1.6. Non-crossing partitions arising in representation theory

In this section we explain how non-crossing partitions arise naturally in representation theory. For any finite dimensional algebra A over a field k we consider the category $\operatorname{mod} A$ of finite dimensional (right) A-modules and denote by $K_0(A)$ its *Grothendieck group.* This group is free abelian of finite rank, and a representative set of simple A-modules $S_1, \ldots, S_n$ provides a basis $e_1, \ldots, e_n$ if one sets $e_i = [S_i]$ for all i. As usual, we denote for any A-module X by $[X]$ the corresponding class in $K_0(A)$. The Grothendieck group comes equipped with the *Euler form* $K_0(A) \times K_0(A) \to \mathbb{Z}$ given by

$$\langle [X], [Y] \rangle = \sum_{n \geqslant 0} (-1)^n \dim_k \operatorname{Ext}^n_A(X, Y)$$

which is bilinear and non-degenerate (assuming that A is of finite global dimension). The corresponding symmetrised form is given by $(x, y) = \langle x, y \rangle + \langle y, x \rangle$. For a class $x = [X]$ given by a module X, one defines the reflection

$$s_x \colon K_0(A) \longrightarrow K_0(A), \quad a \mapsto a - 2\frac{(a, x)}{(x, x)} x, \tag{1.6.1}$$

assuming that $(x, x) \neq 0$ divides (e_i, x) for all i. Let us denote by $W(A)$ the group of automorphisms of $K_0(A)$ that is generated by the set of simple reflections $S(A) = \{s_{e_1}, \ldots, s_{e_n}\}$; it is called the *Weyl group* of A.

From now on assume that A is *hereditary*, that is, of global dimension at most one. Then one can show that the Weyl group $W(A)$ is actually a Coxeter group. For example, the path algebra kQ of any quiver Q is hereditary and in that case kQ-modules identify with k-linear representations of Q.

Proposition 1.6.1 ([36, Theorem B.2])**.** *A Coxeter system (W, S) is of the form $(W(A), S(A))$ for some finite dimensional hereditary algebra A if and only if it is crystallographic in the following sense:*

(1) $m_{st} \in \{2, 3, 4, 6, \infty\}$ *for all $s \neq t$ in S, and*
(2) *in each circuit of the Coxeter graph not containing the edge label ∞, the number of edges labelled* 4 *(resp.* 6*) is even.* □

We may assume that the simple A-modules are numbered in such a way that $\langle e_i, e_j \rangle = 0$ for $i > j$, and we set $c = s_{e_1} \cdots s_{e_n}$. Note that $c = c(A)$ is a *Coxeter element* which is determined by the formula

$$\langle x, y \rangle = -\langle y, c(x) \rangle \qquad \text{for} \qquad x, y \in K_0(A).$$

We are now in a position to formulate a theorem which provides an explicit bijection between certain subcategories of $\operatorname{mod} A$ and the non-crossing partitions in $\operatorname{NC}(W(A), c)$. Call a full subcategory $\mathcal{C} \subseteq \operatorname{mod} A$ *thick* if it is closed under direct summands and satisfies the following two-out-of-three property: any exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules lies in $\mathcal{C}$ if two of $\{X, Y, Z\}$ are in $\mathcal{C}$. A subcategory is *coreflective* if the inclusion functor admits a right adjoint.

Theorem 1.6.2. *Let A be a hereditary finite dimensional algebra. Then there is an order preserving bijection between the set of thick and coreflective subcategories of* $\operatorname{mod} A$ *(ordered by inclusion) and the partially ordered set of non-crossing partitions* $\mathrm{NC}(W(A), c)$. *The map sends a subcategory which is generated by an exceptional sequence $E = (E_1, \dots, E_r)$ to the product of reflections $s_E = s_{E_1} \cdots s_{E_r}$.* □

The rest of this article is devoted to explaining this result. In particular, the crucial notion of an exceptional sequence will be discussed.

This result goes back to beautiful work of Ingalls and Thomas [38]. It was then established for arbitary path algebras by Igusa, Schiffler, and Thomas [37], and we refer to [36] for the general case. Observe that path algebras of quivers cover only the Coxeter groups of simply laced type (via the correspondence $A \mapsto W(A)$); so there are further hereditary algebras.

We may think of Theorem 1.6.2 as a *categorification* of the poset of non-crossing partitions. There is an immediate (and easy) consequence which is not obvious at all from the original definition of non-crossing partitions; the first (combinatorial) proof required a case by case analysis.

Corollary 1.6.3. *For a finite crystallographic Coxeter group the corresponding poset of non-crossing partitions is a lattice.*

Proof. Any finite Coxeter group can be realised as the the Weyl group $W(A)$ of a hereditary algebra of finite representation type. In that case any thick subcategory is coreflective. On the other hand, it is clear from the definition that the intersection of any collection of thick subcategories is again thick. This yields the join, but also the meet operation; so the poset of thick and coreflective subcategories is actually a lattice; see Remark 1.1.1 □

This categorification provides some further insight into the *collection* of all posets of non-crossing partitions. This is based on the simple observation that any thick and coreflective subcategory $\mathcal{C} \subseteq \operatorname{mod} A$ (given by an exceptional sequence $E = (E_1, \dots, E_r)$) is again the module category of a finite dimensional hereditary algebra, say $\mathcal{C} = \operatorname{mod} B$. Then the inclusion $\operatorname{mod} B \to \operatorname{mod} A$ induces not only an inclusion $K_0(B) \to K_0(A)$, but also an inclusion $W(B) \to W(A)$ for the corresponding Weyl groups, which identifies $W(B)$ with the subgroup of $W(A)$ generated by $s_{E_1}, \dots, s_{E_r}$, and identifies the Coxeter element $c(B)$ with the non-crossing partition s_E in $W(A)$. Moreover, the inclusion $W(B) \to W(A)$ induces an isomorphism

$$\mathrm{NC}(W(B), c(B)) \xrightarrow{\sim} \{x \in \mathrm{NC}(W(A), c(A)) \mid x \leqslant s_E\}.$$

The following result summarises this discussion; it reflects the fact that there is a *category of non-crossing partitions*. This means that we consider a poset of non-crossing partitions not as a single object but look instead at the relation with other posets of non-crossing partitions.

Corollary 1.6.4 ([36, Corollary 5.8]). *Let* $\mathrm{NC}(W,c)$ *be the poset of non-crossing partitions given by a crystallographic Coxeter group* W*. Then any element* $x \in \mathrm{NC}(W,c)$ *is the Coxeter element of a subgroup* $W' \leqslant W$ *that is again a crystallographic Coxeter group. Moreover*

$$\mathrm{NC}(W',x) = \{y \in \mathrm{NC}(W,c) \mid y \leqslant x\}. \qquad \square$$

1.7. Generalised Cartan lattices

Coxeter groups and non-crossing partitions are closely related to root systems. The approach via representation theory provides a natural setting, because the Grothendieck group equipped with the Euler form determines a root system; we call this a *generalised Cartan lattice* and refer to [36] for a detailed study.

The following definition formalises the properties of the Grothendieck group $K_0(A)$. A *generalised Cartan lattice* is a free abelian group $\Gamma \cong \mathbb{Z}^n$ with an ordered standard basis $e_1,\dots,e_n$ and a bilinear form $\langle -,-\rangle\colon \Gamma\times\Gamma \to \mathbb{Z}$ satisfying the following:

(1) $\langle e_i,e_i\rangle > 0$ and $\langle e_i,e_i\rangle$ divides $\langle e_i,e_j\rangle$ for all i,j.
(2) $\langle e_i,e_j\rangle = 0$ for all $i > j$.
(3) $\langle e_i,e_j\rangle \leqslant 0$ for all $i < j$.

The corresponding *symmetrised form* is

$$(x,y) = \langle x,y\rangle + \langle y,x\rangle \quad \text{for } x,y \in \Gamma.$$

The ordering of the basis yields the *Coxeter element*

$$\mathrm{cox}(\Gamma) := s_{e_1}\cdots s_{e_n}.$$

We can define reflections s_x as in (1.6.1) and denote by $W = W(\Gamma)$ the corresponding *Weyl group*, which is the subgroup of $\mathrm{Aut}(\Gamma)$ generated by the simple reflections $s_{e_1},\dots,s_{e_n}$. We write $\mathrm{NC}(\Gamma) = \mathrm{NC}(W,c)$ with $c = \mathrm{cox}(\Gamma)$ for the poset of non-crossing partitions, and the set of *real roots* is

$$\Phi(\Gamma) := \{w(e_i) \mid w \in W(\Gamma),\ 1 \leqslant i \leqslant n\} \subseteq \Gamma.$$

A *real exceptional sequence* of Γ is a sequence $(x_1,\dots,x_r)$ of elements that can be extended to a basis $x_1,\dots,x_n$ of Γ consisting of real roots and satisfying $\langle x_i,x_j\rangle = 0$ for all $i > j$. A *morphisms* $\Gamma' \to \Gamma$ of generalised Cartan lattices is given by an isometry (morphism of abelian groups preserving the bilinear form $\langle -,-\rangle$) that maps the standard basis of Γ' to a real exceptional sequence of Γ. This yields a *category of generalised Cartan lattices*.

What is this category good for? One of the basic principles of category theory is *Yoneda's lemma* which tells us that we understand an object Γ by looking at the representable functor $\mathrm{Hom}(-,\Gamma)$ which records all morphisms that are received by Γ. In our category all morphisms are monomorphisms, so $\mathrm{Hom}(-,\Gamma)$ amounts to the poset of subobjects (equivalence classes of monomorphisms $\Gamma' \to \Gamma$).

Theorem 1.7.1 ([36, Theorem 5.6]). *The poset of subobjects of a generalised Cartan lattice Γ is isomorphic to the poset of non-crossing partitions $\mathrm{NC}(\Gamma)$. The isomorphism sends a monomorphism $\phi\colon \Gamma' \to \Gamma$ to $s_{\phi(e_1)} \cdots s_{\phi(e_r)}$ where $\mathrm{cox}(\Gamma') = s_{e_1} \cdots s_{e_r}$. Moreover, the assignment $w \mapsto w|_{\Gamma'}$ induces an isomorphism*

$$W(\Gamma) \supseteq \langle s_{\phi(e_1)}, \dots, s_{\phi(e_r)} \rangle \xrightarrow{\sim} W(\Gamma'). \qquad \square$$

1.8. Braid group actions on exceptional sequences

The link between representation theory and non-crossing partitions is based on the notion of an exceptional sequence and the action of the braid group on the collection of complete exceptional sequences. This will be explained in the following section.

There are two sorts of abelian categories that we need to consider. This follows from a theorem of Happel [33, 34] which we now explain. Fix a field k and consider a connected hereditary abelian category $\mathcal{A}$ that is k-linear with finite dimensional Hom and Ext spaces. Suppose in addition that $\mathcal{A}$ admits a *tilting object*. This is by definition an object T in $\mathcal{A}$ with $\mathrm{Ext}^1_{\mathcal{A}}(T,T) = 0$ such that $\mathrm{Hom}_{\mathcal{A}}(T,A) = 0$ and $\mathrm{Ext}^1_{\mathcal{A}}(T,A) = 0$ imply $A = 0$. Thus the functor $\mathrm{Hom}_{\mathcal{A}}(T,-)\colon \mathcal{A} \to \mod \Lambda$ into the category of modules over the endomorphism algebra $\Lambda = \mathrm{End}_{\mathcal{A}}(T)$ induces an equivalence

$$\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mod \Lambda)$$

of derived categories [3]. There are two important classes of such hereditary abelian categories admitting a tilting object: module categories over hereditary algebras, and categories of coherent sheaves on weighted projective lines in the sense of Geigle and Lenzing [29]. Happel's theorem then states that there are no further classes.

Theorem 1.8.1 (Happel). *A hereditary abelian category with a tilting object is, up to a derived equivalence, either of the form $\mod A$ for some finite dimensional hereditary algebra A or of the form $\mathrm{coh}\,\mathbb{X}$ for some weighted projective line $\mathbb{X}$.* $\square$

It is interesting to observe that these abelian categories form a category: Any thick and coreflective subcategory is again an abelian category of that type; so the morphisms are given by such inclusion functors.

Now fix an abelian category $\mathcal{A}$ which is either of the form $\mathcal{A} = \mod A$ or $\mathcal{A} = \mathrm{coh}\,\mathbb{X}$, as above. Note that in both cases the Grothendieck group $K_0(\mathcal{A})$

is free of finite rank and equipped with an Euler form, as explained before. An object X in $\mathcal{A}$ is called *exceptional* if it is indecomposable and $\mathrm{Ext}^1_{\mathcal{A}}(X,X)=0$. A sequence $(X_1,\dots,X_r)$ of objects is called *exceptional* if each X_i is exceptional and $\mathrm{Hom}_{\mathcal{A}}(X_i,X_j)=0=\mathrm{Ext}^1_{\mathcal{A}}(X_i,X_j)$ for all $i>j$. Such a sequence is *complete* if r equals the rank of the Grothendieck group $K_0(\mathcal{A})$. Let n denote rank of $K_0(\mathcal{A})$. Then the braid group $\mathcal{B}_n$ on n strands is acting on the collection of isomorphism classes of complete exceptional sequences in $\mathcal{A}$ via mutations, and it is an important theorem that this action is transitive (due to Crawley-Boevey [24] and Ringel [46] for module categories, and Kussin–Meltzer [42] for coherent sheaves).

Any tilting object T admits a decomposition $T=\bigoplus_{i=1}^n T_i$ such that $(T_1,\dots,T_n)$ is a complete exceptional sequence. We denote by $W(\mathcal{A})$ the group of automorphisms of $K_0(\mathcal{A})$ that is generated by the corresponding reflections $s_{T_1},\dots,s_{T_n}$; it is the *Weyl group* with Coxeter element $c=s_{T_1}\cdots s_{T_n}$ and does not depend on the choice of T. Thus we can consider the poset of non-crossing partitions and we have the Hurwitz action on factorisations of the Coxeter element as product of reflections. But it is important to note that $W(\mathcal{A})$ is not always a Coxeter group when $\mathcal{A}=\operatorname{coh}\mathbb{X}$, and it is an open question whether the Hurwitz action is transitive.

The key observation is now the following.

Proposition 1.8.2. *The map*

$$(E_1,\dots,E_r)\longmapsto s_{E_1}\cdots s_{E_r}$$

which assigns to an exceptional sequence in $\mathcal{A}$ the product of reflections in $W(\mathcal{A})$ is equivariant for the action of the braid group $\mathcal{B}_r$. □

The proof is straightforward. But a priori it is not clear that the product $s_{E_1}\cdots s_{E_r}$ is a non-crossing partition. In fact, the proof of Theorem 1.6.2 hinges on the transitivity of the Hurwitz action on factorisations of the Coxeter element. So the analogue of Theorem 1.6.2 for categories of type $\mathcal{A}=\operatorname{coh}\mathbb{X}$ remains open. A proof would provide an interesting extension of the theory of crystallograpic Coxeter groups and non-crossing partitions, which seems very natural in view of Happel's theorem since the Grothendieck group $K_0(\mathcal{A})$ is a derived invariant.

Partial results were obtained recently by Wegener in his thesis [51]. In fact, when a weighted projective line $\mathbb{X}$ is of tubular type (that is, the weight sequence is up to permutation of the form $(2,2,2,2),(3,3,3),(2,4,4)$ or $(2,3,6)$), then the Grothendieck group gives rise to a tubular elliptic root system [47, 48]. Wegener showed the transitivity of the Hurwitz action in this case. Thus, one has in particular the analogue of Theorem 1.6.2 for $\operatorname{coh}\mathbb{X}$ in the tubular case.

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