Lemma. *Let C and D be two closed and unbounded $\mathcal{M}$-classes in $\mathbb{O}$ and $f\colon \mathbb{O} \to \mathbb{O}$ be an $\mathcal{M}$-class function. Then the following hold:*

(1) $C \cap D$ is closed and unbounded in $\mathbb{O}$.
(2) $C_f = \{\alpha \in \mathbb{O} : f(\gamma) < \alpha \text{ for all } \gamma < \alpha\}$ is closed and unbounded in $\mathbb{O}$.

Proof. (1) Clearly, $C \cap D$ is closed in $\mathbb{O}$.

To verify that it is unbounded in $\mathbb{O}$ choose $\varepsilon \in \mathbb{O}$. Since C and D are both unbounded we can recursively construct two sequences $\langle \gamma_\alpha \rangle_{\alpha<\omega}$ in C and $\langle \delta_\alpha \rangle_{\alpha<\omega}$ in D with the property $\varepsilon \leq \gamma_\alpha \leq \delta_\alpha \leq \gamma_{\alpha+\underline{1}}$ for all $\alpha < \omega$. Then $\varepsilon \leq \bigsqcup_{\alpha<\omega} \gamma_\alpha = \bigsqcup_{\alpha<\omega} \delta_\alpha \in C \cap D$.

(2) For all $X \sqsubseteq C_f$ and $\gamma < \bigsqcup X$ there is $\alpha \sqin X$ with $\gamma < \alpha$, so $f(\gamma) < \alpha \leq \bigsqcup X$, hence $\bigsqcup X \in C_f$. This shows that C_f is closed in $\mathbb{O}$.

To prove unboundedness of C_f choose $\varepsilon \in \mathbb{O}$. Then $\varepsilon \leq \alpha \in C_f$ where $\alpha = \bigsqcup_{\beta<\omega} \eta_\beta$ with $\langle \eta_\beta \rangle_{\beta<\omega}$ recursively defined as $\eta_{\underline{0}} = \varepsilon$ and $\eta_{\beta+\underline{1}} = \bigsqcup_{\gamma<\eta_\beta} f(\gamma) + \underline{1}$ for all $\beta < \omega$. Indeed, for all $\gamma < \alpha$ there is some $\beta < \omega$ with $\gamma < \eta_\beta$ such that $f(\gamma) < \eta_{\beta+\underline{1}} \leq \alpha$. $\square$

1. In order to be able to use structural induction on π, we will prove more generally for all S^{Set}-formulas π (instead of only for sentences) the existence of a closed and unbounded $\mathcal{M}$-class C_π in $\mathbb{O}$ such that for every $\alpha \in C_\pi$

$$\mathcal{M}|_{\mathbb{V}} \models \pi\big[\vec{x}/\vec{X}\big] \;\Leftrightarrow\; \mathcal{M}|_{\mathbb{V}_\alpha} \models \pi\big[\vec{x}/\vec{X}\big] \quad \text{for all } \vec{X} = (X_1, \ldots, X_n) \in \mathbb{V}_\alpha^n,$$

where the tuple $\vec{x} = (x_1, \ldots, x_n)$ consists of the distinct free variables of π.

If π is atomic, we can take $C_\pi = \mathbb{O}$.

If $\pi = (\varphi \to \psi)$, we can take $C_\pi = C_\varphi \cap C_\psi$.

If $\pi = \bigwedge_x \varphi$, consider the $\mathcal{M}$-class function $f\colon \mathbb{O} \to \mathbb{O}$ defined as

$$f(\gamma) \;=\; \bigsqcup\Big[\alpha_{\vec{X}} : \vec{X} \in \mathbb{V}_\gamma^n\Big]$$

where – using the convention $\min \emptyset = \underline{0}$ –

$$\alpha_{\vec{X}} \;=\; \min\Big\{\alpha \in \mathbb{O} : \mathcal{M}|_{\mathbb{V}} \not\models \varphi\big[(\vec{x}, x)/(\vec{X}, X)\big] \text{ for some } X \in \mathbb{V}_\alpha\Big\}.$$

The lemma and the fact that $\mathbb{O}_{\lim}$ is closed and unbounded in $\mathbb{O}$ imply that

$$C_\pi \;=\; \{\alpha \in \mathbb{O}_{\lim} \cap C_\varphi : f(\gamma) < \alpha \text{ for all } \gamma < \alpha\}$$

is closed and unbounded in $\mathbb{O}$. It remains to check the equivalence $\Leftrightarrow$ claimed above. Here, $\Rightarrow$ follows from $\mathbb{V}_\alpha \subseteq \mathbb{V}$ and $\alpha \in C_\varphi$. To verify $\Leftarrow$ assume $\mathcal{M}|_{\mathbb{V}} \not\models \pi\big[\vec{x}/\vec{X}\big]$. Then there is $X \in \mathbb{V}_{\alpha_{\vec{X}}}$ such that

$$\mathcal{M}|_{\mathbb{V}} \not\models \varphi\big[(\vec{x}, x)/(\vec{X}, X)\big].$$

Now, $\alpha \in \mathbb{O}_{\lim}$ and $\underline{n} < \omega$ ensures $\vec{X} \in \mathbb{V}_\gamma^n$ for some $\gamma < \alpha$. From $\alpha_{\vec{X}} \leq f(\gamma) < \alpha$ we get $\mathbb{V}_{\alpha_{\vec{X}}} \subseteq \mathbb{V}_\alpha$ and therefore $X \in \mathbb{V}_\alpha$. Finally, $\alpha \in C_\varphi$ implies

$$\mathcal{M}|_{\mathbb{V}_\alpha} \not\models \varphi\big[(\vec{x}, x)/(\vec{X}, X)\big],$$

which shows $\mathcal{M}|_{\mathbb{V}_\alpha} \not\models \pi\big[\vec{x}/\vec{X}\big]$.

2.

(a) The "only if" part holds since $\mathrm{rk}_{C,\prec}$ is $\prec$-ranking. For the "if" part let $r\colon C \to \mathbb{O}$ be a $\prec$-ranking $\mathcal{M}$-class function. Then, for every non-empty $\mathcal{M}$-class $D \subseteq C$, every element $X \in D$ with $r(X) = \min_\prec r(D)$ is $\prec$-minimal in D.

(b) Let us first check that $\mathrm{rk} = \mathrm{rk}_{C,\prec}$ has this property. If it hadn't, the $\mathcal{M}$-class

$$\{Y \in C : \text{there is } \alpha < \mathrm{rk}(Y) \text{ such that } \mathrm{rk}(X) \neq \alpha \text{ for all } X \sqsubset C_{\preceq^\infty Y}\}$$

would have a minimal element Y and there would be $\alpha < \mathrm{rk}(Y)$ such that $\mathrm{rk}(X) \neq \alpha$ for all $X \sqsubset C_{\preceq^\infty Y}$. Since $\mathrm{rk}(Y) = \bigsqcup[\mathrm{rk}(W){+}\underline{1} : W \in C_{\prec Y}]$, there would be $W \in C_{\prec Y}$ with $\alpha \leq \mathrm{rk}(W)$. Then $\alpha < \mathrm{rk}(W)$ because of $W \sqsubset C_{\preceq^\infty Y}$. Due to the minimality of Y there would finally be $X \sqsubset C_{\preceq^\infty W} \sqsubseteq C_{\preceq^\infty Y}$ with $\mathrm{rk}(X) = \alpha$, a contradiction.

Now let $r\colon C \to \mathbb{O}$ be any $\prec$-ranking $\mathcal{M}$-class function. On the one hand, $\alpha_Y \leq r(Y)$ for each $Y \in C$ where $\alpha_Y = \bigsqcup[r(W){+}\underline{1} : W \in C_{\prec Y}]$. On the other hand, $r(X) < \alpha_Y$ for every $W \in C_{\prec Y}$ and $X \sqsubset C_{\preceq^\infty W}$, as is easily seen by induction. Combining these two observations with the fact $C_{\preceq^\infty Y} = [Y] \sqcup \bigsqcup[C_{\preceq^\infty W} : W \in C_{\prec Y}]$ shows that for every $\mathcal{M}$-ordinal α with $\alpha_Y \leq \alpha < r(Y)$ there can be no $X \sqsubset C_{\preceq^\infty Y}$ with $r(X) = \alpha$. Hence, if r has the property that for all $Y \in C$ and $\alpha < r(Y)$ there exists $X \sqsubset C_{\preceq^\infty Y}$ with $r(X) = \alpha$, we must have $r(Y) = \alpha_Y$ for all $Y \in C$, which means $r = \mathrm{rk}$.

(c) Let $t = t_{C,\prec}$.

Firstly, we claim that $t(X) \in \mathbb{O}$ for every $X \in C$. To prove this, we can assume by induction $t(W) \in \mathbb{O}$ for every $W \in C_{\prec X}$. Given that $t(X) = [t(W) : W \in C_{\prec X}]$ and using Exercise 1 on Problem Set 4, it is thus enough to check that $t(X)$ is transitive. For $A \sqsubset B \sqsubset t(X)$ there are $W \sqsubset C_{\prec X}$ with $B = t(W)$ and $V \sqsubset C_{\prec W}$ with $A = t(V)$. Since $\prec$ is transitive on C, we get $V \sqsubset C_{\prec X}$, so $A = t(V) \sqsubset t(X)$.

Secondly, t is $\prec$-ranking by Lemma 2.5.5.

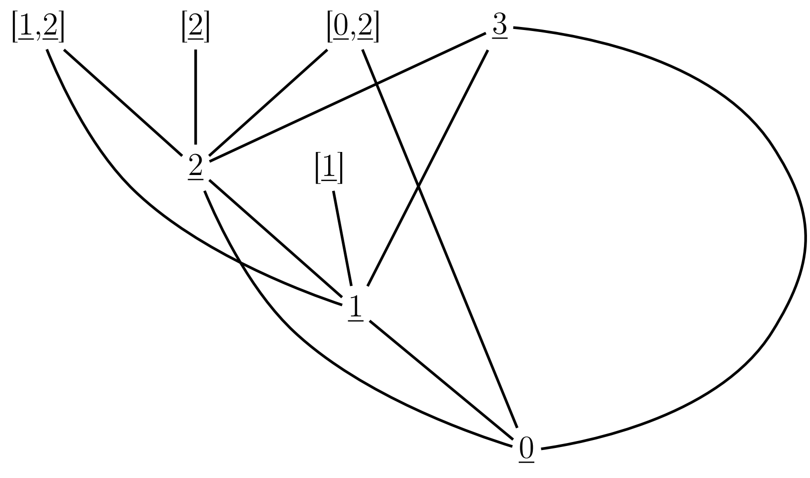
Thirdly, for every $Y \in C$ and $\alpha \sqsubset t(Y)$ there obviously is $X \in C_{\prec Y}$ with $t(X) = \alpha$.

All in all, we can conclude with (b) that $t = \mathrm{rk}$.

(d) It is $X = \langle \underline{2}, \underline{1} \rangle = [\underline{2}, [\underline{1}, \underline{2}]]$. So we easily deduce $t_{X,\sqsubset}(\underline{2}) = \underline{0}$ and $t_{X,\sqsubset}([\underline{1}, \underline{2}]) = \underline{1}$. Because of $X \sqsubseteq \mathtt{P}(\underline{3})$ the rank $\mathrm{rk}_{X^\infty,\sqsubset} = \mathrm{rk}_{\mathtt{P}(\underline{3})^\infty,\sqsubset}|_{X^\infty}$ will be computed in (f).

(e) It is $X = \underline{2} * \underline{1} = [\langle \underline{0}, \underline{0} \rangle, \langle \underline{1}, \underline{0} \rangle] = [\underline{2}, [\underline{1}, \underline{2}]]$. So that's the same X as in (d).

(f) Since $\underline{3}$ is transitive, so is $X = \mathtt{P}(\underline{3})$. Hence, $t_{X,\sqsubset}$ is the identity. The rank $\mathrm{rk}_{X^\infty,\sqsubset}$ can be read off as the height of the vertices in the diagram depicting $X^\infty \smallsetminus [X]$:



3. We begin with an easy observation:

Lemma. *Let T be a transitive $\mathcal{M}$-class such that we have $\varnothing^{\mathcal{M}}, [X,Y]^{\mathcal{M}}, \bigsqcup^{\mathcal{M}} X \in T$ for all $X, Y \in T$. Then $\mathcal{M}|_T \models \mathtt{EXT} \cup \mathtt{EMP} \cup \mathtt{PAI} \cup \mathtt{UNI}$ and moreover:*

- *(5) $f[X]^{\mathcal{M}} \in T$ for all $X \in T$, $f\colon \sqsubseteq^{-1}(X) \twoheadrightarrow T$ definable in $\mathcal{M} \Rightarrow \mathcal{M}|_T \models \mathtt{REP}$.*
- *(6) $\mathtt{P}^{\mathcal{M}}(X)$ exists and $\mathtt{P}^{\mathcal{M}}(X) \sqcap T \in T$ for all $X \in T \Rightarrow \mathcal{M}|_T \models \mathtt{POW}$.*
- *(7) $T \subseteq \mathbb{W}^{\mathcal{M}}$ and $\omega^{\mathcal{M}} \in T \Rightarrow \mathcal{M}|_T \models \mathtt{INF}$.*
- *(8) $\mathcal{M} \models \mathtt{CHO}$ and $Z \sqcap X \in T$ for all $X \in T$ and $\mathcal{M}$-sets $Z \Rightarrow \mathcal{M}|_T \models \mathtt{CHO}$.*
- *(9) $T \subseteq \mathbb{W}^{\mathcal{M}} \Rightarrow \mathcal{M}|_T \models \mathtt{REG}$ and $\mathbb{O}^{\mathcal{M}|_T} = \mathbb{O}^{\mathcal{M}} \cap T$ and $\mathbb{N}^{\mathcal{M}|_T} = \mathbb{N}^{\mathcal{M}}$.*

Proof. Similar to the proof of Lemma 2.7.3 and Theorem 2.7.4. □

Because of $\mathbb{HF} = \bigcup_{\alpha \in \mathbb{N}^{\mathcal{M}}} \mathbb{W}^{\mathcal{M}}_{\alpha}$ Corollary 2.6.7 implies that every $X \in \mathbb{HF}$ is a finite $\mathcal{M}$-set and so has a power set $\mathtt{P}^{\mathcal{M}}(X)$, which again belongs to $\mathbb{HF}$.

The above lemma applies to $T = \mathbb{HF}$ and the premises in (5), (6), (9) are satisfied such that we can conclude that $\mathbb{HF}$ is a $(\mathrm{ZF}^{\circ} \cup \mathtt{POW} \cup \mathtt{REG})$-universe with $\mathbb{N}^{\mathcal{M}|_T} = \mathbb{N}^{\mathcal{M}}$.

Moreover, an $\mathcal{M}|_{\mathbb{HF}}$-set is finite if and only if it is finite when regarded as an $\mathcal{M}$-set, as follows from the observation $[X \rightarrowtail Y]^{\mathcal{M}|_{\mathbb{HF}}} = [X \rightarrowtail Y]^{\mathcal{M}}$ for all $\mathcal{M}|_{\mathbb{HF}}$-sets X, Y.

We are left with proving $\mathcal{M}|_{\mathbb{HF}} \models \mathtt{CHO}$. Given that $\mathop{\ast}^{\mathcal{M}|_{\mathbb{HF}}}_{i \sqsubseteq I} X_i = \mathop{\ast}^{\mathcal{M}}_{i \sqsubseteq I} X_i$ for every family $\langle X_i \rangle_{i \sqsubseteq I}$ of $\mathcal{M}|_{\mathbb{HF}}$-sets with $I \in \mathbb{HF}$, this is a consequence of the following fact:

Lemma (Finite Choice)**.** *Let $\mathcal{N}$ be a ZF°-universe. Then for every family $\langle X_i \rangle_{i \sqsubseteq I}$ of $\mathcal{N}$-sets with finite I the cartesian product $\mathop{\ast}_{i \sqsubseteq I} X_i$ exists and is non-empty.*

Proof. Let $\beta \in \mathbb{N}^{\mathcal{N}}$ such that there is a bijective $f\colon \beta \rightarrowtail I$. Consider the $\mathcal{N}$-class

$$C \;=\; \left\{ \alpha \in \mathbb{N}^{\mathcal{N}} : \alpha \leq \beta \Rightarrow \mathop{\ast}_{\gamma < \alpha} X_{f(\gamma)} \text{ exists and is non-empty} \right\}.$$

Clearly, $\underline{0} \in C$. For $\alpha \in \mathbb{N}^{\mathcal{N}}$ with $\alpha - \underline{1} \in C$, we have $\alpha \in C$ trivially in case $\alpha > \beta$ and otherwise because of $\mathop{\ast}_{\gamma < \alpha} X_{f(\gamma)} \simeq X \ast X_{f(\alpha)} \neq \varnothing$ since $X = \mathop{\ast}_{\gamma < \alpha - \underline{1}} X_{f(\gamma)} \neq \varnothing$ and $X_{f(\alpha)} \neq \varnothing$. By induction $C = \mathbb{N}^{\mathcal{N}}$, so $\beta \in C$, which gives the lemma. □

4. Let Y be an $\mathcal{M}$-set. By Lemma 2.8.16 there exists an $\mathcal{M}$-ordinal α with $\alpha \not\preceq Y$. It suffices to show $Y \preceq \alpha$ because any injective $Y \rightarrowtail \alpha$ yields a well-order on Y.

We may assume without loss of generality that Y and α are disjoint. Let $X = \alpha \sqcup Y$. By assumption there exists an injective $\mathcal{M}$-function $f\colon X \ast X \rightarrowtail X$. For each $y \sqsubseteq Y$ the $\mathcal{M}$-class $C_y = \{\gamma < \alpha : f(\langle y, \gamma \rangle) \sqsubseteq \alpha\}$ must be non-empty, since otherwise the rule $\gamma \mapsto f(\langle y, \gamma \rangle)$ would define an injective $\mathcal{M}$-function $\alpha \rightarrowtail Y$. Therefore we have an injective $\mathcal{M}$-function $Y \rightarrowtail \alpha$ given by $y \mapsto f(\langle y, \min C_y \rangle)$.