15

Gorenstein algebras, approximations, and Serre duality

This chapter discusses the homological theory of modules over Gorenstein rings. A characteristic feature is the decomposition of the module category into two orthogonal subcategories: the Gorenstein projective (or maximal Cohen-Macaulay) modules and the modules of finite projective dimensions. These subcategories are glued together via certain approximation sequences. Moreover, the Gorenstein projective modules carry modulo projectives a natural triangulated structure, and the corresponding stable category admits a Serre functor when it is Hom-finite. This Serre duality specialises to Tate duality when the algebra is self-injective. Also, the category of perfect complexes admits a Serre functor.

15.1 Approximations

We establish the existence of approximations in exact categories and use the concept of a cotorsion pair.

15.1.1 Cotorsion pairs

Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory.

A *finite* $\mathcal{C}$*-resolution* of an object A in $\mathcal{A}$ is an admissible exact sequence (that is, an acyclic complex)

$$0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

such that $X_i \in \mathcal{C}$ for all i. We write $\operatorname{Res}(\mathcal{C})$ for the full subcategory of objects in $\mathcal{A}$ that admit a finite $\mathcal{C}$-resolution.

For a class of objects $\mathcal{C} \subseteq \mathcal{A}$ we set

$${}^{\perp}\mathcal{C} = \{X \in \mathcal{A} \mid \operatorname{Ext}^n_{\mathcal{A}}(X,Y) = 0 \text{ for all } Y \in \mathcal{C},\, n > 0\}$$

and

$$\mathcal{C}^{\perp} = \{Y \in \mathcal{A} \mid \operatorname{Ext}^n_{\mathcal{A}}(X,Y) = 0 \text{ for all } X \in \mathcal{C},\, n > 0\}.$$

Let $\mathcal{A}$ be an exact category and $\mathcal{X}, \mathcal{Y}$ full subcategories of $\mathcal{A}$. Then $(\mathcal{X}, \mathcal{Y})$ is a (hereditary and complete) *cotorsion pair* for $\mathcal{A}$ if

$$\mathcal{X}^{\perp} = \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} = {}^{\perp}\mathcal{Y}$$

and every object $A \in \mathcal{A}$ fits into admissible exact sequences

$$0 \longrightarrow Y_A \longrightarrow X_A \longrightarrow A \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow A \longrightarrow Y^A \longrightarrow X^A \longrightarrow 0 \quad (15.1)$$

with $X_A, X^A \in \mathcal{X}$ and $Y_A, Y^A \in \mathcal{Y}$.

The sequences (15.1) are called *approximation sequences*, because every morphism $X \to A$ with $X \in \mathcal{X}$ factors through $X_A \to A$ and every morphism $A \to Y$ with $Y \in \mathcal{Y}$ factors through $A \to Y^A$.

Remark 15.1.1 Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair for $\mathcal{A}$ and set $\mathcal{C} = \mathcal{X} \cap \mathcal{Y}$.

(1) We have $X_A \in \mathcal{C}$ if $A \in \mathcal{Y}$, and $Y^A \in \mathcal{C}$ if $A \in \mathcal{X}$. In particular, any morphism from $\mathcal{X}$ to $\mathcal{Y}$ factors through an object in $\mathcal{C}$.

(2) The exact sequences in (15.1) are uniquely determined up to isomorphism in the quotient category $\mathcal{A}/\mathcal{C}$. In fact, the assignment $A \mapsto X_A$ gives a right adjoint of the inclusion $\mathcal{X}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$, while the assignment $A \mapsto Y^A$ gives a left adjoint of the inclusion $\mathcal{Y}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$.

15.1.2 A decomposition theorem

The following result establishes a procedure for constructing cotorsion pairs; it is the basis for the existence of approximations.

Proposition 15.1.2 *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory. Set $\mathcal{X} = {}^{\perp}\mathcal{C}$ and let $\mathcal{Y}$ be the closure under direct summands of* $\operatorname{Res}(\mathcal{C})$. *Suppose that $\mathcal{A} = \operatorname{Res}(\mathcal{X})$ and that $\mathcal{C}$ cogenerates $\mathcal{X}$, that is, every object $X \in \mathcal{X}$ fits into an admissible exact sequence $0 \to X \to Y \to Z \to 0$ with $Y \in \mathcal{C}$ and $Z \in \mathcal{X}$. Then $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair for $\mathcal{A}$.*

Proof Let $A \in \mathcal{A}$ and choose an admissible exact sequence

$$0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

with $X_i \in \mathcal{X}$ for all i. We need to construct the sequences (15.1) and use induction on r. The case $r = 0$ is clear. Now suppose $r > 0$ and let B denote the image of $X_1 \to X_0$. By the inductive hypothesis there is an exact sequence $0 \to B \to Y^B \to$

$X^B \to 0$ with $X^B \in \mathcal{X}$ and $Y^B \in \mathcal{Y}$. We form the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y^B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X^B & = & X^B & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and obtain an exact sequence $0 \to Y^B \to X \to A \to 0$ with $X \in \mathcal{X}$ and $Y^B \in \mathcal{Y}$. This gives the first approximation sequence. Now take this sequence and form the pushout with the sequence $0 \to X \to C \to X' \to 0$. This yields the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^B & \longrightarrow & C & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X' & = & X' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and the sequence $0 \to A \to Y \to X' \to 0$ has $X' \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Thus we have constructed the second approximation sequence.

It remains to show that $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{X}^{\perp} = \mathcal{Y}$. The first equality is clear since

$$\mathcal{X} = {}^{\perp}\mathcal{C} = {}^{\perp}\operatorname{Res}(\mathcal{C}).$$

Also, the inclusion $\mathcal{X}^{\perp} \supseteq \mathcal{Y}$ is clear, since $\mathcal{X}^{\perp} \supseteq \mathcal{C}$. For the other inclusion, let $A \in \mathcal{X}^{\perp}$ and consider the sequence $0 \to A \to Y^A \to X^A \to 0$ which splits. Thus $A \in \mathcal{Y}$. □

15.1.3 Self-orthogonal subcategories

For applications of the decomposition theorem, we are interested in the case that the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is given by a self-orthogonal subcategory $\mathcal{C}$.

Let $\mathcal{A}$ be an exact category. A full additive subcategory $\mathcal{C} \subseteq \mathcal{A}$ is *self-orthogonal* if it is closed under direct summands and $\operatorname{Ext}^n_{\mathcal{A}}(X, Y) = 0$ for all X, Y in $\mathcal{C}$ and $n > 0$.

Lemma 15.1.3 *Let $\mathcal{A}$ be an exact category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive and self-orthogonal subcategory. Then* $\operatorname{Res}(\mathcal{C})$ *is closed under direct summands and*

$$^{\perp}\mathcal{C} \cap \operatorname{Res}(\mathcal{C}) = \mathcal{C}.$$

Proof One can show that $\operatorname{Res}(\mathcal{C}) = \operatorname{Thick}(\mathcal{C}) \cap \mathcal{C}^{\perp}$ (Lemma 16.1.4). Clearly, $\operatorname{Thick}(\mathcal{C})$ and $\mathcal{C}^{\perp}$ are closed under direct summands. It follows that $\operatorname{Res}(\mathcal{C})$ is closed under direct summands.

The inclusion $^{\perp}\mathcal{C} \cap \operatorname{Res}(\mathcal{C}) \supseteq \mathcal{C}$ is clear. Thus we fix $A \in {}^{\perp}\mathcal{C} \cap \operatorname{Res}(\mathcal{C})$, and an induction on the length n of a $\mathcal{C}$-resolution shows that A is in $\mathcal{C}$. The case $n = 0$ is clear. If $n > 0$, consider an exact sequence $\eta\colon 0 \to A' \to C \to A \to 0$ with $C \in \mathcal{C}$. Then $A' \in {}^{\perp}\mathcal{C} \cap \operatorname{Res}(\mathcal{C})$, and $A' \in \mathcal{C}$ by the inductive hypothesis. Thus the sequence η splits, and A is in $\mathcal{C}$. □

15.2 Gorenstein rings

Let Λ be a ring and suppose that Λ is two-sided noetherian. The ring Λ is called *Gorenstein* if the injective dimension of Λ is finite as a left and as a right module over itself. In that case one can show that both dimensions coincide [109, Lemma A]. We denote this dimension by d and say Λ is Gorenstein *of dimension d*.

15.2.1 Gorenstein projective modules

A Λ-module X is called *Gorenstein projective* (or *maximal Cohen-Macaulay*) if $\operatorname{Ext}^i_{\Lambda}(X, \Lambda) = 0$ for all $i \neq 0$. We set

$$\operatorname{Gproj} \Lambda = \{X \in \operatorname{mod} \Lambda \mid X \text{ is Gorenstein projective}\}.$$

Now fix a finitely presented Λ-module X and a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow X \longrightarrow 0.$$

For $n \geq 1$ set $\Omega^n X = \operatorname{Im} d^n$ and $X^* = \operatorname{Hom}_{\Lambda}(X, \Lambda)$.

Lemma 15.2.1 *Let Λ be a Gorenstein ring of dimension d and X a finitely presented Λ-module. Then the following holds:*

(1) *The module $\Omega^n X$ is Gorenstein projective for all $n \geq d$.*
(2) *If X is Gorenstein projective, then $\Omega^n X$ is Gorenstein projective for all $n \geq 1$.*
(3) *If X is Gorenstein projective, then the sequence $0 \to X^* \to P_0^* \to P_1^* \to \cdots$ is exact and X^* is Gorenstein projective as Λ^{op}-module.*
(4) *The functor* $\operatorname{Hom}_{\Lambda}(-, \Lambda)$ *induces an exact duality* $(\operatorname{Gproj} \Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Gproj}(\Lambda^{\mathrm{op}})$.

Proof We apply the dimension shift formula

$$\mathrm{Ext}^p_\Lambda(\Omega^q X, -) \cong \mathrm{Ext}^{p+q}_\Lambda(X, -) \qquad (p, q \geq 1).$$

Then (1) and (2) are clear. From this we obtain the exactness of

$$0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots$$

and therefore X^* is a syzygy of arbitrarily high order. Thus X^* is Gorenstein projective by (1), and this completes (3). The assertion in (4) is then a consequence. □

15.2.2 Gorenstein approximations

For Gorenstein rings there is a good approximation theory. The category of finitely presented modules decomposes into two orthogonal subcategories which are glued together via approximation sequences.

Theorem 15.2.2 (Auslander–Buchweitz) *Let Λ be a Gorenstein ring. Set $\mathcal{X} = \operatorname{Gproj} \Lambda$ and write $\mathcal{Y}$ for the category of finitely presented Λ-modules of finite projective dimension. Then $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair for* $\operatorname{mod} \Lambda$ *with* $\mathcal{X} \cap \mathcal{Y} = \operatorname{proj} \Lambda$.

Proof We apply Proposition 15.1.2. Thus we set $\mathcal{A} = \operatorname{mod} \Lambda$ and $\mathcal{C} = \operatorname{proj} \Lambda$. This gives $\mathcal{X} = {}^\perp\mathcal{C}$ and $\mathcal{Y} = \operatorname{Res}(\mathcal{C})$. The assumption on Λ implies that $\mathcal{A} = \operatorname{Res}(\mathcal{X})$ and that $\mathcal{C}$ cogenerates $\mathcal{X}$; this follows from Lemma 15.2.1. More precisely, if Λ is Gorenstein of dimension d, then any Λ-module X admits a resolution

$$0 \longrightarrow \Omega^d X \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that $P_0, \ldots, P_{d-1}$ are projective Λ-modules. Thus $X \in \operatorname{Res}(\mathcal{X})$, since $\Omega^d X$ is Gorenstein projective. If X is Gorenstein projective, choose an exact sequence $0 \to Y \to P \to X^* \to 0$ in $\operatorname{mod} \Lambda^{\mathrm{op}}$ such that P is projective. This yields an exact sequence $0 \to X \to P^* \to Y^* \to 0$ in $\operatorname{Gproj} \Lambda$, since $X \xrightarrow{\sim} X^{**}$.

The final assertion $\mathcal{X} \cap \mathcal{Y} = \operatorname{proj} \Lambda$ follows from Lemma 15.1.3. □

15.2.3 The stable category

For a noetherian ring Λ we consider the derived category $\mathbf{D}^b(\operatorname{mod} \Lambda)$ and obtain the *singularity category* (or *stabilised derived category*) by forming the triangulated quotient

$$\mathbf{D}_{\mathrm{sg}}(\Lambda) = \frac{\mathbf{D}^b(\operatorname{mod} \Lambda)}{\mathbf{D}^b(\operatorname{proj} \Lambda)}.$$

An exact category $\mathcal{A}$ is a *Frobenius category* if $\mathcal{A}$ has enough projective objects and enough injective objects, and if projective and injective objects in $\mathcal{A}$ coincide. The *stable category* of $\mathcal{A}$ is obtained by annihilating all morphisms that factor through a projective object. The exact structure of $\mathcal{A}$ induces a triangulated structure for the stable category.

Theorem 15.2.3 (Buchweitz) *Let Λ be Gorenstein. Then the Gorenstein projective Λ-modules form a Frobenius category. Writing $\underline{\mathrm{Gproj}}\,\Lambda$ for its stable category, the composition*

$$F\colon \operatorname{Gproj}\Lambda \rightarrowtail \mathbf{D}^b(\operatorname{mod}\Lambda) \twoheadrightarrow \mathbf{D}_{\mathrm{sg}}(\Lambda)$$

induces a triangle equivalence

$$\underline{\mathrm{Gproj}}\,\Lambda \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda).$$

Proof It follows from Lemma 15.2.1 that $\operatorname{Gproj}\Lambda$ is a Frobenius category. The projective Λ-modules form the subcategory of objects that are projective and injective.

The functor F is exact: it takes an exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Gproj}\Lambda$ to an exact triangle $F(X) \to F(Y) \to F(Z) \to F(X)[1]$. Also, F annihilates all projective Λ-modules and yields therefore an exact functor $\bar{F}\colon \underline{\mathrm{Gproj}}\,\Lambda \to \mathbf{D}_{\mathrm{sg}}(\Lambda)$. The suspension in $\underline{\mathrm{Gproj}}\,\Lambda$ takes X to $\Omega^{-1}X$, and

$$F(\Omega^{-1}X) \cong F(X)[1].$$

We construct an inverse for $\bar{F}$ as follows.

Consider the category of complexes $\mathbf{K}(\operatorname{proj}\Lambda)$ of finitely generated projective Λ-modules up to homotopy. We identify the subcategories

$$\mathbf{K}^b(\operatorname{proj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{proj}\Lambda) \quad \text{and} \quad \mathbf{K}^{-,b}(\operatorname{proj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda).$$

For a complex X and $n \in \mathbb{Z}$ we use the following truncation:

$$\begin{array}{ccccccccccccc}
X & & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^{n} & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
\sigma^{\leq n}X & & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^{n} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Now fix a complex X in $\mathbf{K}^{-,b}(\operatorname{proj}\Lambda)$ and choose $n \in \mathbb{Z}$ such that $H^i(X) = 0$ for all $i \leq n+d$. Then $\operatorname{Coker}(X^{n-1} \to X^n)$ is Gorenstein projective, by Lemma 15.2.1. Note that the cone of $X \to \sigma^{\leq n}X$ belongs to $\mathbf{K}^b(\operatorname{proj}\Lambda)$. Thus $X \cong \sigma^{\leq n}X$ in $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ and the assignment

$$X \longmapsto \Omega^n \operatorname{Coker}(X^{n-1} \to X^n)$$

yields a functor $G\colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \to \underline{\mathrm{Gproj}}\,\Lambda$ which does not depend on n. It is not difficult to check that $G \circ \bar{F} \cong \mathrm{Id}$ and $\bar{F} \circ G \cong \mathrm{Id}$. □

15.3 Gorenstein artin algebras

Let k be a commutative artinian ring and Λ an artin k-algebra. We write $D = \operatorname{Hom}_k(-, E)$ for the Matlis duality over k, given by an injective k-module E.

The derived category $\mathbf{D}^b(\operatorname{mod}\Lambda)$ of a Gorenstein algebra Λ 'decomposes' into the category of perfect complexes

$$\mathbf{D}^{\mathrm{per}}(\Lambda) = \mathbf{D}^b(\operatorname{proj}\Lambda)$$

and the singularity category

$$\underline{\operatorname{Gproj}}\,\Lambda \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda).$$

In this section we establish Serre duality for both categories.

Let $\mathcal{C}$ be a k-linear and Hom-finite additive category. A *Serre functor* is an equivalence $F\colon \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms

$$D\operatorname{Hom}_{\mathcal{C}}(X, Y) \cong \operatorname{Hom}_{\mathcal{C}}(Y, FX)$$

for all objects X, Y in $\mathcal{C}$.

15.3.1 Auslander-Reiten duality

Let $\underline{\operatorname{mod}}\,\Lambda$ denote the stable category of finitely presented Λ-modules *modulo projectives*, and let $\overline{\operatorname{mod}}\,\Lambda$ denote the stable category *modulo injectives*. The functors $D\operatorname{Tr}$ and $\operatorname{Tr} D$ induce mutually inverse equivalences:

$$\underline{\operatorname{mod}}\,\Lambda \underset{\operatorname{Tr}}{\overset{\operatorname{Tr}}{\rightleftarrows}} \underline{\operatorname{mod}}(\Lambda^{\mathrm{op}}) \underset{D}{\overset{D}{\rightleftarrows}} \overline{\operatorname{mod}}\,\Lambda$$

Moreover, we have for all $X \in \operatorname{mod}\Lambda$ natural isomorphisms

$$D\operatorname{Ext}^1_\Lambda(X, -) \cong \overline{\operatorname{Hom}}_\Lambda(-, D\operatorname{Tr} X) \quad \text{and} \quad D\,\underline{\operatorname{Hom}}_\Lambda(X, -) \cong \operatorname{Ext}^1_\Lambda(-, D\operatorname{Tr} X).$$

15.3.2 Gorenstein projective and injective modules

Let Λ be Gorenstein. A Λ-module X is *Gorenstein projective* if $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for $i \neq 0$. Dually, X is *Gorenstein injective* if $\operatorname{Ext}^i_\Lambda(D(\Lambda), X) = 0$ for $i \neq 0$. We set

$$\operatorname{Ginj}\Lambda = \{X \in \operatorname{mod}\Lambda \mid X \text{ is Gorenstein injective}\}.$$

The duality induces an equivalence

$$(\operatorname{Gproj}\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Ginj}(\Lambda^{\mathrm{op}}).$$

Observe that a Λ-module has finite projective dimension if and only if it has finite injective dimension, because Λ is Gorenstein. Then Theorem 15.2.2 yields two cotorsion pairs

$$(\operatorname{Gproj}\Lambda, \mathcal{Y}) \qquad \text{and} \qquad (\mathcal{Y}, \operatorname{Ginj}\Lambda)$$

for $\operatorname{mod}\Lambda$, where $\mathcal{Y}$ denotes the subcategory of modules having finite projective and finite injective dimension.

The following lemma collects the basic properties of Gorenstein projective and injective modules. We consider the full subcategories

$$\underline{\mathrm{Gproj}}\,\Lambda \subseteq \underline{\mathrm{mod}}\,\Lambda \qquad \text{and} \qquad \overline{\mathrm{Ginj}}\,\Lambda \subseteq \overline{\mathrm{mod}}\,\Lambda.$$

Lemma 15.3.1 *Let Λ be Gorenstein and $X, Y \in \operatorname{mod}\Lambda$. Then the following holds:*

(1) *The inclusion $\underline{\mathrm{Gproj}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ admits a right adjoint, taking X to $\mathrm{GP}(X)$.*
(2) *The inclusion $\overline{\mathrm{Ginj}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda$ admits a left adjoint, taking X to $\mathrm{GI}(X)$.*
(3) *If X is Gorenstein projective, then $D\operatorname{Tr} X$ is Gorenstein injective.*
(4) *If X is Gorenstein injective, then $\operatorname{Tr} DX$ is Gorenstein projective.*
(5) *If X is Gorenstein projective, then $\mathrm{GP}(\mathrm{GI}(X)) \cong X$ in $\underline{\mathrm{mod}}\,\Lambda$.*
(6) *If X is Gorenstein injective, then $\mathrm{GI}(\mathrm{GP}(X)) \cong X$ in $\overline{\mathrm{mod}}\,\Lambda$.*
(7) *If X is Gorenstein projective and Y is Gorenstein injective, then*

$$\underline{\mathrm{Hom}}_\Lambda(X, Y) = \overline{\mathrm{Hom}}_\Lambda(X, Y).$$

Proof We prove one half, while (2), (4), and (6) follow by duality.

(1) The existence of the adjoint follows from Theorem 15.2.2, using also Remark 15.1.1. Using the notation of the approximation sequence (15.1), the right adjoint sends a Λ-module A to X_A.

(3) If X is Gorenstein projective, then $\operatorname{Tr} X$ is a Gorenstein projective Λ^{op}-module by Lemma 15.2.1. Thus $D\operatorname{Tr} X$ is Gorenstein injective.

(5) Consider the approximation sequence $0 \to X \to \mathrm{GI}(X) \to Y \to 0$, where Y is of finite injective dimension and therefore of finite projective dimension. Let $P \to Y$ be a projective cover and form the following pullback.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & X \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & \mathrm{GI}(X) & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

The morphism $X \oplus P \to \mathrm{GI}(X)$ is a Gorenstein projective approximation, since its kernel has finite projective dimension. Thus we have $\mathrm{GP}(\mathrm{GI}(X)) \cong X$ stably.

(7) Fix a morphism $\phi\colon X \to Y$ that factors through an injective module. Let $X \to P$ denote the approximation with P of finite projective dimension, which exists by Theorem 15.2.2. Then P is projective, by Remark 15.1.1, and ϕ factors through Y, since any injective module has finite projective dimension. The dual argument shows that ϕ factors through an injective module when one assumes that it factors through a projective module. □

15.3.3 Serre duality for the stable category

Auslander-Reiten duality translates into Serre duality for the stable category of Gorenstein projective Λ-modules.

Proposition 15.3.2 *Let Λ be Gorenstein. For Gorenstein projective Λ-modules X, Y there are natural isomorphisms*

$$\underline{\mathrm{Hom}}_\Lambda(\mathrm{Tr}\, D(\mathrm{GI}\,\Omega Y), X) \cong D\,\underline{\mathrm{Hom}}_\Lambda(X, Y) \cong \underline{\mathrm{Hom}}_\Lambda(Y, \Omega^{-1}\,\mathrm{GP}(D\,\mathrm{Tr}\, X)).$$

Proof We have

$$\begin{aligned} D\,\underline{\mathrm{Hom}}_\Lambda(X, Y) &\cong D\,\mathrm{Ext}^1_\Lambda(X, \Omega Y) \\ &\cong \overline{\mathrm{Hom}}_\Lambda(\Omega Y, D\,\mathrm{Tr}\, X) \\ &\cong \overline{\mathrm{Hom}}_\Lambda(\mathrm{GI}\,\Omega Y, D\,\mathrm{Tr}\, X) \\ &\cong \underline{\mathrm{Hom}}_\Lambda(\mathrm{Tr}\, D(\mathrm{GI}\,\Omega Y), X). \end{aligned}$$

The first iso is induced by an exact sequence $0 \to \Omega Y \to P \to Y \to 0$ with P projective. The second iso is Auslander-Reiten duality. The third iso is induced by $\Omega Y \to \mathrm{GI}(\Omega Y)$; see Lemma 15.3.1. The last iso is obtained by applying $\mathrm{Tr}\, D$.

A similar sequence of arguments yields

$$\begin{aligned} D\,\underline{\mathrm{Hom}}_\Lambda(X, Y) &\cong D\,\mathrm{Ext}^1_\Lambda(X, \Omega Y) \\ &\cong \overline{\mathrm{Hom}}_\Lambda(\Omega Y, D\,\mathrm{Tr}\, X) \\ &\cong \underline{\mathrm{Hom}}_\Lambda(\Omega Y, D\,\mathrm{Tr}\, X) \\ &\cong \underline{\mathrm{Hom}}_\Lambda(\Omega Y, \mathrm{GP}(D\,\mathrm{Tr}\, X)) \\ &\cong \underline{\mathrm{Hom}}_\Lambda(Y, \Omega^{-1}\,\mathrm{GP}(D\,\mathrm{Tr}\, X)). \end{aligned}$$ □

Corollary 15.3.3 *Let Λ be Gorenstein. The assignments*

$$X \mapsto \Omega^{-1}\,\mathrm{GP}(D\,\mathrm{Tr}\, X) \quad \text{and} \quad Y \mapsto \mathrm{Tr}\, D(\mathrm{GI}\,\Omega Y)$$

yield mutually inverse equivalences $\underline{\mathrm{Gproj}}\, A \xrightarrow{\sim} \underline{\mathrm{Gproj}}\, A$. *In particular, the composition* $\Omega^{-1} \circ \mathrm{GP} \circ D\,\mathrm{Tr}$ *is a Serre functor for* $\underline{\mathrm{Gproj}}\, A$.

Proof We have $\Omega^{-1} \circ \Omega \cong \mathrm{Id} \cong \Omega \circ \Omega^{-1}$ since $\underline{\mathrm{Gproj}}\,\Lambda$ is a Frobenius category; see Theorem 15.2.3. Also, $D\,\mathrm{Tr} \circ \mathrm{Tr}\, D \cong \mathrm{Id}$ and $\mathrm{Tr}\, D \circ D\,\mathrm{Tr} \cong \mathrm{Id}$. Finally, $\mathrm{GP} \circ \mathrm{GI} \cong \mathrm{Id}$ and $\mathrm{GI} \circ \mathrm{GP} \cong \mathrm{Id}$ by Lemma 15.3.1. The isomorphism in Proposition 15.3.2 then shows that $\Omega^{-1} \circ \mathrm{GP} \circ D\,\mathrm{Tr}$ is a Serre functor. □

For $X \in \mathrm{mod}\,\Lambda$ we set

$$\nu X := X \otimes_\Lambda D(\Lambda) \cong D\,\mathrm{Hom}_\Lambda(X, \Lambda).$$

Remark 15.3.4 If X is a Gorenstein projective Λ-module, then a projective presentation $P_1 \to P_0 \to X \to 0$ induces an exact sequence

$$0 \longrightarrow D\,\mathrm{Tr}\, X \longrightarrow \nu P_1 \longrightarrow \nu P_0 \longrightarrow \nu X \longrightarrow 0$$

and therefore

$$\Omega^{-2}(D \operatorname{Tr} X) \cong \nu X.$$

It follows that for self-injective Λ the Serre functor is given by

$$X \longmapsto \Omega X \otimes_\Lambda D(\Lambda).$$

In particular, Serre duality equals Tate duality in this case.

15.3.4 Serre duality for the category of perfect complexes

Next we establish Serre duality for the category of perfect complexes

$$\mathbf{D}^{\mathrm{per}}(\Lambda) = \mathbf{D}^b(\operatorname{proj} \Lambda).$$

We need the following standard isomorphisms.

Lemma 15.3.5 *Let $(A_\Lambda, {}_\Gamma B_\Lambda, {}_\Gamma C)$ be modules and suppose that A_Λ is finitely generated projective. Then there are natural isomorphisms*

$$B \otimes_\Lambda \operatorname{Hom}_\Lambda(A, \Lambda) \xrightarrow{\sim} \operatorname{Hom}_\Lambda(A, B)$$

and

$$A \otimes_\Lambda \operatorname{Hom}_\Gamma(B, C) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(A, B), C).$$

Proof Straightforward. □

Theorem 15.3.6 (Happel) *Let Λ be Gorenstein. Then the derived Nakayama functor*

$$X \longmapsto X \otimes^{\mathbf{L}}_\Lambda D(\Lambda)$$

provides a Serre functor for the category of perfect complexes $\mathbf{D}^{\mathrm{per}}(\Lambda)$.

Proof Let us denote by $\operatorname{inj} \Lambda$ the full subcategory of injective objects in $\operatorname{mod} \Lambda$. We have equivalences

$$\mathbf{K}^b(\operatorname{proj} \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{proj} \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Thick}(\operatorname{proj} \Lambda))$$

and analogously

$$\mathbf{K}^b(\operatorname{inj} \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{inj} \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Thick}(\operatorname{inj} \Lambda)).$$

The Nakayama functor $- \otimes_\Lambda D(\Lambda)$ and its adjoint $\operatorname{Hom}_\Lambda(D(\Lambda), -)$ induce mutually inverse equivalences:

$$\mathbf{D}^b(\operatorname{Thick}(\operatorname{proj} \Lambda)) \xleftarrow{\sim} \mathbf{K}^b(\operatorname{proj} \Lambda) \underset{\operatorname{Hom}_\Lambda(D(\Lambda), -)}{\overset{- \otimes_\Lambda D(\Lambda)}{\rightleftarrows}} \mathbf{K}^b(\operatorname{inj} \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Thick}(\operatorname{inj} \Lambda))$$

If Λ is Gorenstein, then we have

$$\operatorname{Thick}(\operatorname{proj} \Lambda) = \operatorname{Thick}(\operatorname{inj} \Lambda).$$

Thus the Nakayama functor gives an equivalence $\mathbf{D}^{\mathrm{per}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}(\Lambda)$, and this is a Serre functor since for complexes X, Y and perfect X there are natural isomorphisms

$$\begin{aligned} D\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X,Y) &= \operatorname{Hom}_k(\operatorname{Hom}_{\mathbf{K}(\Lambda)}(X,Y),E) \\ &\cong \operatorname{Hom}_k(H^0\operatorname{Hom}_\Lambda(X,Y),E) \\ &\cong H^0\operatorname{Hom}_k(\operatorname{Hom}_\Lambda(X,Y),E) \\ &\cong H^0\operatorname{Hom}_k(Y\otimes_\Lambda \operatorname{Hom}_\Lambda(X,\Lambda),E) \\ &\cong H^0\operatorname{Hom}_\Lambda(Y,\operatorname{Hom}_k(\operatorname{Hom}_\Lambda(X,\Lambda),E)) \\ &\cong H^0\operatorname{Hom}_\Lambda(Y,X\otimes_\Lambda \operatorname{Hom}_k(\Lambda,E)) \\ &\cong H^0\operatorname{Hom}_\Lambda(Y,X\otimes_\Lambda D(\Lambda)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\Lambda)}(Y,X\otimes_\Lambda D(\Lambda)). \end{aligned}$$

Here, we compute morphisms in $\mathbf{K}(\Lambda) = \mathbf{K}(\operatorname{mod}\Lambda)$ and use the standard isomorphisms from Lemma 15.3.5. □

The derived Nakayama functor $X \mapsto X \otimes^{\mathbf{L}}_\Lambda D(\Lambda)$ provides an equivalence

$$\nu\colon \mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$$

satisfying $\nu(\mathbf{D}^{\mathrm{per}}(\Lambda)) = \mathbf{D}^{\mathrm{per}}(\Lambda)$ when Λ is Gorenstein. Thus there is an equivalence $\bar{\nu}$ making the following square commutative

$$\begin{array}{ccc} \underline{\operatorname{Gproj}}\,\Lambda & \xrightarrow[p]{\sim} & \mathbf{D}_{\mathrm{sg}}(\Lambda) \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} \\ \overline{\operatorname{Ginj}}\,\Lambda & \xrightarrow[q]{\sim} & \mathbf{D}_{\mathrm{sg}}(\Lambda) \end{array}$$

where the horizontal equivalences are from Theorem 15.2.3.

Lemma 15.3.7 *The Gorenstein projective approximation* GP *induces a triangle equivalence* $\overline{\operatorname{Ginj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$*. Moreover*

$$q \cong p \circ \mathrm{GP} \qquad \textit{and} \qquad \bar{\nu} \circ p \cong q \circ \nu \cong q \circ \Omega^{-2} \circ D\operatorname{Tr}.$$

Proof For any Λ-module X there is an exact sequence $0 \to X' \to \mathrm{GP}(X) \to X \to 0$ such that X' has finite projective dimension, by Theorem 15.2.2. Thus $q \cong p \circ \mathrm{GP}$. It follows that GP induces a triangle equivalence $\overline{\operatorname{Ginj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$ since p and q are triangle equivalences. The isomorphism $\bar{\nu} \circ p \cong q \circ \nu$ is clear and the last one follows from Remark 15.3.4. □

Corollary 15.3.8 *Let Λ be Gorenstein. Then*

$$\Sigma^{-1} \circ \bar{\nu}\colon \mathbf{D}_{\mathrm{sg}}(\Lambda) \xrightarrow{\sim} \mathbf{D}_{\mathrm{sg}}(\Lambda) \quad \textit{and} \quad \Omega \circ \mathrm{GP} \circ \nu\colon \underline{\operatorname{Gproj}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$$

are Serre functors.

Proof We apply Proposition 15.3.2 and and Lemma 15.3.7. Thus for Gorenstein projective Λ-modules X, Y we obtain

$$\begin{aligned} D\operatorname{Hom}(pX, pY) &\cong \operatorname{Hom}(pY, p\Omega^{-1}\operatorname{GP}(D\operatorname{Tr}X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1}p\operatorname{GP}\Omega^{-2}(D\operatorname{Tr}X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1}q\Omega^{-2}(D\operatorname{Tr}X)) \\ &\cong \operatorname{Hom}(pY, \Sigma^{-1}\bar{\nu}pX). \end{aligned}$$

Thus $\Sigma^{-1}\circ\bar{\nu}$ is a Serre functor for $\mathbf{D}_{\mathrm{sg}}(\Lambda)$. We have $\bar{\nu}\circ p \cong p\circ\operatorname{GP}\circ\nu$, and it follows that $\Omega\circ\operatorname{GP}\circ\nu$ is a Serre functor for $\underline{\operatorname{Gproj}}\,\Lambda$. □

Notes

The decomposition theorem and the existence of approximations for modules over Gorenstein rings were established by Auslander and Buchweitz [10]. The notion of a cotorsion pair provides a convenient language; it was introduced by Salce in the context of abelian groups [98]. The basic properties of the stable category of Gorenstein projective modules are discussed in [27]. Serre duality for the derived category of a finite dimensional algebra of finite global dimension is due to Happel [65], while the notion of a Serre functor was introduced later by Bondal and Kapranov [23]. The Auslander-Reiten theory for Gorenstein projective modules was developed by Auslander and Reiten in [15].