

STRATIFYING TRIANGULATED CATEGORIES, I SUPPORT AND COSUPPORT OF COMPLEXES

Henning Krause

Universität Bielefeld

Derived Categories in Algebraic Geometry
Steklov Mathematical Institute, Moscow
September 5–10, 2011

www.math.uni-bielefeld/~hkrause

Support and cosupport provide a link between

ALGEBRA AND GEOMETRY.

I discuss two papers of [Amnon Neeman](#) involving these concepts:

- *The chromatic tower of $D(R)$* , Topology (1992).
- *Colocalizing subcategories of $D(R)$* , Crelles Journal (2011).

Applications in representation theory of finite groups follow at the end.

All this is part of a joint project with [D. Benson](#) and [S. Iyengar](#).

Here is the setup:

- $R =$ a commutative noetherian ring
- $\text{Mod } R =$ the category of R -modules
- $D(R) =$ the (unbounded) derived category of $\text{Mod } R$
- $\text{Spec } R =$ the set of prime ideals of R

$D(R)$ is a triangulated category with set-indexed (co)products.

DEFINITION

A triangulated subcategory $C \subseteq D(R)$ is called

- **localizing** if C is closed under taking all coproducts,
- **colocalizing** if C is closed under taking all products.

For any class $S \subseteq D(R)$ write:

$\text{Loc}(S) =$ the smallest localizing subcategory containing S

$\text{Coloc}(S) =$ the smallest colocalizing subcategory containing S

THEOREM (NEEMAN, 1992)

The assignment

$$\text{Spec } R \supseteq \mathcal{U} \longmapsto \text{Loc}(\{k(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{U}\}) \subseteq D(R)$$

induces a bijection between

- the collection of *subsets* of $\text{Spec } R$, and
- the collection of *localizing subcategories* of $D(R)$.

Notation: $k(\mathfrak{p}) =$ the residue field $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$

THEOREM (NEEMAN, 2011)

The assignment

$$\text{Spec } R \supseteq \mathcal{U} \longmapsto \text{Coloc}(\{k(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{U}\}) \subseteq D(R)$$

induces a bijection between

- the collection of *subsets* of $\text{Spec } R$, and
 - the collection of *colocalizing subcategories* of $D(R)$.
-
- This is surprising because products tend to be complicated!
 - How are the results from '92 and '11 related to each other?
 - Is there a common proof?

A CONSEQUENCE / REFORMULATION

For $\mathcal{C} \subseteq D(R)$ write:

$$\mathcal{C}^\perp = \{X \in D(R) \mid \text{Hom}_{D(R)}(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$$

$${}^\perp\mathcal{C} = \{X \in D(R) \mid \text{Hom}_{D(R)}(X, C) = 0 \text{ for all } C \in \mathcal{C}\}$$

- If \mathcal{C} is localizing, then \mathcal{C}^\perp is colocalizing.
- If \mathcal{C} is colocalizing, then ${}^\perp\mathcal{C}$ is localizing.
- If \mathcal{C} is localizing, then ${}^\perp(\mathcal{C}^\perp) = \mathcal{C}$ [Neeman 1992].

COROLLARY (NEEMAN, 2011)

The assignment $\mathcal{C} \mapsto \mathcal{C}^\perp$ induces a bijection between

- *the collection of **localizing subcategories** of $D(R)$, and*
- *the collection of **colocalizing subcategories** of $D(R)$.*

THE SUPPORT OF A COMPLEX

DEFINITION (FOXBY, 1979)

For $X \in D(R)$ define the **support**

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X \otimes_R^{\mathbf{L}} k(\mathfrak{p}) \neq 0\}.$$

Some examples:

- If $X \in D^b(\operatorname{mod} R)$, then

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \neq 0\} = \bigcup_{n \in \mathbb{Z}} \operatorname{supp} H^n(X).$$

- Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\operatorname{supp} E(R/\mathfrak{p}) = \operatorname{supp} k(\mathfrak{p}) = \{\mathfrak{p}\}$.

COROLLARY (NEEMAN, 1992)

For $X, Y \in D(R)$ we have

$$\operatorname{supp} X \subseteq \operatorname{supp} Y \iff \operatorname{Loc}(X) \subseteq \operatorname{Loc}(Y).$$

A TEST QUESTION

Let us test our understanding of localizing subcategories:

QUESTION

Let $X \in D(R)$. Is it true that

$$\text{Loc}(X) = \text{Loc}(H^*(X))?$$

AN EXAMPLE

$R = k[[x, y]]$ (k a field)

$\mathfrak{m} = (x, y)$ the maximal ideal of R

$Q =$ the field of fractions of R

The minimal injective resolution of R :

$$\cdots \longrightarrow 0 \longrightarrow Q \longrightarrow \bigoplus_{\text{ht } \mathfrak{p}=1} E(R/\mathfrak{p}) \longrightarrow E(R/\mathfrak{m}) \longrightarrow 0 \longrightarrow \cdots$$

$$X: \quad \cdots \longrightarrow 0 \longrightarrow Q \longrightarrow \bigoplus_{\text{ht } \mathfrak{p}=1} E(R/\mathfrak{p}) \longrightarrow 0 \longrightarrow \cdots$$

In $D(R)$ one has

$$\text{supp } X = (\text{Spec } R) \setminus \{\mathfrak{m}\} \quad \text{supp } H^*(X) = \text{Spec } R$$

Thus $\text{Loc}(X) \neq \text{Loc}(H^*X)$.

THE COSUPPORT OF A COMPLEX

DEFINITION

For $X \in D(R)$ define the **cosupport**

$$\text{cosupp } X = \{\mathfrak{p} \in \text{Spec } R \mid \mathbf{R}\text{Hom}_R(k(\mathfrak{p}), X) \neq 0\}.$$

This seems hard to compute, even for 'simple' objects:

- Let $R = \mathbb{Z}$. Then $\text{cosupp } X = \text{supp } X$ for $X \in D^b(\text{mod } R)$.
- Let (R, \mathfrak{m}) be complete local. Then $\text{cosupp } R = \{\mathfrak{m}\}$.

PROPOSITION

For a complex X in $D(R)$ we have

$$\text{Max}(\text{supp } X) = \text{Max}(\text{cosupp } X).$$

Notation: $\text{Max } \mathcal{U} = \{\mathfrak{p} \in \mathcal{U} \mid \mathfrak{p} \subseteq \mathfrak{q} \in \mathcal{U} \implies \mathfrak{p} = \mathfrak{q}\}$.

FOUR FUNDAMENTAL FUNCTORS

Four fundamental (idempotent) functors $\text{Mod } R \rightarrow \text{Mod } R$:

- localization $M \longrightarrow M \otimes_R R_{\mathfrak{p}}$
- colocalization $\text{Hom}_R(R_{\mathfrak{p}}, M) \longrightarrow M$
- torsion $\Gamma_{\mathfrak{a}}M = \varinjlim \text{Hom}(R/\mathfrak{a}^n, M) \longrightarrow M$
- completion $M \longrightarrow \Lambda_{\mathfrak{a}}M = \varprojlim M \otimes_R R/\mathfrak{a}^n$

Their derived functors $D(R) \rightarrow D(R)$:

- localization $X \longrightarrow X \otimes_R^{\mathbf{L}} R_{\mathfrak{p}}$
- colocalization $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, X) \longrightarrow X$
- local cohomology $\mathbf{R}\Gamma_{\mathfrak{a}}X \longrightarrow X$ [Grothendieck, 1967]
- local homology $X \longrightarrow \mathbf{L}\Lambda_{\mathfrak{a}}X$ [Greenlees–May, 1992]

Note:

- The functor $\mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, -)$ is a right adjoint of $- \otimes_R^{\mathbf{L}} R_{\mathfrak{p}}$.
- The functor $\mathbf{L}\Lambda_{\mathfrak{a}}$ is a right adjoint of $\mathbf{R}\Gamma_{\mathfrak{a}}$.

LOCAL (CO)HOMOLOGY

DEFINITION

Fix $\mathfrak{p} \in \text{Spec } R$ and define (by abuse of notation):

- **local cohomology** $\Gamma_{\mathfrak{p}} = \mathbf{R}\Gamma_{\mathfrak{p}}(- \otimes_R^{\mathbf{L}} R_{\mathfrak{p}})$,
- **local homology** $\Lambda_{\mathfrak{p}} = \mathbf{R}\text{Hom}_R(R_{\mathfrak{p}}, \mathbf{L}\Lambda_{\mathfrak{p}}-)$.

These are idempotent functors $D(R) \rightarrow D(R)$, and $\Lambda_{\mathfrak{p}}$ is a right adjoint of $\Gamma_{\mathfrak{p}}$.

We consider their essential images:

- $\text{Im } \Gamma_{\mathfrak{p}} =$ **local cohomology objects** (a localizing subcategory)
- $\text{Im } \Lambda_{\mathfrak{p}} =$ **local homology objects** (a colocalizing subcategory)

Note: $\Lambda_{\mathfrak{p}}$ induces an equivalence $\text{Im } \Gamma_{\mathfrak{p}} \xrightarrow{\sim} \text{Im } \Lambda_{\mathfrak{p}}$.

An alternative description of (co)support:

- $\text{supp } X = \{\mathfrak{p} \in \text{Spec } R \mid \Gamma_{\mathfrak{p}} X \neq 0\}$.
- $\text{cosupp } X = \{\mathfrak{p} \in \text{Spec } R \mid \Lambda_{\mathfrak{p}} X \neq 0\}$.

The following are equivalent:

- $H^n(X)$ is \mathfrak{p} -local and \mathfrak{p} -torsion for all $n \in \mathbb{Z}$.
- $\text{supp } X \subseteq \{\mathfrak{p}\}$.
- X lies in $\text{Im } \Gamma_{\mathfrak{p}}$.

There seems to be no analogue for $\Lambda_{\mathfrak{p}}$.

STRATIFICATION OF $D(R)$

PROPOSITION

The assignment

$$D(R) \supseteq \mathcal{C} \longmapsto (\mathcal{C} \cap \text{Im } \Gamma_p)_{p \in \text{Spec } R}$$

induces a bijection between

- *the collection of localizing subcategories of $D(R)$, and*
- *the collection of families $(\mathcal{C}_p)_{p \in \text{Spec } R}$ with each $\mathcal{C}_p \subseteq \text{Im } \Gamma_p$ a localizing subcategory.*

Analogously, the assignment

$$D(R) \supseteq \mathcal{C} \longmapsto (\mathcal{C} \cap \text{Im } \Lambda_p)_{p \in \text{Spec } R}$$

classifies the colocalizing subcategories of $D(R)$.

(CO)LOCALIZING SUBCATEGORIES OF $D(R)$

PROPOSITION

Let $\mathfrak{p} \in \text{Spec } R$.

- $\text{Im } \Gamma_{\mathfrak{p}}$ has no proper localizing subcategories.
- $\text{Im } \Lambda_{\mathfrak{p}}$ has no proper colocalizing subcategories.

PROOF.

For each $0 \neq X \in \text{Im } \Gamma_{\mathfrak{p}}$, one shows that

$$\text{Loc}(X) = \text{Loc}(k(\mathfrak{p})) = \text{Im } \Gamma_{\mathfrak{p}}.$$

Analogously, $\text{Coloc}(Y) = \text{Im } \Lambda_{\mathfrak{p}}$ for each $0 \neq Y \in \text{Im } \Lambda_{\mathfrak{p}}$. □

The classifications of [Neeman, 1992] and [Neeman, 2011] are immediate consequences.

A GENERALIZATION AND AN APPLICATION

The above proof allows to generalize Neeman's results to the derived category of a **differential graded algebra** A such that

- A is **formal**, i.e. quasi-isomorphic to its cohomology $H^*(A)$,
- $H^*(A)$ is **graded-commutative** and **noetherian**.

An application to the study of **modular representations of finite groups** goes as follows:

Let G be a finite group and k a field of characteristic $p > 0$. We consider modules over the group algebra kG and classify the (co)localizing subcategories of the stable category $\text{StMod } kG$.

MODULAR REPRESENTATIONS OF FINITE GROUPS

Take as example $G = (\mathbb{Z}/2\mathbb{Z})^r$ and a field k of characteristic 2.

Group algebra $kG \cong k[x_1, \dots, x_r]/(x_1^2, \dots, x_r^2)$

Group cohomology $H^*(G, k) = \text{Ext}_{kG}^*(k, k) \cong k[\xi_1, \dots, \xi_r]$

$K(\text{Inj } kG) =$ category of complexes of injective kG -modules / htpy.

$\mathbf{i}k =$ an injective resolution of the trivial representation k

$\text{End}_{kG}(\mathbf{i}k) =$ the endomorphism dg algebra of $\mathbf{i}k$ (is formal)

$$\begin{aligned} \text{StMod } kG &\xrightarrow{\sim} K_{\text{ac}}(\text{Inj } kG) \hookrightarrow K(\text{Inj } kG) \\ &\xrightarrow[\text{Hom}_{kG}(\mathbf{i}k, -)]{\sim} D(\text{End}_{kG}(\mathbf{i}k)) \xrightarrow{\sim} D(k[\xi_1, \dots, \xi_r]) \end{aligned}$$

COROLLARY

There are canonical bijections between

- *(co)localizing subcategories of $\text{StMod } kG$, and*
- *sets of graded non-maximal prime ideals of $H^*(G, k)$.*

WANT TO READ MORE?



D. J. Benson, S. B. Iyengar, H. Krause, Representations of finite groups: Local cohomology and support, Notes from an Oberwolfach Seminar, Birkhäuser Verlag, arXiv:1107.4815.