REPRESENTATION THEORY
EXERCISES 6

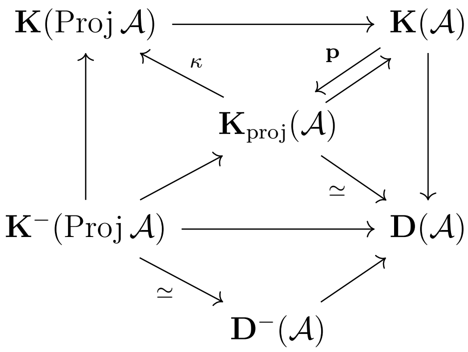
HENNING KRAUSE
JAN GEUENICH

1. Let $\mathcal{A}$ be an abelian category with exact countable coproducts and with enough projectives. Convince yourself of the correctness of the facts below, partly dual to results from the lecture:

(a) The inclusion $\mathbf{K}_{\mathrm{proj}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ admits a right adjoint $\mathbf{p}$ with image in $\mathbf{K}(\operatorname{Proj}\mathcal{A})$.

(b) For each $X \in \mathbf{K}(\mathcal{A})$ the cone of the counit map $\mathbf{p}X \to X$ of this adjunction is acyclic.

(c) We have the following commutative diagram of functors:



Apart from κ and $\mathbf{p}$ all the arrows are the canonical functors. The left and the upper triangle commute up to natural isomorphism and the rest of the diagram commutes on the nose.

(d) If $\mathcal{A}$ has finite projective dimension, then κ is an equivalence.

Conclude that for every ring Λ of finite global dimension there are exact equivalences:

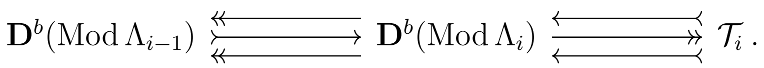
$$\mathbf{K}(\operatorname{Inj}\Lambda) \;\cong\; \mathbf{D}(\operatorname{Mod}\Lambda) \;\cong\; \mathbf{K}(\operatorname{Proj}\Lambda)$$

2. (a) Find an example of a (necessarily non-commutative) noetherian ring Λ and a Serre subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ such that the induced functor $\mathbf{D}^b(\mathcal{C}) \to \mathbf{D}^b(\operatorname{mod}\Lambda)$ is not fully faithful.

(b) Let Λ be a right coherent ring. Prove that the canonical functor $\mathbf{D}^b(\operatorname{mod}\Lambda) \to \mathbf{D}^b(\operatorname{Mod}\Lambda)$ is fully faithful with essential image formed by those X with $H^nX \in \operatorname{mod}\Lambda$ for all $n \in \mathbb{N}$.

3. Let Λ be a quasi-hereditary ring with heredity chain $\Lambda = \Lambda_n \to \Lambda_{n-1} \to \cdots \to \Lambda_1 \to \Lambda_0 = 0$. Recall that this means that Λ is semiprimary and that the kernels of $\Lambda_i \to \Lambda_{i-1}$ are heredity ideals.

Recall or verify that $\Lambda_i \to \Lambda_{i-1}$ are homological epimorphisms and conclude that they give rise to recollements of triangulated categories



Prove that the triangulated categories $\mathcal{T}_i$ are abelian.

To be handed in via email by June 1, 2020 (Whit Monday), 2 p.m.