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SELFSIMILAR TILINGS WITH INTEGER FACTOR

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Abstract: *There are nonperiodic selfsimilar tilings with integer factor, which share many important properties — for example being a projection set or being almost periodic — with the quasiperiodic[1] tilings (e.g. the famous Penrose tilings). Quasiperiodic tilings always have an irrational similarity factor. A nonperiodic selfsimilar tiling need not to be symmetric in the classical sense, i.e. it need not have a rotational or mirror symmetry. But selfsimilarity itself can be viewed as a kind of symmetry, too. Furthermore, there are additional concepts of symmetries, which are used in tiling theory and can enlarge the concept of symmetry (like inflation-deflation symmetry, limitperiodicity etc.). We will explain these concepts and give some relations between them.*

1 NONPERODICITY and SELFSIMILARITY

A *tiling* is, as one would expect, a collection of *tiles*, such that the interiors of the tiles do not intersect and the union of the tiles fill the whole space. Here we restrict ourselves to the plane, but all the concepts and facts stated here work in higher dimensions, too. Thus a *tile* is a bounded set $T \subset \mathbb{R}^2$ with nonempty interior, and a *tiling* is a countable collection of tiles $\{T_1, T_2, T_3, \ldots\}$, such that the union $\bigcup_{i\in\mathbb{N}} T_i$ gives $\mathbb{R}^2$. Figure 1 shows a part of a tiling.

A tiling P is called *selfsimilar with factor* η, if:
1. For every tile $T \in P$ there is a set of tiles $\{T_1, \ldots, T_k\}$ such that the union $\bigcup_{i=1}^{k} T_i$ is equal to ηT, and
2. If the tiles $T \in P$ and $S \in P$ are equal up to translation, the relating sets of tiles $\{T_1, \ldots, T_k\}$ and $\{S_1, \ldots, S_\ell\}$ are equal up to translation.

The thick lines in figure 1 indicate the selfsimilarity. The fat point in the middle is the origin, the factor is 2, and one can find scaled copies of the central hexagon H — namely $2H, 4H, 8H, \ldots$ — built of small tiles. One can now find the rule, how the large hexagons are composed of smaller hexagons and half hexagons. This rule allows us to fill larger and larger regions of the plane, thus giving an infinite tiling of the plane.

Selfsimilarity can be seen as a kind of symmetry: If one scales a tiling P by a factor η, the union of edges of the scaled tiling ηP is a subset of the union of the edges of P. Because of point 2. of the definition above we know locally, how to add edges to get the original tiling P. This scaling and adding of edges together gives an action, lets say Φ, such that $\Phi(P) = P$. This last equation would also hold if P has a 'classical' symmetry, such as a rotation symmetry with rotation Φ.

A tiling P is called *nonperiodic*, if $P = P + t$ forces $t = 0$. That means, if we move the tiling around, the

[1] It is not easy to give a good definition for 'quasiperiodicity'. So we state only, that quasiperiodic tilings are of special interest for many scientists in many fields, and gave rise to a lot of mathematical investigations. For more information, cf. [Baa]

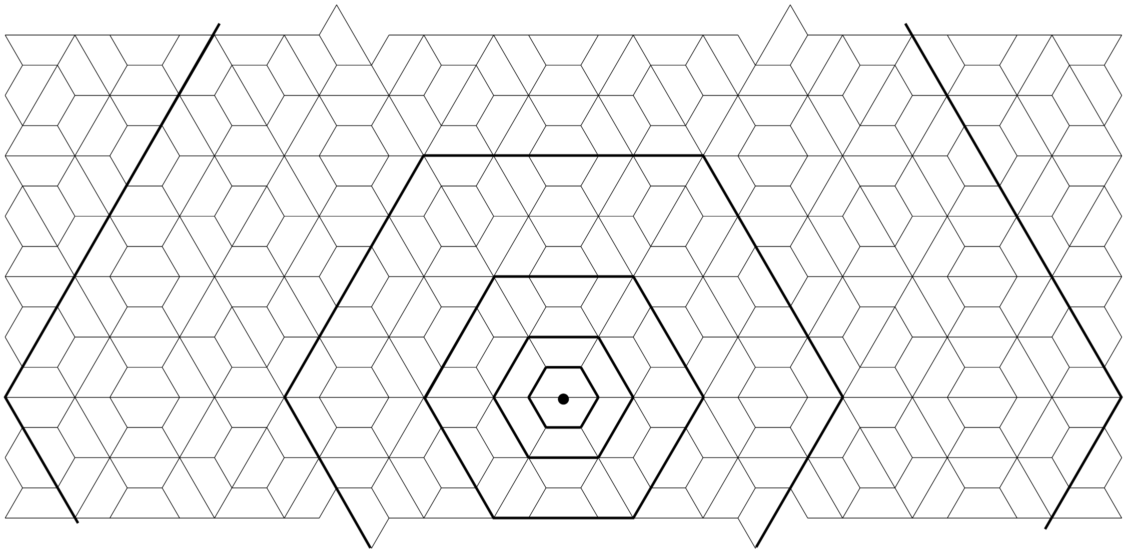

Figure 1: A part of a selfsimilar nonperiodic tiling by half hexagons [G-S].

moved tiling will never match perfectly the original tiling, unless we move it back to the original position. The tiling shown in figure 1 is nonperiodic, as we will see later.

If a periodic tiling with 2 linearly independent period vectors has a point of selfsimilarity (like the fat point in figure 1), it has infinitely many such points. A nonperiodic tiling can have at most one point of selfsimilarity.

2 INFLATION and DEFLATION

The previous facts about selfsimilarity leads to a powerful concept of generating nonperiodic selfsimilar tilings: inflation. The idea is shown in figure 2: We choose a finite number of tiles, called *prototiles*, together with a factor η and a rule, how to dissect the tiles scaled by factor η into tiles congruent to prototiles.

If we start with one tile (or a small number of tiles), we first scale the tile(s) with factor η. Then we dissect the scaled tile(s) according to the given rule. The resulting configuration of tiles we scale again as a whole ('inflate'). The scaled tiles in this scaled configuration will be dissected according to the given rule, and so on. In this way we fill larger and larger parts of the plane, getting a tiling in the limit. The process of scaling and dissecting together is called *inflation*.

The connection to selfsimilarity is:

Theorem 2.1 (*see* [Fre]) *The family of all tilings generated by an inflation contains a selfsimilar tiling. Every selfsimilar tiling gives rise to an inflation.*

By inflation one can generate both periodic and nonperiodic tilings. Inflation rules for tilings with an integer factor are easy to find. The author found the rule in figure 2(d) in some minutes by playing tangram. Every reader can try to find another one. In the following we show how to decide if an inflation rule gives rise to a periodic or a nonperiodic tiling.

An inflation rule can be applied to tiles, to finite sets of tiles, and also to tilings. In such an inflation tiling one can always find a 'previous level tiling': If P is an inflation tiling, one can find a tiling Q with the property, that P is the inflation of Q. (The reader can try with figure 1, and also with the trivial 2–periodic square tiling.) If the process of finding this Q is unique, we call the inverse operation of inflation 'deflation'. The existence of a deflation is an important property:

Theorem 2.2 (*see* [Sol]) *An inflation tiling P with finitely many local configurations up to translation is nonperiodic, if and only if it has a deflation.*

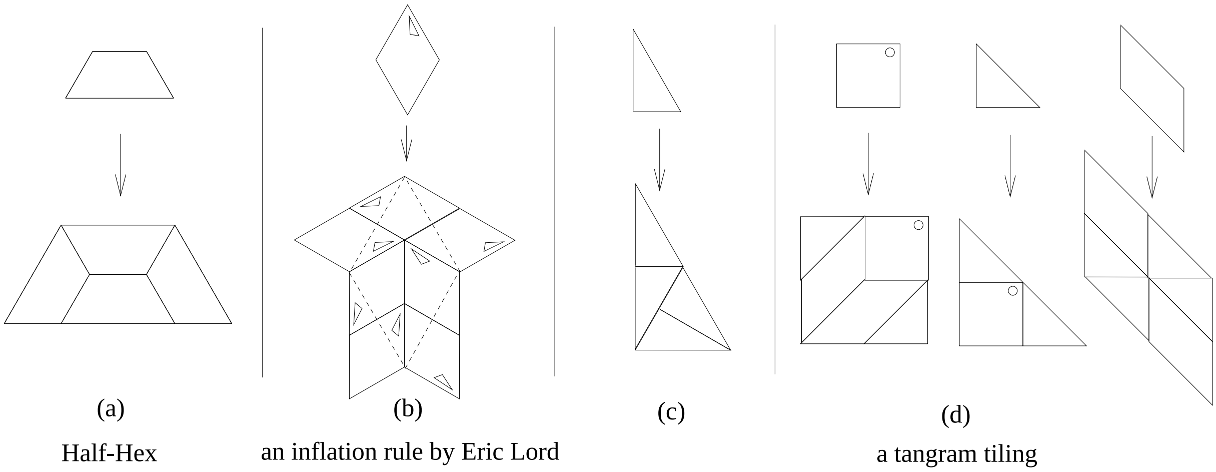

Figure 2: Four inflation rules for nonperiodic tilings.

In the tiling in figure 1 we can find the previous level configuration in a unique way. Hence the related tilings are nonperiodic. In a trivial square tiling with factor 2 there are always four different possibilities for a previous level configuration. Clearly the trivial square tiling is 2–periodic.

The property of an inflation tiling to have a deflation can be seen as a symmetry, too (see [Baa], where it is called 'inflation–deflation symmetry'). It is related to selfsimilarity, but it is not the same. There are selfsimilar tilings without inflation–deflation symmetry (for example periodic inflation tilings), and there are tilings with inflation–deflation symmetry, which are not selfsimilar.

3 LIMITPERIODICITY and PROJECTION

An amazing property of nonperiodic selfsimilar tilings with integer factor is, that some of them contain 2–periodic subsets. (That is impossible for quasiperiodic tilings.) In figure 1 one can see, that the tiling has a honeycomb–like structure. This is more clearly shown in figure 3: the grey shaded tiles on the right half of figure 3 are part of a honeycomb–shaped pattern, that continues to the whole tiling. This set — call it M — of tiles is obviously 2–periodic. It covers 75 % of the tiling.

Because of the selfsimilarity, the set $2M$ (sketched on the left side of figure 3) is also 2–periodic. M and $2M$ together cover 93,75 % of the tiling. (Only parts of them can be shown in figure 3.) Furthermore, every set $2^k M$, $k \in \mathbb{N}$ is a periodic subset of the tiling, and the union $\bigcup_{k \geq 0} 2^k M$ gives the tiling itself (except the two tiles in the center). So this nonperiodic tiling is a countable union of periodic sets of tiles. This property is called *limitperiodic*. Again, limitperiodicity is strongly related to selfsimilarity, but it is not the same. It can be viewed as another kind of symmetry of a tiling.

Another powerful tool to generate nonperiodic tilings — beneath inflation — is the projection method. The general idea is to project a part of a n–periodic structure in n–dimensional space ($n \geq 3$) into 2–dimensional space. A formal description is:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^{d+e} & \xrightarrow{\pi_2} & \mathbb{R}^e \\ \cup & & \cup & & \cup \\ V & & L & & W \end{array} \tag{1}$$

with L a $(d+e)$–dimensional lattice, $W \subset \mathbb{R}^e$ a compact set and π_1, π_2 orthogonal projections. Under some additional conditions gives

$$V = \{\pi_1(x) \,|\, x \in L, \pi_2(x) \in W\}$$

rise to a quasiperiodic tiling. In this way one cannot get a nonperiodic tiling with integer factor. But if one

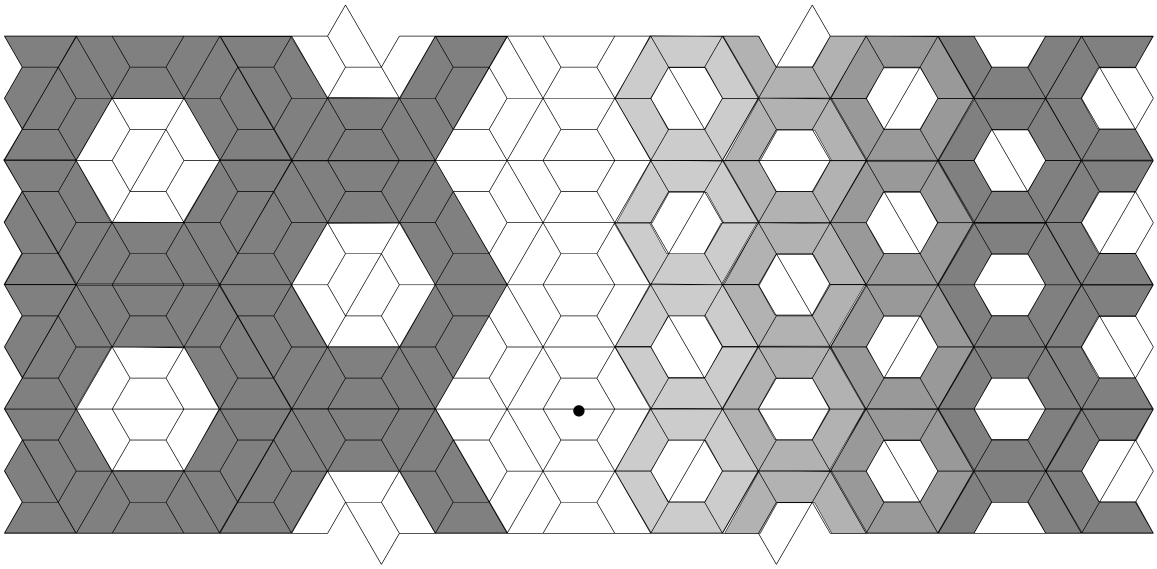

Figure 3: Periodic structures in a nonperiodic tiling

choose instead of (1) the setting

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times G & \xrightarrow{\pi_2} & G \\ \cup & & \cup & & \cup \\ V & & L & & W \end{array} \tag{2}$$

with $G = \widehat{\mathbb{Z}_p} \times \cdots \times \widehat{\mathbb{Z}_p}$ ($\widehat{\mathbb{Z}_p}$ the ring of p–adic integers), one can generate limitperiodic tilings, thus tilings with an integer factor. This was discovered by BAAKE, MOODY and SCHLOTTMANN and is described in [BMS] and [LM]. In [LM] and [FRE] criteria are given, if a tiling is limitperiodic or not, and thus can be obtained or not by a project scheme (2).

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