TEN COLOURS IN QUASIPERIODIC AND REGULAR HYPERBOLIC TILINGS

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Abstract. Colour symmetries with ten colours are presented for different tilings. In many cases, the existence of these colourings were predicted by group theoretical methods. Only in a few cases explicit constructions were known, sometimes using combination of two-colour and five-colour symmetries. Here we present explicit constructions of several of the predicted colourings for the first time, and discuss them in contrast to already known colourings with ten colours.

1. Introduction

The discovery of quasicrystals generated interest in colour symmetries of quasicrystallographic structures, see [1], [9] and references therein. In [1] and [2], the existence and the number of colour symmetries using k colours is determined for a large class of quasicrystallographic structures. Despite of the richness of regular structures in hyperbolic plane, colour symmetries of regular tilings in the hyperbolic plane are not widely studied. For an exception see [4] and references therein. Recently a method was presented to determine the possible number of colours for which a colour symmetry of any regular tiling exists, either in the Euclidean, spherical or hyperbolic plane [5]. This purely algebraic method does not yield the colouring itself. Here we focus on colour symmetries with ten colours. We present explicit colourings with ten colours which are predicted by [5].

Basic definitions about regular tilings are quite standard and can be found in [6], for instance, from where we also adopt the notation. A tiling of the plane is said to be *regular*, if all tiles are regular polygons, and all vertex figures are regular polygons. Any ordered pair of integers (p, q), where $p, q \geq 3$, defines a regular tiling by regular p-gons, where q tiles meet at each vertex. Following [6], such a tiling is denoted by (p^q). The value $d := 2(\frac{1}{p} + \frac{1}{q})$ determines in which plane (p^q) lives: If $d > 1$ then (p^q) is a regular spherical tiling (which can be regarded as a Platonic solid). If $d = 1$ then (p^q) is a regular Euclidean tiling, and if $d < 1$ it is a regular tiling of the hyperbolic plane. Figures in this paper represent regular tilings of the hyperbolic plane, using the Poincaré disc model.

2. Colour symmetries of regular tilings with ten colours

A symmetry of a tiling is any isometry which maps the tiling to itself. In the sequel we distinguish two cases: Either we consider all isometries (reflections, rotations, and their products), or direct isometries only (no reflections, only rotations and their products). By colour symmetry we denote here any colouring of a tiling such that any symmetry of the uncoloured tiling acts as a global permutation of colours. Such a colour symmetry is called 'perfect colouring' in [6]. In Table 1 we list the regular tilings which are known to possess colour symmetries with ten colours, first with respect to all isometries, then with respect to direct isometries only.

There is only one 10-colour symmetry for regular spherical tilings: for (3^5), which can be regarded as a coloured icosahedron. Moreover, there is only one 10-colour symmetry for regular euclidean tilings: for the square tiling (4^4). In contrast, there are many 10-colour symmetries with ten colours for regular hyperbolic tilings. For instance, there are four of those for the tiling (4^5) with respect to all isometries. These are shown in Figure 1. (Note, that all figures show the tiling (p^q) superimposed with its dual tiling (q^p)). Two of ten colours are marked by black and gray, the other colours are omitted for the sake of clarity and are resulting from fivefold rotation around the

	spher	eucl	hyp									
all isometries	(3^5)		$(3^8)_2$		$(3^{10})_3$	$(4^5)_4$	(4^6)	$(6^4)_2$	$(6^5)_6$			
direct isom.	(3^5)	(4^4)	$(3^8)_3$	$(3^9)_3$	$(3^{10})_6$	$(4^5)_6$	$(4^6)_3$	$(6^4)_6$	$(6^5)_{15}$	$(4^9)_8$	(8^3)	$(8^4)_7$

TABLE 1. Regular tilings which have a 10-colour symmetry. A lower index denotes the number of different colour symmetries for that tiling, where no lower index means 'one'.

centre of the figure. Combining black and gray yields a five-colour symmetry. Figures (a) - (c) are based on the same five-colour symmetry. Figure (d) is based on a second five-colour symmetry. A full-colour version of Fig. 1 is given in Fig. 5 of the on-line version of the present paper. For the same tiling, there are six colour symmetries with respect to direct isometries, including the former four. An additional one is shown in Figure 2. A full colour version is represented in Fig. 6 in the online version. It is based on the second five-colour symmetry mentioned above. The figure represents one partner of an enantiomorphic pair. Note, that a reflection in the horizontal axis does not permute entire colour classes, which shows that this one is not a colour symmetry with respect to all isometries.

According to Table 1, there are two different perfect colourings with ten colours for the regular tiling (3^8). Figure 3 represents both examples for an eightfold centre. The gray colour represents one of eight colours in the orbit around the eightfold centre in both 3(a) and 3(b). The black colour in 3(a) stands for a single colour in two orbits around the eightfold centre, in 3(b) it stands for two colours in a single orbit.

Figure 3(a) is based on a coincidence site lattice (CSL, see [3]). This can be seen if the threefold centre is transformed to the centre of the figure (see Figure 4) and if the pattern is rotated by 60° or 180°: points which are vertices in both the rotated and the unrotated image are elements of the CSL. In Figure 4 the triangles surrounding a threefold coinciding vertex are coloured.

3. Colour symmetries of quasiperiodic tilings with ten colours

Ten colours in n-fold symmetric Euclidean tilings are predicted only for the periodic square lattice [1], [2]. In several 10-fold tilings, including Penrose tilings, synthetic combinations of five-colour symmetry with two-colour symmetry is possible. The colour symmetry with five colours has been discussed several times ([9] and references therein), a two-colour symmetry for decagonal tilings was presented by Li et al. [8]. The combination of both colour symmetries in decagonal tilings is not straightforward. Both colour symmetries are restricted to special cases [9] and often these cases exclude each other. A possible combination is represented by the 'orientated points' in the Penrose pentagon pattern as published by Gummelt [7].

4. Concluding remarks

Colour symmetries of regular Euclidean patterns are composed of operations based on prime powers [1] and their sequence can usually be permuted. (Usually means, that the prime does not ramify over the underlying integer ring of n-fold cyclotomic integers.) In particular, if there are k colourings with p colours and m colourings with q colours, where p and q are distinct primes, then there are km colour symmetries with pq colours. The composition of colour symmetries in hyperbolic space can be quite different: a genuine colour symmetry is not necessarily based on a prime power. Composition of several numbers of colours is often found. In contrast to the Euclidean case, the operations can not be permuted in general. However, in some cases they can. For instance, the decoration of (4^6) with all isometries according to the table can be composed from a two-colour and a five-colour symmetry.

The role of coincidence site lattices is also different. In planar Euclidean space the CSLs exist as enantiomorphic pairs and force a colouring which is only perfect with respect to rotations. For the hyperbolic space, we found enantiomorphic pairs which are not based on CSLs, and CSLs which are not related to a colour symmetry.

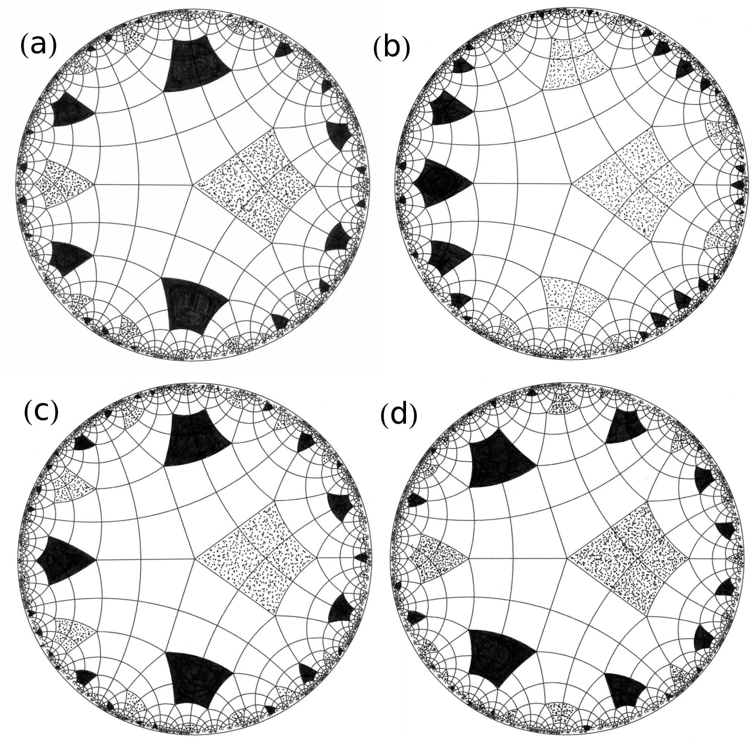

FIGURE 1. Two out of ten colours of a perfect colour symmetry in the hyperbolic tiling (4^5), indicated by black and gray. Identifying black and gray yields a five-colour symmetry. (a) - (c) are based on the same five-colour symmetry, (d) is based on a second five-colour symmetry. See Fig. 5 for a colour version.

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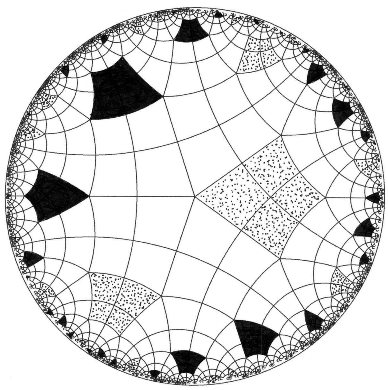

FIGURE 2. Example of a 10-colour symmetry in the hyperbolic tiling (4^5) which is only perfect with respect to rotational symmetry. Only two out of ten colours are marked. The combination of black and gray results in the five-colour symmetry of the second type. See Fig. 6 for a colour version.

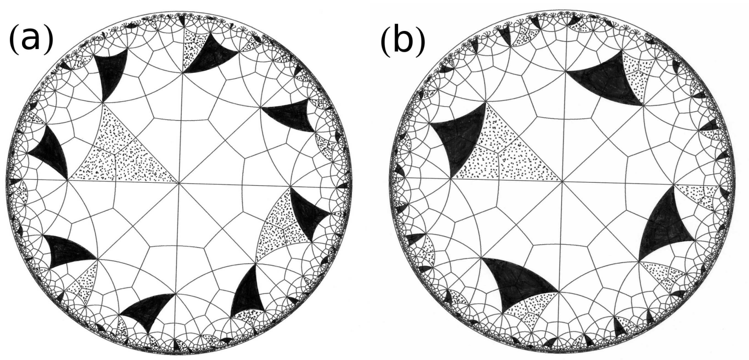

FIGURE 3. Perfect colour symmetry with ten colours in the hyperbolic tiling (3^8). The gray colour symbolyses the symmetry of eight colours in the orbit around the eightfold centre in both figures (a) and (b). The black colour in Fig. (a) stands for a single colour in two orbits around the eightfold centre, in Fig. (b) it stands for two colours in a single orbit.

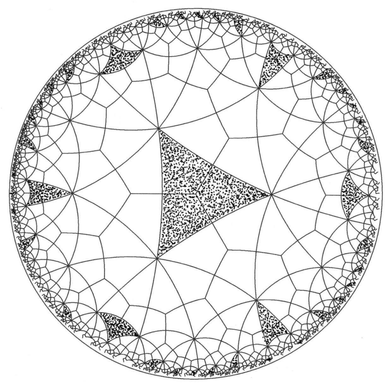

FIGURE 4. A single colour of Fig. 3 a represented for a threefold centre. This figure results also as a coincidence site lattice of threefold vertices for 60° or 180° rotations around the figure centre.

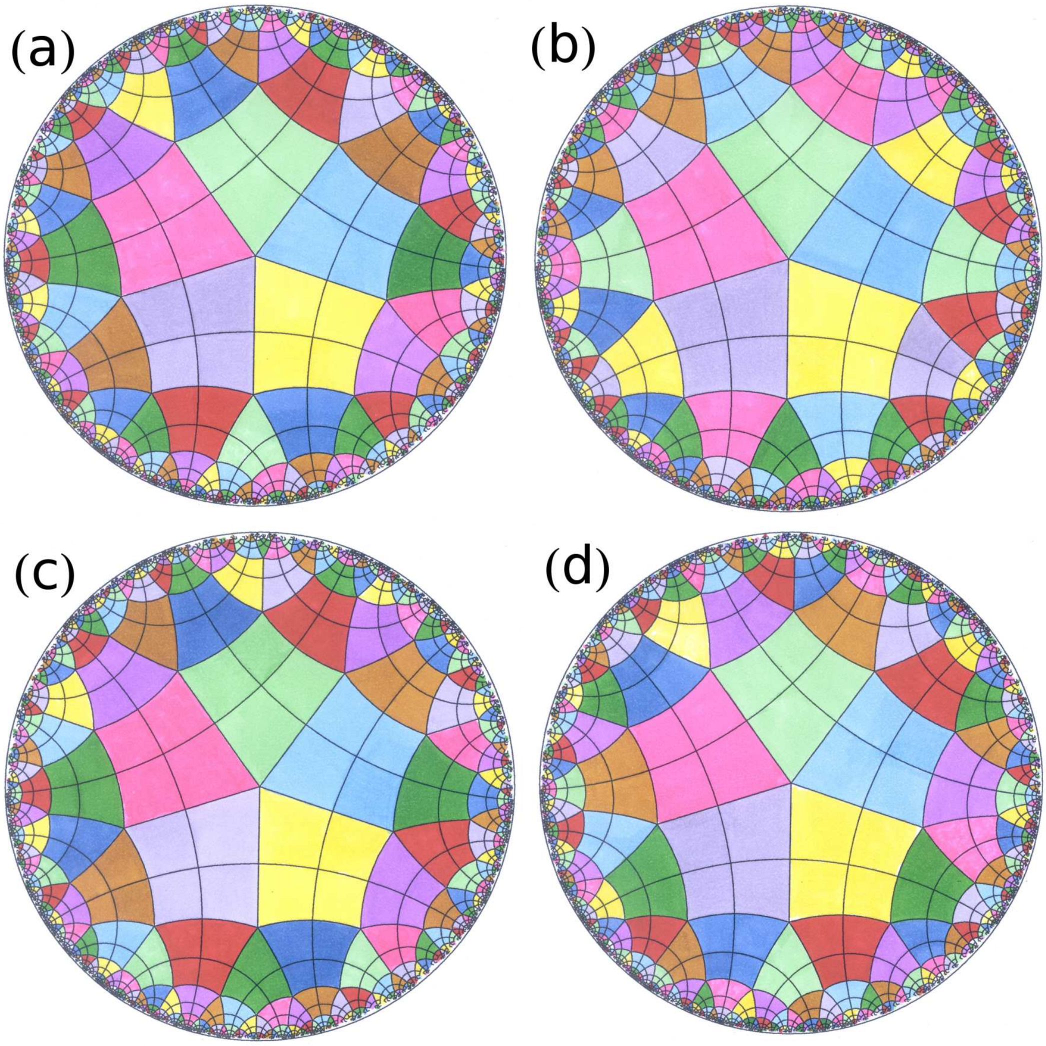

FIGURE 5. Perfect colour symmetry in the hyperbolic tiling (4^5) designated by 10 colours. The combinations of dark and light colours represent a five-colour symmetry. (a) - (c) are based on a first type of a five-colour decoration, (d) is based on a second type of a five-colour decoration (compare Fig. 1).

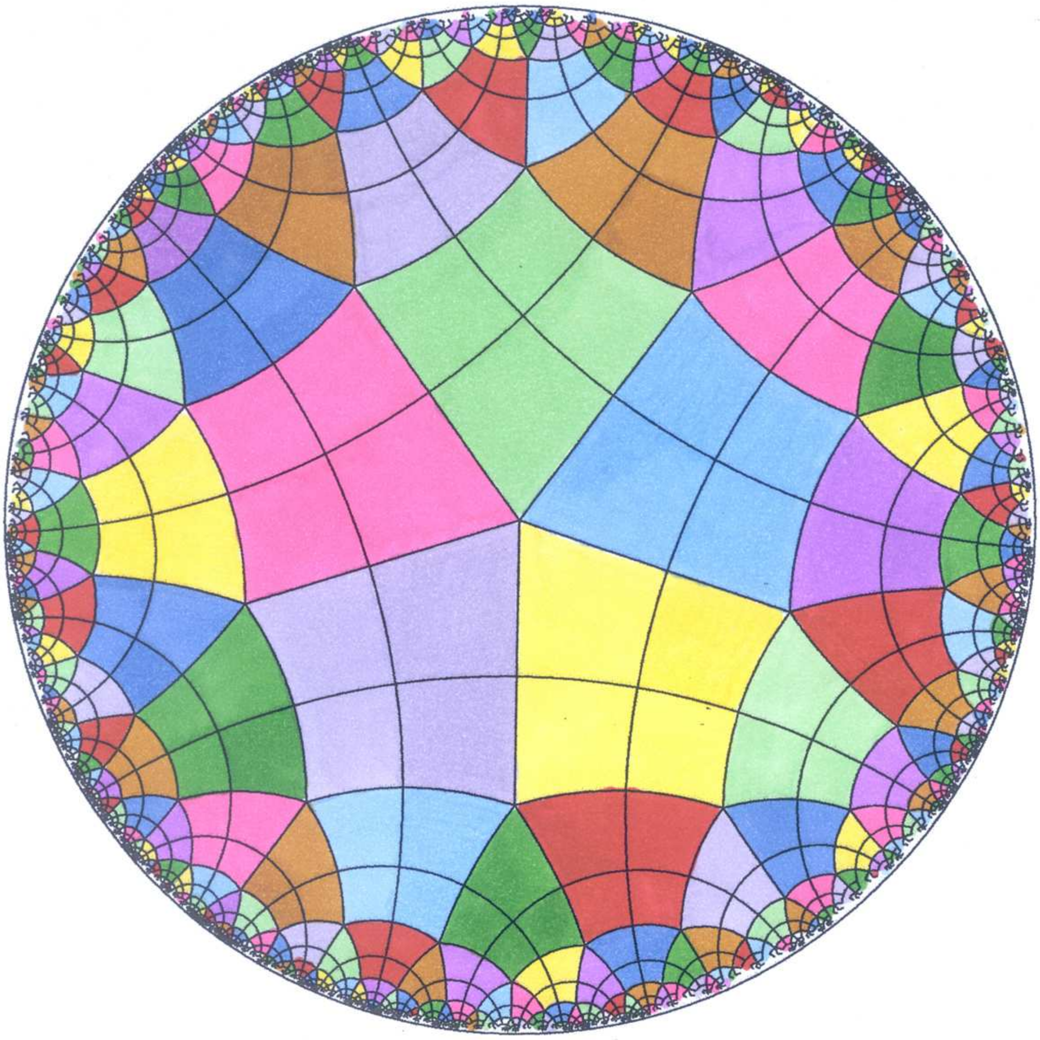

FIGURE 6. Example of a 10-colour symmetry in the hyperbolic tiling (4^5) which is only perfect with respect to rotational symmetry. The combination of dark and light colours results in the five-colour symmetry of the second type (compare Fig. 2).