COLOURINGS OF CYCLOTOMIC INTEGERS WITH CLASS NUMBER ONE

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Abstract. This paper continues the study on colourings of the sets of cyclotomic integers $\mathcal{M}_n = \mathbb{Z}[\xi_n]$, ($\xi_n = e^{2\pi i/n}$, a primitive nth root of unity) with class number one. We present results for the colour symmetry group and colour preserving group for a given ideal colouring of $\mathcal{M}_n$, with $\phi(n) = 8$ and 10, thus completing the characterisation of the colour preserving group for the cases $\phi(n) \leq 10$, where ϕ is Euler's totient function.

1. Introduction

The classification of colour symmetries for periodic crystals has been investigated in much detail in existing literature [1, 2, 3, 4, 5]. With the advent of quasicrystals, the problem has been extended to include the study of colour symmetry groups of quasiperiodic and non-periodic structures. In our work, the problem on colour symmetries of periodic and non-periodic structures, including quasiperiodic structures, is analysed by studying the sets of cyclotomic integers $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ (where, $\xi_n = e^{2\pi i/n}$) following the setting given in [6, 7, 8]. We consider values of n for which $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ is a principal ideal domain and has class number one, namely

$$(*) \qquad \begin{aligned} n \;=\; & 3,4,5,7,8,9,11,12,13,15,16,17,19,20,21,24,\\ & 25,27,28,32,33,35,36,40,44,45,48,60,84. \end{aligned}$$

The symmetry cases are grouped into classes with equal value of the Euler ϕ-function

$$\phi(n) = \big|\{1 \leq k \leq n | \gcd(k,n) = 1\}\big|.$$

If n is odd, we have $\mathcal{M}_n = \mathcal{M}_{2n}$, and $\mathcal{M}_n$ thus has $2n$-fold symmetry. To avoid duplication of results, values of $n \equiv 2 \mod 4$ do not appear in (*). The smallest value n not covered in this work is $n = 23$, involving 46-fold symmetry, as our methods involve unique factorization in $\mathbb{Z}[\xi_n]$, possible only for the values given in (*). For a discussion of other n than in (*), see [9].

Our previous work derived the colour symmetry group for any n and focused on determining the colour preserving group for the crystallographic cases: $n = 3, 4$ [1]; as well as the non-periodic cases: $n = 5, 8, 12$ (involving standard quasicrystallographic symmetries) [10] and $n = 7, 9$ [11].

In this note, we enumerate the colour preserving group for the cases $n = 11, 15, 16, 20, 24$; thus completing the characterisation of colour symmetries of ideal colourings associated with $\mathcal{M}_n$ for $\phi(n) \leq 10$. We outline the results that facilitate our calculations.

2. Ideal colourings of $\mathcal{M}_n$

In studying the colourings of $\mathcal{M}_n$, we consider colourings of $\mathcal{M}_n$ which are compatible with its underlying symmetry. A restriction to be imposed is that a colour occupies a subset which is of the same Bravais type as the original set, while the other colours code the cosets. Assuming this compatibility requirement, a Bravais colouring of $\mathcal{M}_n$ is arrived at by considering a colouring using cosets of a principal ideal of $\mathcal{M}_n$. Given an ideal of index ℓ in $\mathcal{M}_n$, each element of $\mathcal{M}_n$ is assigned a colour from a set of ℓ distinct colours. A Bravais colouring c of $\mathcal{M}_n$ with ℓ colors is a surjective map $c : \mathcal{M}_n \to \{1, 2, \ldots, \ell\}$.

We also refer to c as an ideal colouring of $\mathcal{M}_n$ which we define formally as follows:

Definition 2.1. Let (q) denote the ideal generated by $q \in M_n$. An *ideal colouring* c of $\mathcal{M}_n$ with ℓ colours is defined as: Let $q \in \mathcal{M}_n$ such that $[\mathcal{M}_n : (q)] = \ell$. For each $z \in (q) = q\mathcal{M}_n$, let $c(z) = 1$. Let the other cosets of (q) be $(q) + t_2, \ldots, (q) + t_\ell$. For each $z \in (q) + t_i$, let $c(z) = i$.

If (q) is given explicitly, we also call such an ideal colouring as a *colouring induced by* (q).

In the analysis of an ideal colouring c of $\mathcal{M}_n$, we consider the *symmetry group* G of $\mathcal{M}_n$. This group is *symmorphic*, that is, $G = \mathcal{M}_n \rtimes D_N$, the semi-direct product of its translation group with its point group, the dihedral group D_N ($N = n$ if n is even, and $N = 2n$ if n is odd). For a given colouring, we also consider the following subgroup of G, the group

$$H = \{h \in G \mid \exists\, \pi \in \mathcal{S}_\ell \;\forall x \in \mathcal{M}_n : \; c(h(x)) = \pi(c(x))\},$$

where $\mathcal{S}_\ell$ denotes the symmetric group on ℓ letters. The elements of H are called the *colour symmetries* of $\mathcal{M}_n$ and H is the *colour symmetry group* of the corresponding colouring c of $\mathcal{M}_n$.

By the requirement $\pi c = ch$, each $h \in H$ determines a unique permutation $\pi = \pi_h$. This also defines a map

$$P : H \to \mathcal{S}_\ell, \quad P(h) := \pi_h.$$

Let $g, h \in H$. Because of $c(hg(x)) = ch(g(x)) = \pi_h c(g(x)) = \pi_h(\pi_g(c(x))) = \pi_h\pi_g(c(x))$, we obtain the following lemma.

Lemma 2.2. *P is a group homomorphism.* □

Another group of interest is the subgroup of H which consists of elements which fix the colours of the colouring c of $\mathcal{M}_n$, the *colour preserving group*

$$K := \{k \in H \mid c(k(x)) = c(x),\, x \in \mathcal{M}_n\}.$$

In other words, K is the kernel of P, a normal subgroup of H. The aim of our study is to deduce the nature of the groups H and K for the ideal colourings of $\mathcal{M}_n$.

3. The structure of the colour symmetry group H

Definition 3.1. A colouring c of $\mathcal{M}_n$ is called *perfect*, if its colour symmetry group H equals G. It is called *chirally perfect*, if $H = G'$, where G' is the index 2 subgroup of G consisting of the orientation preserving isometries in G.

In [11], it is shown that all chirally perfect colourings of $\mathcal{M}_n$ arise from the principal ideals $(q) = q\mathcal{M}_n, q \in \mathcal{M}_n$. Consequently, there exists a chirally perfect colouring of $\mathcal{M}_n$ with ℓ colours, if and only if there is a q such that $N_n(q) = [\mathcal{M}_n : (q)] = \ell$, where $N_n(q)$ is the algebraic norm of q.

The unique factorisation of q over $\mathcal{M}_n$ (with class number one) reads

$$q = \varepsilon \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i} \prod_{p_j \in \mathcal{C}} \omega_{p_j}^{\beta_j} \overline{\omega_{p_j}}^{\gamma_j} \prod_{p_k \in \mathcal{R}} p_k^{\delta_k},$$

where ε is a unit in $\mathcal{M}_n$ and $\omega_{p_j}\overline{\omega}_{p_j} = p_j$. Here, $\mathcal{P}$ (resp. $\mathcal{C}$, resp. $\mathcal{R}$) denotes the set of inert (resp. complex splitting, resp. ramified) primes over $\mathcal{M}_n$. The generator q is called *balanced* if $\beta_j = \gamma_j$ for all j. In other words: q is balanced if it is of the form

$$q = \varepsilon x p,$$

where ε is a unit in $\mathcal{M}_n$, x is a real number in $\mathcal{M}_n$, and p is a product of ramified primes. By the definition of a ramified prime p (see [12]), $\overline{p} \in (p)$ holds in $\mathcal{M}_n$. (Equivalently, $p/\overline{p}$ is a unit in $\mathcal{M}_n$.) Recall that all units ε in $\mathbb{Z}[\xi_n]$ are of the form $\varepsilon = \pm\lambda\xi_n^k$, where $\lambda \in \mathbb{Z}[\xi + \overline{\xi}]$. See [11].

In [11], it is shown that an ideal colouring of $\mathcal{M}_n$ induced by (q) is perfect (that is, $H = G$) if and only if q is balanced. Otherwise, $H = G'$.

For an ideal (q) of $\mathcal{M}_n$ of index $\ell = 2^{\phi(n)}$, the following assertions are true and allow us to conclude directly the structure of H.

Lemma 3.2. *Suppose $\ell = 2^{\phi(n)}$. Then q is balanced if and only if $(q) = (2)$. Otherwise, $(q) \neq (2)$.*

Proof. If $(q) = (2)$, then $(\overline{2}) = (2)$, and so q is balanced. Conversely, suppose q is balanced and $N_n(q) = \ell = 2^{\phi(n)}$. From the unique factorisation of q, we write $q = \varepsilon y p$, where ε is a unit in $\mathcal{M}_n$, y being real is a product of complex splitting primes (and/or inert primes) and p is a product of ramified primes. We then observe what happens to p, noting that only either $N_n(p) = 1$ or $N_n(p) = 2^{\phi(n)}$, since y is real.

If $N_n(p) = 1$, then $N_n(\varepsilon y) = 2^{\phi(n)}$, implying $y = 2$. Thus $q = 2\varepsilon$, and so $(q) = (2)$. Suppose $N_n(p) = 2^{\phi(n)}$. This forces $(y) = (1)$. More so, p, being the product of ramified primes, can only take factors of (2), or else $N_n(p) \neq 2^{\phi(n)}$. This implies $(p) = (2)$. Since $q = \varepsilon p$, then $(q) = (p) = (2)$. □

Corollary 3.3. *If $\ell = 2^{\phi(n)}$ but $(q) \neq (2)$, then q is not balanced and so $H = G'$.*

For $\mathcal{M}_{15}$ for example, there are three colourings with $2^{\phi(15)} = 256$ colours. From the above corollary it follows that one colouring is perfect ($H = G$) and the other two colourings are chirally perfect ($H = G'$). See Table 2.

4. The structure of the colour preserving group K

We now give a series of results that will be used in the determination of the structure of K. These results complement those given in [11] and serve as additional tools in arriving at the calculations carried out in this work.

The following notations will be used to denote particular isometries in $\mathcal{M}_n$: Let S denote the reflection $z \mapsto \overline{z}$, and R_k the k-fold rotation about 0. In particular R_2 is the rotation by π about 0. We denote the group of translations by elements in (q) by $T_{(q)}$. Note that $T_{(q)}$ is always contained in K. (See [11].)

Lemma 4.1. *If $\ell \nmid 2^{\phi(n)}$, then $R_2 \notin K$.*

Proof. If $R_2 \in K$, then $c(1) = c(-1)$. This implies $2 \in (q)$. Thus, $N_n(q)|N_n(2)$ or $\ell | 2^{\phi(n)}$. □

Lemma 4.2. *For n prime in $\mathbb{Z}$: If $\ell \neq n^j$ for any j, and $\ell \nmid 2^{\phi(n)}$ then $K = T_{(q)}$.*

Proof. This follows immediately from Lemma 5.10 in [11], which states that if $\ell \nmid 2^{\phi(n)}$ and $\ell \nmid n^{\phi(n)}$, then $K = T_{(q)}$. □

Lemma 4.3. *For $n > 2$ prime in $\mathbb{Z}$: If $\ell = n^j$ for $j > 1$, then $K = T_{(q)}$.*

Proof. Since n is an odd prime, there are four elements of $D_N = D_{2n}$ that may generate a subgroup which fixes the colours. These are R_2, R_n, R_{2n} and S.

If $c(1) = c(-1)$ then $\ell \mid 2^{\phi(n)}$, which is a contradiction. So R_2 does not fix the colours. Similarly, if $c(1) = c(\xi)$, then $1 - \xi \in (q)$ which is again a contradiction since $N_n(1-\xi) = n < n^j$, and so $N_n(q) \nmid N_n(1-\xi)$. Hence, R_n and thus also R_{2n} are not in K. Finally, $c(\xi) = c(\xi^{n-1})$ implies $c(1) = c(\xi^2) = c(\xi^4) = \ldots = c(\xi^{n-1}) = c(\xi)$, which is another contradiction, and so S cannot fix the colours. □

Lemma 4.4. *For $n > 2$ prime in $\mathbb{Z}$: $\ell = n$ if and only if $K = T_{(q)} \rtimes D_n$.*

Proof. Note that in the case when n is an odd prime, the symmetry group of (q) contains $D_N = D_{2n}$.

Let $2 < \ell = n$ and n prime in $\mathbb{Z}$. Then the unique factorisation of $\ell = n$ in $\mathcal{M}_n$ is $\ell = \prod_{i=1}^{n-1}(1 - \xi^i)$. Thus ℓ ramifies, and the possible generators of the ideal (q) are exactly the $1 - \xi^i$. Hence each colouring is perfect: $H = G$. In fact, there is only one such colouring, since for all

$1 \leq j \leq n$ holds: $1-\xi^j \in (1-\xi)$. (This follows from $\xi^k(1-\xi) \in (q)$, thus $\sum_{k=0}^{j-1} \xi^k(1-\xi) = 1-\xi^j \in (q)$.) Moreover, it follows that $c(1) = c(\xi^j)$ for all j.

Since ℓ is prime in $\mathbb{Z}$, we have $\mathcal{M}_n/(q) \cong C_\ell$ (the cyclic group of order ℓ), and so the ℓ distinct cosets can be expressed as $(q), (q)+1, (q)+2, \ldots, (q)+\ell-1$. Each coset is invariant under multiplication by ξ^j, but not under multiplication by $-\xi$. Thus, $K = T_{(q)} \rtimes D_n$.

Conversely, compare the proof of the last lemma. Notice that if $K = T_{(q)} \rtimes D_n$ then $1-\xi \in (q)$ and so $\ell \mid n$. Since n is prime in $\mathbb{Z}$ then either $\ell = 1$ or $\ell = n$. The case $\ell = 1$ is trivial and so $K = G$ and is a contradiction. Thus, $\ell = n$. □

It is well known that $\prod_{j=1}^{p^r-1}(1-\xi_{p^r}^j) = p^r$, for p prime. This implies the following:

$$(**) \qquad \prod_{(p^r,j)=1} (1-\xi_{p^r}^j) = p, \quad \text{for any } r > 0.$$

Now, recall that the algebraic norm of $q \in M_n$ is defined as follows:

$$N_n(q) = \prod_{j=1}^{\phi(n)} \sigma_j(q), \quad \text{where } \sigma_j \in Gal(\mathbb{Q}(\xi_n), \mathbb{Q}),$$

and that $N_n(q) = N_n(\sigma_j(q))$ for all j. The following results characterise the norm $N_n(1-\xi_s^r)$.

Lemma 4.5. *For $n = p^r$, p prime and $r > 0$, $N_{p^r}(1-\xi_{p^r}) = p$ for any $r > 0$.*

Proof. This follows from (**), since $N_{p^r}(1-\xi_{p^r}) = \prod_{j=1}^{\phi(p^r)} \sigma_j(1-\xi_{p^r}) = \prod_{(p^r,j)=1}(1-\xi_{p^r}^j) = p$. □

Lemma 4.6. *For prime p dividing n: $N_n(1-\xi_p) = p^{\phi(n)/\phi(p)}$.*

Proof. Since $N_n(1-\xi_p) = \prod_{j=1}^{\phi(n)} \sigma_j(1-\xi_p) = \prod_{j=1}^{\phi(n)/\phi(p)} \prod_{k=1}^{\phi(p)} \sigma'_k(1-\xi_p)$, where $\sigma_j \in Gal(\mathbb{Q}(\xi_n), \mathbb{Q})$ and $\sigma'_k \in Gal(\mathbb{Q}(\xi_p), \mathbb{Q})$, then $N_n(1-\xi_p) = \prod_{j=1}^{\phi(n)/\phi(p)} N_p(1-\xi_p) = p^{\phi(n)/\phi(p)}$. □

Lemma 4.7. *If two distinct primes p and q divide n, then $N_n(1-\xi_n) = 1$.*

Proof. Let pq divide n. Then $(1-\xi_p) \subset (1-\xi_n)$ and $(1-\xi_q) \subset (1-\xi_n)$. Thus $N_n(1-\xi_n) \mid N_n(1-\xi_p)$ and $N_n(1-\xi_n) \mid N_n(1-\xi_q)$, and so $N_n(1-\xi_n) = 1$. □

Lemma 4.8. *$N_n(1-\xi_n^m) = N_n(1-\xi_n)$ whenever $(m,n) = 1$.*

Proof. This follows from the fact that $N_n(q) = N_n(\sigma_j(q))$, where $\sigma_j \in Gal(\mathbb{Q}(\xi_n), \mathbb{Q})$. □

Theorem 4.9. *Whenever $(r,s)=1$, for integers r and s, the following is true: $N_n(1-\xi_s^r) = N_n(1-\xi_s) = p^{\phi(n)/\phi(s)}$, if $s=p^j$ for some $j>0$; otherwise, $N_n(1-\xi_s^r) = N_n(1-\xi_s) = 1$.*

Proof. From the previous lemma, $N_n(1-\xi_s^r) = N_n(1-\xi_s)$. Now, recall that $N_n(1-\xi_s) = \prod_{j=1}^{\phi(n)} \sigma_j(1-\xi_s) = \prod_{j=1}^{\phi(n)/\phi(s)} \prod_{k=1}^{\phi(s)} \sigma'_k(1-\xi_s)$, where $\sigma_j \in Gal(\mathbb{Q}(\xi_n),\mathbb{Q})$ and $\sigma'_k \in Gal(\mathbb{Q}(\xi_s),\mathbb{Q})$. This implies that $N_n(1-\xi_s) = \prod_{j=1}^{\phi(n)/\phi(s)} N_s(1-\xi_s)$. Finally, from Lemmas 4.5 and 4.7, the assertion follows. □

The lemma that follows suggests that for any non-trivial colouring of M_n, for which n is odd, the $2n$-fold rotation cannot preserve the colours, and so $R_{2n} \notin K$.

Lemma 4.10. *For n odd, if $R_{2n} \in K$, then $\ell = 1$.*

Proof. Since R_{2n} fixes the colours, then in particular, $c(1) = c(\xi_n) = c(-1)$. The last equality implies that $\ell \mid N_n(1-\xi_n)$ and $\ell \mid N_n(2)$. From the previous lemmas, $N_n(1-\xi_n)$ is only either 1 or p^j for some j, where p is a prime dividing n. Since n is odd, then p is also odd. A number dividing 2 and an odd prime can only be 1. Thus, $\ell = 1$. □

With the addition of the next result, we can then enumerate all permissible indices for a rotation R_k (or S) to be contained in the colour preserving group K given an ideal colouring of $\mathcal{M}_n$.

Lemma 4.11. *For $2 < j \leq n$: If $R_j \in K$, then $\ell \mid N_n(1-\xi_n^{n/j})$. When $j=2$, then $\ell \mid 2^{\phi(n)}$.*

Proof. From the previous lemma, if n is odd, $R_{2n} \notin K$, hence, we only need to check until $j=n$. If $R_j \in K$, then $c(1) = c(\xi_n^{n/j})$, which implies that $\ell \mid N_n(1-\xi_n^{n/j})$. □

Theorem 4.9 facilitates the computation of $N_n(1-\xi_n^{n/j})$. Table 1 gives a comprehensive list of permissible indices for a particular element of D_N to fix the colours given a coset colouring of $\mathcal{M}_n$ induced by (q). For example, consider the ideal colouring of $\mathcal{M}_{15}$ induced by (q). Suppose R_3, which is a 3-fold rotation, fixes all the colours, then $c(1) = c(\xi_{15}^5)$. Thus, $(1-\xi_{15}^5) \in (q)$ and so, $\ell \mid [\mathcal{M}_{15} : (1-\xi_{15}^5)] = N_n(1-\xi_{15}^5) N_n(1-\xi_3) = 3^4 = 81$. Hence, if $\ell \nmid 81$, then we have $R_3 \notin K$.

In this table, $s \lrcorner t$ denotes that an s-fold rotation may only preserve all the colours if the index ℓ divides t. This implies that if ℓ does not divide t, then an s-fold rotation cannot fix the colours. Note that $\overline{2}$ in the table pertains to the reflection along the horizontal axis.

We enumerate our calculations for the groups H and K as shown in Tables 2, 3 and 4. In these tables, many entries under H and K follow immediately from the results we have established (also, see [11]). Entries in brackets require further computations (compare [10, 13]), depending on the case. For instance, if $n = 16$, $\ell = 4$ (see Table 2), one checks that the four cosets are invariant under R_8 but not R_{16}. On the other hand, entries without brackets are immediate. Entries with an asterisk mean that there are multiple different possibilities. The last column lists one (out of possibly more than one) generator q of a corresponding ℓ-colouring.

5. Conclusion

In this work we present a series of results that strengthens our understanding of the colour symmetry group and colour preserving group of ideal colourings of $\mathcal{M}_n$. The results given here provide us with more insight on the colour preserving group K and facilitate the characterisation of its structure in addition to what has already been determined in previous works [10, 11, 13]. Moreover, the results make the derivation of K more straightforward for a large number of cases. As an example, consider the case when $n = 11$. From [11], we learn the following: $K = T_{(q)} \rtimes D_{11}$ when $\ell = 11$; $K = T_{(q)} \rtimes C_2$ when $\ell = 2^{10} = 1024$; and $K = T_{(q)}$ for all ℓ-colourings whenever $\ell > 1024$. Using Lemmas 4.2 and 4.3, we learn that all the other ℓ-colourings yield $K = T_{(q)}$ (see Table 4).

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References

[1] M.L.A.N. de las Peñas and R.P. Felix, *Color groups associated with square and hexagonal lattices*, Z. Krist. 222 (2007) p. 505.

[2] M.L.A.N. de las Peñas and M.V.P. Quilinguin, *A framework for coloring symmetrical patterns*, in *Algebras and Combinatorics: An International Congress - ICAC '97*, Hongkong, (Eds. K.P. Shum, E.J. Taft and Z.X.Wan), Springer Verlag, Singapore, 1999, pp. 159-175.

[3] R. Lifshitz, *Theory of color symmetry for periodic and quasiperiodic crystals*, Rev. Modern Phys. 69 (1997) p. 1181-1218.

[4] M. Senechal, *Color groups*, Discrete Appl. Math. 1 (1979) p. 51-73.

[5] R.L.E. Schwarzenberger, *Colour symmetry*, Bull. Lond. Math. Soc. 16 (1984) p. 209-240.

[6] M. Baake, *Combinatorial aspects of colour symmetries*, J. Phys. A 30 (1997) p. 2687-2698; `mp_arc/02-323`.

[7] M. Baake and U. Grimm, *Bravais colourings of planar modules with N-fold symmetry*, Z. Krist. 219 (2004) p. 72-80; `math.CO/0301021`.

[8] M. Baake, U. Grimm and M. Scheffer, *Colourings of planar quasicrystals*, J. Alloys. Compd. 342 (2002) p. 195-197; `cond-mat/0110654`.

[9] N. D. Mermin, D. S. Rokhsar and D. C. Wright, *Beware of 46-fold symmetry: The classification of two-dimensional quasicrystallographic lattices* Phys. Rev. Lett. 58 (1987) p. 2099-2101.
[10] E.P. Bugarin, M.L.A.N. de las Peñas, I. Evidente, R.P. Felix and D. Frettlöh, *On color groups of Bravais colorings of planar modules with quasicrystallographic symmetry*, Z. Krist. 223 (2008) p. 785-790.
[11] E.P. Bugarin, M.L.A.N. de las Peñas and D. Frettlöh, *Perfect colourings of cyclotomic integers*, submitted (preprint 2009), submitted; arxiv.org/pdf/0905.4048.
[12] L.C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1996.
[13] M.L.A.N. de las Peñas and E.P. Bugarin, *Bravais colorings associated with periodic and non-periodic crystals*, (preprint 2010), submitted.
[14] J. Neukirch, *Algebraic number theory*, Springer, Berlin, 1999.
[15] The GAP Group, *GAP – Groups, Algorithms, and Programming ??Version 4.4.12.*, 2008. Available at http://www.gap-system.org.
[16] J. Dräger and N.D. Mermin, *Superspace Groups without the Embedding: The Link between Superspace and Fourier-Space Crystallography*, Phys. Rev. Lett. 76 (1996) p. 1489.-1492.
[17] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, Freeman, New York, 1987.
[18] R. Lück, *Colour symmetry of 25 colours in quasiperiodic patterns*, Phil. Mag. 88 (2008) p. 2049.-2058.
[19] N.D. Mermin, *Copernican crystallography*, Phys. Rev. Lett. 68 (1992) p. 1172.-1175.
[20] R.V. Moody and J. Patera, *Colourings of quasicrystals*, Can. J. Phys. 72 (1994) pp. 442-452.
[21] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, 1995.
[22] B.L. van der Waerden and J.J. Burckhardt: *Farbgruppen*, Z. Krist. 115 (1961) p. 231-234.

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Table 1. Divisibility conditions. $s \lrcorner t$ denotes an s-fold rotation may only fix the colours of an ℓ-colouring if ℓ divides t.

$\phi(n)$	n	Divisibility conditions
2	3	$2\lrcorner 4$ $3\lrcorner 3$ $\overline{2}\lrcorner 3$
	4	$2\lrcorner 4$ $4\lrcorner 2$ $\overline{2}\lrcorner 4$
4	5	$2\lrcorner 16$ $5\lrcorner 5$ $\overline{2}\lrcorner 5$
	8	$2\lrcorner 16$ $4\lrcorner 4$ $8\lrcorner 2$ $\overline{2}\lrcorner 4$
	12	$2\lrcorner 16$ $3\lrcorner 9$ $4\lrcorner 4$
6	7	$2\lrcorner 64$ $7\lrcorner 7$ $\overline{2}\lrcorner 7$
	9	$2\lrcorner 64$ $3\lrcorner 27$ $9\lrcorner 3$ $\overline{2}\lrcorner 3$
8	15	$2\lrcorner 256$ $3\lrcorner 81$ $5\lrcorner 25$
	16	$2\lrcorner 256$ $4\lrcorner 16$ $8\lrcorner 4$ $16\lrcorner 2$ $\overline{2}\lrcorner 4$
	20	$2\lrcorner 256$ $4\lrcorner 16$ $5\lrcorner 25$
	24	$2\lrcorner 256$ $3\lrcorner 81$ $4\lrcorner 16$ $8\lrcorner 4$
10	11	$2\lrcorner 1024$ $11\lrcorner 11$ $\overline{2}\lrcorner 11$
12	13	$2\lrcorner 4096$ $13\lrcorner 13$ $\overline{2}\lrcorner 13$
	21	$2\lrcorner 4096$ $3\lrcorner 729$ $7\lrcorner 49$
	28	$2\lrcorner 4096$ $4\lrcorner 64$ $7\lrcorner 49$
	36	$2\lrcorner 4096$ $3\lrcorner 729$ $4\lrcorner 64$ $9\lrcorner 9$
16	17	$2\lrcorner 65536$ $17\lrcorner 17$ $\overline{2}\lrcorner 17$
	32	$2\lrcorner 65536$ $4\lrcorner 256$ $8\lrcorner 16$ $16\lrcorner 4$ $32\lrcorner 2$ $\overline{2}\lrcorner 4$
	40	$2\lrcorner 65536$ $4\lrcorner 256$ $5\lrcorner 625$ $8\lrcorner 16$
	48	$2\lrcorner 65536$ $3\lrcorner 6561$ $4\lrcorner 256$ $8\lrcorner 16$
	60	$2\lrcorner 65536$ $3\lrcorner 6561$ $4\lrcorner 256$ $5\lrcorner 625$
18	19	$2\lrcorner 262144$ $19\lrcorner 19$ $\overline{2}\lrcorner 19$
	27	$2\lrcorner 262144$ $3\lrcorner 19683$ $9\lrcorner 27$ $27\lrcorner 3$ $\overline{2}\lrcorner 3$
20	25	$2\lrcorner 1048576$ $5\lrcorner 3125$ $25\lrcorner 5$ $\overline{2}\lrcorner 5$
	33	$2\lrcorner 1048576$ $3\lrcorner 59049$ $11\lrcorner 121$
	44	$2\lrcorner 1048576$ $4\lrcorner 1024$ $11\lrcorner 121$
24	35	$2\lrcorner 16777216$ $5\lrcorner 15625$ $7\lrcorner 2401$
	45	$2\lrcorner 16777216$ $3\lrcorner 531441$ $5\lrcorner 15625$ $9\lrcorner 81$
	84	$2\lrcorner 16777216$ $3\lrcorner 531441$ $4\lrcorner 4096$ $7\lrcorner 2401$

TABLE 2. The cases $n = 15, 16$ ($\phi(n) = 8$). Here, j denotes the number of colourings with ℓ colours.

$\phi(n)$	n	ℓ	j	H	K	q
8	15	16	2	$\{G'\}$	$\{T_{(q)} \rtimes C_2\}$	$1 + \xi_{15} + \xi_{15}^4$
		25	1	G	$\{T_{(q)} \rtimes C_5\}$	$1 - \xi_{15}^3$
		31	8	G'	$T_{(q)}$	$1 + \xi_{15} + \xi_{15}^3$
		61	8	G'	$T_{(q)}$	$1 + \xi_{15}^3 + \xi_{15}^5 + \xi_{15}^7$
		81	1	G	$\{T_{(q)} \rtimes C_3\}$	$1 - \xi_{15}^5$
		121	4	$\{G'\}$	$T_{(q)}$	$2 + \xi_{15}^3 + \xi_{15}^6$
		151	8	G'	$T_{(q)}$	$1 - 2\xi_{15}$
		181	8	G'	$T_{(q)}$	$1 + \xi_{15} + \xi_{15}^3 + \xi_{15}^5 + \xi_{15}^7$
		211	8	G'	$T_{(q)}$	$1 + \xi_{15}^2 + 2\xi_{15}^3$
		241	8	G'	$T_{(q)}$	$1 - \xi_{15}^6 + 2\xi_{15}^7 - \xi_{15}^9$
		256	1	G	$T_{(q)} \rtimes C_2$	2
			2	G'	$T_{(q)}$	$(1 + \xi_{15} + \xi_{15}^4)^2$
		> 256	*	*	$T_{(q)}$	*
8	16	2	1	G	$T_{(q)} \rtimes D_{16}$	$1 - \xi_{16}$
		4	1	G	$\{T_{(q)} \rtimes D_8\}$	$1 - \xi_{16}^2$
		8	1	G	$\{T_{(q)} \rtimes C_4\}$	$(1 - \xi_{16})^3$
		16	1	G	$\{T_{(q)} \rtimes C_4\}$	$1 - \xi_{16}^4$
		17	8	G'	$T_{(q)}$	$1 - \xi_{16} + \xi_{16}^3$
		32	1	G	$\{T_{(q)} \rtimes C_2\}$	$(1 - \xi_{16})^5$
		34	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})(1 - \xi_{16} + \xi_{16}^3)$
		49	4	$\{G'\}$	$T_{(q)}$	$1 - \xi_{16} - \xi_{16}^2$
		64	1	G	$\{T_{(q)} \rtimes C_2\}$	$(1 - \xi_{16}^2)^3$
		68	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16}^2)(1 - \xi_{16} + \xi_{16}^3)$
		81	2	G'	$T_{(q)}$	$1 + \xi_{16}^4 + \xi_{16}^6$
		97	8	G'	$T_{(q)}$	$1 + 2\xi_{16}^3 + \xi_{16}^5 + \xi_{16}^7$
		98	4	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})(1 - \xi_{16} - \xi_{16}^2)$
		113	8	G'	$T_{(q)}$	$2 - 2\xi_{16} + \xi_{16}^5$
		128	1	G	$\{T_{(q)} \rtimes C_2\}$	$(1 - \xi_{16})^7$
		136	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})^3(1 - \xi_{16} + \xi_{16}^3)$
		162	2	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})(1 + \xi_{16}^4 + \xi_{16}^6)$
		193	8	G'	$T_{(q)}$	$1 - \xi_{16}^2 - \xi_{16}^3 + \xi_{16}^4 + \xi_{16}^7$
		194	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})(1 + 2\xi_{16}^3 + \xi_{16}^5 + \xi_{16}^7)$
		196	4	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16}^2)(1 - \xi_{16} - \xi_{16}^2)$
		226	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{16})(2 - 2\xi_{16} + \xi_{16}^5)$
		241	8	G'	$T_{(q)}$	$1 - \xi_{16} - \xi_{16}^2 + \xi_{16}^3 + \xi_{16}^5$
		256	1	G	$T_{(q)} \rtimes C_2$	2
		> 256	*	*	$T_{(q)}$	*

TABLE 3. The cases $n = 20, 24$ ($\phi(n) = 8$). Here, j denotes the number of colourings with ℓ colours.

$\phi(n)$	n	ℓ	j	H	K	q
8	20	5	2	$\{G'\}$	$\{T_{(q)} \rtimes C_5\}$	$1 + \xi_{20} - \xi_{20}^3$
		16	1	G	$\{T_{(q)} \rtimes C_4\}$	$1 - \xi_{20}^5$
		25	1	$\{G\}$	$\{T_{(q)} \rtimes C_5\}$	$1 - \xi_{20}^4$
			2	$\{G'\}$	$\{T_{(q)}\}$	$(1 + \xi_{20} - \xi_{20}^3)^2$
		41	8	G'	$T_{(q)}$	$1 + \xi_{20} - \xi_{20}^5$
		61	8	G'	$T_{(q)}$	$1 + \xi_{20} + \xi_{20}^2$
		80	2	$\{G'\}$	$\{T_{(q)}\}$	$(1 + \xi_{20} - \xi_{20}^3)(1 - \xi_{20}^5)$
		81	2	$\{G'\}$	$T_{(q)}$	$1 + \xi_{20} + \xi_{20}^2 + \xi_{20}^3 + \xi_{20}^8$
		101	8	G'	$T_{(q)}$	$2 + \xi_{20} - \xi_{20}^3$
		121	2	$\{G'\}$	$T_{(q)}$	$2 - \xi_{20}^2$
		125	2	$\{G\}$	$\{T_{(q)}\}$	$(1 + \xi_{20} - \xi_{20}^3)^3$
			2	$\{G'\}$	$\{T_{(q)}\}$	$(1 + \xi_{20} - \xi_{20}^3)(1 - \xi_{20}^4)$
		181	8	G'	$T_{(q)}$	$2 + 2\xi_{20} - \xi_{20}^3$
		205	16	$\{G'\}$	$T_{(q)}$	$(1 + \xi_{20} - \xi_{20}^3)(1 + \xi_{20} - \xi_{20}^5)$
		241	8	G'	$T_{(q)}$	$1 - \xi_{20} + \xi_{20}^4 - 2\xi_{20}^6$
		256	1	G	$T_{(q)} \rtimes C_2$	2
		> 256	*	*	$T_{(q)}$	*
8	24	4	1	G	$\{T_{(q)} \rtimes C_8\}$	$1 - \xi_{24}^3$
		9	2	$\{G'\}$	$\{T_{(q)} \rtimes C_3\}$	$1 - \xi_{24} - \xi_{24}^3$
		16	1	G	$\{T_{(q)} \rtimes C_4\}$	$1 - \xi_{24}^6$
		25	4	$\{G'\}$	$T_{(q)}$	$1 + \xi_{24} - \xi_{24}^3$
		36	2	$\{G'\}$	$\{T_{(q)}\}$	$(1 - \xi_{24}^3)(1 - \xi_{24} - \xi_{24}^3)$
		49	4	$\{G'\}$	$T_{(q)}$	$1 - \xi_{24} - \xi_{24}^6$
		64	1	G	$\{T_{(q)} \rtimes C_2\}$	$(1 - \xi_{24}^3)^3$
		73	8	G'	$T_{(q)}$	$1 - \xi_{24} + \xi_{24}^4$
		81	1	$\{G\}$	$\{T_{(q)} \rtimes C_3\}$	$1 - \xi_{24}^8$
			2	$\{G'\}$	$\{T_{(q)}\}$	$(1 - \xi_{24} - \xi_{24}^3)^2$
		97	8	G'	$T_{(q)}$	$1 - \xi_{24} - \xi_{24}^2 + \xi_{24}^3 + \xi_{24}^5 + \xi_{24}^6$
		100	4	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{24}^3)(1 + \xi_{24} - \xi_{24}^3)$
		121	4	$\{G'\}$	$T_{(q)}$	$2 + \xi_{24} - \xi_{24}^2 - \xi_{24}^4 - \xi_{24}^7$
		144	2	$\{G'\}$	$\{T_{(q)}\}$	$(1 - \xi_{24} - \xi_{24}^3)(1 - \xi_{24}^6)$
		169	4	$\{G'\}$	$T_{(q)}$	$2 - \xi_{24}^2$
		193	8	G'	$T_{(q)}$	$2 - 2\xi_{24} + \xi_{24}^7$
		196	4	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{24}^3)(1 - \xi_{24} - \xi_{24}^6)$
		225	8	$\{G'\}$	$T_{(q)}$	$(1 - \xi_{24} - \xi_{24}^3)(1 + \xi_{24} - \xi_{24}^3)$
		241	8	G'	$T_{(q)}$	$2 - \xi_{24}$
		256	1	G	$T_{(q)} \rtimes C_2$	2
		> 256	*	*	$T_{(q)}$	*

Table 4. The case $\phi(n) = 10$. Here, j denotes the number of colourings with ℓ colours.

$\phi(n)$	n	ℓ	j	H	K	q
10	11	11	1	G	$T_{(q)} \rtimes D_{11}$	$1-\xi_{11}$
		23	10	G'	$T_{(q)}$	$1-\xi_{11}+\xi_{11}^3$
		67	10	G'	$T_{(q)}$	$1-\xi_{11}-\xi_{11}^3$
		89	10	G	$T_{(q)}$	$1-\xi_{11}^3+\xi_{11}^5+\xi_{11}^6$
		121	1	G	$T_{(q)}$	$(1-\xi_{11})^2$
		199	10	G'	$T_{(q)}$	$1+2\xi_{11}-\xi_{11}^3$
		243	2	$\{G'\}$	$T_{(q)}$	$2-\xi_{11}^2+\xi_{11}^5+\xi_{11}^7-\xi_{11}^8$
		253	10	$\{G'\}$	$T_{(q)}$	$(1-\xi_{11})(1-\xi_{11}+\xi_{11}^3)$
		331	10	G'	$T_{(q)}$	$2+\xi_{11}^2+\xi_{11}^4+\xi_{11}^7+\xi_{11}^8$
		353	10	G'	$T_{(q)}$	$2+\xi_{11}^3+\xi_{11}^5+\xi_{11}^6$
		397	10	G'	$T_{(q)}$	$1+\xi_{11}+\xi_{11}^2-\xi_{11}^3$
		419	10	G'	$T_{(q)}$	$2+\xi_{11}^8+\xi_{11}^9$
		463	10	G'	$T_{(q)}$	$2+\xi_{11}+\xi_{11}^3+\xi_{11}^4$
		529	55	$\{G'\}$	$T_{(q)}$	$(1-\xi_{11}+\xi_{11}^3)^2$
		617	10	G'	$T_{(q)}$	$2+\xi_{11}-\xi_{11}^3$
		661	10	G'	$T_{(q)}$	$2-\xi_{11}+\xi_{11}^5-\xi_{11}^7$
		683	10	G'	$T_{(q)}$	$2+\xi_{11}$
		727	10	G'	$T_{(q)}$	$1-\xi_{11}+\xi_{11}^3+\xi_{11}^4-\xi_{11}^5$
		737	10	$\{G'\}$	$T_{(q)}$	$(1-\xi_{11})(1-\xi_{11}-\xi_{11}^3)$
		859	10	G'	$T_{(q)}$	$1-\xi_{11}-\xi_{11}^3-\xi_{11}^4+\xi_{11}^5$
		881	10	G'	$T_{(q)}$	$1-\xi_{11}+2\xi_{11}^3+\xi_{11}^5+\xi_{11}^8+\xi_{11}^9$
		947	10	G'	$T_{(q)}$	$2+\xi_{11}^4-\xi_{11}^8-\xi_{11}^9$
		979	10	$\{G'\}$	$T_{(q)}$	$(1-\xi_{11})(1-\xi_{11}^3+\xi_{11}^5+\xi_{11}^6)$
		991	10	G'	$T_{(q)}$	$2+\xi_{11}+\xi_{11}^3$
		1013	10	G'	$T_{(q)}$	$2-\xi_{11}-\xi_{11}^2-\xi_{11}^9$
		1024	1	G	$T_{(q)} \rtimes C_2$	2
		> 1024	*	*	$T_{(q)}$	*