisfies

$$(\lambda I - \mathcal{L}_B)w = 0, \quad \text{in } L^p_{\mathrm{loc}}.$$

The resolvent estimate implies $\|w\|_{L^p_{\theta_1}} = 0$, thus $v_1 = v_2$ in $L^p_{\theta_1}$, hence in $W^{2,p}_{\mathrm{loc}} \cap L^p_{\theta_1}$.

Proof of Theorem 1

Step 6:

$$\frac{\mathrm{Re}\lambda + c_B}{2}\left\|v\right\|_{L^p_{\theta_1}}^p + \frac{\gamma_A}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\theta_1\left|v\right|^{p-2}\left|D_j v\right|^2 \mathbb{1}_{\{v\neq 0\}} \leqslant C^{\frac{1}{p}}\left\|v\right\|_{L^p_{\theta_1}}^{p-1}\left\|g\right\|_{L^p_{\theta_2}}$$

Step 7:

$$\left\|v\right\|_{L^p_{\theta_1}} \leqslant \frac{2C^{\frac{1}{p}}}{\mathrm{Re}\lambda + c_B}\left\|g\right\|_{L^p_{\theta_2}}.$$

Step 8: Step 6 implies for any $j = 1, \ldots, m$

$$\int_{\mathbb{R}^d}\theta_1\left|v\right|^{p-2}\left|D_j v\right|^2 \mathbb{1}_{\{v\neq 0\}} \leqslant \frac{2C^{\frac{1}{p}}}{\gamma_A}\left\|v\right\|_{L^p_{\theta_1}}^{p-1}\left\|g\right\|_{L^p_{\theta_2}}.$$

Since $|D_j v| = |D_j v|\mathbb{1}_{\{v\neq 0\}}$ a.e. we deduce from Hölder's inequality for $1 < p \leqslant 2$

$$\left\|D_j v\right\|_{L^p_{\theta_1}}^p = \int_{\mathbb{R}^d}\theta_1\left|D_j v\right|^p \mathbb{1}_{\{v\neq 0\}} = \int_{\mathbb{R}^d}\theta_1^{\frac{p}{2}}\left|D_j v\right|^p\left|v\right|^{-\frac{p(2-p)}{2}}\mathbb{1}_{\{v\neq 0\}}\theta_1^{\frac{2-p}{2}}\left|v\right|^{\frac{p(2-p)}{2}}$$

$$\leqslant\left(\int_{\mathbb{R}^d}\theta_1\left|D_j v\right|^2\left|v\right|^{p-2}\mathbb{1}_{\{v\neq 0\}}\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^d}\theta_1\left|v\right|^p\right)^{\frac{2-p}{2}} \leqslant \left(\frac{4C^{\frac{2}{p}}}{(\mathrm{Re}\lambda + c_B)\gamma_A}\right)^{\frac{p}{2}}\left\|g\right\|_{L^p_{\theta_2}}^p.$$

Sum up $j = 1, \ldots, d$ and taking pth root yields the $W^{1,p}_{\theta_1}$-resolvent estimate

$$\left|v\right|_{W^{1,p}_{\theta_1}} = \left(\sum_{j=1}^{d}\left\|D_j v\right\|_{L^p_{\theta_1}}^p\right)^{\frac{1}{p}} \leqslant \frac{2dC^{\frac{1}{p}}\gamma_A^{-\frac{1}{2}}}{(\mathrm{Re}\lambda + c_B)^{\frac{1}{2}}}\left\|g\right\|_{L^p_{\theta_2}}.$$

Applications of Theorem 1

Some applications of Theorem 1:

1. $B(x) = B_\infty$, $\theta_1(x) = \theta_2(x) = 1$:
   Identification problem of $\mathcal{L}_\infty$ in L^p (unweighted L^p-spaces)
2. $B(x) = B_\infty - Q_{\mathrm{s}}(x)$:
   A-priori estimates for solutions $v \in L^p_{\theta_1}$ of $(\lambda I - \mathcal{L}_Q)v = g$ for $g \in L^p_{\theta_2}$
   (necessary for proving exponential decay).
3. $B(x) = B_\infty$, $\theta_1(x) = \theta_2(x)$:
   Identification problem of $\mathcal{L}_\infty$ in $L^p_{\theta_1}$ (weighted L^p-spaces)

L^p**-dissipativity condition**:

$$\exists\, \gamma_A > 0 :\ |z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\, \mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \ \forall\, w, z \in \mathbb{K}^m$$

Question: Can we express L^p-dissipativity by spectral properties of A?
Answer: Yes, in terms of antieigenvalues of A.

Outline

L^p-dissipativity condition vs. L^p-antieigenvalue bound

Theorem 2: (L^p-dissipativity condition vs. L^p-antieigenvalue bound)

Let $A \in \mathbb{K}^{m,m}$ for $\mathbb{K} = \mathbb{R}$ if $m \geqslant 2$ and $\mathbb{K} = \mathbb{C}$ if $m \geqslant 1$, and let $b \in \mathbb{R}$, $b > -1$.

1. Given some $\gamma_A > 0$, then the following statements are equivalent:

   (6) $|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + b\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \;\forall\, w, z \in \mathbb{K}^m$,

   (7) $\left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{K}^m,\ |w| = 1.$

2. Moreover, the following statements are equivalent:

   (8) $\exists\, \gamma_A > 0 : \left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \quad \forall\, w \in \mathbb{K}^m,\ |w| = 1,$

   (9) A invertible and $\mu_1(A) > \frac{|b|}{2+b}$,

Here, $\mu_1(A)$ denotes the first antieigenvalue of A

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^m \\ w \neq 0 \\ Aw \neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} \quad \text{with} \quad \langle w, z\rangle := \overline{w}^\top z.$$

Apply Theorem 2 for $b = p - 2$ with $1 < p < \infty$.

Outline of proof: Theorem 2

1. Given some $\gamma_A > 0$, then the following statements are equivalent:

(1) $|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + b\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2\ \forall\, w, z \in \mathbb{K}^m$,

(2) $\left(1 + \frac{b}{2}\right)\mathrm{Re}\,\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{K}^m,\ |w| = 1.$

Note: Dividing (1) by $|z|^2|w|^2$ implies equivalence of (1) with

(1') $\mathrm{Re}\,\langle w, Aw\rangle + b\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A\ \forall\, w, z \in \mathbb{K}^m,\ |w| = |z| = 1.$

Case 1: $(\mathbb{K} = \mathbb{R})$. Let $m \geqslant 2$. For $\gamma_A > 0$ given, show equivalence of

(1') $\langle w, Aw\rangle + b\,\langle w, z\rangle\,\langle z, Aw\rangle \geqslant \gamma_A \qquad \forall\, w, z \in \mathbb{R}^m,\ |w| = |z| = 1,$

(2) $\left(1 + \frac{b}{2}\right)\langle w, Aw\rangle - \frac{|b|}{2}\,|Aw| \geqslant \gamma_A \qquad \forall\, w \in \mathbb{R}^m,\ |w| = 1.$

Optimization problem: For any fixed $w \in \mathbb{R}^m$, $|w|^2 = 1$, solve

$$\min_{z \in \mathbb{R}^m} f_w(z) \quad \text{subject to} \quad |z|^2 = 1, \quad f_w(z) = \langle w, Aw\rangle + b\langle w, z\rangle\langle z, Aw\rangle - \gamma_A.$$

Existence of minimum due to boundedness

$$|f_w(z)| \leqslant |w||Aw| + |b||w||z|^2|Aw| + |\gamma_A| = (1 + |b|)|Aw| + |\gamma_A| < \infty.$$

Outline

Explicit representations of the first antieigenvalue

Recall: Theorem 2 shows that

L^p-dissipativity condition: There is $\gamma_A > 0$ such that

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \;\forall\, w, z \in \mathbb{K}^m,$$

and

L^p-antieigenvalue condition:

$$A \text{ invertible} \quad \text{and} \quad \mu_1(A) := \inf_{\substack{w\in\mathbb{K}^m \\ w\neq 0 \\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}$$

are equivalent.

Questions:

1. Are there explicit formulas of $\mu_1(A)$ (e.g. in terms of the eigenvalues of A)?
2. What are the minimizers $w \in \mathbb{K}^m$? And how does one obtain them?

Answer:

- In general no explicit formula, neither for $\mu_1(A)$ nor for $w \in \mathbb{K}^m$
- In some special cases they are obtained by the method of Lagrange multipliers

CASE 1: $(\mathbb{K} = \mathbb{R},\ m = 1)$.

L^p**-dissipativity condition:** There is $\gamma_A > 0$ such that

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \;\forall\, w, z \in \mathbb{K}^m,$$

is equivalent with $(z^2 w^2 A + (p-2) w^2 z^2 A \geqslant \gamma_A z^2 w^2,\ z, w \in \mathbb{R},\ 1 < p < \infty)$

Positivity condition:

$$A > 0$$

CASE 2: $(\mathbb{K} = \mathbb{C},\ m = 1)$.

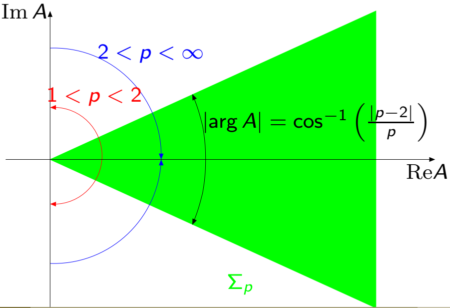
L^p-antieigenvalue bound:

$$\mu_1(A) = \inf_{\substack{w \in \mathbb{C} \\ w \neq 0 \\ Aw \neq 0}} \frac{\mathrm{Re}\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}$$

is equivalent with $\left(\frac{\mathrm{Re}\langle w, Aw\rangle}{|w||Aw|} = \frac{\mathrm{Re}A}{|A|}\right)$

Cone conditions:

$$\frac{|p-2|}{2\sqrt{p-1}} |\mathrm{Im}\,A| < \mathrm{Re}A \quad \text{or} \quad |\arg A| < \cos^{-1}\left(\frac{|p-2|}{p}\right) = \arctan\left(\frac{2\sqrt{p-1}}{|p|}\right).$$



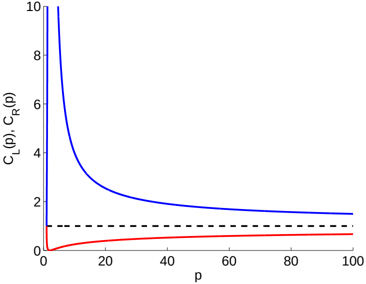
CASE 3: ($\mathbb{K} = \mathbb{C}$, $m \geqslant 2$, A Hermitian positive definite).

L^p-antieigenvalue bound:

$$\mu_1(A) = \frac{\sqrt{\lambda_1^A \lambda_m^A}}{\frac{1}{2}\left(\lambda_1^A + \lambda_m^A\right)} = \frac{2\sqrt{\kappa_A}}{\kappa_A + 1} = \frac{\mathrm{GeometricMean}(\lambda_1^A, \lambda_m^A)}{\mathrm{ArithmeticMean}(\lambda_1^A, \lambda_m^A)} > \frac{|p-2|}{p},$$

Minimizer: $w = \sqrt{\lambda_m^A} w_1 + \sqrt{\lambda_1^A} w_m$, $w_1 \perp w_m$, $Aw_1 = \lambda_1^A w_1$, $Aw_m = \lambda_m^A w_m$.

- $0 < \lambda_1^A \leqslant \cdots \leqslant \lambda_m^A$ eigenvalues
- $\kappa_A = \frac{\lambda_m^A}{\lambda_1^A}$ spectral condition number
- $\sqrt{\lambda_1^A \lambda_m^A}$ geometric mean
- $\frac{1}{2}(\lambda_1^A + \lambda_m^A)$ arithmetic mean



L^p-spectral condition number bound:

$$C_L(p) = \frac{p^2 + 4p - 4 - 4p\sqrt{p-1}}{(p-2)^2} < \kappa_A < \frac{p^2 + 4p - 4 + 4p\sqrt{p-1}}{(p-2)^2} = C_R(p)$$

CASE 4: ($\mathbb{K} = \mathbb{C}$, $m \geqslant 2$, A normal accretive).

L^p-antieigenvalue bound:

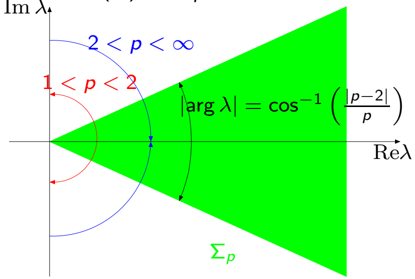
(4) $$\mu_1(A) = \min(E \cup F) > \frac{|p-2|}{p},$$

$$E = \left\{ \frac{a_j^A}{|\lambda_j^A|} : j \in \{1,\ldots,m\} \right\}, \qquad F = \left\{ \frac{2\sqrt{(a_j - a_i)(a_i|\lambda_j^A|^2 - a_j|\lambda_i^A|^2)}}{|\lambda_j^A|^2 - |\lambda_i^A|^2} : \right.$$

$$\left. 0 < \frac{a_j|\lambda_j^A|^2 - 2a_i|\lambda_j^A| + a_j|\lambda_i^A|^2}{(|\lambda_i^A|^2 - |\lambda_j^A|^2)(a_i - a_j)} < 1, |\lambda_i^A| \neq |\lambda_j^A|,\ i,j \in \{1,\ldots,m\} \right\}, \qquad a_j^A := \mathrm{Re}\lambda_j^A$$

- $\min E > \frac{|p-2|}{p}$ is equivalent with **cone condition** $\sigma(A) \subseteq \Sigma_p$ with **conic section**

$$\Sigma_p := \left\{ \lambda \in \mathbb{C} : \frac{|p-2|}{2\sqrt{p-1}} |\mathrm{Im}\,\lambda| < \mathrm{Re}\lambda \right\}$$
$$= \left\{ \lambda \in \mathbb{C} : |\arg \lambda| < \cos^{-1}\left(\frac{|p-2|}{p}\right) \right\}.$$



Minimizer:

- $\mu_1(A) = \frac{a_j^A}{|\lambda_j^A|}$, $w \in \mathbb{C}^m$, $|w_j| = 1$, $|w_k| = 0$, $k \in \{1,\ldots,m\}$, $k \neq j$.

CASE 4: ($\mathbb{K} = \mathbb{C}$, $m \geqslant 2$, A normal accretive).

L^p-antieigenvalue bound:

$$\mu_1(A) = \min(E \cup F) > \frac{|p-2|}{p}, \tag{4}$$

$$E = \left\{ \frac{a_j^A}{|\lambda_j^A|} : j \in \{1,\ldots,m\} \right\}, \qquad F = \left\{ \frac{2\sqrt{(a_j - a_i)(a_i|\lambda_j^A|^2 - a_j|\lambda_i^A|^2)}}{|\lambda_j^A|^2 - |\lambda_i^A|^2} : \right.$$

$$\left. 0 < \frac{a_j|\lambda_j^A|^2 - 2a_i|\lambda_j^A| + a_j|\lambda_i^A|^2}{(|\lambda_i^A|^2 - |\lambda_j^A|^2)(a_i - a_j)} < 1, |\lambda_i^A| \neq |\lambda_j^A|,\ i,j \in \{1,\ldots,m\} \right\}, \qquad a_j^A := \mathrm{Re}\lambda_j^A$$

- $\min F > \frac{|p-2|}{p}$ is equivalent with a **semi-ellipse condition**:

$$\frac{2\sqrt{(a_j - a_i)(a_i|\lambda_j^A|^2 - a_j|\lambda_i^A|^2)}}{|\lambda_j^A|^2 - |\lambda_i^A|^2} > \frac{|p-2|}{p}$$

Note:

$$\frac{2\sqrt{(a_j - a_i)(a_i|\lambda_j^A|^2 - a_j|\lambda_i^A|^2)}}{|\lambda_j^A|^2 - |\lambda_i^A|^2} = \frac{2\sqrt{\frac{|\lambda_j^A|}{|\lambda_i^A|}[(\frac{a_i}{|\lambda_i^A|})(\frac{|\lambda_j^A|}{|\lambda_i^A|}) - \frac{a_j}{|\lambda_j^A|}][(\frac{a_j}{|\lambda_j^A|})(\frac{|\lambda_j^A|}{|\lambda_i^A|}) - \frac{a_i}{|\lambda_i^A|}]}}{(\frac{|\lambda_j^A|}{|\lambda_i^A|})^2 - 1}$$

$$= \frac{2\sqrt{(r_i\rho_{ij} - r_j)(r_j\rho_{ij} - r_i)\rho_{ij}}}{\rho_{ij}^2 - 1}, \quad \rho_{ij} := \frac{|\lambda_j^A|}{|\lambda_i^A|}, \quad r_k := \mathrm{Re}\frac{\lambda_k^A}{|\lambda_k^A|} = \frac{a_k}{|\lambda_k^A|},\ k \in \{i,j\}$$