PERFECT COLOURINGS OF SIMPLICES AND HYPERCUBES IN DIMENSION FOUR AND FIVE WITH FEW COLOURS

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Abstract. A vertex colouring of some graph is called *perfect* if each vertex of colour i has the same number a_{ij} of neighbours of colour j. Here we determine all perfect colourings of the edge graphs of the hypercube in dimensions 4 and 5 by two and three colours, respectively. For comparison we list all perfect colourings of the edge graphs of the simplex in dimensions 4 and 5, respectively.

1. Introduction

Perfect colourings of graphs are colourings with the following property: If some vertex of colour i has exactly a_{i1} neighbours of colour 1, exactly a_{i2} neighbours of colour 2 and so on, then *all* vertices of colour i have exactly a_{i1} neighbours of colour 1, exactly a_{i2} neighbours of colour 2 and so on. Figure 1 shows a perfect colouring of the edge graph of the cube with three colours. The matrix $(a_{ij})_{ij}$ is the *colour adjacency* of the perfect colouring.

Perfect colourings appear throughout the literature under several different names: equitable partitions, completely regular vertex sets, distance partitions, association schemes, etc; and in in several contexts: algebraic graph theory, combinatorial designs, coding theory, finite geometry. For instance, each distance partition of a distance regular graph is a perfect colouring, but not vice versa. Similarly, any subgroup of the automorphism group of a graph G induces a perfect colouring of G by considering the vertex orbits of the subgroup [9, Sec. 9.3]. However, not every perfect colouring arises from a graph automorphism. For a broader overview see [7, 10].

Perfect 2-colourings of hypercube graphs were already studied in [6], with emphasis on existence in arbitrary dimension, without aiming for determining all perfect colourings. In particular, in [6] perfect 2-colourings of the hypercube in dimension $d = 2^k - 1$ are derived from Hamming codes, and others were derived from the observation in the first part of the proof of Theorem 6.1. Some concrete perfect colourings for small graphs were constructed for instance in [2, 1, 3, 8, 11]. In [5] we generalized several results from those papers. In particular we determined all colour adjacency matrices of perfect colourings with 2, 3, or 4 colours for all 3-, 4-, and 5-regular graphs, and list all perfect colourings of the edge graph of the Platonic solids with 2, 3, or 4 colours. In the present paper we apply and extend the results of [5] to find all perfect colourings of the edge graphs of the simplices in dimension four and five, and all perfect 2-colourings and 3-colourings of the edge graphs of the hypercubes in dimension four and five. For the sake of brevity, let us denote the edge graph of the simplex in $\mathbb{R}^d$ by *d-simplex*, and the edge graph of the hypercube in $\mathbb{R}^d$ by *d-cube*.

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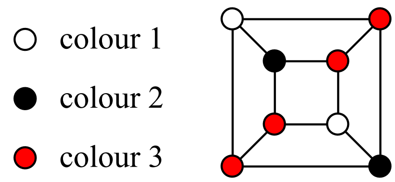


Figure 1. A perfect colouring of the edge graph of the cube with three colours. The corresponding colour adjacency matrix is $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

2. Preliminaries

Throughout the paper let $G = (V, E)$ be a finite, undirected, simple, loop-free graph. A partition of V into disjoint nonempty sets $V_1, \ldots, V_m$ is called an *m-colouring* of G. Note that we do not require adjacent vertices to have different colours. Let us recall the definition of a perfect colouring and make it precise.

Definition 2.1. A colouring of the vertex set V of some graph $G = (V, E)$ with m colours is called *perfect* if (1) all colours are used, and (2) for all i, j the number of neighbours of colours j of any vertex v of colour i is a constant a_{ij}. The matrix $A = (a_{ij})_{1 \le i,j \le m}$ is called the *colour adjacency matrix* of the perfect colouring.

Two trivial cases are $m = 1$, and $m = |V|$. In the latter case the colour adjacency matrix equals the adjacency matrix of G.
An early study of perfect colourings is [13], where the colour adjacency matrix was introduced to study spectral properties of certain graphs. The original paper is hard to find, but these results are contained in the textbook [4]. In particular, the following result was shown in [4][Theorem 4.5], see also [9, Theorem 9.3.3].

Theorem 2.2. *Let M be the adjacency matrix of some graph G and let A be the colour adjacency matrix of some perfect colouring of G. Then the characteristic polynomial of A divides the characteristic polynomial of M. In particular, each eigenvalue of A is an eigenvalue of M (with multiplicities).*

One main result of [5] is the following.

Theorem 2.3. *Suppose $A = (a_{ij}) \in \mathbb{N}^{m \times m}$. Then A is a colour adjacency matrix for a perfect m-colouring of some graph $G = (V, E)$ if and only if the following hold:*

1. *(Weak symmetry) For all $1 \le i, j \le m$ holds: $a_{ij} = 0$ if and only if $a_{ji} = 0$.*
2. *(Consistency) For any nontrivial cycle $(n_1\, n_2\, \ldots\, n_t)$ in the symmetric group S_m on the set $\{1, 2, \ldots, m\}$ holds:*

$$a_{n_1,n_2} a_{n_2,n_3} \cdots a_{n_{t-1},n_t} a_{n_t,n_1} = a_{n_2,n_1} a_{n_3,n_2} \cdots a_{n_t,n_{t-1}} a_{n_1,n_t}.$$

Moreover, there is a connected graph G with a perfect colouring corresponding to A if and only if A fulfills (1) and (2), and A is irreducible.

By a nontrivial cycle we mean a cyclic permutation in S_m of length at least three. (If the cycle is of length two, it might not deserve to be called trivial, but condition (2) becomes trivial in this case.) Recall that a symmetric matrix M is called *irreducible* if it is not conjugate via a permutation matrix to a block diagonal matrix having more than one block. (By "block diagonal matrix" we mean a square matrix having square matrices on its main diagonal, and all other entries being zero.) It is well-known that a directed graph G is connected if and only if its adjacency matrix is irreducible. A weaker statement is true here: if a graph G is connected then its colour adjacency matrix is irreducible. (Because one can travel from any colour to any other colour.) The generalization of the theorem to non-connected graphs is straight forward.

The proof of Theorem 2.3 is not hard. It is easy to see that (1) and (2) are necessary conditions for adjacency matrices for perfect colourings. (2) is due to an elaborate double counting argument. In order to illustrate condition (2), let us consider the perfect 3-colouring in Figure 1: for the nontrivial cycle $(1\,2\,3) \in S_3$ must hold that $a_{12}a_{23}a_{31} = 1 \cdot 2 \cdot 1 = 2 \cdot 1 \cdot 1 = a_{21}a_{32}a_{13}$, which is true here. For all other (trivial and nontrivial) cycles — for instance, $(1\,2)$ and $(1\,3\,2)$ — the condition is fulfilled as well.

The sufficiency part of the proof is achieved by providing a construction that yields a graph with a perfect m-colouring for any given matrix satisfying the conditions of Theorem 2.3.

In [5] we proceeded by using Theorem 2.3 to obtain the lists of all adjacency matrices for all perfect colourings of any regular graph of degree 3, 4, or 5 with two, three or four colours, respectively. These computations were carried out in `sagemath` [14]. The code and the lists are available online [15]. (Unfortunately, newer versions of `sagemath` seem to be unable to open these. Still you can see the code and the lists as plain text, and may import the lists by copy-and-paste into a sagemath notebook.) In [5] we then used Theorem 2.2 to determine those matrices in the lists whose eigenvalues match the eigenvalues of the edge graphs of the Platonic solids.

For the present paper we follow the same plan. We use Theorem 2.2 to determine all matrices in the lists mentioned above whether their eigenvalues match the spectrum of the edge graphs of the d-simplex, respectively the d-cube. This is easily done in `sagemath`. We end up with a list of candidates for perfect colourings. This is done in the next section. It remains to show the existence, respectively non-existence, of a perfect colouring for each of the candidates, which is done in Sections 4, 5, and 6.

3. Colour adjacency matrices of hypercubes and simplices

As just mentioned, we extract from the respective list those matrices whose eigenvalues are in the spectrum of the the 4-simplex, the 5-simplex, the 4-cube, or the 5-cube, respectively. The eigenvalues of these graphs are given in Table 1. These values can be found for instance in [4]. An entry a^n means that a is an eigenvalue of algebraic multiplicity n.

We used `sagemath` to check which of the matrices in the corresponding list have eigenvalues in the respective spectrum of the graph under consideration. The lists of these candidates are given in Sections 4, 5, and 6 below. Table 2 gives a summary by providing the number of these candidates in each case.

The table is to be read as follows. Each entry in row 5 and 6 just shows the number of perfect colourings. An entry in rows 1 to 4 is of the form a/b, where b is the number of candidates and a is the number of actual perfect colourings. For instance, the entry 5/10 (4-cube with 3 colours) in the table tells us: among all 64 matrices that are possible for perfect 3-colourings of 4-regular graphs (see [5] or [15]) only 10

G	eigenvalues
4-simplex	$-1^4, 4$
5-simplex	$-1^5, 5$
4-cube	$-4, -2^4, 0^6, 2^4, 4$
5-cube	$-5, -3^5, -1^{10}, 1^{10}, 3^5, 5$

TABLE 1. The eigenvalues of the graphs of the d-simplex and the d-cube for $d \in \{4, 5\}$. A superscript denotes the multiplicity of the respective eigenvalue.

$m \setminus G$	4-simplex	5-simplex	4-cube	5-cube
2	2/2	3/3	5/6	6/9
3	2/2	3/3	5/10	8/16
4	1/1	2/2	?/23	?/57
5	1	1	1	?
6	0	1	1	?

TABLE 2. The number of actual perfect colourings compared to the number of possible colour adjacency matrices (that is, correct eigenvalues) for perfect m-colourings of the 4-simplex, 5-simplex, 4-cube, 5-cube, respectively.

have all their eigenvalues in $\{-4, -2, 0, 2, 4\}$. These are the candidates for perfect 3-colourings of the 4-cube. Only 5 of them correspond to actual perfect colourings of the 4-cube, as we will see in Section 5.

4. All perfect colourings of the 4-simplex and the 5-simplex

By the procedure described above we obtain the following matrices as candidates for perfect colourings of the 4-simplex and the 5-simplex.

(1) 2 colours: $\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

(2) 3 colours: $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

(3) 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

5-simplex:

(1) 2 colours: $\begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

(2) 3 colours: $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

(3) 4 colours: $\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

In principle it remains to find a perfect colouring of the simplex for each of these matrices. However, since the edge graph of the d-simplex is just the complete graph

K_{d+1}, it gets much simpler. Because of the following result all these candidates correspond to perfect actual colourings.

Lemma 4.1. *Any colouring of K_{d+1} is perfect. Any integer partition of $d+1$ into m summands gives a perfect m-colouring of K_{d+1}, and vice versa.*

Proof. This is immediate. Since each vertex in K_{d+1} has all other vertices as neighbours there is no further restriction imposed by the combinatorics of the neighbours. Indeed, any arbitrary m-colouring of the vertices just corresponds to any partition of the vertex set V into m sets of size $a_1, \ldots, a_m$. Then $a_1 + \cdots + a_m = d + 1$. □

For instance, the three perfect colourings of the 5-simplex with two colours are in one-to-one correspondence with the integer partition of 6 into two summands: $6 = 5 + 1 = 4 + 2 = 3 + 3$, the three perfect colourings of the 5-simplex with three colours are in one-to-one correspondence with the integer partition of 6 into three summands: $6 = 4 + 1 + 1 = 3 + 2 + 1 = 2 + 2 + 2$, and the two perfect colourings of the 5-simplex with four colours are in one-to-one correspondence with the integer partition of 6 into four summands: $6 = 3 + 1 + 1 + 1 = 2 + 2 + 1 + 1$.
This observation allows us to determine all other perfect colourings of the 4-simplex and 5-simplex with five or more colours in Table 2. Of course there cannot be more colours than vertices, hence Table 2 contains all perfect colourings of the 4-simplex and the 5-simplex. The only remaining case in the list that is not entirely trivial is to colour the 5-simplex (having six vertices) with five colours. This is obtained from the integer partition $6 = 2 + 1 + 1 + 1 + 1$.
Having said all this all the computations in `sagemath` yielding the matrices above seem unnecessary. Still we carried them out to serve as a sanity check of the software.

5. Perfect colourings of the 4-cube

Applying the analogous procedure we obtain a list of all candidates for colour adjacency matrices for 2-, 3-, and 4-colourings of the 4-cube, respectively. Here they are:

(1) 2 colours: $\begin{pmatrix} 0&4\\2&2 \end{pmatrix}, \begin{pmatrix} 0&4\\4&0 \end{pmatrix}, \begin{pmatrix} 1&3\\1&3 \end{pmatrix}, \begin{pmatrix} 1&3\\3&1 \end{pmatrix}, \begin{pmatrix} 2&2\\2&2 \end{pmatrix}, \begin{pmatrix} 3&1\\1&3 \end{pmatrix}$

(2) 3 colours: $\begin{pmatrix} 0&0&4\\0&0&4\\1&1&2 \end{pmatrix}, \begin{pmatrix} 0&0&4\\0&0&4\\1&3&0 \end{pmatrix}, \begin{pmatrix} 0&0&4\\0&0&4\\2&2&0 \end{pmatrix}, \begin{pmatrix} 0&2&2\\2&0&2\\1&1&2 \end{pmatrix}, \begin{pmatrix} 0&2&2\\2&0&2\\2&2&0 \end{pmatrix}, \begin{pmatrix} 0&2&2\\2&1&1\\2&1&1 \end{pmatrix}, \begin{pmatrix} 0&2&2\\2&2&0\\2&0&2 \end{pmatrix}$
$\begin{pmatrix} 1&1&2\\1&1&2\\1&1&2 \end{pmatrix}, \begin{pmatrix} 2&0&2\\0&2&2\\1&1&2 \end{pmatrix}, \begin{pmatrix} 2&2&0\\1&0&3\\0&2&2 \end{pmatrix}$

(3) 4 colours: $\begin{pmatrix} 0&0&0&4\\0&0&0&4\\0&0&0&4\\1&1&2&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&4\\0&0&4&0\\0&2&0&2\\2&0&2&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&3\\0&0&1&3\\1&1&2&0\\1&1&0&2 \end{pmatrix}, \begin{pmatrix} 0&0&1&3\\0&0&3&1\\1&3&0&0\\3&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&2\\0&0&2&2\\1&1&0&2\\1&1&2&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&2\\0&0&2&2\\1&1&1&1\\1&1&1&1 \end{pmatrix},$
$\begin{pmatrix} 0&0&2&2\\0&0&2&2\\1&1&2&0\\1&1&0&2 \end{pmatrix}, \begin{pmatrix} 0&0&2&2\\0&0&2&2\\2&2&0&0\\2&2&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&3\\1&0&3&0\\0&1&0&3\\1&0&3&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&3\\1&0&3&0\\0&1&2&1\\1&0&1&2 \end{pmatrix}, \begin{pmatrix} 0&1&1&2\\1&0&2&1\\1&2&0&1\\2&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&2\\1&1&1&1\\1&1&1&1\\2&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&2\\1&2&0&1\\1&0&2&1\\2&1&1&0 \end{pmatrix},$
$\begin{pmatrix} 0&2&0&2\\2&0&0&2\\0&0&2&2\\1&1&2&0 \end{pmatrix}, \begin{pmatrix} 0&2&2&0\\1&0&0&3\\1&0&0&3\\0&2&2&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&3\\0&1&3&0\\0&1&1&2\\1&0&2&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&2\\0&1&2&1\\1&2&1&0\\2&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&3&0\\0&1&0&3\\1&0&0&3\\0&1&1&2 \end{pmatrix}, \begin{pmatrix} 1&1&0&2\\1&1&0&2\\0&0&2&2\\1&1&2&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{pmatrix},$
$\begin{pmatrix} 1&1&1&1\\1&1&1&1\\1&1&2&0\\1&1&0&2 \end{pmatrix}, \begin{pmatrix} 2&0&0&2\\0&2&0&2\\0&0&2&2\\1&1&2&0 \end{pmatrix}, \begin{pmatrix} 2&0&1&1\\0&2&1&1\\1&1&2&0\\1&1&0&2 \end{pmatrix}$

Unlike in the previous section not all of these candidates give rise to actual perfect colourings. In the following result we list all of the matrices that correspond to actual perfect colourings of the 4-cube with two and three colours. We leave to check the existence of the 23 distinct 4-colourings as a challenge to the reader.

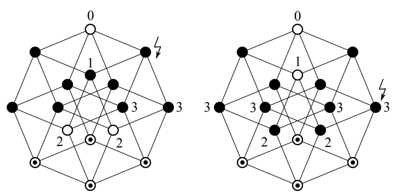
Theorem 5.1. *All perfect 2-colourings of the 4-cube are the five ones corresponding to*

$$\left(\begin{smallmatrix}0&4\\4&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&3\\1&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&3\\3&1\end{smallmatrix}\right),\left(\begin{smallmatrix}2&2\\2&2\end{smallmatrix}\right),\left(\begin{smallmatrix}3&1\\1&3\end{smallmatrix}\right)$$

All perfect 3-colourings of the 4-cube are the five ones corresponding to

$$\begin{pmatrix}0&0&4\\0&0&4\\1&3&0\end{pmatrix},\begin{pmatrix}0&0&4\\0&0&4\\2&2&0\end{pmatrix},\begin{pmatrix}0&2&2\\2&0&2\\1&1&2\end{pmatrix},\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix},\begin{pmatrix}2&0&2\\0&2&2\\1&1&2\end{pmatrix}.$$

Proof. Perfect colourings corresponding to the colour adjacency matrices above are shown in Figure 2. It remains to show that there are no further perfect colourings of the 4-cube corresponding to the other candidates.
Two colours, matrix $\left(\begin{smallmatrix}0&4\\2&2\end{smallmatrix}\right)$: This is not possible, as one can easily see as follows: a white vertex (without loss of generality vertex 0 in this diagram)



has four black neighbours. If the vertex labelled 1 is black then the two vertices labelled 2 are necessarily white. This in turn forces the vertices labelled 3 to be black. Now the black vertex labelled with a lightning bolt has three black neighbours. Contradiction.
If vertex 1 is white, then the vertices labelled 2 are black. Moreover, the topmost two black vertices already have two white neighbours, hence all vertices labelled 3 are necessarily black. Now the black vertex labelled with a lightning bolt has three black neighbours. Contradiction.
Three colours: we need to exclude the five matrices

$$\begin{pmatrix}0&0&4\\0&0&4\\1&1&2\end{pmatrix},\begin{pmatrix}0&2&2\\2&0&2\\2&2&0\end{pmatrix},\begin{pmatrix}0&2&2\\2&1&1\\2&1&1\end{pmatrix},\begin{pmatrix}0&2&2\\2&2&0\\2&0&2\end{pmatrix},\begin{pmatrix}2&2&0\\1&0&3\\0&2&2\end{pmatrix}.$$

This is easy: For instance in a perfect 3-colouring corresponding to $\begin{pmatrix}0&0&4\\0&0&4\\1&1&2\end{pmatrix}$ we can identify the colours number 1 and 2 (white and black). We obtain a perfect 2-colouring with colour adjacency matrix $\left(\begin{smallmatrix}0&4\\2&2\end{smallmatrix}\right)$. But we just saw that such a perfect 2-colouring does not exist. All remaining matrices can be excluded in this manner. Note that for the last one we need to identify colours 1 and 3 in order to obtain $\left(\begin{smallmatrix}2&2\\4&0\end{smallmatrix}\right)$, which is just a permutation of $\left(\begin{smallmatrix}0&4\\2&2\end{smallmatrix}\right)$, respectively a permutation of colours, hence also impossible. □

6. Perfect colourings of the 5-cube

Applying the analogous procedure we obtain a list of all candidates for colour adjacency matrices for 2-, 3-, and 4-colourings of the 5-cube, respectively.

(1) 2 colours $\left(\begin{smallmatrix}0&5\\1&4\end{smallmatrix}\right),\left(\begin{smallmatrix}0&5\\3&2\end{smallmatrix}\right),\left(\begin{smallmatrix}0&5\\5&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&4\\4&1\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\1&4\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\3&2\end{smallmatrix}\right),\left(\begin{smallmatrix}3&2\\2&3\end{smallmatrix}\right),\left(\begin{smallmatrix}4&1\\1&4\end{smallmatrix}\right)$

(2) 3 colours $\begin{pmatrix}0&1&4\\1&0&4\\1&1&3\end{pmatrix},\begin{pmatrix}0&1&4\\1&0&4\\2&2&1\end{pmatrix},\begin{pmatrix}0&2&3\\1&1&3\\1&2&2\end{pmatrix},\begin{pmatrix}0&2&3\\2&1&2\\3&2&0\end{pmatrix},\begin{pmatrix}0&3&2\\3&0&2\\1&1&3\end{pmatrix},\begin{pmatrix}0&5&0\\1&0&4\\0&2&3\end{pmatrix},\begin{pmatrix}1&0&4\\0&1&4\\1&1&3\end{pmatrix},\begin{pmatrix}1&0&4\\0&1&4\\1&3&1\end{pmatrix},$
$\begin{pmatrix}1&0&4\\0&1&4\\2&2&1\end{pmatrix},\begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix},\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix},\begin{pmatrix}1&2&2\\2&2&1\\2&1&2\end{pmatrix},\begin{pmatrix}1&2&2\\2&3&0\\2&0&3\end{pmatrix},\begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix},\begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix},\begin{pmatrix}3&2&0\\1&1&3\\0&2&3\end{pmatrix}$

(3) 4 colours: see Appendix B.

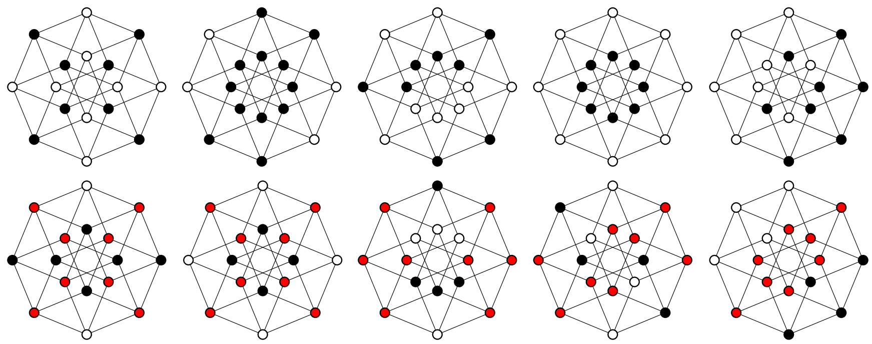


FIGURE 2. The perfect colourings of the 4-cube with two and three colours.

In the following result we list all of the matrices that correspond to actual perfect colourings of the 4-cube with two and three colours. We leave to check the existence of the 57 distinct 4-colourings to the reader as a challenging exercise.

Theorem 6.1. *All perfect 2-colourings of the 5-cube are the six ones corresponding to*

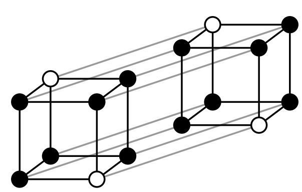
$$\left(\begin{smallmatrix}0&5\\5&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&4\\4&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&3\\1&4\end{smallmatrix}\right), \left(\begin{smallmatrix}2&3\\3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}3&2\\2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}4&1\\1&4\end{smallmatrix}\right).$$

All perfect 3-colourings of the 5-cube are the eight ones corresponding to

$$\begin{pmatrix}0&1&4\\1&0&4\\2&2&1\end{pmatrix}, \begin{pmatrix}0&3&2\\3&0&2\\1&1&3\end{pmatrix}, \begin{pmatrix}0&5&0\\1&0&4\\0&2&3\end{pmatrix}, \begin{pmatrix}1&0&4\\0&1&4\\1&3&1\end{pmatrix}, \begin{pmatrix}1&0&4\\0&1&4\\2&2&1\end{pmatrix}, \begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix}, \begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix}, \begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix}.$$

Proof. **Two colours:** It can be fun to find the particular perfect colourings by hand for each case. But the following argument from [6] proves the existence in a simpler manner.

Consider two copies of a perfect colouring of a 3-cube and draw an edge between corresponding vertices. Here is an example for the perfect 2-colouring with colour adjacency matrix $\left(\begin{smallmatrix}0&3\\1&2\end{smallmatrix}\right)$.



The result is a perfect 2-colouring of the 4-cube with colour adjacency matrix $\left(\begin{smallmatrix}1&3\\1&3\end{smallmatrix}\right)$. In a similar manner, each perfect m-colouring of a d-cube with colour adjacency matrix A yields a m-perfect colouring of a $(d+1)$-cube with colour adjacency matrix $A+I$, where I is the identity matrix.

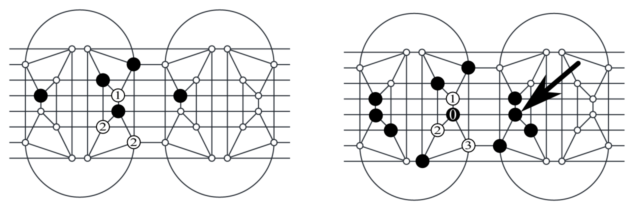
In the same way each of the five perfect 2-colourings of the 4-cube gives rise to one perfect 2-colouring of the 5-cube. The only remaining case is the matrix $\left(\begin{smallmatrix}0&5\\5&0\end{smallmatrix}\right)$. The corresponding perfect 2-colouring comes from the fact that the d-cube is a bipartite graph for any d.

It remains to show that the three other matrices in the list of candidates above do not yield perfect colourings. In order to exclude $\left(\begin{smallmatrix}0&5\\1&4\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&4\\2&3\end{smallmatrix}\right)$ we apply a result from [5]:

Lemma 6.2. *Let $A = (a_{ij}) \in \mathbb{N}^{2\times 2}$ be a colour adjacency matrix of some connected graph $G = (V, E)$. Then a_{12} and a_{21} are both nonzero, and if v_i denotes the number of vertices of colour i, then*

$$v_1 = \frac{|V|}{1 + \frac{a_{12}}{a_{21}}}, v_2 = \frac{|V|}{\frac{a_{21}}{a_{12}} + 1}.$$

Applied to the two matrices above for the particular case of the 5-cube, where $|V| = 32$, we obtain $v_1 = \frac{16}{3}$ for $\left(\begin{smallmatrix}0&5\\1&4\end{smallmatrix}\right)$ and $v_1 = \frac{32}{3}$ for $\left(\begin{smallmatrix}1&4\\3&2\end{smallmatrix}\right)$. Since these values are not integers there is no corresponding perfect colouring of the 5-cube.
For the matrix $\left(\begin{smallmatrix}0&5\\3&2\end{smallmatrix}\right)$ the lemma yields $v_1 = 12$ and $v_2 = 20$. This does not help to exclude this case. But it can be excluded as follows:
Without loss of generality let the vertex labelled 1 in the following diagram be white.



This is indeed the 5-cube graph, see [12]. (The edges that leave to the left are connected to those that enter on the right on the same height. You may imagine the image wrapped on a cylinder, or you may draw the missing connections.) Because of the matrix all five neighbours of 1 have to be black. Without loss of generality, let the two vertices 2 and 3 be the other two white neighbours of the black vertex labelled 0. (Because of the high symmetry of the 5-cube it does not matter which two are chosen.) All neighbours of vertices 2 and 3 are black. The other two neighbours of 0 are black as well. Now the black vertex labelled by an arrow has four black neighbours, contradiction.
Three colours: For the five matrices $\begin{pmatrix}1&0&4\\0&1&4\\1&3&1\end{pmatrix}$, $\begin{pmatrix}1&0&4\\0&1&4\\2&2&1\end{pmatrix}$, $\begin{pmatrix}1&2&2\\2&1&2\\1&1&3\end{pmatrix}$, $\begin{pmatrix}2&1&2\\1&2&2\\1&1&3\end{pmatrix}$, and $\begin{pmatrix}3&0&2\\0&3&2\\1&1&3\end{pmatrix}$, the observation above works: Each of the five perfect 3-colourings of a 4-cube with colour adjacency matrix A yields a perfect 3-colouring of a 5-cube with colour adjacency matrix $A + I$.
For two of the remaining matrices, $\begin{pmatrix}0&1&4\\1&0&4\\2&2&1\end{pmatrix}$ and $\begin{pmatrix}0&3&2\\3&0&2\\1&1&3\end{pmatrix}$, a variation of the same argument works. Since

$$\begin{pmatrix}0&1&4\\1&0&4\\2&2&1\end{pmatrix} = \begin{pmatrix}0&0&4\\0&0&4\\2&2&0\end{pmatrix} + \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \quad \text{and } \begin{pmatrix}0&3&2\\3&0&2\\1&1&3\end{pmatrix} = \begin{pmatrix}0&2&2\\2&0&2\\2&1&1\end{pmatrix} + \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$

two copies of the corresponding perfect 3-colourings of the 4-cube yield perfect 3-colourings of the 5-cube. Here we need to use two slightly different copies: in one colouring the roles of black and white are swapped. So each white vertex gains a new black neighbour and vice versa. (This trick works whenever the number of black vertices equals the number white of vertices, and when they are distributed in the same manner on the cube. Both conditions are fulfilled here, compare Figure 2 bottom row, second and third from left.)
The last matrix $\begin{pmatrix}0&5&0\\1&0&4\\0&2&3\end{pmatrix}$ corresponds to the perfect 3-colouring shown in Figure 3.
The remaining matrices that do not correspond to perfect colourings of the 5-cube can all be excluded by the last argument from the proof of Theorem 5.1. For

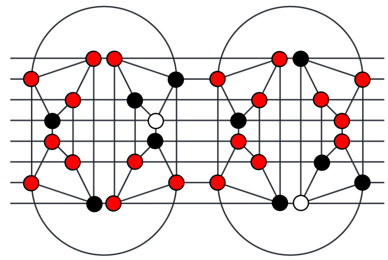


FIGURE 3. A perfect 3-colouring of the 5-cube with colour adjacency matrix $\begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}$. This is indeed the graph of the 5-cube if we imagine the image wrapped on a cylinder, so edges that leave to the left are connected to those that enter on the right on the same height.

instance, let us assume there is a perfect colouring corresponding to either one of $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}$. By identifying colours 2 and 3 we obtain a perfect 2-colouring with colour adjacency matrix $\left(\begin{smallmatrix} 0 & 5 \\ 1 & 4 \end{smallmatrix}\right)$. But we saw already that such a perfect 2-colouring does not exist.

All remaining matrices can be excluded in this way, reducing them to an impossible perfect 2-colouring with matrix $\left(\begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix}\right)$ (or $\left(\begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix}\right)$, which is the same up to swapping colours). In most remaining cases this is pretty obvious. In the case $\begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix}$ we need to identify colours 1 and 3 in order to obtain $\left(\begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix}\right)$. $\square$

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Appendix: possible colour adjacency matrices for perfect 4-colourings of the 5-cube

All possible colour adjacency matrices A for perfect 4-colourings of the edge graph of the 5-dimensional cube. It is not known which correspond to actual perfect colourings. In principle all of them can be studied by the methods of this paper.

$$\begin{pmatrix}0&0&0&5\\0&0&3&2\\0&3&0&2\\1&2&2&0\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&0&3&2\\0&3&2&0\\1&2&0&2\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&0&5&0\\0&1&0&4\\1&0&4&0\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&0&5&0\\0&3&0&2\\3&0&2&0\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&1&2&2\\0&2&1&2\\1&2&2&0\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&2&1&2\\0&1&2&2\\1&2&2&0\end{pmatrix}, \begin{pmatrix}0&0&0&5\\0&3&0&2\\0&0&3&2\\1&2&2&0\end{pmatrix}, \begin{pmatrix}0&0&1&4\\0&0&4&1\\1&4&0&0\\4&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&2&3\\0&0&3&2\\2&3&0&0\\3&2&0&0\end{pmatrix}, \begin{pmatrix}0&0&5&0\\0&0&2&3\\1&1&0&3\\0&1&2&2\end{pmatrix}, \begin{pmatrix}0&0&5&0\\0&3&2&0\\1&1&0&3\\0&0&2&3\end{pmatrix}, \begin{pmatrix}0&1&0&4\\1&0&0&4\\0&0&1&4\\1&1&2&1\end{pmatrix}, \begin{pmatrix}0&1&0&4\\1&0&4&0\\0&2&0&3\\2&0&3&0\end{pmatrix}, \begin{pmatrix}0&1&0&4\\1&0&4&0\\0&2&2&1\\2&0&1&2\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&0&1&3\\1&1&0&3\\1&1&1&2\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&0&1&3\\1&1&3&0\\1&1&0&3\end{pmatrix},$$

$$\begin{pmatrix}0&1&1&3\\1&0&3&1\\1&3&0&1\\3&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&1&0&3\\1&0&1&3\\1&1&1&2\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&1&2&1\\1&2&1&1\\3&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&2&1&1\\1&1&2&1\\3&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&3\\1&3&0&1\\1&0&3&1\\3&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&2&2\\1&0&2&2\\1&1&0&3\\1&1&3&0\end{pmatrix}, \begin{pmatrix}0&1&2&2\\1&0&2&2\\1&1&1&2\\1&1&2&1\end{pmatrix}, \begin{pmatrix}0&1&2&2\\1&0&2&2\\1&1&2&1\\1&1&1&2\end{pmatrix},$$

$$\begin{pmatrix}0&1&2&2\\1&0&2&2\\1&1&3&0\\1&1&0&3\end{pmatrix}, \begin{pmatrix}0&1&2&2\\1&0&2&2\\2&2&0&1\\2&2&1&0\end{pmatrix}, \begin{pmatrix}0&1&2&2\\1&0&2&2\\2&2&1&0\\2&2&0&1\end{pmatrix}, \begin{pmatrix}0&2&0&3\\2&0&3&0\\0&1&0&4\\1&0&4&0\end{pmatrix}, \begin{pmatrix}0&2&0&3\\2&0&3&0\\0&1&2&2\\1&0&2&2\end{pmatrix}, \begin{pmatrix}0&3&0&2\\3&0&0&2\\0&0&3&2\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&0&0&4\\0&1&0&4\\0&0&1&4\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&0&0&4\\0&1&4&0\\0&2&1&2\\2&0&2&1\end{pmatrix},$$

$$\begin{pmatrix}1&0&1&3\\0&1&1&3\\1&1&3&0\\1&1&0&3\end{pmatrix}, \begin{pmatrix}1&0&1&3\\0&1&3&1\\1&3&1&0\\3&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&2&2\\0&1&2&2\\1&1&0&3\\1&1&3&0\end{pmatrix}, \begin{pmatrix}1&0&2&2\\0&1&2&2\\1&1&1&2\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&0&2&2\\0&1&2&2\\1&1&2&1\\1&1&1&2\end{pmatrix}, \begin{pmatrix}1&0&2&2\\0&1&2&2\\1&1&3&0\\1&1&0&3\end{pmatrix}, \begin{pmatrix}1&0&2&2\\0&1&2&2\\2&2&1&0\\2&2&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&3\\1&1&3&0\\0&1&1&3\\1&0&3&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&0&3\\1&1&3&0\\0&1&3&1\\1&0&1&3\end{pmatrix}, \begin{pmatrix}1&1&1&2\\1&1&2&1\\1&2&1&1\\2&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&2\\1&2&1&1\\1&1&2&1\\2&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&2\\1&3&0&1\\1&0&3&1\\2&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&0&2\\2&1&0&2\\0&0&3&2\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&2&2&0\\1&0&1&3\\1&1&0&3\\0&2&2&1\end{pmatrix}, \begin{pmatrix}1&2&2&0\\1&1&0&3\\1&0&1&3\\0&2&2&1\end{pmatrix}, \begin{pmatrix}1&4&0&0\\2&0&0&3\\0&0&1&4\\0&1&2&2\end{pmatrix},$$

$$\begin{pmatrix}2&0&0&3\\0&2&3&0\\0&1&2&2\\1&0&2&2\end{pmatrix}, \begin{pmatrix}2&0&1&2\\0&2&2&1\\1&2&2&0\\2&1&0&2\end{pmatrix}, \begin{pmatrix}2&0&3&0\\0&2&0&3\\1&0&1&3\\0&1&1&3\end{pmatrix}, \begin{pmatrix}2&1&0&2\\1&2&0&2\\0&0&3&2\\1&1&2&1\end{pmatrix}, \begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix}, \begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&3&0\\1&1&0&3\end{pmatrix}, \begin{pmatrix}2&3&0&0\\1&0&0&4\\0&0&3&2\\0&2&2&1\end{pmatrix}, \begin{pmatrix}3&0&0&2\\0&3&0&2\\0&0&3&2\\1&1&2&1\end{pmatrix},$$

$$\begin{pmatrix}3&0&1&1\\0&3&1&1\\1&1&3&0\\1&1&0&3\end{pmatrix}$$

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