REPRESENTATION THEORY
EXERCISES 11

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1. Let Λ be a ring with Jacobson radical J and set $\overline{\Lambda} = \Lambda/J$. Prove the following:

(a) $P \mapsto P/PJ$ for $P \in \operatorname{proj} \Lambda$ induces a monomorphism $K_0(\Lambda) \xrightarrow{\iota} K_0(\overline{\Lambda})$ of abelian groups.

(b) If Λ is semiperfect, then ι is an isomorphism and the Grothendieck group $K_0(\Lambda)$ is a finitely generated free abelian group.

2. For exact categories $\mathcal{A}$ denote by $\operatorname{tilt} \mathcal{A}$ the poset of equivalence classes of tilting objects in $\mathcal{A}$.

Let k be a field and let Q and Q' be two finite acyclic quivers whose underlying graphs coincide. Prove that there exists a bijection

$$\operatorname{tilt} \operatorname{mod} kQ \xrightarrow{\simeq} \operatorname{tilt} \operatorname{mod} kQ' .$$

Give an explicit description how the structures of these two posets are related to each other in the case that the quiver Q' is obtained from Q by changing the orientation of all arrows at a sink.

Illustrate this description pictorially for a linearly oriented quiver Q of type A_n.

3. Nagata's famous example of a commutative noetherian ring R of infinite Krull dimension arises from the following construction:

(i) Fix a field k and a strictly increasing sequence $d\colon \mathbb{N} \to \mathbb{N}$.

(ii) Take for R the localization of $k[x_0, x_1, x_2, \ldots]$ at the complement of the union of the infinite set of prime ideals $(x_0, \ldots, x_{d_0})$, $(x_{d_0+1}, \ldots, x_{d_1})$, $(x_{d_1+1}, \ldots, x_{d_2})$, $\ldots$.

Verify the two statements below:

(a) R is noetherian.

(b) R has Krull dimension $\dim R = \sup\{d_{n+1} - d_n : n \in \mathbb{N}\}$.

In particular, $\dim R$ is infinite whenever the differences $d_{n+1} - d_n$ are unbounded.

Recall that for right coherent rings Λ the *little* and the *big finitistic dimension* are defined as

$$\begin{aligned} \operatorname{fin.dim} \Lambda &= \{\operatorname{proj.dim} M : M \in \operatorname{mod} \Lambda \text{ with } \operatorname{proj.dim} M < \infty\}, \\ \operatorname{Fin.dim} \Lambda &= \{\operatorname{proj.dim} M : M \in \operatorname{Mod} \Lambda \text{ with } \operatorname{proj.dim} M < \infty\}. \end{aligned}$$

Conclude that for commutative noetherian rings R the following observations are true:

(c) $\operatorname{fin.dim} R$ is not finite in general.

(d) $\operatorname{fin.dim} R$ is finite if R is assumed to be local.

(e) The inequality $\operatorname{fin.dim} R \leq \operatorname{Fin.dim} R$ may be strict even for local R.

Hint for (d) and (e): Use the identity $\operatorname{Fin.dim} R = \dim R$ and the Auslander–Buchsbaum formula.

Finally, consult the literature to find an example for the next statement:

(f) The inequality $\operatorname{fin.dim} \Lambda \leq \operatorname{Fin.dim} \Lambda$ can be strict even for finite-dimensional algebras Λ.

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