Dynamic Patterns in PDEs

Technische Universität Dresden, July 4, 2013

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July 4, 2013

CRC 701

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Outline

1. Relative equilibria in reaction-diffusion systems
   - Traveling waves
   - Rotating waves
   - Phase-rotating waves
2. Computation of relative equilibria and their interaction
   - The freezing method
   - Interaction and multisolitons
3. Rotating patterns in parabolic systems
   - Exponential decay of rotating patterns
   - Spectra of linearization about rotating patterns

Outline

Traveling waves

Consider a **system of reaction-diffusion equations**

$$\begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)), \; x\in\mathbb{R}^d,\; t>0,\; d\geqslant 1,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\quad , \; x\in\mathbb{R}^d,\; t=0. \end{aligned}$$

with diffusion matrix $A\in\mathbb{R}^{N,N}$, smooth nonlinearity $f:\mathbb{R}^N\to\mathbb{R}^N$, initial data $u_0:\mathbb{R}^d\to\mathbb{R}^N$ and solution $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$.

Special solutions:

1. **traveling waves**: $u_\star(x,t) = v_\star(x-\mu_\star t)$, $x\in\mathbb{R}$, $t\geqslant 0$, $d=1$,

$$\lim_{\xi\to-\infty} v_\star(\xi) = u_-, \quad \lim_{\xi\to\infty} v_\star(\xi) = u_+, \quad f(u_\pm)=0$$

$u_-\neq u_+$: **traveling front**, $u_-=u_+$: **traveling pulse**, wave moves to the left/right if $\mu_\star>0/\mu_\star<0$.

Notation:

$v_\star:\mathbb{R}\to\mathbb{R}^N$ **profile (pattern)**

$\mu_\star\in\mathbb{R}$ **translational velocity**

Example

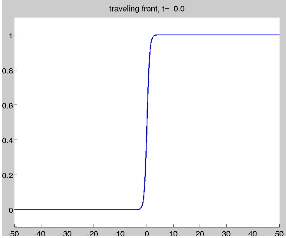
Consider the **Nagumo equation**:

$$u_t = u_{xx} + u(1-u)(u-a), \quad u = u(x,t) \in \mathbb{R}$$

with $u : \mathbb{R} \times [0,\infty[\to \mathbb{R}$, $0 < a < 1$. This equation has **traveling front** solutions

$$v_\star(\xi) = \frac{1}{1+\exp\left(-\frac{\xi}{\sqrt{2}}\right)}, \quad \mu_\star = \sqrt{2}\left(a - \frac{1}{2}\right),$$

called **Huxley wave (front)**. For the parameter $a = \frac{1}{4}$ we have $\mu_\star = -\frac{\sqrt{2}}{4}$



[NAY62] J. Nagumo, S. Arimoto, S. Yoshizawa. 1962

Example

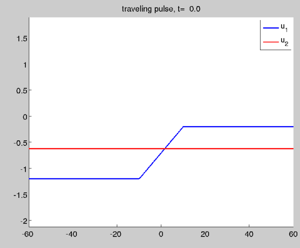
Consider the **FitzHugh-Nagumo system**:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{xx} + \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi\left(u_1 + a - bu_2\right) \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u = u(x,t) \in \mathbb{R}^2$$

with $u : \mathbb{R} \times [0,\infty[\to \mathbb{R}^2$, $0 \leqslant D << 1$, $\phi, a, b > 0$. For the parameters

$$D = 0.1,\ a = 0.7,\ b = 3,\ \phi = 0.08$$

this system exhibits **traveling pulse** solutions.



[F61] R. FitzHugh. 1961

Rotating waves

Consider a **system of reaction-diffusion equations**

$$u_t(x,t) = A\triangle u(x,t) + f(u(x,t)),\ x\in\mathbb{R}^d,\ t>0,\ d\geqslant 1,$$
$$u(x,0) = u_0(x) \qquad ,\ x\in\mathbb{R}^d,\ t=0.$$

with diffusion matrix $A\in\mathbb{R}^{N,N}$, smooth nonlinearity $f:\mathbb{R}^N\to\mathbb{R}^N$, initial data $u_0:\mathbb{R}^d\to\mathbb{R}^N$ and solution $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$.

Special solutions:

1. **traveling waves**: $u_\star(x,t) = v_\star(x-\mu_\star t),\ x\in\mathbb{R},\ t\geqslant 0,\ d=1,$
2. **rotating waves**: $u_\star(x,t) = v_\star(e^{-tS_\star}x),\ x\in\mathbb{R}^d,\ t\geqslant 0,\ d\geqslant 2,$

   $0\neq S_\star\in\mathbb{R}^{d,d}$, $S_\star$ skew-symmetric, i.e. $S_\star^T = -S_\star$, $e^{-tS_\star}$ **rotational matrix**

Notation:

$v_\star:\mathbb{R}^d\to\mathbb{R}^N$ **profile (pattern)**

$\mu_\star\in\mathbb{R}$ **translational velocity**

$S_\star\in\mathbb{R}^{d,d}$ **rotational velocity matrix**

Example

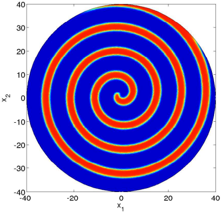
Consider the **Barkley model**

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} u_1(1-u_1)(u_1 - \frac{u_2+b}{a}) \\ u_1 - u_2 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u(x,t) \in \mathbb{R}^2$$

with $u : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}^2$, $0 \leqslant D << 1$, $\varepsilon, a, b > 0$. For the parameters

$$D = 0,\ \varepsilon = 0.02,\ a = 0.75,\ b = 0.01$$

this system exhibits (rigidly) **rotating spiral** solutions.



[B91] D. Barkley. 1991, 1994

[BB04] M. Bär, L. Brusch. 2004

Example

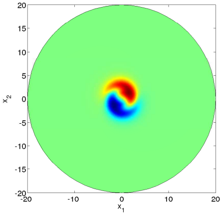
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

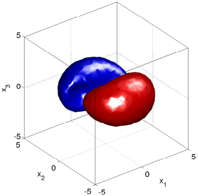
$$u_t = \alpha \triangle u + u \left(\mu + \beta |u|^2 + \gamma |u|^4 \right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{C}$, $d \in \{2, 3\}$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\mu \in \mathbb{R}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2} i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10} i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





[CMM01] L.-C. Crasovan, B.A. Malomed, D. Mihalache. 2001

Example

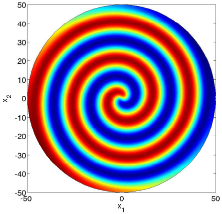
Consider the λ-ω **system**:

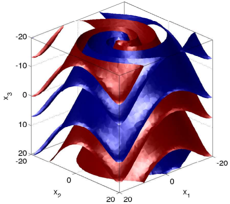
$$u_t = \alpha \triangle u + \left(\lambda(|u|^2) + i\omega(|u|^2)\right) u, \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $\lambda, \omega : [0,\infty[\to \mathbb{R}$. For the parameters

$$\alpha = 1,\ \lambda(|u|^2) = 1 - |u|^2,\ \omega(|u|^2) = -|u|^2$$

this system exhibits (rigidly) **rotating spiral** and (untwisted) **scroll ring** solutions.





[KK81] Y. Kuramoto, S. Koga. 1981

[M04] J. D. Murray. 2004

Phase-rotating waves

Consider a **system of reaction-diffusion equations**

$$\begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ x \in \mathbb{R}^d,\ t > 0,\ d \geqslant 1, \\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ x \in \mathbb{R}^d,\ t = 0. \end{aligned}$$

with diffusion matrix $A \in \mathbb{R}^{N,N}$, smooth nonlinearity $f : \mathbb{R}^N \to \mathbb{R}^N$, initial data $u_0 : \mathbb{R}^d \to \mathbb{R}^N$ and solution $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{R}^N$.

Special solutions:

1. **traveling waves**: $u_\star(x,t) = v_\star(x - \mu_\star t),\ x \in \mathbb{R},\ t \geqslant 0,\ d = 1,$
2. **rotating waves**: $u_\star(x,t) = v_\star(e^{-tS_\star}x),\ x \in \mathbb{R}^d,\ t \geqslant 0,\ d \geqslant 2,$
3. **phase-rotating waves**: $u_\star(x,t) = e^{-i\theta_\star t} v_\star(x),\ x \in \mathbb{R}^d,\ t \geqslant 0,\ d \geqslant 1.$

Notation:

$v_\star : \mathbb{R}^d \to \mathbb{R}^N$ **profile (pattern)**

$\mu_\star \in \mathbb{R}$ **translational velocity**

$S_\star \in \mathbb{R}^{d,d}$ **rotational velocity matrix**

$\theta_\star \in \mathbb{R}$ **phase velocity**

Example

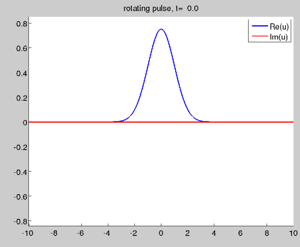
Consider the **Gross-Pitaevskii equation**:

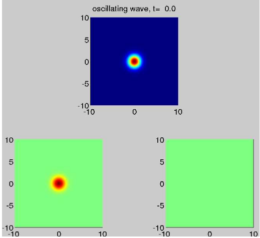
$$u_t = ia\triangle u + \mu V(x)u + \beta\,|u|^2\,u, \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{1,2,3\}$, $0 \neq a \in \mathbb{R}$, $\beta, \mu \in \mathbb{C}$, $V : \mathbb{R}^d \to \mathbb{R}$. For the parameters

$$\alpha = \frac{i}{2},\ \mu = -i,\ \beta = i,\ V(x) = \frac{|x|^2}{2}$$

this equation exhibits **phase-rotating wave** solutions (**solitary oscillons**).





$\mu = 0$: **Schrödinger equation**

Example

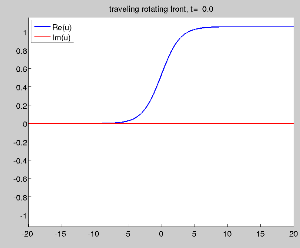
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha u_{xx} + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R} \times [0,\infty[\to \mathbb{C}$, $d = 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\,\alpha > 0$, $\mu \in \mathbb{R}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation has **traveling and phase-rotating front** solutions.



Coherent structures

Consider a **system of reaction-diffusion equations**

$$\begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ x\in\mathbb{R}^d,\ t>0,\ d\geqslant 1,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ x\in\mathbb{R}^d,\ t=0. \end{aligned}$$

with diffusion matrix $A\in\mathbb{R}^{N,N}$, smooth nonlinearity $f:\mathbb{R}^N\to\mathbb{R}^N$, initial data $u_0:\mathbb{R}^d\to\mathbb{R}^N$ and solution $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$.

Special solutions:

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3. **phase-rotating waves**: $u_\star(x,t)=e^{-i\theta_\star t}v_\star(x),\ x\in\mathbb{R}^d,\ t\geqslant 0,\ d\geqslant 1.$

Coherent structures:

$$u_\star(x,t)=e^{-i\theta_\star t}v_\star(e^{-tS_\star}(x-\mu_\star t)),\ x\in\mathbb{R}^d,\ t\geqslant 0.$$

Topics

- simultaneously computation of profile and velocity ($\rightarrow$ freezing method)
- asymptotic stability with asymptotic phase, nonlinear stability
- spectral properties of linearization ($\rightarrow$ point spectra and essential spectra)
- truncation to bounded domains

Outline

The freezing method

Consider the **Cauchy Problem** for $u(x,t) \in \mathbb{R}^N$

(PDE)
$$\begin{aligned} u_t &= Au_{xx} + f(u), & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) &= u_0(x), & x \in \mathbb{R},\ t = 0 \end{aligned}$$

Aim: Approximation of traveling wave $u_\star(x,t) = v_\star(x - \mu_\star t)$, $x \in \mathbb{R}$, $t \geqslant 0$.
Ansatz: Introduce new functions $\gamma(t) \in \mathbb{R}$ (**position**), $v(x,t) \in \mathbb{R}^N$ (**profile**) via

(TWA)
$$u(x,t) = v(x - \gamma(t), t),\ x \in \mathbb{R},\ t \geqslant 0$$

Insert (TWA) into (PDE) yields

$$v_t = Av_{xx} + f(v) + \gamma_t v_x,\ x \in \mathbb{R},\ t > 0.$$

Introduce $\mu(t) \in \mathbb{R}$ (**velocity**) via $\gamma_t(t) = \mu(t)$ and obtain

(PDE2)
$$\begin{aligned} v_t &= Av_{xx} + f(v) + \mu v_x, & v(\cdot,0) &= u_0 \\ \gamma_t &= \mu, & \gamma(0) &= 0 \end{aligned}$$

Too many unknowns, not yet well posed! ($\rightarrow$ phase conditions for $\mu(t)$)

Phase conditions

Type 1 (**Fixed phase condition**):
$\hat{v} : \mathbb{R} \to \mathbb{R}^N$ template, e.g. $\hat{v} = u_0$. Choose $v(\cdot, t)$ such that

$$\min_{g \in \mathbb{R}} \left\| v(\cdot, t) - \hat{v}(\cdot - g) \right\|_{L^2} = \left\| v(\cdot, t) - \hat{v}(\cdot) \right\|_{L^2}, \; t \geqslant 0.$$

Require $v(\cdot, t)$ to stay as close as possible to the template $\hat{v}$

$$\begin{aligned} 0 &= \frac{d}{dg} \left(v(\cdot, t) - \hat{v}(\cdot - g), v(\cdot, t) - \hat{v}(\cdot - g) \right)_{L^2} \Big|_{g=0} \\ &= 2 \left(v(\cdot, t) - \hat{v}, \hat{v}_x \right)_{L^2} \end{aligned}$$

This leads to

$$\text{(PDAE2)} \qquad \begin{aligned} v_t &= A v_{xx} + f(v) + \mu v_x, & v(\cdot, 0) &= u_0 \\ 0 &= \left(v - \hat{v}, \hat{v}_x \right)_{L^2} & & \\ \gamma_t &= \mu(t), & \gamma(0) &= 0 \end{aligned}$$

Type 2 (**Orthogonal phase condition**): On demand.

Frozen system

Frozen system:

$$\text{(PDAE2)}\qquad \begin{aligned} v_t &= Av_{xx} + f(v, v_x) + \mu v_x, & v(\cdot, 0) &= u_0 \\ 0 &= (v - \hat{v}, \hat{v}_x)_{L^2} & & \\ \gamma_t &= \mu(t), & \gamma(0) &= 0 \end{aligned}$$

This is a **partial differential algebraic equation** (**PDAE**) of index 2. Differentiate the algebraic constraint with respect to t and insert the PDE

$$0 = (v_t, \hat{v}_x)_{L^2} = \mu\,(v_x, \hat{v}_x)_{L^2} + (Av_{xx} + f(v, v_x), \hat{v}_x)_{L^2} = \psi_{fix}(v, \mu)$$

$$\text{(PDAE1)}\qquad \begin{aligned} v_t &= Av_{xx} + f(v) + \mu v_x, & v(\cdot, 0) &= u_0 \\ 0 &= \mu\,(v_x, \hat{v}_x)_{L^2} + (Av_{xx} + f(v), \hat{v}_x)_{L^2} = \psi_{fix}(v, \mu) & & \\ \gamma_t &= \mu(t), & \gamma(0) &= 0 \end{aligned}$$

yields a PDAE of index 1 if $(v_x, \hat{v}_x)_{L^2} \neq 0$. Solve the second equation for μ.

Frozen systems for traveling and rotating waves

Ansatz for traveling waves: $u(x,t) = v(x-\gamma(t))$, $x \in \mathbb{R}$, $t \geqslant 0$.

Frozen system for traveling waves ($d = 1$)

$$v_t = Av_{xx} + f(v) + \mu v_x,\ v(\cdot,0) = u_0$$
$$0 = (v - \hat{v}, \hat{v}_x)_{L^2}$$
$$\gamma_t = \mu(t),\ \gamma(0) = 0.$$

Ansatz for rotating waves:
$u(x,t) = v(e^{-S(t)}(x - \xi(t)))$, $x \in \mathbb{R}^2$, $t \geqslant 0$, $-S = S^T$.

Frozen system for rotating waves ($d = 2$)

$$v_t = A\triangle v + f(v) + \mu_1 D_\phi v + \mu_2 D_1 v + \mu_3 D_2 v,\ v(\cdot,0) = u_0$$
$$0 = (v - \hat{v}, D_1\hat{v})_{L^2} = (v - \hat{v}, D_2\hat{v})_{L^2} = (v - \hat{v}, D_\phi\hat{v})_{L^2}$$
$$\gamma_t = \begin{pmatrix}\phi\\ \tau\end{pmatrix}_t = \begin{pmatrix}1 & 0\\ 0 & R_\phi\end{pmatrix}\mu,\ \gamma(0) = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$R_\phi = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix},\ D_\phi = x_2 D_1 - x_1 D_2$$

Example

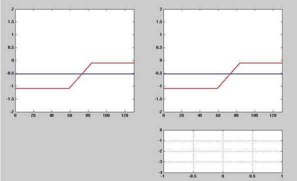
Consider the **frozen version** of the **FitzHugh-Nagumo system**:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{xx} + \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi\,(u_1 + a - bu_2) \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u = u(x,t) \in \mathbb{R}^2$$

with $u : \mathbb{R} \times [0,\infty[\to \mathbb{R}^2$, $0 \leqslant D << 1$, $\phi, a, b > 0$. For the parameters

$$D = 0.1,\ a = 0.7,\ b = 3,\ \phi = 0.08$$

this system exhibits **traveling pulse** solutions.



[F61] R. FitzHugh. 1961

Example

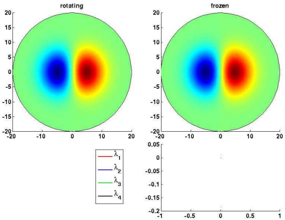
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

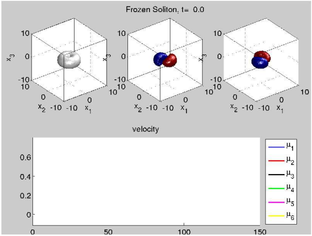
$$u_t = \alpha \triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $\mu \in \mathbb{R}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





[CMM01] L.-C. Crasovan, B.A. Malomed, D. Mihalache. 2001

References

Freezing method:

- [BT] W.-J. Beyn, V. Thümmler. 2004, 2007, 2009
- [T] V. Thümmler. 2006, 2008
- [BOR13] W.-J. Beyn, D. O., J. Rottmann-Matthes. 2013

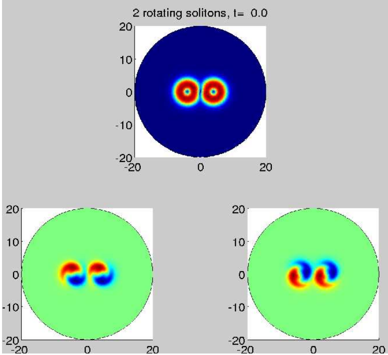
Multisolitons: Interaction of 2 spinning solitons

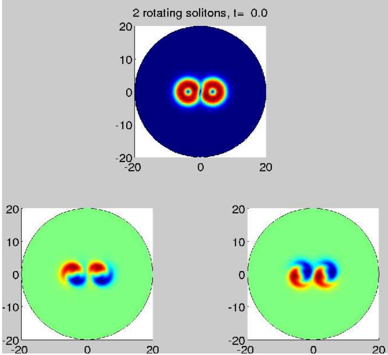
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

$\alpha = \frac{(1+i)}{2}$, $\delta = -\frac{1}{2}$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{i}{10}$.

- **Two solitons**: weak interaction (left), strong interaction (right)





Center of solitons initially at $\pm(4,0)$ (left) and at $\pm(3.75,0)$ (right).

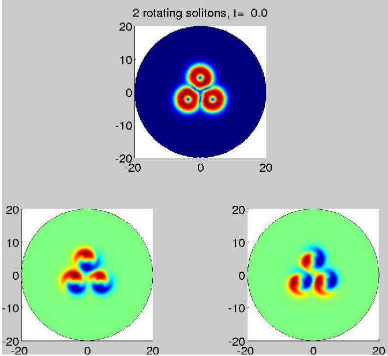
Multisolitons: Interaction of 3 spinning solitons

Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

$\alpha = \frac{(1+i)}{2}$, $\delta = -\frac{1}{2}$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{i}{10}$.

- **Three solitons**: strong interaction



Centers on a quilateral triangle with radius of circumcircle 3.75.

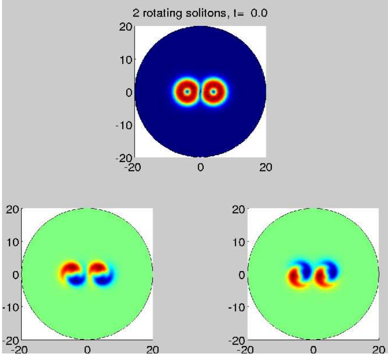
Decompose and freeze: Interaction of 2 spinning solitons

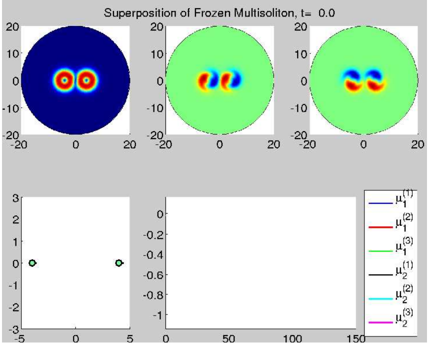
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

$\alpha = \frac{(1+i)}{2}$, $\delta = -\frac{1}{2}$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{i}{10}$.

- **Weak interaction**: without freezing (left), with decompose and freeze (right)





Center of solitons initialy at $\pm(4,0)$. Longtime behavior: collison in the frozen system, slow repulsion in the nonfrozen system.

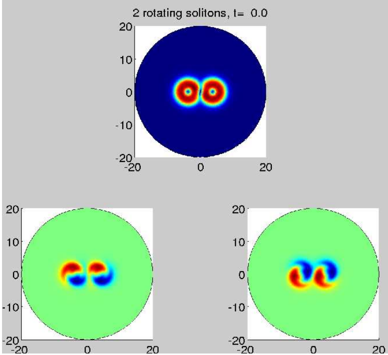
Decompose and freeze: Interaction of 2 spinning solitons

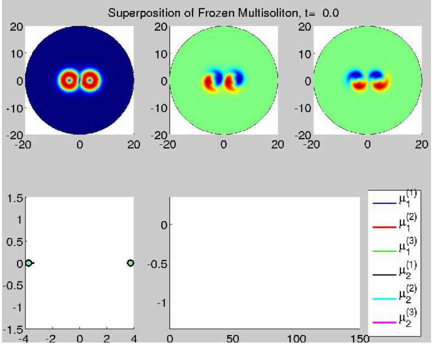
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

$\alpha = \frac{(1+i)}{2}$, $\delta = -\frac{1}{2}$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{i}{10}$.

- **Strong interaction**: without freezing (left), with decompose and freeze (right)





Center of solitons at $\pm(3.75, 0)$.

Decompose and freeze: Interaction of 3 spinning solitons

Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

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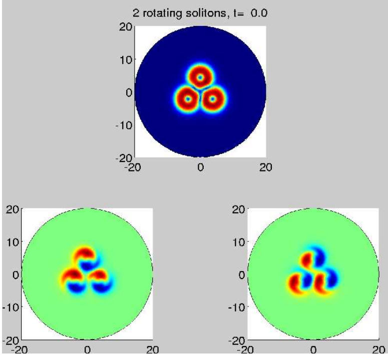
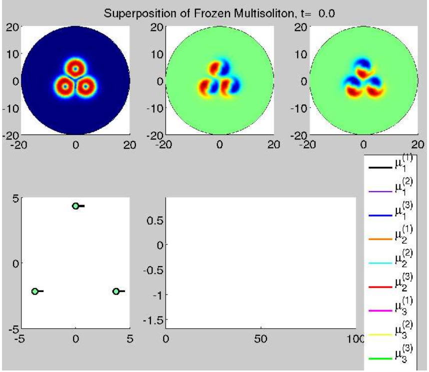
- **Strong interaction**: without freezing (left), with decompose and freeze (right)





Centers on a equilateral triangle with radius of circumcircle 3.75.

References

Decompose and freeze method:

- [BST08] W.-J. Beyn, S. Selle, V. Thümmler. 2008
- [S09] S. Selle. 2009
- [BOR13] W.-J. Beyn, D. O., J. Rottmann-Matthes. 2013

Outline

Rotating Patterns in Parabolic Systems

Consider a **reaction diffusion system**

$$(1) \qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ x\in\mathbb{R}^d,\ t>0,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ x\in\mathbb{R}^d,\ t=0. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$, $A\in\mathbb{R}^{N,N}$, $f\in C^2(\mathbb{R}^N,\mathbb{R}^N)$.
Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^N$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.
Transformation (into a rotating frame): $v(x,t)=u(e^{tS}x,t)$ solves

$$(2) \qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx,\nabla v(x)\rangle = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right) v(x) \ (\textbf{rotational term}).$$

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Rotating Patterns in Parabolic Systems

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$v_\star$ is a stationary solution of (2).

Question: How to show exponential decay of $v_\star$ at $|x| = \infty$?

Consequence: Exponentially small error on truncation to bounded domain.

Rotating Patterns in Parabolic Systems

Consider a **reaction diffusion system**

$$(1)\quad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ x\in\mathbb{R}^d,\ t>0,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\quad ,\ x\in\mathbb{R}^d,\ t=0. \end{aligned}$$

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$v_\star$ is a stationary solution of (2).
$d=2$: Spectral stability implies nonlinear stability.

[BL08] W.-J. Beyn, J. Lorenz. 2008.

Example

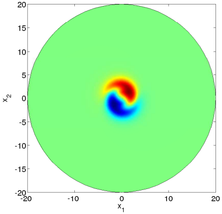
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

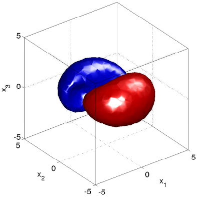
$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





[CMM] L.-C. Crasovan, B.A. Malomed, D. Mihalache. 2001

Main result: Exponential decay of $v_\star$

Theorem: (Exponential Decay of $v_\star$)

Let $f(v_\infty) = 0$ and $\mathrm{Re}\,\sigma(Df(v_\infty)) < 0$. Under further assumptions holds:
For every $1 < p < \infty$, $0 < \vartheta < 1$ and for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ with

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

there exists $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ with the following property:
Every classical solution $v_\star$ of

$$A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,$$

such that $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and

$$\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N) \text{ (**weighted Sobolev space**).}$$

Exponential decay

- A positive function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a **weight function of exponential growth rate** $\eta \geqslant 0$ provided that

$$\exists\, C_\theta > 0: \ \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \quad \forall\, x, y \in \mathbb{R}^d.$$

[ZM09] S. Zelik, A. Mielke. 2009.

Examples: $\mu \in \mathbb{R}$, $x \in \mathbb{R}^d$

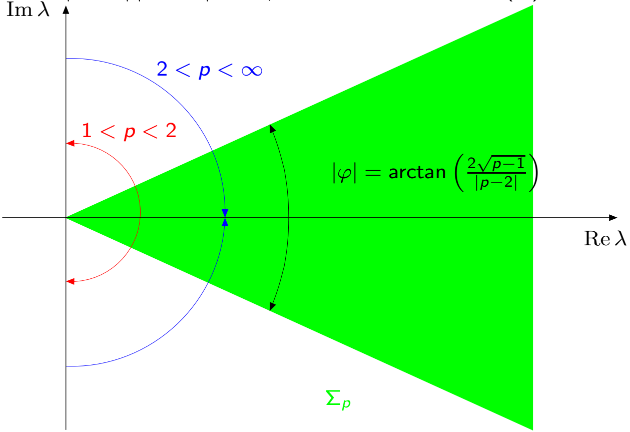
$$\theta_1(x) = \exp\left(-\mu|x|\right), \quad \theta_3(x) = \exp\left(-\mu\sqrt{|x|^2+1}\right),$$
$$\theta_2(x) = \cosh\left(\mu|x|\right), \quad \theta_4(x) = \cosh\left(\mu\sqrt{|x|^2+1}\right).$$

- **Exponentially weighted Sobolev spaces**: $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$

$$L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^N) \mid \|\theta v\|_{L^p} < \infty \right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \ \forall\, |\beta| \leqslant k \right\}.$$

The assumptions

- $\operatorname{Re}\lambda > 0$ and $|\operatorname{Im}\lambda|\,|p-2| \leqslant 2\sqrt{p-1}\operatorname{Re}\lambda \quad \forall\,\lambda\in\sigma(A)$



$A, Df(v_\infty) \in \mathbb{R}^{N,N}$ simultaneously diagonalizable over $\mathbb{C}$

- $a_0 \leqslant \operatorname{Re}\lambda, \quad |\lambda| \leqslant a_{\max} \quad \forall\,\lambda\in\sigma(A)$
  $\operatorname{Re}\mu \leqslant -b_0 < 0 \quad \forall\,\mu\in\sigma(Df(v_\infty))$

Outline of proof: Exponential Decay of $v_\star$

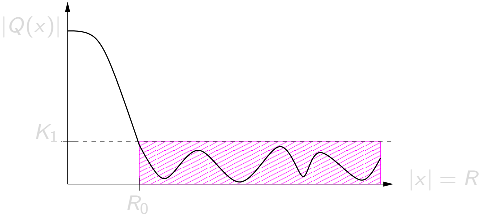
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

1. Far-Field Linearization: $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + a(x)w(x) = 0,\ x \in \mathbb{R}^d.$$



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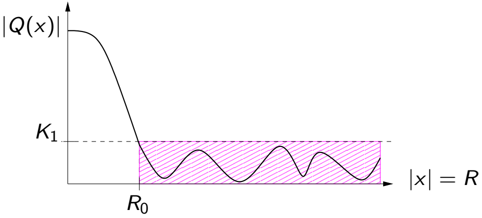
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

2. Decomposition of a:

$$Df(v_\infty) + Q(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q(x)\right) w(x) = 0,\ x \in \mathbb{R}^d.$$



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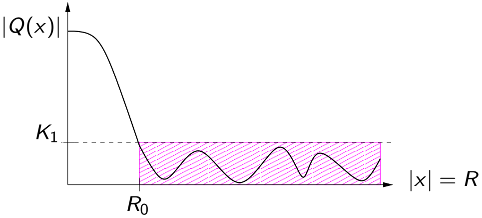
$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) w(x) = 0,\ x \in \mathbb{R}^d.$$



3. Decomposition of Q:

$Q(x) = Q_\varepsilon(x) + Q_c(x)$,

$Q, Q_\varepsilon, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

Q_ε small, i.e. $\|Q_\varepsilon\|_{L^\infty} < K_1$,

Q_c compactly supported.

Outline of proof: Exponential Decay of $v_\star$

Consider the nonlinear problem

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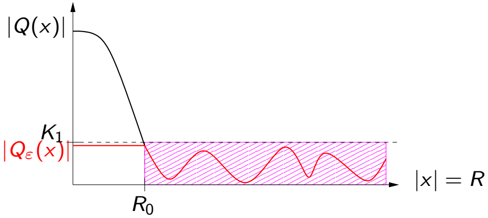

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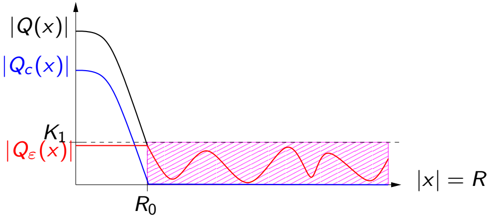

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investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

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Operators: Study the following operators

$$\begin{aligned}
\mathcal{L}_Q v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v + Q_c v,\\
\mathcal{L}_{Q_\varepsilon} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v,\\
\mathcal{L}_\infty v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v,\\
\mathcal{L}_0 v :=& A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator).} \quad \text{(max. domain)}
\end{aligned}$$

Ornstein-Uhlenbeck Operator

Let $P, B \in \mathbb{R}^{d,d}$, $P = P^T$, $P > 0$ and $B \neq 0$.

$$\nabla^T P \nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij} D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x \in \mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2$$

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\mathcal{L}_{Q_\varepsilon} v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v,\\
\mathcal{L}_\infty v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v,\\
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investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

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\mathcal{L}_0 v &:= A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator)}. && \text{(max. domain)}
\end{aligned}$$

Ornstein-Uhlenbeck Operator

Let $P, B \in \mathbb{R}^{d,d}$, $P = P^T$, $P > 0$ and $B \neq 0$.

$$\nabla^T P \nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij} D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x \in \mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

Exponential Decay: To show exponential decay for the solution $v_\star$ of

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[MPV05] G. Metafune, D. Pallara, V. Vespri. 2005.

[M01] G. Metafune. 2001.

The operator $\mathcal{L}_0$: An overview

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

$\downarrow$

Heat kernel matrix

$$H_0(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right), \; x,\xi \in \mathbb{R}^d, \; t>0.$$

$\downarrow$

Semigroup in $L^p(\mathbb{R}^d,\mathbb{R}^N)$, $1 \leqslant p \leqslant \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t>0.$$

strong $\downarrow$ continuity

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \; 1 \leqslant p < \infty.$$

semigroup theory $\swarrow$ | A-priori $\downarrow$ estimates | $\searrow$ $\mathcal{L}_0$: L^p-resolvent est.

unique solv. of resolvent equ., $1 \leqslant p < \infty$

$$(\lambda I - A_p)\, v_\star = g \in L^p.$$

exponential decay, $1 \leqslant p < \infty$

$$v_\star \in W^{1,p}_\theta.$$

max. domain and max. realization, $1 < p < \infty$

$$A_p = \mathcal{L}_0 \text{ on } \mathcal{D}(A_p) = \mathcal{D}^p(\mathcal{L}_0).$$

Spectra of linearization about rotating patterns

Motivation: Stability is determined by spectral properties of linearization $\mathcal{L}$.
Linearization about the profile $v_\star$ of the rotating wave:

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Eigenvalue problem:

$$[\mathcal{L}v](x) = \lambda v(x),\ x \in \mathbb{R}^d,\ d \geqslant 2,\ \lambda \in \mathbb{C}.$$

Definition: (Spectral stability)

A **rotating wave** solution $u_\star(x,t) = v_\star\left(e^{-tS}x\right)$ is called **spectrally stable** if

$$\sigma(\mathcal{L}) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant 0\}.$$

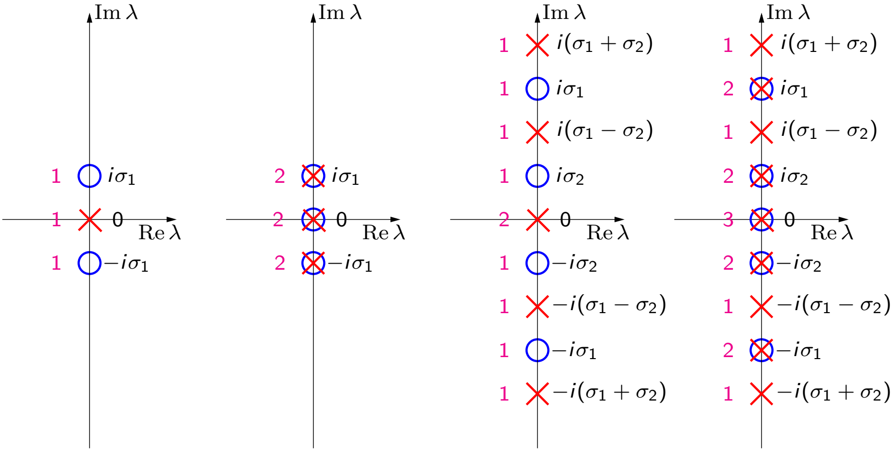
Decompose the **spectrum** $\sigma(\mathcal{L})$ into

$$\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}) \mathbin{\dot\cup} \sigma_{\mathrm{pt}}(\mathcal{L}),$$

$\sigma_{\mathrm{ess}}(\mathcal{L})$ (**essential spectrum**), $\sigma_{\mathrm{pt}}(\mathcal{L})$ (**point spectrum**).

Illustration: Point spectrum of $\mathcal{L}$

$\lambda \in (\sigma(S) \cup \{\lambda+\mu \mid \lambda,\mu\in\sigma(S),\ \lambda\neq\mu\}) \subseteq \sigma_{\mathrm{pt}}(\mathcal{L})$ with algebraic multiplicity



$d=2$, $\dim \mathrm{SE}(2)=3$ | $d=3$, $\dim \mathrm{SE}(3)=6$ | $d=4$, $\dim \mathrm{SE}(4)=10$ | $d=5$, $\dim \mathrm{SE}(5)=15$

Point spectrum of $\mathcal{L}$

Theorem: (Point spectrum of $\mathcal{L}$ on the imaginary axis)

Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of $[\mathcal{L}_0 v](x) + f(v(x)) = 0$. Then

$$v(x) = \left\langle C^{rot}x + C^{tra}, \nabla v_\star(x) \right\rangle, \; x \in \mathbb{R}^d, \; C^{rot} \in \mathrm{so}(d), \; C^{tra} \in \mathbb{R}^d$$

solves $\mathcal{L}v = \lambda v$, whenever $(\lambda, (C^{rot}, C^{tra}))$ solves

$$\begin{aligned} \lambda C^{rot} &= -SC^{rot} + \left(SC^{rot}\right)^T, \\ \lambda C^{tra} &= -SC^{tra}. \end{aligned}$$

Note: Explicit formula for $\lambda, C^{rot}, C^{tra}$ available.

Consequences:

- $\dim \mathrm{SE}(d)$ eigenfunctions of $\mathcal{L}$ and their explicit representation,
- $\sigma(S) \cup \{\lambda + \mu \mid \lambda, \mu \in \sigma(S), \lambda \neq \mu\} \subseteq \sigma_{\mathrm{pt}}(\mathcal{L})$,
- $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geqslant 2$,
- point spectrum on imaginary axis is determined by the group action,
- Theorem also valid for spiral waves, scroll waves, scroll rings.

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Exponential decay of eigenfunctions

Theorem: (Exponential decay of eigenfunctions)

Let the assumptions of the main result be satisfied. Given a classical solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,$$

such that $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and

$$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0.$$

Then every classical solution $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d,$$

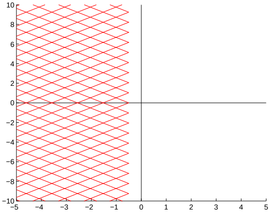
with $\lambda \in \mathbb{C}$ and $\mathrm{Re}\,\lambda \geqslant -\frac{b_0}{3}$ satisfies

$$v \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N).$$

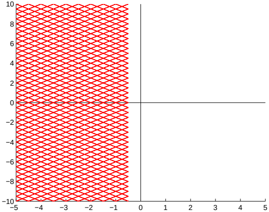
- $v_\star$ exp. localized $\Rightarrow$ v exp. localized (with same rate)

Illustration: Essential spectrum of $\mathcal{L}$

$$\left\{-\lambda(\omega) + i\sum_{l=1}^{k} n_l\sigma_l \mid \lambda(\omega) \text{ eigenvalue of } \omega^2 A - Df(v_\infty)\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$



$d = 2$ or 3



$d = 4$ (not dense)

$d = 4$ (dense)

Essential spectrum of $\mathcal{L}$

Theorem: (Essential spectrum of v)

Let the assumptions of the main result be satisfied. Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S and let $\lambda(\omega)$ denote an eigenvalue of $\omega^2 A - Df(v_\infty)$ for some $\omega \in \mathbb{R}$. Then

$$\left\{ \lambda = -\lambda(\omega) - i \sum_{l=1}^{k} n_l \sigma_l \in \mathbb{C} \;\middle|\; n_l \in \mathbb{Z}, \, \omega \in \mathbb{R} \right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.

- essential spectrum is determined by the Far-field linearization
- only for exponentially localized rotating waves, but **not** for nonlocalized waves (e.g. spiral waves, sroll waves)
- theory e.g. for spiral waves much more involved ($\rightarrow$ Floquet theory)

Dispersion relation: $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$ if

$$\det\left(\lambda I_N + \omega^2 A + i \sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty) \right) = 0 \text{ for some } \omega \in \mathbb{R}.$$

Example

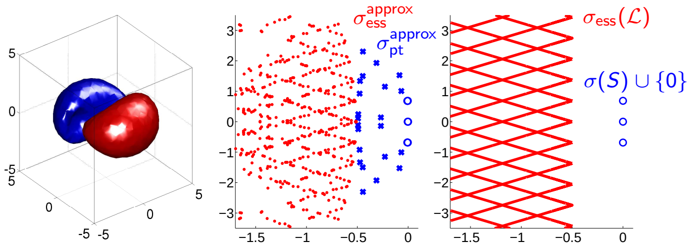
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

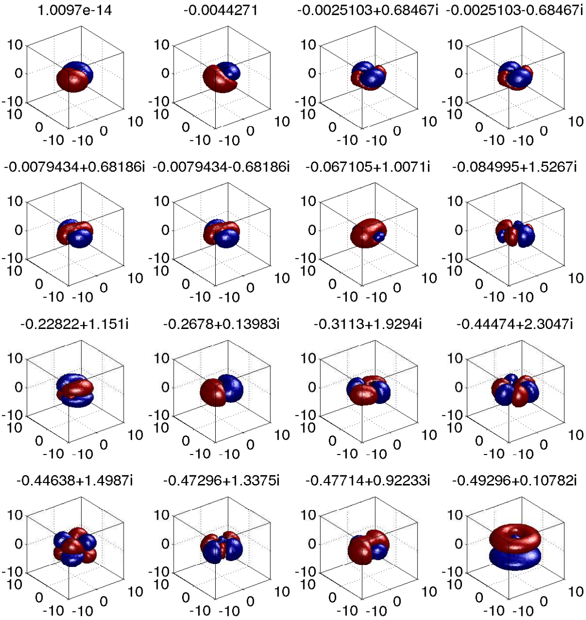
with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.
Spectrum of linearization at a spinning soliton with $d = 3$:



Eigenfunctions of QCGL for a spinning soliton with $d = 3$:



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Spectrum for 2**-dimensional localized rotating waves**:

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Spectrum for rotational term:

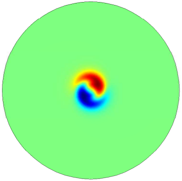
[M01] G. Metafune. 2001.

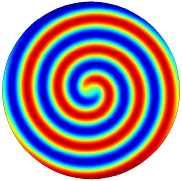
Spectrum for spiral and scroll waves:

[SaSc00] B. Sandstede, A. Scheel. 2000.

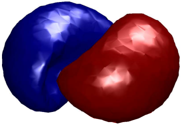
[ScF03] A. Scheel, B. Fiedler. 2003.

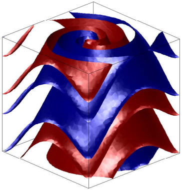
Work in progress





- exponential decay in **space of continuous functions**
- rotating waves in **bounded domains**
- **approximation theorem** for rotating waves
- **numerical computations** (interaction of multisolitons for $d = 3$)
- freezing method for **damped wave equations**
- freezing method for **Hamiltonian systems** (S. Dieckmann)





The freezing method: An abstract framework

Equivariant evolution equation:

$$\text{(EV)}\qquad \begin{aligned} u_t(t) &= F(u(t)), \quad && 0 < t < T, \\ u(0) &= u_0, && t = 0, \end{aligned}$$

$$F : X \supset Y \to X, \quad u \mapsto F(u), \quad (X, \|\cdot\|) \text{ Banach space}, \quad Y \text{ dense}.$$

Lie group: $(G, \circ)$ finite-dimensional, generally noncompact Lie group with group operation

$$\circ : G \times G \to G, \quad (\gamma_1, \gamma_2) \mapsto \gamma_1 \circ \gamma_2,$$

$\mathbb{1} \in G$ unit element, $\dim G = p < \infty$, $\mathfrak{g} = T_{\mathbb{1}} G$ Lie algebra (tangent space at $\mathbb{1}$).
Left multiplication by $\gamma \in G$ on G

$$L_\gamma : G \to G, \quad g \mapsto L_\gamma(g) = \gamma \circ g$$

with derivative

$$dL_\gamma(\mathbb{1}) : T_{\mathbb{1}} G \to T_\gamma G, \quad \mu \mapsto dL_\gamma(\mathbb{1})\mu.$$

Group action of G on X

$$a : G \times X \to X, \quad (\gamma, u) \mapsto a(\gamma)u.$$

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with the following properties:

- Homomorphism:

$$a(\gamma) \in GL(X),\ a(\mathbb{1}) = I$$
$$a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2)$$

- Equivariance:

$$F(a(\gamma)u) = a(\gamma)F(u),\ u \in Y,\ \gamma \in G$$
$$a(\gamma)Y \subset Y,\ \gamma \in G$$

- Smoothness:

$$a(\cdot)v : G \to X, \quad \gamma \to a(\gamma)v$$

  is

  - continuous for $v \in X$
  - continuously differentiable for $v \in Y$, derivative $d[a(\gamma)v] : T_\gamma G \to X$

Abstract freezing approach

Equivariant evolution equation:

(EV)
$$\begin{aligned} u_t(t) &= F(u(t)), \quad && 0 < t < T, \\ u(0) &= u_0, && t = 0, \end{aligned}$$

$$F : X \supset Y \to X, \quad u \mapsto F(u), \quad (X, \|\cdot\|) \text{ Banach space}, \quad Y \text{ dense}.$$

Ansatz: Introduce new functions $\gamma(t) \in G$ (**position**), $v(t) \in Y$ (**profile**) via

(AF)
$$u(t) = a(\gamma(t))v(t), \quad 0 \leqslant t < T.$$

Derive modified equation: Insert (AF) into (EV)

$$a(\gamma)F(v) = F(a(\gamma)v) = F(u) = u_t = a(\gamma)v_t + d[a(\gamma)v]\gamma_t$$

and apply $a(\gamma^{-1})$ we obtain

$$v_t(t) = F(v(t)) - a(\gamma^{-1}(t))d[a(\gamma(t))v(t)]\gamma_t(t), \quad 0 < t < T$$

$$v_t(t) = F(v(t)) - a(\gamma^{-1}(t))d[a(\gamma(t))v(t)]\gamma_t(t), \quad 0 < t < T$$

Introduce $\mu(t) \in \mathfrak{g} = T_{\mathbb{1}} G$ via

$$\gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu(t),\ 0 < t < T.$$

Take derivative of $a(\gamma)a(g)v = a(\gamma \circ g)v = a(L_\gamma g)v$ w.r.t. g at $g = \mathbb{1}$

$$\begin{aligned} a(\gamma)d[a(\mathbb{1})v]\mu &= d[a(\gamma)v]dL_\gamma(\mathbb{1})\mu \\ d[a(\mathbb{1})v]\mu &= a(\gamma)^{-1}d[a(\gamma)v]\gamma_t \end{aligned}$$

New system: (not yet well posed!)

$$\text{(EV2)} \quad \begin{aligned} v_t &= F(v) - d[a(\mathbb{1})v]\mu &&, v(0) = u_0 \text{ PDE on } Y \\ \gamma_t &= dL_\gamma \mu &&, \gamma(0) = \mathbb{1} \text{ ODE on } G \end{aligned}$$

Phase conditions: To compensate extra variable μ, add $\dim \mathfrak{g} = \dim G$ phase conditions

$$\psi(v, \mu) = 0,\ \psi : Y \times \mathfrak{g} \to \mathfrak{g}^*$$

Differential algebraic evolution equation (DAEV):

$$\text{(DAEV)} \qquad \begin{aligned} v_t &= F(v) - d[a(\mathbb{1})v]\mu &&, \ v(0) = u_0 \\ 0 &= \psi(v, \mu) && \\ \gamma_t &= dL_\gamma \mu &&, \ \gamma(0) = \mathbb{1} \end{aligned}$$

- $\gamma_t = dL_\gamma(\mathbb{1})\mu$ is called the **reconstruction equation** (Mardsen 2003), it decouples from the DAE and is needed for the reconstruction of $u(t) = a(\gamma(t))v(t)$

Relative equilibria:

Definition: (Relative equilibrium)

A classical solution $u_\star$ of (EV) on $[0, T[$ is called a **relative equilibrium** (w.r.t. the action a of G on X) if it has the form

$$u_\star(t) = a(\gamma_\star(t))v_\star, \quad 0 \leqslant t < T$$

for some $v_\star \in Y$ and for some $\gamma_\star \in C^1(]0, T[, G) \cap C([0, T[, G)$.

Note: $u_\star$ relative equilibrium of (EV) $\Rightarrow$ $v_\star$ steady state of (DAEV)!

Definition: (Asymptotic stability with asymptotic phase)

A relative equilibrium $u_\star$ of (EV) on $[0,\infty[$ with $u_\star(t) = a(\gamma_\star(t))v_\star$ is called **asymptotically stable** if there exists some $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ there exists $\delta > 0$ with the following property:

For every $u_0 \in Y$ with $\|u_0 - v_\star\|_Y \leqslant \delta$ the equation (EV) admits a unique classical solution $u \in C^1(]0,T[,X) \cap C([0,T[,Y)$ and there exists an orbit $\gamma(t) \in G$, $t \geqslant 0$, such that

$$\|u(t) - a(\gamma(t) \circ \gamma_\star(t))v_\star\|_Y \begin{cases} \leqslant \varepsilon \; \forall\, t \geqslant 0, \\ \to 0 \text{ as } t \to \infty. \end{cases}$$

If, in addition, $\gamma(t)$ converges as $t \to \infty$ to an element $\gamma_\infty \in G$ in the ε-neighborhood of $\mathbb{1}$, then γ_∞ is called the **asymptotic phase** and the relative equilibrium $u_\star$ is called **asymptotically stable with asymptotic phase**.

Note: Stability is determined by spectral properties of the linearization.

References

Freezing method:

[BT] W.-J. Beyn, V. Thümmler. 2004, 2007, 2009

[T] V. Thümmler. 2006, 2008

[BOR13] W.-J. Beyn, D. O., J. Rottmann-Matthes. 2013

Asymptotic stability:

[BL08] W.-J. Beyn, J. Lorenz. 2008.

The decompose and freeze method: An abstract framework

Module: E module acting on X via multiplication

$$\bullet : E \times X \to X, \quad (\varphi, u) \mapsto \varphi \bullet u.$$

Group action of G on E

$$b : G \times E \to E, \quad (\gamma, \varphi) \mapsto b(\gamma)\varphi$$

with the following properties

$$\begin{aligned} a(\gamma)(\varphi \bullet u) &= (b(\gamma)\varphi) \bullet (a(\gamma)u), && \gamma \in G,\, \varphi \in E,\, u \in X, \\ b(\gamma)(\varphi\psi) &= (b(\gamma)\varphi)(b(\gamma)\psi), && \gamma \in G,\, \varphi, \psi \in E. \end{aligned}$$

Decompose and freeze approach

Equivariant evolution equation:

(EV)
$$\begin{aligned} u_t(t) &= F(u(t)), \quad & 0 < t < T, \\ u(0) &= u_0, & t = 0, \end{aligned}$$

$$F : X \supset Y \to X, \quad u \mapsto F(u), \quad (X, \|\cdot\|) \text{ Banach space}, \quad Y \text{ dense}.$$

Ansatz: Introduce new functions $\gamma_j(t) \in G$ (**positions**), $v_j(t) \in Y$ (**profiles**) via

(ADF)
$$u(t) = \sum_{j=1}^{m} a(\gamma_j(t)) v_j(t), \quad 0 \leqslant t < T.$$

(decomposition into m single profiles)

(ADF) $$u(t) = \sum_{j=1}^{m} a(\gamma_j(t)) v_j(t), \quad 0 \leqslant t < T.$$

Derive modified system: Insert (ADF) into (EV), $\gamma_j^k := \gamma_j^{-1} \circ \gamma_k$,

$$\sum_{j=1}^{m} \left[a(\gamma_j) v_{j,t} + d\left[a(\gamma_j) v_j\right] \gamma_{j,t} \right] = \frac{d}{dt} \sum_{j=1}^{m} a(\gamma_j) v_j = u_t = F(u)$$

$$= \sum_{j=1}^{m} \left[F(a(\gamma_j) v_j) + \frac{b(\gamma_j)\varphi}{\sum_{k=1}^{m} b(\gamma_k)\varphi} \left(F\left(\sum_{k=1}^{m} a(\gamma_k) v_k \right) - \sum_{k=1}^{m} F(a(\gamma_k) v_k) \right) \right]$$

$$= \sum_{j=1}^{m} \left[a(\gamma_j) F(v_j) + a(\gamma_j) \left(\frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k)\varphi} \left(F\left(\sum_{k=1}^{m} a(\gamma_j^k) v_k \right) - \sum_{k=1}^{m} F(a(\gamma_j^k) v_k) \right) \right) \right]$$

Require equality of summands $[\cdots]$ in $\sum_{j=1}^{m}$, add initial and phase conditions for each v_j.

Coupled nonlinear system of differential algebraic evolution equations:

$$
\begin{aligned}
v_{j,t} =& F(v_j) - d\left[a(\mathbb{1})v_j\right]\mu_j + \frac{\varphi}{\sum_{k=1}^{m} b(\gamma_j^k)\varphi} && , v_j(0) = 0, \\
& \bullet\left[F\left(\sum_{k=1}^{m} a(\gamma_j^k)v_k\right) - \sum_{k=1}^{m} F(a(\gamma_j^k)v_k)\right] \; 0 = \quad \Psi\left(v_j, \mu_j\right), \\
\gamma_{j,t} =& dL_{\gamma_j}(\mathbb{1})\mu_j && , \gamma_j(0) = \gamma_j^0.
\end{aligned}
$$

for $j = 1, \ldots, m$.

References

- [BST08] W.-J. Beyn, S. Selle, V. Thümmler. 2008
- [S09] S. Selle. 2009
- [BOR13] W.-J. Beyn, D. O., J. Rottmann-Matthes. 2013

Outline of proof: Point spectrum of $\mathcal{L}$

(3) $$0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\; x \in \mathbb{R}^d,\; d \geqslant 2.$$

1. Group action: Apply $a(R,\tau)$ to (3)

$$0 = a(R,\tau)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$

2. Derivative $\frac{d}{d(R,\tau)}$ at $(R,\tau) = (I_d, 0)$ leads to $\frac{d(d+1)}{2}$ equations

$$0 = (x_j D_i - x_i D_j)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$
$$0 = D_l\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right)$$

for $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$, $l = 1,\ldots,d$.

3. Commutator relations for differential operators yield, $D^{(ij)} := x_j D_i - x_i D_j$

$$0 = \mathcal{L}\left(D^{(ij)} v_\star(x)\right) + \sum_{\substack{n=1\\ n\neq j}}^{d} S_{in} D^{(jn)} v_\star(x) - \sum_{\substack{n=1\\ n\neq i}}^{d} S_{jn} D^{(in)} v_\star(x)$$

$$0 = \mathcal{L}\left(D_l v_\star(x)\right) - \sum_{n=1}^{d} S_{ln} D_n v_\star(x)$$

Outline of proof: Point spectrum of $\mathcal{L}$

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4. Finite-dimensional eigenvalue problem: The ansatz

$$v(x) = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} C_{ij}^{rot}(x_j D_i - x_i D_j) v_\star(x) + \sum_{l=1}^{d} C_l^{tra} D_l v_\star(x),\ C_{ij}^{rot}, C_l^{tra} \in \mathbb{C}$$

reduces $\mathcal{L}v = \lambda v$ to a

$$\lambda C^{rot} = -SC^{rot} + \left(SC^{rot}\right)^T,$$
$$\lambda C^{tra} = -SC^{tra}.$$

Note: S is unitary diagonalizable.

Outline of proof: Point spectrum of $\mathcal{L}$

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$$\lambda C^{rot} = -SC^{rot} + \left(SC^{rot}\right)^T,$$
$$\lambda C^{tra} = -SC^{tra}.$$

Note: S is unitary diagonalizable.

Outline of proof: Essential spectrum of $\mathcal{L}$

Linearization at the profile $v_\star$:

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\infty)v(x) + Q(x)v(x)$$

$$Q(x) := Df(v_\star(x)) - Df(v_\infty), \qquad \sup_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R \to \infty$$

1. Orthogonal transformation: $S \in \mathbb{R}^{d,d}$, $S^T = -S$, $S = P\Lambda^S_{\mathrm{block}}P^T$.
$T_1(x) = Px$ yields

$$[\mathcal{L}_1 v](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

with

$$\left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l}D_{2l-1} - x_{2l-1}D_{2l}\right) v(x).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Orthogonal transformation:

$$[\mathcal{L}_1 v](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x) \right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

$$\left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x) \right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l} D_{2l-1} - x_{2l-1} D_{2l} \right) v(x)$$

2. Several planar polar coordinates: Transformation

$$\begin{pmatrix} x_{2l-1} \\ x_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l \cos\phi_l \\ r_l \sin\phi_l \end{pmatrix}, \; l = 1, \ldots, k, \; \phi_l \in]-\pi, \pi], \; r_l > 0.$$

yields for $\xi = (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d)$ with total transformation $T_2(\xi)$, $Q(\xi) := Q(T_1(T_2(\xi)))$

$$[\mathcal{L}_2 v](\xi) = A\left[\sum_{l=1}^{k} \left(\partial^2_{r_l} + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial^2_{\phi_l} \right) + \sum_{l=2k+1}^{d} \partial^2_{x_l}\right] v(\xi) - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Several planar polar coordinates:

$$[\mathcal{L}_2 v](\xi) = A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),$$

$$\xi = (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d), \quad Q(\xi) := Q(T_1(T_2(\xi)))$$

3. Simplified operator (far-field linearization): Neglecting $\mathcal{O}\left(\frac{1}{r}\right)$–terms yields

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](\xi) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Simplified operator (far-field linearization):

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](\xi) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi)$$

4. Angular Fourier decomposition:

$$v(\xi) = \exp\left(i\omega\sum_{l=1}^{k} r_l\right)\exp\left(i\sum_{l=1}^{k} n_l\phi_l\right)\hat{v},\ n_l\in\mathbb{Z},\ \omega\in\mathbb{R},\ \hat{v}\in\mathbb{C}^N,\ |\hat{v}|=1$$

$$\phi_l \in]-\pi,\pi],\ r_l > 0,\ l = 1,\ldots,k,$$

yields

$$\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right)v\right](\xi) = \left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k} n_l\sigma_l I_N - Df(v_\infty)\right)v(\xi).$$

Outline of proof: Essential spectrum of $\mathcal{L}$

Angular Fourier decomposition:

$$\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right) v\right](\xi) = \left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty)\right) v(\xi).$$

$$n_l \in \mathbb{Z}, \quad \omega \in \mathbb{R}, \quad \pm i\sigma_l \text{ nonzero eigenvalues of } S \in \mathbb{R}^{d,d}$$

5. Finite-dimensional eigenvalue problem: $\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right) v\right](\xi) = 0$ for every ξ if $\lambda \in \mathbb{C}$ satisfies

$$\left(\omega^2 A - Df(v_\infty)\right)\hat{v} = -\left(\lambda + i\sum_{l=1}^{k} n_l \sigma_l\right)\hat{v}, \text{ for some } \omega \in \mathbb{R}.$$