ABOUT THE BOOK
HOMOLOGICAL THEORY OF REPRESENTATIONS

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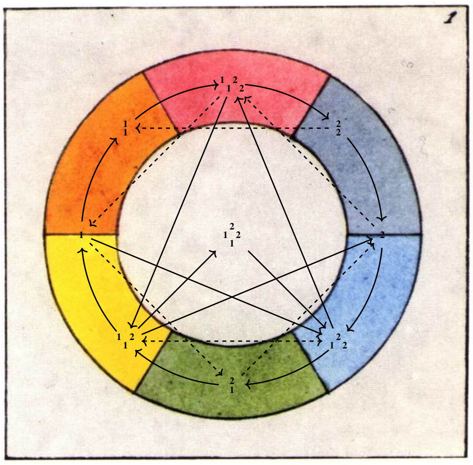
The book is devoted to representations of associative algebras and their homological theory. What does that mean? One way to approach the subject of this book is to explain the following picture and how it relates to the material of the book.



There are two main directions in the development of representation theory of finite dimensional algebras: the study of representations of finite groups and the study of representations of finite quivers. This leads to group algebras and path algebras of quivers. The methods one uses to study their representations are usually quite different, but there is a common generalisation of both classes of algebras: these are the *Gorenstein algebras*; one chapter of the book is devoted this class.

The smallest non-trivial example of a group algebra over a field k is the algebra $k[\varepsilon]$ of dual numbers, given by the group with two elements when the field has characteristic two. On the other hand, the smallest non-trivial example of a path algebra identifies with the matrix algebra $\left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$ and is given by the quiver $\circ \to \circ$. The tensor product of both algebras is the finite dimensional algebra

$$\Lambda = k[\varepsilon] \otimes_k \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$$

which is Gorenstein of dimension one and isomorphic to the path algebra of the quiver

$$\varepsilon_1 \circlearrowright 1 \xrightarrow{\alpha} 2 \circlearrowleft \varepsilon_2$$

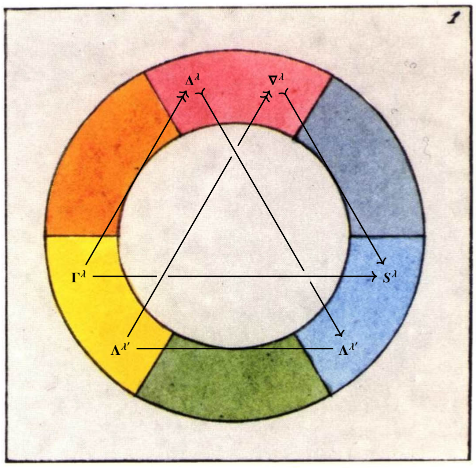
modulo the relations

$$\varepsilon_1^2 = 0 = \varepsilon_2^2 \qquad \text{and} \qquad \alpha\varepsilon_1 = \varepsilon_2\alpha.$$

The algebra Λ is of finite representation type; it has precisely 9 indecomposable representations which are represented by their dimension vectors. These provide the vertices of the Auslander-Reiten quiver of Λ which is precisely the underlying diagram of the above picture.

The other component of the picture is Goethe's Farbkreis which he added to his monumental treatise on the theory of colours [J. W. von Goethe, Zur Farbenlehre, Erster Band, Nebst einem Hefte mit sechzehn Kupfertafeln, Tübingen, 1810]. This Farbkreis relates to the notion of a *spectrum* which arises in various forms in this book. Indecomposable representations are often viewed as points of some space. The analogue in commutative algebra is the Zariski spectrum, consisting by definition of the set of prime ideals. The spectrum of indecomposable injective objects of a *Grothendieck category* is the more general concept which plays a significant role throughout the book. In fact, a substantial part of homological algebra is developed in the context of Grothendieck categories, including derived categories and the theory of purity.

Possibly the most challenging chapter of the book is devoted to *polynomial representations of general linear groups*; it combines some intricate multilinear algebra with the homological theory of highest weight categories. The following picture illustrates some of the essential ingredients.



The category of polynomial representations of fixed degree $d \geq 0$ admits a canonical stratification which is parametrised by the integer partitions λ of d. There is a natural choice of projective generators Γ^λ (given by symmetric tensors) and injective cogenerators S^λ (given by symmetric powers). The exterior powers Λ^λ play an intermediate role and their direct sum provides a canonical tilting object. Finally, the standard objects Δ^λ (given by Weyl modules) and the costandard objects (given by Schur modules) determine the highest weight structure. These objects are related to each other via various canonical morphisms; and they are associated with the three primary colours of Goethe's Farbkreis. The sign representation of the symmetric group on d elements induces an adjoint pair of functors on the category of polynomial representations which maps for each colour the corresponding pair of objects to each other (modulo conjugation $\lambda \leftrightarrow \lambda'$).