Counting Tiles in Substitution Tilings

Dirk Frettlöh

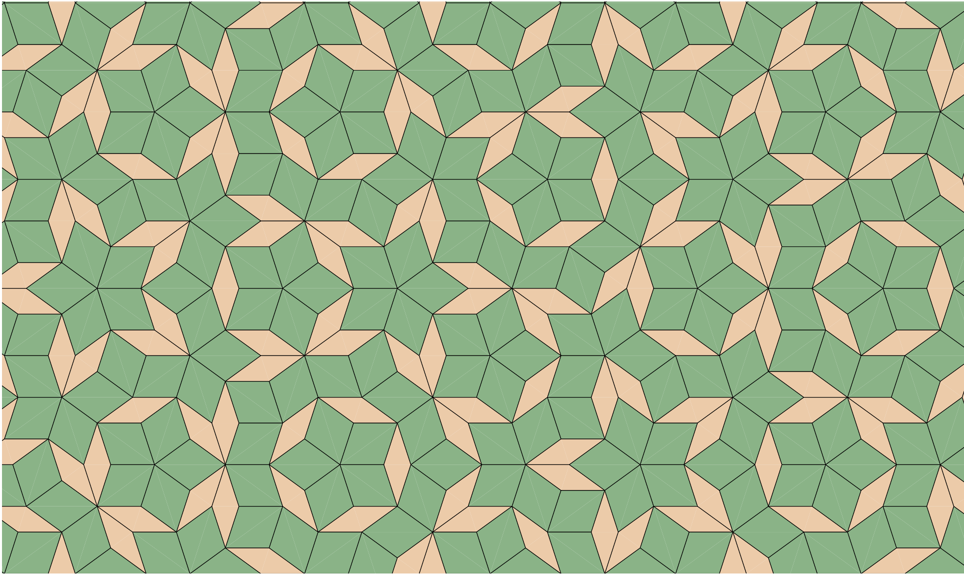
Joint work with Alexey Garber and Neil Mañibo

Technische Fakultät
Universität Bielefeld

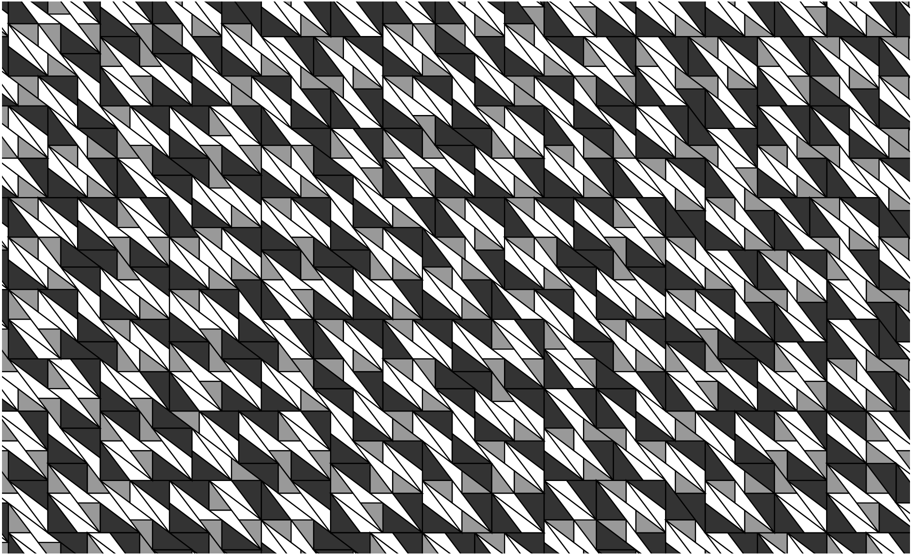
Kolloquium über Kombinatorik

Heidelberg, 20. Oktober 2023

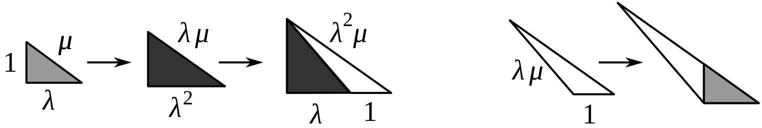
A famous aperiodic tiling: the Penrose Tiling



Another aperiodic tiling:



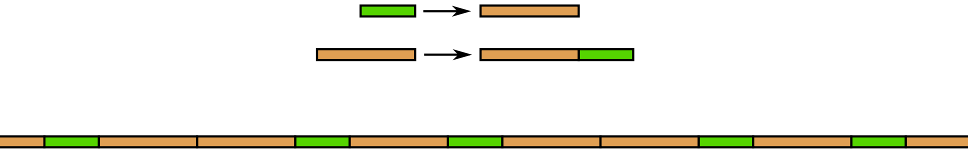
Obtained by a *tile substitution*:



- substitution matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$,
- inflation factor $\lambda = 1.3247\ldots$ (the *plastic number*),
- characteristic polynomial $x^3 - x - 1$.

Visit our zoo: `https:tilings.math.uni-bielefeld.de`

Substitution tiling in dimension $d = 1$ (Fibonacci substitution):



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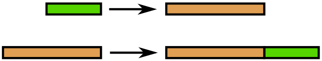
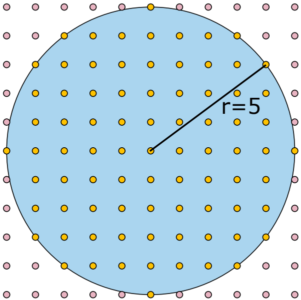


- substitution matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$,
- inflation factor $\lambda = \frac{1}{2}(1 + \sqrt{5})$,
- characteristic polynomial $x^2 - x - 1$.

Counting Meta-Question: Given a point set in $\mathbb{R}^2$, how many points are contained in a ball of radius r?

For instance,

- Gauss Circle Problem $\rightarrow$
- Point Density
- Mass Transport
- ...



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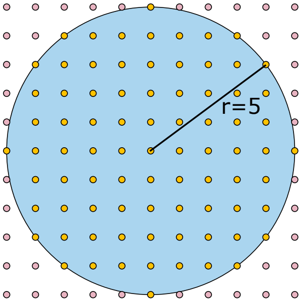

We are interested in the asymptotics and/or the error term.

For instance, Gauss circle problem: $\pi r^2 + O(r^{1/2+\varepsilon})$.

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Let us take a closer look.

- Denote the substitution by ϱ,
- the "tiles" by $a, b, \ldots$.
- What is $L(n) = \#\varrho^n(a)$? We know:

$$L(n) = \Theta(\lambda^n) + \text{"error term"}$$

In dimension 1 we can represent tiles by letters. Some examples:

Thue-Morse Substitution:

$$\varrho = \begin{cases} a & \to & ab \\ b & \to & ba \end{cases}$$

Here

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$$\varrho = \begin{cases} a & \to & ab \\ b & \to & ba \end{cases}$$

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- $L(n) = \#\varrho^n(a) = 2^n \qquad (\lambda = 2)$
- No error term.

Fibonacci Substitution:

$$\varrho = \begin{cases} a & \to & b \\ b & \to & ba \end{cases}$$

Here

- Here $L(n) = \#\varrho^n(a)$ is the $(n+1)^{st}$ Fibonacci number.
- Hence $L(n) \approx \frac{1}{\sqrt{5}}\lambda^{n+1}, \qquad \lambda = \frac{1}{2}(\sqrt{5}+1)$

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- Hence $L(n) \approx \frac{1}{\sqrt{5}}\lambda^{n+1}$, $\qquad \lambda = \frac{1}{2}(\sqrt{5}+1)$
- More precisely:

$$L(n) = \frac{1}{\sqrt{5}}\lambda^{n+1} + \frac{1}{\sqrt{5}}\Big(\frac{1}{2} - \frac{\sqrt{5}}{2}\Big)^{n+1}$$

Error term $\to 0$.

Weird Substitution:

$$\varrho = \begin{cases} a & \rightarrow & bbbbbb \\ b & \rightarrow & ba \end{cases}$$

- Here
$$L(n) = \#\varrho^n(a) = c_1 \cdot 3^n + c_2 \cdot (-2)^n.$$
- Large error term.

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- The substitution matrix is $M = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$.
- We have
$$L(n) = (1, 1) \cdot M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
- Therefore $L(n)$ can be expressed by the eigenvalues of M!

Crazy Substitution:

$$\varrho = \begin{cases} a \rightarrow abbbccc \\ b \rightarrow abbbbcccccc \\ c \rightarrow aabbbcccccccccc \end{cases}$$

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- Here

$$L(n) = \#\varrho^n(a) = c_1 \cdot 13^n + \text{linear polynomial in } n.$$

- The substitution matrix is $M = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 3 & 6 & 10 \end{pmatrix}$.
- Eigenvalues: 13, 1, 1. Nontrivial Jordan block.

Lesson:

Counting tiles tilings (or letters in words) generated by some substitution ϱ always yields:

$$L(n) = \#\varrho^n(a) = c_1 \cdot \lambda^n + \Theta(\lambda_2^n \cdot \text{polynomial in } n)$$

where

- λ the **Perron-Frobenius** eigenvalue (dominant eigenvalue),
- λ_2 the second largest eigenvalue in modulus.

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This is old stuff (OK, 10 or 20 years old). New game:

Infinitely many tile types (resp letters):

$$\varrho = \begin{cases} a & \to & ab \\ b & \to & aac \\ c & \to & abd \\ d & \to & ace \\ e & \to & adf \\ \dots & & \end{cases}$$

Infinitely many tile types (resp "letters"):

$$\varrho = \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 002 \\ 2 & \rightarrow & 013 \\ 3 & \rightarrow & 024 \\ 4 & \rightarrow & 035 \\ \ldots \end{cases}$$

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An entirely new playground, a lot to explore.

The substitution "matrix":

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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We cannot use just Linear Algebra anymore. Still,

(under much more restrictive conditions)

- There is an inflation factor λ,
- ...that is the dominant "eigenvalue" of M.
- And again we get

$$L(n) = \#\varrho^n(a) = \Theta(\lambda^n) + \text{"error term"}$$

Theorem

(F., Garber, Mañibo) For the example above

$$L(n) = \frac{3}{4}\left(\frac{5}{2}\right)^n + \Theta(n^{-3/2} \cdot 2^n)$$

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Idea: Let us pretend that Linear Algebra works. Then

$$L(n) = \frac{3}{4}\left(\frac{5}{2}\right)^n + r(n) = (1, 1, 1, 1, \cdots) \cdot M^n \cdot (1, 0, 0, 0, \cdots)^T$$

That is, the 0-th entry $\Big((1, 1, 1, \cdots) \cdot M^n\Big)_0$ of $(1, 1, 1, \cdots) \cdot M^n$.

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Eliminate the leading term:

$$2r(n+1) - 5r(n) = \Big((1,1,1,1,\cdots)(2M - 5I)M^n\Big)_0 = \Big((-1,1,1,1,\cdots)M^n\Big)_0$$

Under a change of basis:

$$
\begin{aligned}
e_0 &= (-1, 1, 1, 1, \cdots) \\
e_1 &= (1, -2, 0, 0, \cdots) \\
e_2 &= (0, 1, -2, 0, \cdots) \\
e_3 &= (0, 0, 1, -2, \cdots) \\
\cdots
\end{aligned}
$$

... right multiplication by M has the matrix

$$
B = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & \\
0 & 0 & 1 & 0 & 1 & \\
\vdots & & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
$$

Powers of B:

$$B^1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \\ 0 & 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Powers of B:

$$B^2 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 2 & 0 & 1 & \\ 0 & 1 & 0 & 2 & 0 & \\ 0 & 0 & 1 & 0 & 2 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Powers of B:

$$B^3 = \begin{pmatrix} 3 & 3 & 1 & 1 & 0 & \cdots \\ 3 & 1 & 3 & 0 & 1 & \cdots \\ 1 & 3 & 0 & 3 & 0 & \\ 1 & 0 & 3 & 0 & 3 & \\ 0 & 1 & 0 & 3 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Powers of B:

$$B^4 = \begin{pmatrix} 6 & 4 & 4 & 1 & 1 & \cdots \\ 4 & 6 & 1 & 4 & 0 & \cdots \\ 4 & 1 & 6 & 0 & 4 & \\ 1 & 4 & 0 & 6 & 0 & \\ 1 & 0 & 4 & 0 & 6 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Powers of B:

$$B^5 = \begin{pmatrix} 10 & 10 & 5 & 5 & 1 & \cdots \\ 10 & 5 & 10 & 1 & 5 & \cdots \\ 5 & 10 & 1 & 10 & 0 & \\ 5 & 1 & 10 & 0 & 10 & \\ 1 & 5 & 0 & 10 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Therefore

$$\begin{aligned} 2r(n+1) - 5r(n) &= (-1,1,1,1,\cdots)M^n(1,0,0,0,\cdots)^T \\ &= \Big(e_0 \cdot B^n\Big)_0 \\ &= \begin{cases} \binom{n}{n/2} - \binom{n}{n/2+1} & \text{for } n \text{ even} \\ \binom{n}{(n-1)/2} - \binom{n}{(n+1)/2} & \text{for } n \text{ odd.} \end{cases} \end{aligned}$$

Hence, for n even: **Catalan numbers**, for n odd: 0.

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Hence, for n even: **Catalan numbers**, for n odd: 0.

Using the generating series of Catalan numbers this yields exact values for the error term $\big($In terms of infinite series $\sum_{k=n}^{\infty}(2/5)^{2k}C_k \dots\big)$

One can push it a little further:

Theorem

(F., Garber, Mañibo) There are substitutins with

$$L(n) = \frac{3}{4}\left(\frac{5}{2}\right)^n + \Theta(n^{-3/2-q} \cdot 2^n) \qquad (q \in \mathbb{N})$$

...but it gets technical.

Conclusion

- We have no idea on the general error behaviour.
- D.F., Alexey Garber, Neil Mañibo: Catalan numbers as discrepancies for a family of substitutions on infinite alphabets, *Indagationes Mathematicae* 2023.
- More on the general topic on `arxiv.org`:
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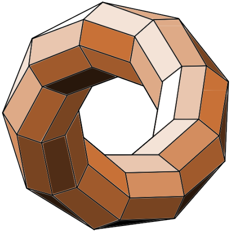

Thank you!