Solutions to Exercises 12 – Representation Theory – SS 2020

Exercise 1.

Note that $F = G = \operatorname{Hom}_\Lambda(D\Lambda, -)$ is just the inverse Nakayama functor. Recall also that the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$ looks as follows:

$$\begin{tikzcd}
 & X \arrow[dr, two heads] & \\
P_2 \arrow[ur, hook] & & I_1 \arrow[ll, dashed]
\end{tikzcd}$$

Clearly, it's enough to check that $\mathbf{R}(G \circ F)$ annihilates the injective Λ-module X but $\mathbf{R}G \circ \mathbf{R}F$ doesn't. This follows from the following straightforward computations:

$$\mathbf{R}(G \circ F)(X) = (G \circ F)(X) = G(P_2) = 0$$

$$\begin{aligned}
\mathbf{R}G(\mathbf{R}F(X)) = \mathbf{R}G(F(X)) &= \mathbf{R}G(P_2) \\
&= G(\mathbf{i}(P_2)) = G(X \twoheadrightarrow I_1) = (P_2 \hookrightarrow X) = \Sigma^{-1} I_1 \neq 0
\end{aligned}$$

Exercise 2.

(b) Obviously, α restricts to an endomorphism of $A' = \operatorname{Im}\alpha$.

The map $\operatorname{Ker}\beta\gamma \xrightarrow{\gamma} A$ factors over $A' = \operatorname{Ker}\beta$ and because of $\gamma\beta = 0$ then induces $X' \xrightarrow{\gamma'} A'$.

For the existence of β' apply the snake lemma to the following diagram:

$$\begin{tikzcd}
 & & & A' \arrow[d, equal] & \\
 & X \arrow[r, "\gamma"] \arrow[d, "\beta\gamma"'] & A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta\gamma \arrow[r, equal] \arrow[d, two heads] & \operatorname{Ker}\beta\gamma \arrow[r] & 0 & \\
 & X' & & &
\end{tikzcd}$$

(f) We should read $\bigcup$ as colim and $\bigcap$ as $\lim$. But to simplify the arguments let's cheat and work with elements so that we can realize the operations $\bigcup$ and $\bigcap$ as set union and intersection.

Unravelling the definitions we then have actual identities

$$E_1^p = Z^p/B^p \text{ where } B^p = \frac{(d(X^p) \cap X^p) + X^{p+1}}{X^{p+1}} \text{ and } Z^p = \frac{(d^{-1}(X^{p+1}) \cap X^p) + X^{p+1}}{X^{p+1}}.$$

Furthermore, the subobjects B_r^p and Z_r^p of E_1^p are actual subsets defined by the equations

$$B_r^p = \frac{(d(X^{p-r}) \cap X^p) + X^{p+1}}{X^{p+1}} \Big/ B^p \quad \text{and} \quad Z_r^p = \frac{(d^{-1}(X^{p+1+r}) \cap X^p) + X^{p+1}}{X^{p+1}} \Big/ B^p.$$

Now it's obvious that the requirements for p-convergence are tantamount to the identities

$$B_\infty^p = \frac{(\operatorname{Im} d \cap X^p) + X^{p+1}}{X^{p+1}} \Big/ B^p \quad \text{and} \quad Z_\infty^p = \frac{(\operatorname{Ker} d \cap X^p) + X^{p+1}}{X^{p+1}} \Big/ B^p.$$

It remains to observe that

$$H(X)^p \;=\; \frac{(\operatorname{Ker} d \cap X^p) + \operatorname{Im} d}{\operatorname{Im} d} \;\cong\; \frac{\operatorname{Ker} d \cap X^p}{\operatorname{Im} d \cap X^p}\,.$$

So in the situation of p-convergence, the isomorphism theorems give canonically

$$H(X)^p/H(X)^{p+1} \xrightarrow{\;\sim\;} Z^p_\infty/B^p_\infty \;=\; E^p_\infty\,.$$

COMPARISON WITH WEIBEL'S SPECTRAL SEQUENCE OF A FILTERED COMPLEX.

Up to index conventions, Weibel's spectral sequence $\widetilde{E}$ associated with the differential object X (cf. Constuction 5.4.6) is defined by

$$\widetilde{E}^p_r \;=\; \widetilde{Z}^p_{r-1}/\widetilde{B}^p_{r-1}$$

where

$$\widetilde{B}^p_r \;=\; \frac{(d(X^{p-r}) \cap X^p) + X^{p+1}}{X^{p+1}} \quad \text{and} \quad \widetilde{Z}^p_r \;=\; \frac{(d^{-1}(X^{p+1+r}) \cap X^p) + X^{p+1}}{X^{p+1}}\,.$$

So clearly, we have canonical identifications $\widetilde{A}^p_r \xrightarrow{\sim} A^p_r$ for all $r > 0$ where A is any of E, Z, B.

Let's now try to compare the different notions of convergence. Since Weibel works with bigraded spectral sequences and compatibly graded filtered objects, we'll implicitly fix such data.

Recall that a graded filtered object F, say with filtration $\cdots \subseteq F^{p+1} \subseteq F^p \subseteq \cdots$, is called

- *exhaustive* iff $\bigcup_p F^p = F$ and
- *complete* iff in each degree n the canonical map $F_n \to \lim_p F_n/F^p_n$ is an isomorphism.

Now in Weibel's terminology the bigraded spectral sequence E

- is *regular* iff in all degrees p, q the identity $Z^{p,q}_r = Z^{p,q}_\infty$ holds for sufficiently large r,
- *weakly converges* to F iff $E^{p,q}_\infty \cong F^p_{p+q}/F^{p+1}_{p+q}$ in all degrees p, q and
- *converges to* F iff it weakly converges to F, is regular and F is exhaustive and complete.

There's no problem to make sense of exhaustiveness, completeness and regularity in our setting (but note that requiring these properties totally instead of degreewise could be asking for more).

Turning to the case of interest $F = H(X)$, the previous considerations show that p-convergence of E is equivalent to having canonical isomorphisms $E^p_\infty \to H(X)^p/H(X)^{p+1}$ for all p.

Consequently, p-convergence of E implies that E weakly converges to $H(X)$. Let's think about the converse, i.e. whether "uncanonical" weak convergence to $H(X)$ implies p-convergence ...

(g) We consider the case $X^p = \operatorname{Tot}^p_\to C$.

The obvious isomorphism $(X^p/X^{p+1}, d_\to + d_\uparrow) \xrightarrow{\sim} (C^{\bullet,p}, d_\to)$ of differential objects yields

(♠) $$_\to E^p_1 \;=\; H(X^p/X^{p+1}) \xrightarrow{\;\sim\;} H_\to(C^{\bullet,p})\,.$$

To check that $_\to d^p_1$ identifies with $H_\to(d^{\bullet,p}_\uparrow)$ take any $x \in \operatorname{Ker} d^{\bullet,p}_\to$. Since $_\to d^p_1$ is the connecting morphism in cohomology composed with the canonical projection, we have

(◇) $$_\to d^p_1([x]_p) \;=\; [d_\uparrow(x) + d_\to(x)]_{p+1} \;=\; [d_\uparrow(x)]_{p+1}\,.$$

So (♠) provides an isomorphism of differential objects $_\to E_1 \to (H_\to(C^{\bullet,\bullet}), H_\to(d^{\bullet,\bullet}_\uparrow))$ and thus

(♣) $$_\to E_2 \xrightarrow{\;\sim\;} H_\uparrow(H_\to(C^{\bullet,\bullet}))\,.$$

Now denote by (α, β, γ) the exact couple from (d) and (e) that induces ${}_{\to}E$. With the choice of the "bigrading" as indicated in the exercise, it is clear that β has degree $(0, 0)$, γ has degree $(1, 0)$ and α has degree $(-1, 1)$. Since the map ${}_{\to}d_r$ is given by $\beta\alpha^{-r+1}\gamma$, it has degree $(r, -r + 1)$.

Considering $\operatorname{Tot} C$ with its grading by total degree, all of (♠), (♣), (♢) restrict to homogeneous components as claimed in the exercise.

Exercise 3.

The canonical embedding $\mathbf{K}^+(\operatorname{Inj}\mathcal{B}) \xrightarrow{\mathbf{q}_\mathcal{B}} \mathbf{D}^+(\mathcal{B})$ induces a natural transformation $G \xrightarrow{\xi^G} \mathbf{R}G$ because of $\mathbf{R}G \circ \mathbf{q}_\mathcal{B} = G \circ \mathbf{i}_\mathcal{B} \circ \mathbf{q}_\mathcal{B} = G$. In turn ξ^G then gives rise to the natural transformation

$$\mathbf{R}(G \circ F) = G \circ F \circ \mathbf{i}_\mathcal{A} \xrightarrow{\quad \xi^{G,F} \quad} \mathbf{R}G \circ F \circ \mathbf{i}_\mathcal{A} = \mathbf{R}G \circ \mathbf{R}F\,.$$

(b) If $F(I)$ is not right G-acyclic for some $I \in \operatorname{Inj}\mathcal{A}$, then $\xi^{G,F}_I = \xi^G_{F(I)}$ is not invertible by (a).

Vice versa, assume now that all objects in $F(\operatorname{Inj}\mathcal{A})$ are right G-acyclic. Take any $X \in \mathbf{D}^+(\mathcal{A})$. We must show that $\xi^{G,F}_X$ is invertible.

Let $Y = F(\mathbf{i}_\mathcal{A}(X))$. Using (c) or just recalling it, we may assume that the unit map $Y \xrightarrow{\eta_Y} \mathbf{i}_\mathcal{B}(Y)$ is induced by a map of complexes. Then we have a short exact sequence of complexes

$$0 \longrightarrow \mathbf{i}_\mathcal{B}(Y) \longrightarrow \operatorname{Cone}(\eta_Y) \longrightarrow \Sigma Y \longrightarrow 0\,.$$

We see that $\operatorname{Cone}(\eta_Y)$ is an acyclic complex of G-acyclic objects, since injective objects in $\mathcal{B}$ are G-acyclic and Y is a complex of G-acyclics by assumption and since η_Y is a quasi-isomorphism.

Clearly, short exact sequences $0 \to U \to V \to W \to 0$ in $\mathcal{B}$ with G-acyclic U stay exact when applying G. Using induction this implies that acyclic complexes in $\mathbf{C}^+(\mathcal{B})$ of G-acyclic objects stay acyclic when applying G. In particular, this means that $G(\operatorname{Cone}(\eta_Y))$ is acyclic and that

$$0 \longrightarrow G(\mathbf{i}_\mathcal{B}(Y)) \longrightarrow G(\operatorname{Cone}(\eta_Y)) \longrightarrow \Sigma G(Y) \longrightarrow 0$$

still is exact. Therefore the map $\mathbf{R}(G\circ F)(X) = G(Y) \longrightarrow G(\mathbf{i}_\mathcal{B}(Y)) = \mathbf{R}G(\mathbf{R}F(X))$ obtained from this short exact sequence is a quasi-isomorphism. But of course this map is just $\xi^{G,F}_X$.

(e) This follows from Exercise 2 (b) if we can show that

$$(H^q_{\to}(F(C^{\bullet,\bullet})), H^q_{\to}(F(d^{\bullet,\bullet}_{\uparrow}))) \;\cong\; (F(H^q_{\to}(C^{\bullet,\bullet})), F(H^q_{\to}(d^{\bullet,\bullet}_{\uparrow})))\,.$$

And this is clear once we know that $(C^{\bullet,p}, d^{\bullet,p}_{\to})$ is split in the sense that it's a direct sum of shifts of objects and isomorphisms in $\mathcal{A}$ viewed as complexes concentrated in at most two degrees.

Indeed, setting $(I^\bullet, d^\bullet) = (C^{\bullet,p}, d^{\bullet,p}_{\to})$ and $K^q = \operatorname{Ker} d^q$, $L^q = \operatorname{Im} d^{q-1}$, $H^q = K^q/L^q$, we note that in particular L^q and K^q are injective since C was chosen as a Cartan-Eilenberg resolution. As a consequence, we can find splittings s_q and t_q for the exact sequences below:

$$\begin{tikzcd} 0 \arrow[r] & L^q \arrow[r] & K^q \arrow[r] & H^q \arrow[l, bend left, "s_q"] \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & K^q \arrow[r] & I^q \arrow[r] & L^{q+1} \arrow[l, bend left, "t_q"] \arrow[r] & 0 \end{tikzcd}$$

We can thus decompose $I^q = L^q \oplus \operatorname{Im} s_q \oplus \operatorname{Im} t_q$ so that the differential d^q takes the form

$$\begin{pmatrix} 0 & 0 & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

As desired, this proves that in $\mathbf{K}^+(\operatorname{Inj}\mathcal{A})$ the complex $(I^\bullet, d^\bullet)$ can be written as the direct sum of appropriate shifts of the objects $\operatorname{Im} s_q$ and the isomorphisms $\operatorname{Im} t_q \to L^{q+1}$.