# REPRESENTATION THEORY
# EXERCISES 10

HENNING KRAUSE
JAN GEUENICH

A map $X \xrightarrow{\alpha} Y$ in a category $\mathcal{C}$ is *left almost split* if it is not a section and all non-sections $X \to Y'$ factor through $\alpha$. *Right almost split* maps are defined dually. If $\mathcal{C}$ is triangulated, a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow \Sigma X$$

in $\mathcal{C}$ is said to be *almost split* whenever $\alpha$ is left almost split and $\beta$ right almost split.

**1.** Recall or verify the following facts in triangulated Krull-Schmidt categories:

(a) For each triangle $\varepsilon\colon X \to Y \to Z \to \Sigma X$ the following are equivalent:
   (i) $\varepsilon$ is almost split.
   (ii) $\alpha$ is left almost split and left minimal.
   (iii) $\alpha$ is left almost split and $\operatorname{End}(Z)$ is local.
   (iv) $\beta$ is right almost split and right minimal.
   (v) $\beta$ is right almost split and $\operatorname{End}(X)$ is local.

(b) An endomorphism $(f, g, h)$ of an almost split triangle is an iso iff any of $f$, $g$, $h$ is an iso.

(c) $X \to Y$ and $Y \to Z$ in an almost split triangle $X \to Y \to Z \to \Sigma X$ are irreducible maps.

(d) For any given indecomposable object $Z$ (resp. $X$) there exists up to isomorphism at most one almost split triangle $X \to Y \to Z \to \Sigma X$.

Compare the notion of almost split triangles with the notion of almost split sequences.

We fix now an artin algebra $\Lambda$ over $k$. As usual, $D = \operatorname{Hom}_k(-, E)$ denotes the Matlis duality, $\nu = - \otimes_\Lambda D\Lambda$ the Nakayama functor and $\tau = D\mathrm{Tr}$ the Auslander-Reiten translation.

**2.** Let $Z$ be an object of $\mathbf{D}^b(\operatorname{mod}\Lambda)$. Recall that, if $Z$ is perfect, we have a natural isomorphism

$$DE_Z \xrightarrow{\phi_Z} \operatorname{Hom}(Z, \nu Z) \quad \text{where} \quad E_Z = \operatorname{End}(Z)\,.$$

Prove the following facts:

(a) If $Z$ is perfect, then there exists an almost split triangle in $\mathbf{D}^b(\operatorname{mod}\Lambda)$

$$\Sigma^{-1}\nu Z \longrightarrow Y \longrightarrow Z \xrightarrow{\gamma} \nu Z$$

where $\gamma = \phi_Z(\eta)$ for any non-zero $\eta \in DE_Z$ with $J(E_Z) \subseteq \operatorname{Ker}\eta$.

(b) If $Z$ is not perfect, there is no almost split triangle $X \to Y \to Z \to \Sigma X$ in $\mathbf{D}^b(\operatorname{mod}\Lambda)$.

(c) $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits almost split triangles iff $\Lambda$ has finite global dimension.

(d) $\mathbf{D}^{\mathrm{per}}(\Lambda)$ admits almost split triangles iff $\Lambda$ is Gorenstein.

*Hint for (b):* Any such $Z \to \Sigma X$ would be a phantom map in contradiction to $\operatorname{Ph}(Z, \Sigma X) = 0$.
*Hint for (d):* Almost split triangles in $\mathbf{D}^{\mathrm{per}}(\Lambda)$ remain almost split in $\mathbf{D}^b(\operatorname{mod}\Lambda)$.

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**3.** Verify the following statements about the derived Nakayama functor $\widehat{\nu}$:

(a) $\Lambda$ is symmetric iff $\widehat{\nu} \simeq \mathrm{id}_{\mathbf{D}^-(\operatorname{mod}\Lambda)}$.

(b) $H^i \circ \widehat{\nu} \simeq D\operatorname{Ext}^i_\Lambda(-,\Lambda)$ on $\operatorname{mod}\Lambda$.

(c) If $\Lambda$ is hereditary, then $\widehat{\nu} \simeq D\operatorname{Hom}_\Lambda(-,\Lambda) \oplus \Sigma D\operatorname{Ext}^1_\Lambda(-,\Lambda) \simeq \nu \oplus \Sigma\tau$ on $\operatorname{mod}\Lambda$.