Fredholm Properties and L^p-Spectra of Localized Rotating Waves in Parabolic Systems

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joint work with: **Wolf-Jürgen Beyn** (Bielefeld University)

- W.-J. Beyn, D. Otten. Fredholm Properties and L^p-Spectra of Localized Rotating waves in Parabolic Systems. Preprint to appear, 2016.
- W.-J. Beyn, D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. *Dyn. Partial Differ. Equ.*, 13(3):191-240, 2016.
- D. Otten. Spatial decay and spectral properties of rotating waves in parabolic systems. PhD thesis, Bielefeld University, *Shaker Verlag*, 2014.

Outline

1. Rotating patterns in $\mathbb{R}^d$
2. Spatial decay of rotating waves
3. Eigenvalue problem for rotating waves and some basic definitions
4. Fredholm properties of linearization in L^p
5. Essential L^p-spectrum and dispersion relation
6. Point L^p-spectrum and shape of eigenfunctions
7. Cubic-quintic complex Ginzburg-Landau equation

Outline

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1)\quad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$, $A\in\mathbb{R}^{m,m}$, $f:\mathbb{R}^m\to\mathbb{R}^m$, $u_0:\mathbb{R}^d\to\mathbb{R}^m$.

Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^m$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x,t)=u(e^{tS}x,t)$ solves

$$(2)\quad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx,\nabla v(x)\rangle = Dv(x)Sx = \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_jD_iv(x) \overset{-S=S^\top}{=} \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_jD_i - x_iD_j\right)v(x)$$

(drift term) **(rotational term)**

Rotating Patterns in $\mathbb{R}^d$

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Note: $v_\star$ is a stationary solution of (2), i.e. $v_\star$ solves the **rotating wave equation**

$$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle$: **Ornstein-Uhlenbeck operator**.

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

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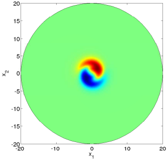
Questions and Ingredients: I1: exp. decay of $v_\star$, I2: spectral properties
Q1: Nonlinear stability of rotating waves on $\mathbb{R}^d$? (**Tools**: I1+I2)
Q2: Truncations of rotating waves to bounded domains? (**Tools**: I1+...)
Q3: Spatial approximation (e.g. with finite element method)? (**open problem**)
Q4: Temporal approximation (e.g. with Euler or BDF)? (**open problem**)

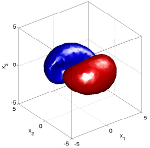
Examples for rotating waves

Cubic-quintic complex Ginzburg-Landau equation: (spinning solitons)

$$u_t = \alpha \triangle u + u\left(\delta + \beta |u|^2 + \gamma |u|^4\right)$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\alpha > 0$, $\delta \in \mathbb{R}$, $d \in \{2,3\}$.

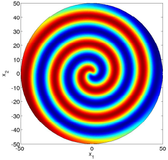


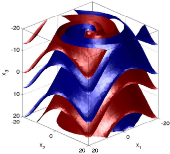


λ-ω system: (spiral waves, scroll waves)

$$u_t = \alpha \triangle u + \left(\lambda(|u|^2) + i\omega(|u|^2)\right) u$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\lambda, \omega : [0,\infty[\to \mathbb{R}$, $\alpha \in \mathbb{C}$, $\mathrm{Re}\alpha > 0$, $d \in \{2,3\}$.

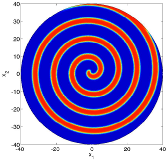




Barkley model: (spiral waves, also scroll waves)

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle u + \begin{pmatrix} \frac{1}{\varepsilon} u_1(1-u_1)(u_1 - \frac{u_2+b}{a}) \\ u_1 - u_2 \end{pmatrix}$$

$u(x,t) \in \mathbb{R}^2$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $0 \leqslant D \ll 1$, $\varepsilon, a, b > 0$, $d \in \{2,3\}$.



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Outline

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$)

Let $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^m)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \left(\begin{array}{rcll} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{array}\right)$$

Spatial decay of rotating waves

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Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ with the following property:

Every classical solution $v_\star \in C^3(\mathbb{R}^d,\mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

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(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0>0$$

satisfies

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$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \left(\begin{array}{rcll} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{array}\right)$$

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$: higher regularity)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^m,\mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k \geqslant 3$).

Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ with the following property:

Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^m)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x)-v_\infty| \leqslant K_1 \text{ for some } R_0>0$$

satisfies

$$v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p}.\qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile $v_\star$: pointwise estimates)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^m,\mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k\geqslant 3$).
Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ with the following property:
Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^m)$ of

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such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x)-v_\infty| \leqslant K_1 \text{ for some } R_0>0$$

satisfies

$$v_\star - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m),\ |D^\alpha(v_\star(x)-v_\infty)| \leqslant C\exp\left(-\mu\sqrt{|x|^2+1}\right)\ \forall x\in\mathbb{R}^d$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p} \qquad \begin{pmatrix} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{pmatrix}$$

and for every multiindex $\alpha\in\mathbb{N}_0^d$ satisfying $d<(k-|\alpha|)p$.

Spatial decay of eigenfunctions

Theorem 2: (Exponential decay of eigenfunctions v)

Let $f\in C^{\max\{2,k\}}(\mathbb{R}^m,\mathbb{R}^m)$, $v_\infty\in\mathbb{R}^m$, $f(v_\infty)=0$, $Df(v_\infty)\leqslant -\beta_\infty I_m<0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta_j(x)=\exp\left(\mu_j\sqrt{|x|^2+1}\right)$ be a weight function for $\mu_j\in\mathbb{R}$, $j=1,2$, $k\in\mathbb{N}$, $p\geqslant\frac{d}{2}$ (if $k\geqslant 2$).
Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ such that for every classical solution $v_\star\in C^{k+1}(\mathbb{R}^d,\mathbb{R}^m)$ of (RWE) satisfying (TC) the following property holds: Every classical solution $v\in C^{k+1}(\mathbb{R}^d,\mathbb{C}^m)$ of

(EVP) $$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+Df(v_\star(x))v(x)=\lambda v(x),\ x\in\mathbb{R}^d,$$

with $\lambda\in\mathbb{C}$, $\mathrm{Re}\lambda\geqslant-(1-\varepsilon)\beta_\infty$, such that

$$v\in L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^m)\quad\text{for **some** exp. growth rate}\quad -\sqrt{\varepsilon\frac{\gamma_A\beta_\infty}{2d|A|^2}}\leqslant\mu_1<0$$

satisfies

$$v\in W^{k,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^m)\quad\text{for **every** exp. decay rate}\quad 0\leqslant\mu_2\leqslant\varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max}p}$$

and

$$|D^\alpha v(x)|\leqslant C\exp\left(-\mu_2\sqrt{|x|^2+1}\right)\ \forall x\in\mathbb{R}^d$$

for every multiindex $\alpha\in\mathbb{N}_0^d$ satisfying $d<(k-|\alpha|)p$.

Exponentially weighted Sobolev spaces and assumptions

Exponentially weighted Sobolev spaces: For $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $1\leqslant p\leqslant\infty$, $k\in\mathbb{N}_0$, and weight function $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ with $\mu\in\mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d,\mathbb{K}^m):=\left\{v\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^m)\mid \|\theta v\|_{L^p}<\infty\right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^m):=\left\{v\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^m)\mid D^\beta u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^m)\;\forall\,|\beta|\leqslant k\right\}.$$

Assumptions:

(A1) (L^p-dissipativity condition): For $A\in\mathbb{R}^{m,m}$, $1<p<\infty$, there is $\gamma_A>0$ with

$$|z|^2\mathrm{Re}\,\langle w,Aw\rangle+(p-2)\mathrm{Re}\,\langle w,z\rangle\,\mathrm{Re}\,\langle z,Aw\rangle\geqslant\gamma_A|z|^2|w|^2\;\forall\,z,w\in\mathbb{R}^m$$

(A2) (System condition): $A, Df(v_\infty)\in\mathbb{R}^{m,m}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) (Rotational condition): $0\neq S\in\mathbb{R}^{d,d}$, $-S=S^\top$

Note: Assumption (A1) is equivalent with

(A1') (L^p-antieigenvalue condition): $A\in\mathbb{R}^{m,m}$ is invertible and

$$\mu_1(A):=\inf_{\substack{w\in\mathbb{R}^m\\ w\neq 0\\ Aw\neq 0}}\frac{\mathrm{Re}\,\langle w,Aw\rangle}{|w||Aw|}>\frac{|p-2|}{p}\text{ for some }1<p<\infty$$

($\mu_1(A)$: first antieigenvalue of A)

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$$L^p_\theta(\mathbb{R}^d, \mathbb{K}^m) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{K}^m) \mid \|\theta v\|_{L^p} < \infty \right\},$$

$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{K}^m) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^m) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^m)\ \forall\, |\beta| \leqslant k \right\}.$$

Assumptions:

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$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\, \mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2\ \forall\, z, w \in \mathbb{R}^m$$

(A2) (System condition): $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) (Rotational condition): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

Additionally:

(A4) (L^q-dissipativity condition): For $A \in \mathbb{R}^{m,m}$, $q = \frac{p}{p-1}$, there is $\delta_A > 0$ with

$$|z|^2 \mathrm{Re}\,\left\langle w, A^H w\right\rangle + (q-2)\mathrm{Re}\,\langle w, z\rangle\, \mathrm{Re}\,\left\langle z, A^H w\right\rangle \geqslant \delta_A |z|^2 |w|^2\ \forall\, z, w \in \mathbb{R}^m$$

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$

Operators: Study the following operators

$$\begin{aligned}
\mathcal{L}_{\mathrm{c}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v + Q_{\mathrm{c}} v, && \text{(exp. decay)}\\
\mathcal{L}_{\mathrm{s}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v, && \text{(exp. decay)}\\
\mathcal{L}_{\infty} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, \quad \text{(far-field operator)} && \text{(exp. decay)}\\
\mathcal{L}_{0} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle. \quad \text{(Ornstein-Uhlenbeck operator)} && \text{(max. domain)}
\end{aligned}$$

D. Otten.
Exponentially weighted resolvent estimates for complex Ornstein-Uhlenbeck systems, 2015.
The identification problem for complex-valued Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, 2016.
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\end{aligned}$$

Maximal domain of $\mathcal{L}_0$ given by

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \left\{v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m) \cap L^p(\mathbb{R}^d,\mathbb{C}^m) :\ \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^m)\right\},\ 1 < p < \infty$$

satisfies $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$.

The operator $\mathcal{L}_0$

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

$\downarrow$

Heat kernel

$$H_0(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right), \; x,\xi \in \mathbb{R}^d, \; t > 0.$$

$\downarrow$

Semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^m)$, $1 \leqslant p \leqslant \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t > 0.$$

strong $\downarrow$ continuity

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \; 1 \leqslant p < \infty.$$

semigroup theory $\swarrow$ $\searrow$ identification problem

unique solv. of resolvent equ. for A_p, $1 \leqslant p < \infty$, $\mathrm{Re}\lambda > 0$

$(\lambda I - A_p)\, v_\star = g \in L^p$.

A-priori $\rightarrow$ estimates

exponential decay, $1 \leqslant p < \infty$

$v_\star \in W^{1,p}_\theta$.

max. domain and max. realization, $1 < p < \infty$

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$.

Identification problem of $\mathcal{L}_0$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m) \cap L^p(\mathbb{R}^d,\mathbb{C}^m) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^m) \right\},\ 1 < p < \infty.$$

Infinitesimal generator

$(A_p, \mathcal{D}(A_p))$, $1 \leqslant p < \infty$.

↓

$\mathcal{S}$ is a core for $(A_p, \mathcal{D}(A_p))$

↓

Identification of $\mathcal{L}_0$

maximal domain and maximal realization for $1 < p < \infty$:

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$

←

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

↓

$\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \to L^p(\mathbb{R}^d,\mathbb{C}^m)$ is a closed operator, $1 < p < \infty$

↓

L^p-resolvent estimates and **unique solv. of resolvent equ.** for $\mathcal{L}_0$ in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$, $1 < p < \infty$

↑

L^p-dissipativity condition: $\exists\, \gamma_A > 0$

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\, \mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2 |w|^2 \ \forall\, z, w \in \mathbb{K}^m$$

↕

L^p-first antieigenvalue condition

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{K}^m\\ w\neq 0\\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}, \quad 1 < p < \infty$$

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Outline

Eigenvalue problem for linearization at rotating waves

Motivation: Stability is determined by spectral properties of linearization $\mathcal{L}$.

Eigenvalue problem:

$$(\lambda I - \mathcal{L})v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2,\ \lambda \in \mathbb{C}.$$

$$\mathcal{L}v(x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Definition 3: (Strongly spectrally stable)

A rotating wave $u_\star(x,t) = v_\star\left(e^{-tS}x\right)$ is called **strongly spectrally stable** iff

1. $\mathrm{Re}\,\sigma(\mathcal{L}) \leqslant 0$ (**spectrally stable**) and
2. $\forall\,\lambda \in \sigma(\mathcal{L}) \cap i\mathbb{R}$: $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$, λ is caused by the $\mathrm{SE}(d)$-group action and

$$\sum_{\lambda\in\sigma(\mathcal{L})\cap i\mathbb{R}} \mathrm{alg}(\lambda) = \frac{d(d+1)}{2} = \mathrm{dim}\mathrm{SE}(d), \quad \mathrm{alg}(\lambda) := \text{algebraic mult. of } \lambda.$$

Recall from spectral theory

Linearized operator is closed and densely defined

$$\mathcal{L}v(x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2,$$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \{v \in W^{2,p}_{\mathrm{loc}} \cap L^p \mid \mathcal{L}_0 v \in L^p\}, \quad \|v\|_{\mathcal{L}_0} := \|v\|_{L^p} + \|\mathcal{L}_0 v\|_{L^p}.$$

Definition 4: (Spectrum of $\mathcal{L}$)

1. **Resolvent set**

   $$\rho(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{L})^{-1} : L^p \to \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \text{ exists and is bounded}\}.$$

2. **Spectrum** $\sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$. $0 \neq v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ is an **eigenfunction** of $\mathcal{L}$ with **eigenvalue** $\lambda \in \sigma(\mathcal{L})$ if $(\lambda I - \mathcal{L})v = 0$. An eigenvalue $\lambda \in \sigma(\mathcal{L})$
   - is **isolated** if $\exists\, \varepsilon > 0\ \forall\, \lambda_0 \in \mathbb{C}$ with $0 < |\lambda - \lambda_0| < \varepsilon : \lambda_0 \in \rho(\mathcal{L})$.
   - has **finite (algebraic) multiplicity** if $\dim(\mathcal{N}(\lambda I - \mathcal{L})) < \infty$ and $\exists\, n_\lambda \in \mathbb{N}$ $\forall\, y \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ s.t. $y(\lambda_0) = \sum_{j=0}^{r}(\lambda_0 - \lambda)^j y_j$ with $y_0 \neq 0$:

     $$[(\lambda I - \mathcal{L})y]^{(\nu)}(\lambda) = 0 \text{ for } \nu = 0, \ldots, n-1 \text{ and } [(\lambda I - \mathcal{L})y]^{(n)}(\lambda) \neq 0.$$

3. **Point spectrum**

   $$\sigma_{\mathrm{pt}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of finite alg. multiplicity}\}.$$

   $\lambda \in \rho(\mathcal{L}) \cup \sigma_{\mathrm{pt}}(\mathcal{L})$ is called a **normal point** of $\mathcal{L}$.

4. **Essential spectrum**

   $$\sigma_{\mathrm{ess}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is not a normal point of } \mathcal{L}\}.$$

Note: $\mathbb{C} = \rho(\mathcal{L}) \,\dot\cup\, \sigma(\mathcal{L})$, $\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}) \,\dot\cup\, \sigma_{\mathrm{point}}(\mathcal{L})$.

Recall from spectral theory

Linearized operator is closed and densely defined

$$\mathcal{L}v(x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \{v\in W^{2,p}_{\mathrm{loc}}\cap L^p \mid \mathcal{L}_0 v\in L^p\},\quad \|v\|_{\mathcal{L}_0} := \|v\|_{L^p} + \|\mathcal{L}_0 v\|_{L^p}.$$

Definition 5: (Fredholm operator)

The linear operator $\lambda I - \mathcal{L} : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)\to L^p$ is called **Fredholm** iff

1. $\lambda I - \mathcal{L}$ is closed,
2. $\dim(\mathcal{N}(\lambda I - \mathcal{L})) < \infty$ and
3. $\mathrm{codim}(\mathcal{R}(\lambda I - \mathcal{L})) < \infty$.

The **index** κ of the Fredholm operator $\lambda I - \mathcal{L}$ is defined by

$$\kappa := \dim(\mathcal{N}(\lambda I - \mathcal{L})) - \mathrm{codim}(\mathcal{R}(\lambda I - \mathcal{L}))$$

with $\mathrm{codim}(\mathcal{R}(\lambda I - \mathcal{L})) := \dim(\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)/\mathcal{R}(\lambda I - \mathcal{L}))$.

Adjoint operator: Let $q = \frac{p}{p-1}$ for $1 < p < \infty$

$$\mathcal{L}^* v(x) = A^H\triangle v(x) + \left\langle S^\top x, \nabla v(x)\right\rangle + Df(v_\star(x))^H v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

$$\mathcal{D}^q_{\mathrm{loc}}(\mathcal{L}_0^*) = \{v\in W^{2,q}_{\mathrm{loc}}\cap L^q \mid \mathcal{L}_0^* v\in L^q\},\quad \|v\|_{\mathcal{L}_0^*} := \|v\|_{L^q} + \|\mathcal{L}_0^* v\|_{L^q}.$$

Outline

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.
Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

1. (**Fredholm properties**).
   $\lambda I - \mathcal{L} : (\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^N), \|\cdot\|_{L^p})$ is Fredholm of index 0.

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.
Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

2. (**Fredholm alternative**). Let in addition to 1, (A4) hold for $q = \frac{p}{p-1}$ and $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ with geom. mult. $1 \leqslant n = \dim \mathcal{N}(\lambda I - \mathcal{L}) < \infty$.
Then, there are exactly n linearly indep. eigenfunctions $v_j \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and adjoint eigenfunctions $\psi_j \in \mathcal{D}^q_{\mathrm{loc}}(\mathcal{L}_0^*)$ with
$$(\lambda I - \mathcal{L})v_j = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^*\psi_j = 0 \quad \text{for} \quad j = 1, \ldots, n.$$
Moreover,
$$\text{(IP)} \qquad (\lambda I - \mathcal{L})v = g, \quad g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$$
has at least one (not necessarily unique) solution $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ iff
$$g \in (\mathcal{N}(\lambda I - \mathcal{L})^*)^\perp, \quad \text{i.e.} \quad \langle \psi_j, g \rangle_{q,p} = 0,\ j = 1, \ldots, n.$$
In this case, one can select a solution $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ of (IP) with
$$\|v\|_{\mathcal{L}_0} \leqslant C\,\|g\|_{L^p} \quad \text{and} \quad \|v\|_{W^{1,p}} \leqslant C\,\|g\|_{L^p}.$$

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.
Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

3. (**Exponential decay**). Let in addition to 2:
$\theta_j(x) = \exp\left(\mu_j\sqrt{|x|^2+1}\right)$, $x \in \mathbb{R}^d$, $\mu_j \in \mathbb{R}$, $j = 1, \ldots, 4$.
Then, every classical solution $v \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ of

$$(\lambda I - \mathcal{L})v = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^*\psi = 0$$

such that $v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in L^q_{\theta_3}(\mathbb{R}^d, \mathbb{C}^m)$ for some exp. growth rate

$$-\sqrt{\varepsilon\frac{\gamma_A(\beta_\infty - b_0 + \gamma)}{2d|A|^2}} \leqslant \mu_1 \leqslant 0 \quad \text{and} \quad -\sqrt{\varepsilon\frac{\delta_A(\beta_\infty - b_0 + \gamma)}{2d|A|^2}} \leqslant \mu_3 \leqslant 0$$

satisfies $v \in W^{1,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in W^{1,q}_{\theta_4}(\mathbb{R}^d, \mathbb{C}^m)$ for every exp. decay rate

$$0 \leqslant \mu_2 \leqslant \varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}p} \quad \text{and} \quad 0 \leqslant \mu_4 \leqslant \varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}q}.$$

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of $\mathcal{L}$)

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.
Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

4. (**Pointwise estimates for** v). Let in addition to 3:
$p \geqslant \frac{d}{2}$, $f \in C^k(\mathbb{R}^m, \mathbb{R}^m)$, $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$, $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$, $2 \leqslant k \in \mathbb{N}$.
Then, $v \in W^{k,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m)$ and
$$|D^\alpha v(x)| \leqslant C \exp\left(-\mu_2 \sqrt{|x|^2 + 1}\right), \; x \in \mathbb{R}^d$$
for any $\mu_2 \in \mathbb{R}$, $0 \leqslant \mu_2 \leqslant \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} p}$ and $\alpha \in \mathbb{N}_0^d$, $d < (k - |\alpha|)p$.

5. (**Pointwise estimates for** ψ). Let in addition to 4:
$\min\{p, q\} \geqslant \frac{d}{2}$, $\psi \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$.
Then, $\psi \in W^{k,q}_{\theta_4}(\mathbb{R}^d, \mathbb{C}^m)$ and
$$|D^\alpha \psi(x)| \leqslant C \exp\left(-\mu_4 \sqrt{|x|^2 + 1}\right), \; x \in \mathbb{R}^d$$
for any $\mu_4 \in \mathbb{R}$, $0 \leqslant \mu_4 \leqslant \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} q}$ and $\alpha \in \mathbb{N}_0^d$, $d < (k - |\alpha|)q$.

Outline of proof: Theorem 6 (Fredholm properties of $\mathcal{L}$)

$$\mathcal{L}v = A\triangle v + \langle Sx, \nabla v\rangle + Df(v_\star(x))v.$$

1. **Splitting off the stable part**: $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ implies

$$\mathcal{L}v = A\triangle v + \langle Sx, \nabla v\rangle + (Df(v_\infty) + Q(x))\, v$$

$$v_\star(x) \to v_\infty \text{ as } |x| \to \infty \quad \Rightarrow \quad \sup_{|x|\geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty$$

2. **Decomposition of** Q:

$$\mathcal{L}v = A\triangle v + \langle Sx, \nabla v\rangle + (Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x))\, v$$

$Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)$, $Q_{\mathrm{s}}, Q_{\mathrm{c}} \in L^\infty$, Q_{s} small w.r.t. $\|\cdot\|_{L^\infty}$, Q_{c} comp. supported

3. **Decomposition of** λ: $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \geqslant -b_0 + \gamma$ for some $\gamma > 0$, then

$$\lambda = \lambda_1 + \lambda_2 \quad \text{with} \quad \lambda_2 := -b_0 + \gamma, \quad \lambda_1 := \lambda - \lambda_2.$$

4. **Decomposition of $\lambda I - \mathcal{L}$:**

$$\lambda I - \mathcal{L} = \left(I - Q_{\mathrm{c}}(\cdot)(\lambda_1 - \tilde{\mathcal{L}}_{\mathrm{s}})^{-1}\right)\left(\lambda_1 I - \tilde{\mathcal{L}}_{\mathrm{s}}\right)$$

$$\tilde{\mathcal{L}}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}} - \lambda_2 I, \quad \mathcal{L}_{\mathrm{s}} v = A\triangle v + \langle Sx, \nabla v\rangle + (Df(v_\infty) + Q_{\mathrm{s}}(x))\, v$$

Outline of proof: Theorem 6 (Fredholm properties of $\mathcal{L}$)

Decomposition of $\lambda I - \mathcal{L}$:

$$\lambda I - \mathcal{L} = \left(I - Q_{\mathrm{c}}(\cdot)(\lambda_1 - \tilde{\mathcal{L}}_{\mathrm{s}})^{-1}\right)\left(\lambda_1 I - \tilde{\mathcal{L}}_{\mathrm{s}}\right)$$

$$\tilde{\mathcal{L}}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}} - \lambda_2 I, \quad \mathcal{L}_{\mathrm{s}} v = A \triangle v + \langle Sx, \nabla v \rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x)\right) v$$

5. Fredholm properties:

- $\lambda_1 I - \tilde{\mathcal{L}}_{\mathrm{s}}$ is Fredholm of index 0:
  - unique solvability of resolvent equation for $\tilde{\mathcal{L}}_{\mathrm{s}}$
- $I - Q_{\mathrm{c}}(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_{\mathrm{s}})^{-1}$ Fredholm of index 0:
  - $Q_{\mathrm{c}}(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_{\mathrm{s}})^{-1}$ is compact
  - compact perturbation of identity
  - unique solvability of resolvent equation for $\tilde{\mathcal{L}}_{\mathrm{s}}$
  - $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^m)$
- $\lambda I - \mathcal{L}$ Fredholm of index 0:
  - Theorem on products of Fredholm operators

Outline

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Eigenvalue problem:

$$(\lambda I - \mathcal{L})v = 0,\; x \in \mathbb{R}^d$$

$$\mathcal{L}v = A\triangle v + \langle Sx, \nabla v\rangle + Df(v_\star(x))v$$

1. Splitting off the stable part: $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ implies

$$(\lambda I - \mathcal{L}_Q)v = 0,\; x \in \mathbb{R}^d$$

$$\mathcal{L}_Q v = A\triangle v + \langle Sx, \nabla v\rangle + (Df(v_\infty) + Q(x))v = \mathcal{L}v$$

$$v_\star(x) \to v_\infty \text{ as } |x| \to \infty \quad \Rightarrow \quad \sup_{|x|\geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Splitting off the stable part:

$$(\lambda I - \mathcal{L}_Q)v = 0,\ x \in \mathbb{R}^d$$

$$\mathcal{L}_Q v = A\triangle v + \langle Sx, \nabla v\rangle + (Df(v_\infty) + Q(x))v$$

$$Q(x) = Df(v_\star(x)) - Df(v_\infty), \qquad \sup_{|x|\geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty$$

2. Orthogonal transformation: $S \in \mathbb{R}^{d,d}$, $S = -S^\top$, implies $S = P\Lambda_{\mathrm{b}}^S P^\top$ with

$$P \in \mathbb{R}^{d,d} \text{ orth.}, \quad \Lambda_{\mathrm{b}}^S = \mathrm{diag}(\Lambda_1^S, \ldots, \Lambda_k^S, \mathbf{0}), \quad \Lambda_j^S = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad \pm i\sigma_j \in \sigma(S).$$

Then, $\tilde{v}(y) = v(T_1(y))$ with $x = T_1(y) = Py$ yields

$$(\lambda I - \mathcal{L}_1)\tilde{v} = 0,\ y \in \mathbb{R}^d$$

$$\mathcal{L}_1 \tilde{v} = A\triangle \tilde{v} + \langle \Lambda_{\mathrm{b}}^S y, \nabla \tilde{v}\rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}$$

$$\langle \Lambda_{\mathrm{b}}^S y, \nabla \tilde{v}\rangle = \sum_{l=1}^{k} \sigma_l \left(y_{2l}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}}\right)\tilde{v}$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Orthogonal transformation:

$$(\lambda I - \mathcal{L}_1)\tilde{v} = 0,\ y \in \mathbb{R}^d$$

$$\mathcal{L}_1\tilde{v} = A\triangle\tilde{v} + \left\langle \Lambda_{\mathrm{b}}^S y, \nabla\tilde{v}\right\rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}$$

$$\left\langle \Lambda_{\mathrm{b}}^S y, \nabla\tilde{v}\right\rangle = \sum_{l=1}^{k} \sigma_l \left(y_{2l}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}}\right)\tilde{v}$$

3. Several planar polar coordinates: For $\phi \in (-\pi,\pi]^k$, $r \in (0,\infty)^k$ define

$$\begin{pmatrix} y_{2l-1} \\ y_{2l} \end{pmatrix} = T(r_l,\phi_l) := \begin{pmatrix} r_l \cos\phi_l \\ r_l \sin\phi_l \end{pmatrix},\ l = 1,\ldots,k,$$

$T_2(\xi) = (T(r_1,\phi_1),\ldots,T(r_k,\phi_k),\tilde{y})$, $\xi = (r_1,\phi_1,\ldots,r_k,\phi_k,\tilde{y})$, $\tilde{y} = (y_{2k+1},\ldots,y_d)$.
Then, $\hat{v}(\xi) = \tilde{v}(T_2(\xi))$ with $y = T_2(\xi)$ and $\mathbf{Q}(\xi) = Q(T_1(T_2(\xi)))$ yields

$$(\lambda I - \mathcal{L}_2)\hat{v} = 0,\ \xi \in \Omega$$

$$\mathcal{L}_2\hat{v} = A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v} - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v} + (Df(v_\infty) + \mathbf{Q}(\xi))\hat{v}.$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Several planar polar coordinates: $\Omega = ((0,\infty)\times(-\pi,\pi])^k \times \mathbb{R}^{d-2k}$

$$(\lambda I - \mathcal{L}_2)\hat{v} = 0,\ \xi \in \Omega$$

$$\mathcal{L}_2\hat{v} = A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v} - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v} + (Df(v_\infty) + \mathbf{Q}(\xi))\hat{v}.$$

$$\mathbf{Q}(\xi) = Q(T_1(T_2(\xi)))$$

4. Limit operator (far-field operator, simplified operator):

Let formally $|x|\to\infty$ (i.e. $r_l\to\infty$) and use $|Q(x)|\to 0$ as $|x|\to\infty$

$$(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})\hat{v} = 0,\ \xi\in\Omega$$

$$\mathcal{L}_\infty^{\mathrm{sim}}\hat{v} = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v} - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v} + Df(v_\infty)\hat{v}$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Limit operator: $\Omega = ((0,\infty)\times(-\pi,\pi])^k \times \mathbb{R}^{d-2k}$

$$(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})\hat{v} = 0,\ \xi \in \Omega$$

$$\mathcal{L}_\infty^{\mathrm{sim}}\hat{v} = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v} - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v} + Df(v_\infty)\hat{v}$$

5. Angular Fourier transform:

For $n\in\mathbb{Z}^k$, $\omega\in\mathbb{R}^k$, $\rho,\tilde{y}\in\mathbb{R}^{d-2k}$, $\underline{v}\in\mathbb{C}^m$, $|\underline{v}|=1$, $\phi\in(-\pi,\pi]^k$, $r\in(0,\infty)^k$.
Inserting

$$\hat{v}(\xi) = \exp\left(i\sum_{l=1}^{k}\omega_l r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\exp\left(i\sum_{l=2k+1}^{d}\rho_l y_l\right)\underline{v},$$
$$= \exp(i\langle\omega,r\rangle + i\langle n,\phi\rangle + i\langle\rho,\tilde{y}\rangle)\underline{v}$$

yields the m-**dimensional eigenvalue problem**

$$\left(\lambda I_m + (|\omega|^2+|\rho|^2)A + i\sum_{l=1}^{k}n_l\sigma_l I_m - Df(v_\infty)\right)\underline{v} = 0.$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

Angular Fourier transform: $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$, $n \in \mathbb{Z}^k$, $\underline{v} \in \mathbb{C}^m$, $|\underline{v}| = 1$

$$\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i\sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty)\right)\underline{v} = 0.$$

6. Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

(DR) $$\det\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i\sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty)\right) = 0$$

for some $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$, $n \in \mathbb{Z}^k$ belongs to $\sigma_{\mathrm{ess}}(\mathcal{L})$.

Dispersion set:

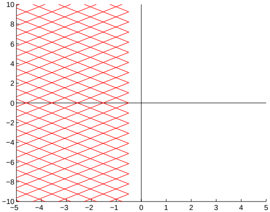
$$\sigma_{\mathrm{disp}}(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k,\ \rho \in \mathbb{R}^{d-2k},\ n \in \mathbb{Z}^k\}.$$

Illustration: Dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$

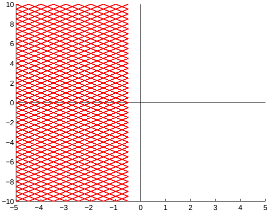
(DR) $$\det\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i\sum_{l=1}^{k} n_l \sigma_l I_m - Df(v_\infty)\right) = 0$$

$$\sigma_{\mathrm{disp}}(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k,\ \rho \in \mathbb{R}^{d-2k},\ n \in \mathbb{Z}^k\}$$

$S \in \mathbb{R}^{d,d}$, $S = -S^\top$, $\pm i\sigma_1, \ldots, \pm i\sigma_k$ nonzero eigenvalues of S, $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$.



$d = 2$ or 3



$d = 4$ (not dense)

$d = 4$ (dense)

Parameters for illustration: $A = \frac{1}{2} + \frac{1}{2}i$, $Df(v_\infty) = -\frac{1}{2}$,

$\sigma_1 = 1.027$ $\qquad$ $\sigma_1 = 1,\ \sigma_2 = 1.5$ $\qquad$ $\sigma_1 = 1,\ \sigma_2 = \frac{\exp(1)}{2}$

$\sigma_{\mathrm{disp}}(\mathcal{L}) \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \leqslant s(Df(v_\infty))\}$ dense $\Longleftrightarrow \exists\, \sigma_n, \sigma_m\colon\ \sigma_n \sigma_m^{-1} \notin \mathbb{Q}$.

Essential L^p-spectrum of $\mathcal{L}$

(DR) $$\det\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i\sum_{l=1}^{k} n_l\sigma_l I_m - Df(v_\infty)\right) = 0$$

$$\sigma_{\mathrm{disp}}(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k,\ \rho \in \mathbb{R}^{d-2k},\ n \in \mathbb{Z}^k\}$$

$S \in \mathbb{R}^{d,d}$, $S = -S^\top$, $\pm i\sigma_1, \ldots, \pm i\sigma_k$ nonzero eigenvalues of S, $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$.

Theorem 7: (Essential L^p-spectrum of $\mathcal{L}$)

Let that assumptions of Theorem 1 (pointwise estimates) be satisfied.
Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:
For every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\mathrm{disp}}(\mathcal{L}) \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}) \quad \text{in} \quad L^p(\mathbb{R}^d, \mathbb{C}^N).$$

- essential spectrum is determined by the far-field linearization
- Thm. 7 holds only for exponentially localized rotating waves, but **not** for nonlocalized rotating waves (e.g. spiral waves, scroll waves)
- essential spectrum for spiral waves much more involved ($\to$ Floquet theory)

Outline

Point spectrum: Derivation of symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$

Rotating wave equation:

(RWE) $$0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\ x \in \mathbb{R}^d$$

$\mathrm{SE}(\mathbf{d})$-group action:

$$[a(R,\tau)v](x) = v(R^{-1}(x-\tau)), \quad x \in \mathbb{R}^d, (R,\tau) \in \mathrm{SE}(d).$$

1. Generators of $\mathrm{SE}(\mathbf{d})$-group action: Applying the generators

$$D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j$$

to (RWE) leads to $\frac{d(d+1)}{2} = d + \frac{d(d-1)}{2}$ equations

$$\begin{aligned} 0 =& D_l \left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \\ 0 =& D^{(i,j)} \left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \end{aligned}$$

for $l = 1,\ldots,d$, $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$.

Point spectrum: Derivation of symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$

Generators of $\mathrm{SE}(\mathbf{d})$-group action:

$$D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j$$

$$0 = D_l \left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$
$$0 = D^{(i,j)} \left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$

for $l = 1,\ldots,d$, $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$.

2. Commutator relations of generators:

$$D_l D_k = D_k D_l,$$
$$D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j,$$
$$D^{(i,j)} D^{(r,s)} = D^{(r,s)} D^{(i,j)} + \delta_{is} D^{(r,j)} - \delta_{ir} D^{(s,j)} - \delta_{js} D^{(r,i)} + \delta_{jr} D^{(s,i)},$$

$$0 = \mathcal{L}(D_l v_\star) - \sum_{n=1}^{d} S_{ln} D_n v_\star,$$

$$0 = \mathcal{L}(D^{(i,j)} v_\star) - \sum_{n=1}^{d} S_{jn} D^{(i,n)} v_\star - \sum_{n=1}^{d} S_{in} D^{(n,j)} v_\star.$$

Point spectrum: Derivation of symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$

Commutator relations of generators: $l=1,\ldots,d$, $i=1,\ldots,d-1$, $j=i+1,\ldots,d$

$$0=\mathcal{L}(D_l v_\star)-\sum_{n=1}^{d}S_{ln}D_n v_\star,$$

$$0=\mathcal{L}(D^{(i,j)}v_\star)-\sum_{n=1}^{d}S_{jn}D^{(i,n)}v_\star-\sum_{n=1}^{d}S_{in}D^{(n,j)}v_\star.$$

3. Finite-dimensional eigenvalue problem: Linear combination of generators

$$v(x)=\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}C_{ij}^{\mathrm{rot}}D^{(i,j)}v_\star(x)+\sum_{l=1}^{d}C_l^{\mathrm{tra}}D_l v_\star(x)=\langle C^{\mathrm{rot}}x+C^{\mathrm{tra}},\nabla v_\star(x)\rangle$$

reduces $\mathcal{L}v=\lambda v$ to the following $\frac{d(d+1)}{2}$-dimensional eigenvalue problem

$$\lambda C^{\mathrm{tra}}=-SC^{\mathrm{tra}},$$

$$\lambda C^{\mathrm{rot}}=S^{\top}C^{\mathrm{rot}}+C^{\mathrm{rot}}S.$$

- **Unknowns**: $\lambda\in\mathbb{C}$, $C^{\mathrm{rot}}\in\mathbb{C}^{d,d}$ skew-symmetric, $C^{\mathrm{tra}}\in\mathbb{C}^{d}$
- EVP appears in **block diagonal form** $\Rightarrow$ solve EVPs **separately**

Point spectrum: Derivation of symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$

Finite-dimensional eigenvalue problem: $S \in \mathbb{R}^{d,d}$, $S = -S^\top$

(1) $$\lambda C^{\mathrm{tra}} = -SC^{\mathrm{tra}},$$

(2) $$\lambda C^{\mathrm{rot}} = S^\top C^{\mathrm{rot}} + C^{\mathrm{rot}} S.$$

Unknowns: $\lambda \in \mathbb{C}$, $C^{\mathrm{rot}} \in \mathbb{C}^{d,d}$ skew-symmetric, $C^{\mathrm{tra}} \in \mathbb{C}^d$.

4. Solution of (1)-(2): S is unitary diagonalizable, i.e.

$\Lambda_S = U^H S U$, $U \in \mathbb{C}^{d,d}$ unitary, $\Lambda_S = \mathrm{diag}(\lambda_1^S, \ldots, \lambda_d^S)$, $\sigma(S) = \{\lambda_1^S, \ldots, \lambda_d^S\}$

A transformation of (1)-(2) implies

$$\lambda = -\lambda_l^S, \quad C^{\mathrm{rot}} = 0, \quad C^{\mathrm{tra}} = Ue_l, \quad (d \text{ solutions}),$$

$$\lambda = -(\lambda_i^S + \lambda_j^S), \quad C^{\mathrm{rot}} = U(I_{ij} - I_{ji})U^\top, \quad C^{\mathrm{tra}} = 0, \quad \left(\frac{d(d-1)}{2} \text{ solutions}\right)$$

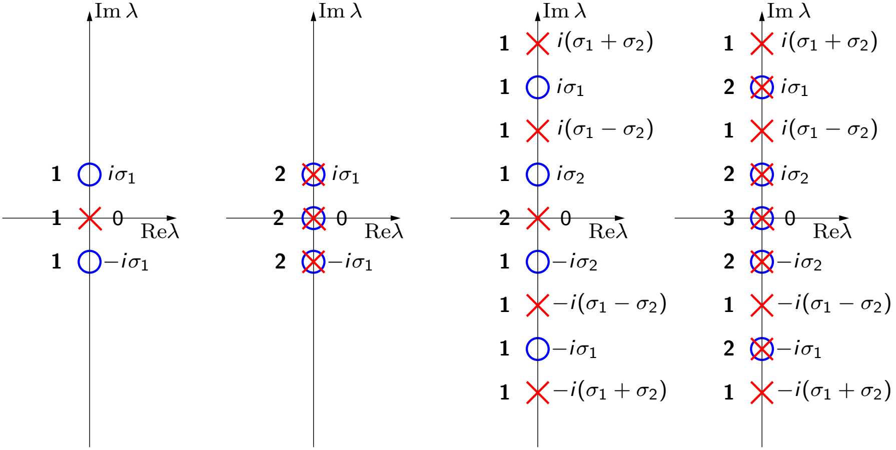
Symmetry set:

$$\sigma_{\mathrm{sym}}(\mathcal{L}) = \sigma(S) \cup \left\{\lambda_i^S + \lambda_j^S \mid 1 \leqslant i < j \leqslant d\right\}$$

Illustration: Symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$

$$\sigma_{\mathrm{sym}}(\mathcal{L}) = \sigma(S) \cup \left\{ \lambda_i^S + \lambda_j^S \mid 1 \leqslant i < j \leqslant d \right\}$$ & **algebraic multiplicities**

Number of elements $\frac{d(d+1)}{2} = d + \frac{d(d-1)}{2}$ equals $\dim \mathrm{SE}(d)$.



$d = 2$, $\dim \mathrm{SE}(2) = 3$ | $d = 3$, $\dim \mathrm{SE}(3) = 6$ | $d = 4$, $\dim \mathrm{SE}(4) = 10$ | $d = 5$, $\dim \mathrm{SE}(5) = 15$

Point L^p-spectrum of $\mathcal{L}$

Theorem 8: (Point L^p-spectrum of $\mathcal{L}$)

Let that assumptions of Theorem 6 ❶ be satisfied.
Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:
For every classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\mathrm{sym}}(\mathcal{L}) \subseteq \sigma_{\mathrm{pt}}(\mathcal{L}) \quad \text{in} \quad L^p(\mathbb{R}^d, \mathbb{C}^N).$$

In particular, Theorem 6 ❷- ❺ implies exponential decay of eigenfunctions and adjoint eigenfunctions.

- point spectrum is determined by the group action
- Thm. 8 even holds for nonlocalized rotating waves (spiral waves, scroll waves)
- $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geqslant 2$

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Outline

Example

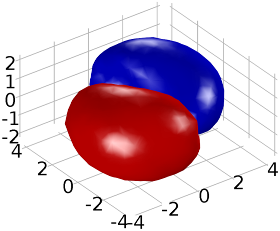
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

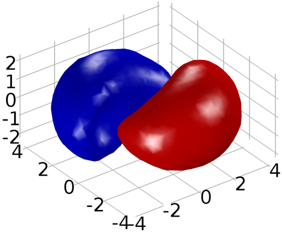
with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2}$$

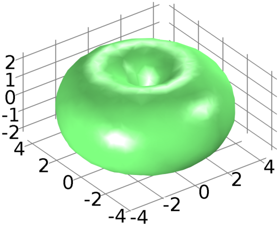
this equation exhibits so called **spinning soliton** solutions.



$\mathrm{Re}\, v_\star(x) = \pm 0.5$



$\mathrm{Im}\, v_\star(x) = \pm 0.5$



$|v_\star(x)| = 0.5$

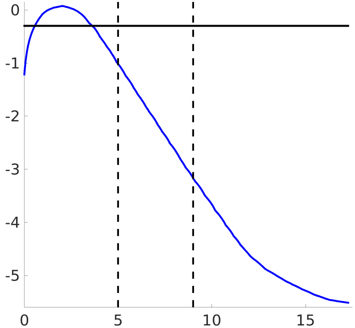
Freezing method implies numerical results for profile $v_\star$ and velocities S.

Spatial decay of a spinning soliton in QCGL for $d=3$: Assume

$$\mathrm{Re}\alpha > 0,\quad \mathrm{Re}\delta < 0,\quad p_{\min} = \frac{2|\alpha|}{|\alpha|+\mathrm{Re}\alpha} < p < \frac{2|\alpha|}{|\alpha|-\mathrm{Re}\alpha} = p_{\max}$$

Decay rate of spinning soliton:

$$0 \leqslant \mu < \frac{\sqrt{-\mathrm{Re}\alpha\mathrm{Re}\delta}}{|\alpha|p} =: \mu^{\mathrm{pro}}(p) < \frac{\sqrt{-\mathrm{Re}\alpha\mathrm{Re}\delta}}{|\alpha|\max\{p_{\min},\frac{d}{2}\}} =: \mu^{\mathrm{pro}}_{\max}.$$



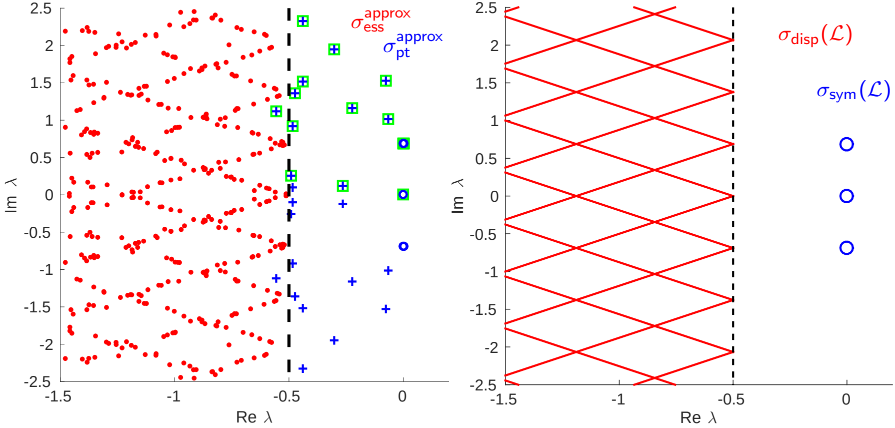
Parameters:

$$\alpha = \frac{1}{2}+\frac{1}{2}i,\ \beta = \frac{5}{2}+i,\ \gamma = -1-\frac{1}{10}i,$$
$$\mu = -\frac{1}{2},\ v_\infty = \begin{pmatrix}0\\0\end{pmatrix},\ a_0 = \mathrm{Re}\alpha,$$
$$a_{\mathrm{max}} = |\alpha|,\ b_0 = \beta_\infty = -\mathrm{Re}\delta = -\frac{1}{2},$$

Numerical vs. theoretical decay rate: $(p=2)$

$$\mathrm{NDR} \approx 0.5387,\quad \mathrm{TDR} = \mu^{\mathrm{pro}}_{\max} = \frac{\sqrt{2}}{4} \approx 0.4714.$$

Spectrum of QCGL for a spinning soliton with $d = 3$: (numerical vs. analytical)



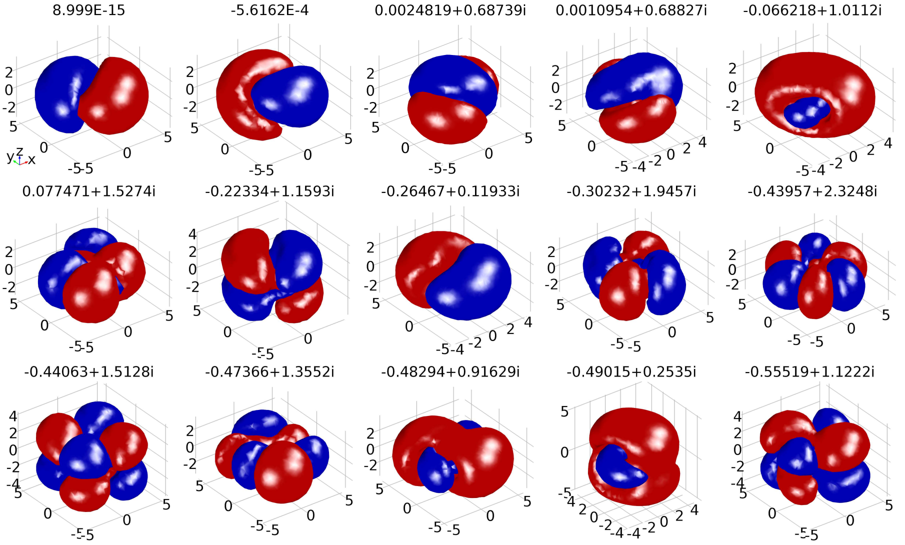
Point spectrum on $i\mathbb{R}$ and **essential spectrum** by dispersion relation:

$$\sigma_{\mathrm{disp}}(\mathcal{L}) = \{\lambda = -\omega^2\alpha_1 + \delta_1 + i(\mp\omega^2\alpha_2 \pm \delta_2 - n\sigma_1) : \omega \in \mathbb{R},\ n \in \mathbb{Z}\},$$

$$\sigma_{\mathrm{sym}}(\mathcal{L}) = \{0, \pm i\sigma_1\}, \quad \sigma_1 = 0.6888$$

for parameters $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{1}{10}i$, $\mu = -\frac{1}{2}$.

Eigenfunctions of QCGL for a spinning soliton with $d = 3$: $\mathrm{Re}\, v(x) = \pm 0.8$

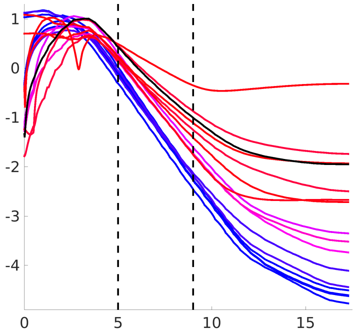


Spatial decay of eigenfunctions of QCGL at a spinning soliton for $d = 3$: Note

$$\mathrm{Re}\lambda \geqslant -(1-\varepsilon)\beta_\infty = -(1-\varepsilon)(-\mathrm{Re}\delta) \quad \Leftrightarrow \quad \varepsilon \leqslant \frac{\mathrm{Re}\lambda - \mathrm{Re}\delta}{-\mathrm{Re}\delta} =: \varepsilon(\lambda).$$

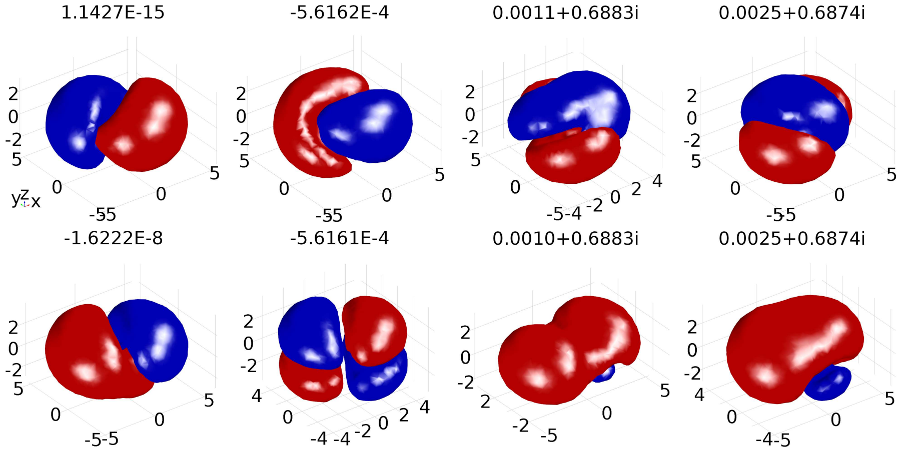
Decay rate of eigenfunctions:

$$0 \leqslant \mu \leqslant \frac{\varepsilon(\lambda)\sqrt{-\mathrm{Re}\alpha\mathrm{Re}\delta}}{|\alpha| p} =: \mu^{\mathrm{eig}}(p,\lambda) < \frac{\varepsilon(\lambda)\sqrt{-\mathrm{Re}\alpha\mathrm{Re}\delta}}{|\alpha| \max\{p_{\min}, \frac{d}{2}\}} =: \mu^{\mathrm{eig}}_{\max}(\lambda).$$



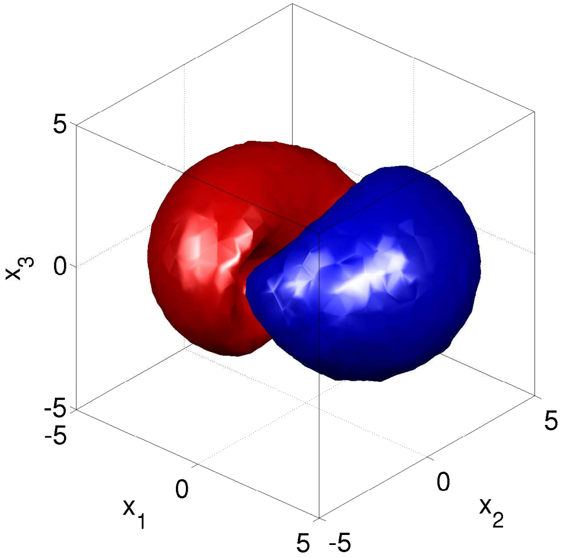
eigenvalue	NDR	TDR
$8.999 \cdot 10^{-15}$	0.5387	0.4714
$-5.6162 \cdot 10^{-4}$	0.5478	0.4714
$0.00110 \pm 0.68827i$	0.5507	0.4714
$0.00248 \pm 0.6874i$	0.5398	0.4714
$-0.06622 \pm 1.0112i$	0.4899	0.4090
$-0.07747 \pm 1.5274i$	0.5355	0.3984
$-0.22334 \pm 1.1593i$	0.4756	0.2608
$-0.26467 \pm 0.1193i$	0.4785	0.2219
$-0.30232 \pm 1.9457i$	0.4649	0.1864
$-0.43957 \pm 2.3248i$	0.3595	0.0570
$-0.44063 \pm 1.5128i$	0.3310	0.0560
$-0.47366 \pm 1.3552i$	0.4781	0.0248
$-0.48294 \pm 0.9163i$	0.4145	0.0161
$-0.48506 \pm 0.0991i$	0.2126	0.0141
$-0.49015 \pm 0.2535i$	0.3307	0.0093
$-0.55519 \pm 1.1222i$	0.3581	—

Eigenfunctions vs. adjoint eigenfunctions of QCGL for a spinning soliton with $d = 3$:



Eigenfunctions (above) and adjoint eigenfunctions (buttom) for $\lambda \in \sigma_{\mathrm{sym}}(\mathcal{L})$

Eigenfunction $\langle Sx, \nabla v_\star(x)\rangle$ of QCGL for a spinning soliton with $d = 3$:



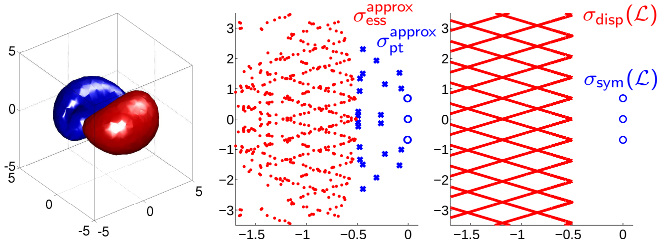
Conclusion:

Theoretical results:

1. spatial decay of rotating waves
2. spectral properties of linearization at localized rotating waves
   - Fredholm properties in L^p
   - symmetry set, point L^p-spectrum, shape of eigenfunctions and spatial decay of eigenfunctions and adjoint eigenfunctions
   - dispersion set, essential L^p-spectrum

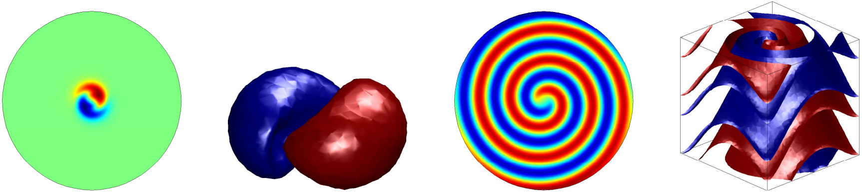
Numerical results:

3. approximation of rotating waves, spectra, eigenfunctions and adjoint eigenfunctions of QCGL (computation: COMSOL, postprocessing: MATLAB)



Open problems and work in progress

- **Fredholm properties and L^{p}-spectra of localized rotating waves**
  (joint work with: W.-J. Beyn)
- **Fourier-Bessel method on $\mathbb{R}^{\mathrm{d}}$ and on circular domains**
  (joint work with: W.-J. Beyn, C. Döding)
- **Nonlinear stability of relative equilibria in evolution equations**
  (joint work with: W.-J. Beyn, C. Döding)
- **Freezing traveling waves in incompressible Navier-Stokes equations**
  (joint work with: W.-J. Beyn, C. Döding)
- **Nonlinear stability of rotating waves for $d \geqslant 3$**
  (joint work with: W.-J. Beyn)
- **Approximation theorem for rotating waves**



Outline

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

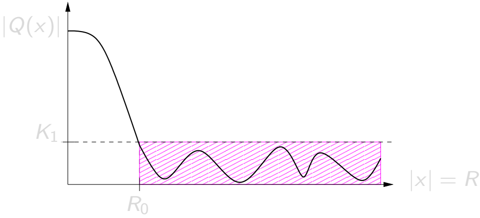
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

1. Far-Field Linearization: $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) := \int_0^1 Df(v_\infty + tw_\star(x))dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + a(x)w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

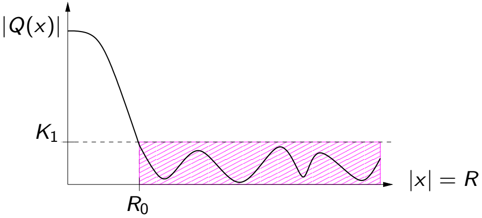
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

2. Decomposition of a: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + (Df(v_\infty) + Q(x))\, w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

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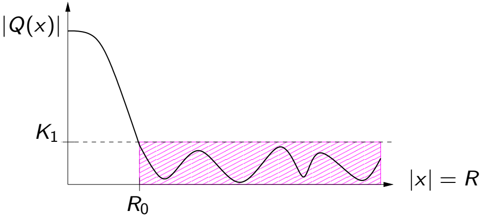
$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\rm s}(x) + Q_{\rm c}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



3. Decomposition of Q:

$Q(x) = Q_{\rm s}(x) + Q_{\rm c}(x)$,

$Q, Q_{\rm s}, Q_{\rm c} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$,

$Q_{\rm s}$ small, i.e. $\|Q_{\rm s}\|_{L^\infty} < K_1$,

$Q_{\rm c}$ compactly supported.

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

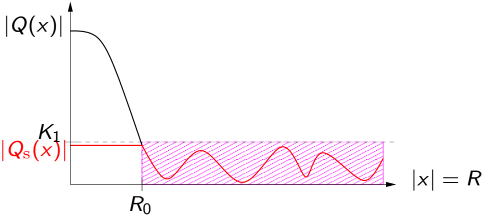
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

2. Decomposition of a: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



3. Decomposition of Q:

$Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)$,

$Q, Q_{\mathrm{s}}, Q_{\mathrm{c}} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$,

Q_{s} small, i.e. $\|Q_{\mathrm{s}}\|_{L^\infty} < K_1$,

Q_{c} compactly supported.

Outline of proof: Theorem 1 (Exponential decay of $v_\star$)

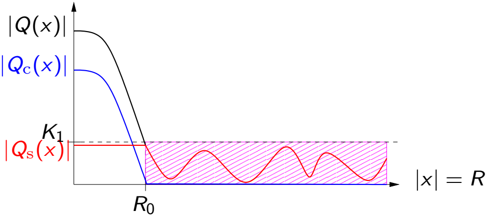
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

2. Decomposition of a: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x\in\mathbb{R}^d.$$



3. Decomposition of Q:

$Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)$,

$Q, Q_{\mathrm{s}}, Q_{\mathrm{c}} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$,

Q_{s} small, i.e. $\|Q_{\mathrm{s}}\|_{L^\infty} < K_1$,

Q_{c} compactly supported.

Outline

Outline of proof: Theorem 2 (Decay of eigenfunctions)

Consider

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+Df(v_\star(x))v(x)=\lambda v(x),\; x\in\mathbb{R}^d.$$

1. Splitting off the stable part:

$$Df(v_\star(x))=Df(v_\infty)+(Df(v_\star(x))-Df(v_\infty))=:Df(v_\infty)+Q(x),\; x\in\mathbb{R}^d,$$

leads to

$$\left[\mathcal{L}_0 v\right](x)+\left(Df(v_\infty)+Q(x)\right)v(x)=\lambda v(x),\; x\in\mathbb{R}^d.$$

2. Decomposition of (the variable coefficient) Q:

$$Q(x)=Q_\varepsilon(x)+Q_{\mathrm{c}}(x),\; Q_\varepsilon\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^{N,N}) \text{ small w.r.t. } \|\cdot\|_{C_{\mathrm{b}}},$$
$$Q_{\mathrm{c}}\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d,$$

leads to

$$\left[\mathcal{L}_0 v\right](x)+\left(Df(v_\infty)+Q_\varepsilon(x)+Q_{\mathrm{c}}(x)\right)v(x)=\lambda v(x),\; x\in\mathbb{R}^d.$$

($\rightarrow$ inhomogeneous Cauchy problem for $\mathcal{L}_{\mathrm{c}}$)

Outline

Outline of proof: Theorem 7 (Essential L^p-spectrum of $\mathcal{L}$)

Choose $R \geqslant 2$ large and cut-off function $\chi_R \in C_{\mathrm{b}}^2$ (bounded indep. on R)

$$\chi_R : [0,\infty) \to [0,1],\ \chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0,1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases}$$

$I_1 = [0, R-1]$, $I_2 = [R-1, R]$, $I_3 = [R, 2R]$, $I_4 = [2R, 2R+1]$, $I_5 = [2R+1, \infty)$.
Introducing

$$v_R(\xi) := \left[\prod_{l=1}^{k} \chi_R(r_l)\right] \chi_R(|\tilde{y}|)\hat{v}(\xi), \qquad w_R := \frac{v_R}{\|v_R\|_{L^p}},$$

we want show that $w_R \in \mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$ and

$$\|(\lambda I - \mathcal{L})w_R\|_{L^p}^p = \frac{\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p} \leqslant \frac{CR^{d-1} + CR^d \eta_R}{CR^d} = \frac{C}{R} + \eta_R \to 0 \text{ as } R \to \infty.$$

Then, $\lambda \notin \rho(\mathcal{L})$ (by continuity of resolvent), i.e. $\lambda \in \sigma(\mathcal{L})$. But $\lambda \notin \sigma_{\mathrm{pt}}(\mathcal{L})$ (since varying ω or ρ shows that λ is not isolated), hence $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$.

Outline of proof: Theorem 7 (Essential L^p-spectrum of $\mathcal{L}$)

$$\chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0,1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases} \quad v_R(\xi) := \left[\prod_{l=1}^{k} \chi_R(r_l)\right] \chi_R(|\tilde{y}|)\hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

$$I_1 = [0, R-1],\ I_2 = [R-1, R],\ I_3 = [R, 2R],\ I_4 = [2R, 2R+1],\ I_5 = [2R+1, \infty).$$

$$\textbf{Aim:} \quad \frac{\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p} \leqslant \frac{CR^{d-1} + CR^d \eta_R}{CR^d} \quad \text{and} \quad w_R \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$$

Show:

1. $\|v_R\|_{L^p}^p \geqslant CR^d$
2. $\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p \leqslant CR^{d-1} + CR^d \eta_R$
3. $|(\lambda I - \mathcal{L}_2)v_R(\xi)| = 0$, if $|\tilde{y}| \in I_1 \cup I_5$ or $r_l \in I_1 \cup I_5$ for some $1 \leqslant l \leqslant k$,
   $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leqslant C\ \forall |\tilde{y}|, r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leqslant l \leqslant k$,
   $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leqslant \left(\sum_{l=1}^{k} \frac{C_l}{r_l} + \eta_R\right)^{\frac{1}{p}}\ \forall |\tilde{y}|, r_l \in I_3$ for all $1 \leqslant l \leqslant k$,
4. $\|(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})v_R\|_{L^p}^p \leqslant CR^{d-1}$
5. $(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})v_R(\xi) = 0$

Outline of proof: Theorem 7 (Essential L^p-spectrum of $\mathcal{L}$)

$$\chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0,1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases} \quad v_R(\xi) := \left[\prod_{l=1}^{k} \chi_R(r_l)\right] \chi_R(|\tilde{y}|)\hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

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   $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leqslant C \;\forall |\tilde{y}|, r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leqslant l \leqslant k$,
   $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leqslant \left(\sum_{l=1}^{k} \frac{C_l}{r_l} + \eta_R\right)^{\frac{1}{p}} \;\forall |\tilde{y}|, r_l \in I_3$ for all $1 \leqslant l \leqslant k$,
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Outline of proof: Theorem 7 (Essential L^p-spectrum of $\mathcal{L}$)

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Aim: $\dfrac{\|(\lambda I-\mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p}\leqslant\dfrac{CR^{d-1}+CR^d\eta_R}{CR^d}$ and $w_R\in\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$

Show:

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   $|(\lambda I-\mathcal{L}_2)v_R(\xi)|\leqslant\left(\sum_{l=1}^{k}\frac{C_l}{r_l}+\eta_R\right)^{\frac{1}{p}}\ \forall\,|\tilde{y}|,r_l\in I_3$ for all $1\leqslant l\leqslant k$,
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$I_1=[0,R-1],\ I_2=[R-1,R],\ I_3=[R,2R],\ I_4=[2R,2R+1],\ I_5=[2R+1,\infty).$

Aim: $\dfrac{\|(\lambda I-\mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p}\leqslant\dfrac{CR^{d-1}+CR^d\eta_R}{CR^d}$ and $w_R\in\mathcal{D}_{\mathrm{loc}}^p(\mathcal{L}_0)$

Show:

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